The local motivic DT/PT correspondence

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Abstract

We show that the Quot scheme $Q_L^n = \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{I}_L, n)$ parameterising length n quotients of the ideal sheaf of a line in $\mathbb{A}^3$ is a global critical locus, and calculate the resulting motivic partition function (varying n), in the ring of relative motives over the configuration space of points in $\mathbb{A}^3$. As in the work of Behrend–Bryan–Szendrői, this enables us to define a virtual motive for the Quot scheme of n points of the ideal sheaf $\mathscr{I}_C \subset \mathscr{O}_Y$, where $C \subset Y$ is a smooth curve embedded in a smooth 3-fold Y, and we compute the associated motivic partition function. The result fits into a motivic wall-crossing type formula, refining the relation between Behrend's virtual Euler characteristic of $\mathrm{Quot}_Y(\mathscr{I}_C, n)$ and of the symmetric product $\mathrm{Sym}^n C$. Our 'relative' analysis leads to results and conjectures regarding the pushforward of the sheaf of vanishing cycles along the Hilbert–Chow map $Q_L^n \to \mathrm{Sym}^n(\mathbb{A}^3)$, and connections with cohomological Hall algebra representations.

Contents

1. Introduction

1.1. Overview

Let C be a smooth curve embedded in a smooth 3-fold Y with ideal sheaf $\mathscr{I}_C \subset \mathscr{O}_Y$. For an integer $n \geqslant 0$, the Quot scheme

$$Q_C^n = \mathrm{Quot}_Y(\mathscr{I}_C, n)$$

parameterises closed subschemes $Z \subset Y$ containing C and differing from it by an effective zero-cycle of length n. The main purpose of this paper is to construct a *virtual motive*

$$\left[Q_C^n\right]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}} \tag{1}$$

for this Quot scheme, that we view as a 1-dimensional analogue of the degree 0 *motivic Donaldson–Thomas invariant* $[\mathrm{Hilb}^n Y]_{\mathrm{vir}}$ defined by Behrend, Bryan and Szendrői [3].

The Quot scheme Q_C^n can be seen as a moduli space of *curves and points* in Y, where the curve C is fixed. This geometric situation presents a new feature that was absent in the purely 0-dimensional case: wall-crossing. More precisely, it is proved in [32, Proposition 5.1] that the generating function of the Behrend weighted Euler characteristics $\widetilde{\chi}(Q_C^n)$ satisfies the wall-crossing type formula

$$\sum_{n \geqslant 0} \widetilde{\chi}(Q_C^n) t^n = M(-t)^{\chi(Y)} \cdot (1+t)^{-\chi(C)}, \tag{2}$$

where $M(t) = \prod_{m \geqslant 1}(1 - t^m)^{-m}$ is the MacMahon function. We show that the motivic partition function encoding the motivic classes (0.1) admits a factorisation similar to (0.2), where the *point contribution*, refining the factor $M(-t)^{\chi(Y)}$, is precisely the motivic partition of the Hilbert schemes $\mathrm{Hilb}^n Y$ computed in [3]. The *curve contribution*, on the other hand, refines the factor $(1+t)^{-\chi(C)}$ and is given by the (shifted) motivic zeta function of the curve C, namely

$$\sum_{n \geqslant 0} \mathbb{L}^{-\frac{n}{2}} \left[\mathrm{Sym}^n C \right]_{\mathrm{vir}} t^n. \tag{3}$$

Our approach to the problem is a natural extension of the approach of Behrend, Bryan and Szendrői, in that our definitions and calculations take place with respect to the natural local model $L \subset \mathbb{A}^3$ given by fixing a line in affine space — since they consider only finite-dimensional quotients of $\mathscr{O}_{\mathbb{A}^3}$, their local model is simply $\mathbb{A}^3$. As in their work, for general embeddings $C \subset Y$, we build $[Q_C^n]_{\mathrm{vir}}$ out of the local model via power structures. We leave for another day the question of whether this virtual motive accords with the virtual motive one obtains from the machinery of (-1)-shifted symplectic stacks, and concentrate on calculating everything in sight for the local theory. Furthermore, since the key to gluing local models appears to be the direct image of the vanishing cycles sheaf to the configuration space of points on $\mathbb{A}^3$, we prove all of our results in the lambda ring of motives relative to this configuration space. We conjecture, moreover, that the wall-crossing type identity (0.2) can be categorified, that is, lifted to an isomorphism between the vanishing cycle cohomologies of the relevant moduli spaces (see Section 5.1).

Just as Behrend, Bryan and Szendrői realise the local model $\mathrm{Hilb}^n \mathbb{A}^3$ as a *critical locus* and show that the associated motivic Donaldson–Thomas invariants $[\mathrm{Hilb}^n \mathbb{A}^3]_{\mathrm{vir}}$ are determined, via power structures, by the motivic weights of the punctual Hilbert schemes $\mathrm{Hilb}^n(\mathbb{A}^3)_0$, we show, for two convenient local models that can also be realised as critical loci, that the induced virtual motives are determined by the motivic contribution of the punctual loci. In our case, we also need to consider the contribution of points *embedded* on the curve $C \subset Y$, and this is what gives rise to the factor (0.3) in our motivic wall-crossing formula.

The appearance of symmetric products is pretty natural and has a neat interpretation in terms of the (C-local) DT/PT correspondence: on a Calabi–Yau 3-fold Y, the symmetric products $\mathrm{Sym}^n C \subset P_{\chi(\mathscr{O}_C)+n}(Y, [C])$ are precisely the C-local moduli spaces in the stable pair theory of Y, just as $Q_C^n \subset I_{\chi(\mathscr{O}_C)+n}(Y, [C])$ are the C-local moduli spaces in Donaldson–Thomas theory.

For a rigid curve $C \subset Y$ in a Calabi–Yau 3-fold, one can interpret the classes (0.1) as motivic Donaldson–Thomas invariants, in the same spirit as in the 0-dimensional case.

We next give an overview of our main results. The main technical tool we use is a *motivic stratification* technique, that we apply to the study of the (relative) motivic Donaldson–Thomas invariants of the Quot schemes $Q_{C_0}^n$, where $C_0 \subset X$ is the exceptional curve in the resolved conifold $X = \mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1))$.

Main results

The first step towards the construction of the motivic classes (0.1) consists in setting up a convenient local model. With respect to the local model

$$L \subset \mathbb{A}^3,$$

we then prove the following as part of Theorem 2.2.

THEOREM A. *The Quot scheme Q_L^n is a global critical locus.*

An analogous statement is proven in [**3**, Proposition 3.1] for the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^3)$, which is realised as the critical locus of a function on the *non-commutative Hilbert scheme*.
   Via the theory of motivic vanishing cycles [**12**], Theorem A produces a relative virtual motive

$$\mathsf{Q}_{L/\mathbb{A}^3}^{\mathrm{rel}} = \sum_{n \geqslant 0} (-1)^n \big[Q_L^n \xrightarrow{\mathbf{hc}_n} \mathrm{Sym}^n \mathbb{A}^3 \big]_{\mathrm{vir}} \in \mathcal{M}_{\mathrm{Sym}(\mathbb{A}^3)},$$

where the maps $\mathbf{hc}_n$ are Hilbert–Chow morphisms. The following result, proven in Section 4.2, follows from Corollary 4.6 and the main calculation of Section 3.

THEOREM B. *There is an identity*

$$\mathsf{Q}_{L/\mathbb{A}^3}^{\mathrm{rel}} = \mathrm{Exp}_{\cup} \left(\sum_{n>0} \Delta_{n\,!} \Big(\big[\mathbb{A}^3 \xrightarrow{\mathrm{id}} \mathbb{A}^3 \big] \boxtimes \Omega_n^{\mathrm{BBS}} \Big) \right) \boxtimes_{\cup} \mathrm{Exp}_{\cup} \left(\Delta_{1\,!} \Big(\big[L \hookrightarrow \mathbb{A}^3 \big] \boxtimes \big(-\mathbb{L}^{-\frac{1}{2}} \big) \Big) \right),$$

$$(4)$$

where $\Delta_n \colon \mathbb{A}^3 \to \mathrm{Sym}^n \mathbb{A}^3$ is the small diagonal, and

$$\Omega_n^{\mathrm{BBS}} = (-1)^n \mathbb{L}^{-\frac{3}{2}} \frac{\mathbb{L}^{\frac{n}{2}} - \mathbb{L}^{-\frac{n}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \in \mathcal{M}_{\mathbb{C}}.$$

Passing to absolute motives, the first factor in (0.4) becomes the (signed) motivic partition function of $\mathrm{Hilb}^n(\mathbb{A}^3)$ computed in [**3**] and reviewed in Section 1.7. The operation $\mathrm{Exp}_{\cup}$ in Theorem B is a lift to the lambda ring $\mathcal{M}_{\mathrm{Sym}(\mathbb{A}^3)}$ of the usual plethystic exponential for power series with coefficients in the ring of absolute motives. These motivic exponentials are reviewed in Section 1.5.1. We let

$$\mathsf{Q}_{L/\mathbb{A}^3}(t) = \sum_{n \geqslant 0} \big[Q_L^n \big]_{\mathrm{vir}} \cdot t^n$$

be the absolute partition function. Up to a sign, it is obtained by pushing (0.4) forward to a point. The absolute version of Theorem B then reads

$$\mathsf{Q}_{L/\mathbb{A}^3}(-t) = \mathrm{Exp} \left(\frac{-\mathbb{L}^{\frac{3}{2}} t}{\big(1 + \mathbb{L}^{-\frac{1}{2}} t \big) \big(1 + \mathbb{L}^{\frac{1}{2}} t \big)} - \mathbb{L}^{\frac{1}{2}} t \right). \tag{5}$$

Let $\mathcal{P}_{\mathrm{crv}}^n \subset Q_L^n$ be the closed subset parameterising quotients $\mathscr{I}_L \twoheadrightarrow \mathscr{F}$ such that $\mathscr{F}$ is entirely supported at the origin $0 \in L$. We assign a motivic weight

$$\big[\mathcal{P}_{\mathrm{crv}}^n \big]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}$$

to this locus. The subscript 'crv' stands for 'curve'. The punctual Hilbert scheme $\mathcal{P}_{\mathrm{pt}}^n \subset Q_L^n$, which we view as parameterising quotients supported at a single point in $\mathbb{A}^3 \setminus L$, also inherits a motivic weight, that agrees (as shown in Proposition 4.3) with the class $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}$ defined in [**3**] starting from the critical structure on $\mathrm{Hilb}^n(\mathbb{A}^3)$. We show in Theorem 4.7 that $[Q_L^n]_{\mathrm{vir}}$ is

determined by these two types of 'punctual' motivic classes. They moreover allow us to define a virtual motive $[Q_C^n]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}$ for every smooth curve $C \subset Y$ in a smooth quasi-projective 3-fold Y. In other words, the class we define satisfies $\chi[Q_C^n]_{\mathrm{vir}} = \widetilde{\chi}(Q_C^n)$.

We then consider the generating function

$$\mathsf{Q}_{C/Y}(t) = \sum_{n \geqslant 0} \left[Q_C^n\right]_{\mathrm{vir}} \cdot t^n. \tag{6}$$

For a smooth quasi-projective variety X of dimension at most 3, let

$$\mathsf{Z}_X(t) = \sum_{n \geqslant 0} \left[\mathrm{Hilb}^n X\right]_{\mathrm{vir}} \cdot t^n$$

be the motivic partition function of the Hilbert scheme of points. In Theorem 4.9, we prove the following explicit formula, generalising (0.5).

THEOREM C. *Let Y be a smooth quasi-projective 3-fold, $C \subset Y$ a smooth curve. Then*

$$\left[Q_C^n\right]_{\mathrm{vir}} = \sum_{j=0}^{n} \left[\mathrm{Hilb}^{n-j} Y\right]_{\mathrm{vir}} \cdot \left[\mathrm{Sym}^j C\right]_{\mathrm{vir}} \tag{7}$$

in $\mathcal{M}_{\mathbb{C}}$. In other words, we have a factorisation

$$\mathsf{Q}_{C/Y} = \mathsf{Z}_Y \cdot \mathsf{Z}_C,$$

that, rewritten in terms of the motivic exponential, reads

$$\mathsf{Q}_{C/Y}(-t) = \mathrm{Exp}\left(-t[Y]_{\mathrm{vir}} \, \mathrm{Exp}(-t[\mathbb{P}^1]_{\mathrm{vir}}) - t[C]_{\mathrm{vir}}\right).$$

In the above formulas, one has $[U]_{\mathrm{vir}} = \mathbb{L}^{-(\dim U)/2}[U] \in \mathcal{M}_{\mathbb{C}}$ for a smooth scheme U. One can view the factorisation $\mathsf{Q}_{C/Y} = \mathsf{Z}_Y \cdot \mathsf{Z}_C$ as a motivic refinement of the identity (0.2). Indeed, we have $M(-t)^{\chi(Y)} = \chi \mathsf{Z}_Y(t)$, and $(1+t)^{-\chi(C)} = \sum_n \widetilde{\chi}(\mathrm{Sym}^n C)t^n = \chi \mathsf{Z}_C(t)$. The relation (0.2) says that Quot schemes and symmetric products are related by a $\widetilde{\chi}$-weighted wall-crossing type formula, and Theorem C upgrades this statement to the motivic level.

Calabi–Yau 3-folds

Let Y be a smooth projective Calabi–Yau 3-fold. For an integer $m \in \mathbb{Z}$ and a homology class $\beta \in H_2(Y, \mathbb{Z})$, the moduli space $I_m(Y, \beta)$ of ideal sheaves $\mathscr{I}_Z \subset \mathscr{O}_Y$ with Chern character $(1, 0, -\beta, -m)$ carries a symmetric perfect obstruction theory and the Donaldson–Thomas invariant $\mathsf{DT}_\beta^m \in \mathbb{Z}$ is by definition the degree of the associated virtual fundamental class. These invariants are related to the stable pair invariants of Pandharipande–Thomas [**25**] by a well-known wall-crossing formula [**6, 41**], and the same is true for the C-local invariants $\mathsf{DT}_C^n \in \mathbb{Z}$. The numbers $\mathsf{DT}_C^\bullet$ represent the *contribution* of C to the full virtual invariants $\mathsf{DT}_{[C]}^\bullet$. The C-local wall-crossing formula [**31**, Theorem 1.1], written term by term, reads

$$\mathsf{DT}_C^n = \sum_{j=0}^{n} \mathsf{DT}_0^{n-j} \cdot \mathsf{PT}_C^j, \tag{8}$$

where $\mathsf{DT}_0^k = \widetilde{\chi}(\mathrm{Hilb}^k Y)$ are the degree 0 DT invariants of Y, $\mathsf{PT}_C^j = n_{g,C} \cdot \widetilde{\chi}(\mathrm{Sym}^j C)$ are the C-local stable pair invariants of Y and $n_{g,C}$ is the BPS number of C (see [**26**] and Section 4.3). There is an identity $\chi[Q_C^n]_{\mathrm{vir}} = \mathsf{DT}_C^n$ when $n_{g,C} = 1$ (Corollary 4.10). Indeed, in this case, equation (0.8) is equivalent to (0.2). This is especially meaningful from the point of view of motivic DT theory in the situation of the following example.

EXAMPLE 0.1. Assume $C \subset Y$ is a smooth *rigid* curve, that is, $H^0(C, N_{C/Y}) = 0$. Then C has BPS number 1, the Quot scheme Q_C^n is a *connected component* of the Hilbert scheme $I_{\chi(\mathscr{O}_C)+n}(Y, [C])$, and the motivic class $[Q_C^n]_{\mathrm{vir}}$ is a *motivic Donaldson–Thomas invariant* in the sense that its Euler characteristic computes the degree of the virtual fundamental class of Q_C^n. In this case, the formula $\mathsf{Q}_{C/Y} = \mathsf{Z}_Y \cdot \mathsf{Z}_C$ of Theorem C can be regarded as a C-local motivic DT/PT correspondence, refining the enumerative correspondence $\mathsf{DT}_C = \mathsf{DT}_0 \cdot \mathsf{PT}_C$ spelled out in (0.8).

Organisation of contents

The paper is organised as follows. In Section 1, we recall foundational material on rings of motivic weights and we revisit the main formula of [3] expressing the virtual motive of $\mathrm{Hilb}^n X$ for 3-folds. In Section 2, we prove Theorem A by restricting the critical structure on $Q_{C_0}^n$, where $C_0 \cong \mathbb{P}^1$ is the exceptional curve in the resolved conifold $X = \mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1))$. In Section 3, we prove that the virtual motives of Q_L^n and $Q_{C_0}^n$ are determined by motivic classes Ω_{pt}^n, Ω_{crv}^n expressing the contributions of 'fully punctual loci' (cf. Definition 3.16 and Theorem 3.17). By explicitly calculating these motives in Section 4.2, we finally prove Theorem B. We then use these classes to *define* (cf. Definition 4.8) a virtual motive of Q_C^n for every smooth curve C in a smooth 3-fold Y, and in Section 4.2 we also prove Theorem C.

2. Background material

In this section, we set up the notation and introduce the main tools that will be used in the rest of the paper.

2.1. Grothendieck rings of varieties

DEFINITION 1.1. Let S be a locally finite type algebraic space over $\mathbb{C}$.

(i) If S is a variety, the *Grothendieck group of S-varieties* is the free abelian group $K_0(\mathrm{Var}_S)$ generated by isomorphism classes $[X \to S]$ of finite type varieties over S, modulo the scissor relations, namely the identities $[p\colon Y \to S] = [p|_X\colon X \to S] + [p|_{Y \setminus X}\colon Y \setminus X \to S]$ whenever $X \hookrightarrow Y$ is a closed S-subvariety of Y. For general S, we impose the *locality* relation $[f\colon X \to S] = [g\colon X' \to S]$ if for all varieties $U \subset S$ there is an identity $[f|_U\colon X \times_S U \to U] = [g|_U\colon X' \times_S U \to U]$ in $K_0(\mathrm{Var}_U)$. The group $K_0(\mathrm{Var}_S)$ is a ring via $[Y \to S] \cdot [Z \to S] = [Y \times_S Z \to S]$.

(ii) We denote by $\mathbb{L} = [\mathbb{A}_S^1] \in K_0(\mathrm{Var}_S)$ the *Lefschetz motive*, the class of the affine line over S.

(iii) The *Grothendieck group of S-stacks* is the free abelian group $K_0(\mathrm{St}_S^{\mathrm{aff}})$ generated by isomorphism classes $[X \to S]$ of locally finite type Artin S-stacks $X \to S$ with affine stabilisers, modulo the scissor and locality relations, and the following additional relation: if $f\colon X \to S$ is an S-stack, such that f factors as $g \circ \pi$ for $g\colon Y \to S$ an S-stack and π the projection from the total space of a rank r vector bundle, then

$$[X \xrightarrow{f} S] = \mathbb{L}^r \cdot [Y \xrightarrow{g} S].$$

(iv) Define the group $K(\mathrm{Var}_S) = \mathrm{Image}(K_0(\mathrm{Var}_S) \to K_0(\mathrm{St}_S^{\mathrm{aff}}))$, and give it the induced ring structure.

Where $S = \coprod_{i \in I} S_i$ is a possibly infinite union of algebraic spaces, we will write

$$\sum_{i \in I} [X_i \xrightarrow{f_i} S_i] := \left[\coprod_{i \in I} (X_i \xrightarrow{f_i} S_i)\right].$$

By results of Kresch [**18**, Section 4], we have

$$K_0(\mathrm{St}_S^{\mathrm{aff}}) = K_0(\mathrm{Var}_S)\big[\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1} \,\big|\, n \geqslant 1\big]$$

and so we can alternatively define $K(\mathrm{Var}_S)$ as the quotient of $K_0(\mathrm{Var}_S)$ by the ideal

$$\mathsf{J}_S = \ker(\cdot\mathbb{L}) + \sum_{n \geqslant 1} \ker(\cdot(\mathbb{L}^n - 1)) \subset K_0(\mathrm{Var}_S). \tag{1.1}$$

For S and S' two varieties, there is an external product

$$K(\mathrm{Var}_S) \times K(\mathrm{Var}_{S'}) \xrightarrow{\boxtimes} K(\mathrm{Var}_{S \times S'}),$$

defined on generators by

$$[g\colon Y \to S] \boxtimes [h\colon Z \to S'] = [g \times h\colon Y \times Z \to S \times S'].$$

In particular, $K(\mathrm{Var}_S)$ is a $K(\mathrm{Var}_{\mathbb{C}})$-module. When we are considering the action of absolute motives on relative motives, we will often abbreviate

$$[X][X' \xrightarrow{f} S'] = [X \to \mathrm{pt}] \boxtimes [X' \xrightarrow{f} S']$$

$$= [X \times X' \xrightarrow{f \circ \pi_{X'}} S'].$$

Often for a relative motive $[X \to S] \in K(\mathrm{Var}_S)$ we will denote it by $[X]_S$, retaining the subscript to at least remind the reader of which motivic ring it lives in.

Given a morphism $f\colon S \to T$ of varieties, there is an induced *pullback* map

$$f^*\colon K(\mathrm{Var}_T) \to K(\mathrm{Var}_S),$$

which is a ring homomorphism given by $f^*[X]_T = [X \times_T S]_S$ on generators. Composition with f defines a *direct image* homomorphism $f_!\colon K(\mathrm{Var}_S) \to K(\mathrm{Var}_T)$, which is $K(\mathrm{Var}_T)$-linear.

If S comes with an associative map $\nu\colon S \times S \to S$, we define the convolution ring structure via $\boxtimes_\nu = \nu_! \circ \boxtimes$, that is, we set

$$A \boxtimes_\nu B = \nu_!(A \boxtimes B) \in K(\mathrm{Var}_S). \tag{1.2}$$

The resulting associative product on $K(\mathrm{Var}_S)$ is commutative if ν commutes with the symmetrising isomorphism.

The ring

$$\mathcal{M}_S = K(\mathrm{Var}_S)\big[\mathbb{L}^{-\frac{1}{2}}\big]$$

is called the *ring of motivic weights* over S. The structures f^*, $f_!$, $\boxtimes$ and $\boxtimes_\nu$ carry over to $\mathcal{M}_S$ without change. When $f\colon S \to \mathrm{Spec}\,\mathbb{C}$ is the structure morphism of S, we use the special notation $\int_S$ for the pushforward $f_!$.

DEFINITION 1.2. We define $S_0(\mathrm{Var}_S) \subset \mathcal{M}_S$ to be the sub semigroup of *effective* motives, that is, the subset of sums of elements of the form

$$(-\mathbb{L}^{\frac{1}{2}})^n [X \to S].$$

REMARK 1.3. By Definition 1.2, the motive $-\mathbb{L}^{1/2}$ is effective, as opposed to $\mathbb{L}^{1/2}$. This is dictated by the fact that in the language of lambda rings (Section 1.3), we make definitions so that $\mathbb{L}^{1/2}$ is not a line element, while $-\mathbb{L}^{1/2}$ is.

2.2. *Equivariant K-groups, quotient and power maps*

Let G be a finite group. A G-action on a variety X is said to be *good* if every point of X has a G-invariant affine open neighbourhood; all actions are assumed to be good throughout. For instance, any G-action on a quasi-projective variety is good. Moreover, for a good G-action, an orbit space X/G exists as a variety.

DEFINITION 1.4. Let S be a variety with good G-action. We let $\widetilde{K}_0^G(\mathrm{Var}_S)$ denote the abelian group generated by isomorphism classes $[X \to S]$ of G-equivariant S-varieties, modulo the G-scissor relations (over S). The *equivariant Grothendieck group* $K_0^G(\mathrm{Var}_S)$ is defined by imposing the further relations $[V \to X \to S] = [\mathbb{A}_X^r]$, whenever $V \to X$ is a G-equivariant vector bundle of rank r, with $X \to S$ a G-equivariant S-variety. The element $[\mathbb{A}_X^r]$ in the right-hand side is taken with the G-action induced by the trivial action on $\mathbb{A}^r$ and the isomorphism $\mathbb{A}_X^r = \mathbb{A}^r \times X$.

There is a natural ring structure on $\widetilde{K}_0^G(\mathrm{Var}_S)$ given by fibre product; if X and Y are G-equivariant S-varieties, we give $X \times_S Y$ the diagonal G-action.

We shall consider the quotient rings

$$\widetilde{K}_0^G(\mathrm{Var}_S) \twoheadrightarrow \widetilde{K}^G(\mathrm{Var}_S), \quad K_0^G(\mathrm{Var}_S) \twoheadrightarrow K^G(\mathrm{Var}_S)$$

obtained by modding out the ideal

$$\widetilde{\mathsf{J}}^G = \ker(\cdot \mathbb{L}) + \sum_{n \geqslant 1} \ker(\cdot(\mathbb{L}^n - 1)) \subset \widetilde{K}_0^G(\mathrm{Var}_S)$$

and its image $\mathsf{J}^G \subset K_0^G(\mathrm{Var}_S)$, respectively. We let

$$\widetilde{\mathcal{M}}_S^G = \widetilde{K}^G(\mathrm{Var}_S)\big[\mathbb{L}^{-\frac{1}{2}}\big], \quad \mathcal{M}_S^G = K^G(\mathrm{Var}_S)\big[\mathbb{L}^{-\frac{1}{2}}\big]$$

be the rings of *G-equivariant motivic weights*.

There exists a natural 'quotient map'

$$\pi_G \colon \widetilde{K}_0^G(\mathrm{Var}_S) \to K_0(\mathrm{Var}_{S/G}), \tag{1.3}$$

defined on generators by taking the orbit space:

$$\pi_G[X \to S] = [X/G \to S/G].$$

If the G-action on S is trivial, $\widetilde{K}_0^G(\mathrm{Var}_S)$ becomes a $K_0(\mathrm{Var}_S)$-algebra, and π_G is $K_0(\mathrm{Var}_S)$-linear. More generally, we have the following:

LEMMA 1.5. *The map* (1.3) *is $K_0(\mathrm{Var}_{S/G})$-linear.*

Proof. The action of a generator $u = [U \to S/G] \in K_0(\mathrm{Var}_{S/G})$ on a G-equivariant motive $x = [h \colon X \to S] \in \widetilde{K}_0^G(\mathrm{Var}_S)$ is given by

$$u \cdot x = h_! h^* q^*(u) = [U \times_{S/G} X \xrightarrow{\mathrm{pr}_2} X \xrightarrow{h} S],$$

where $q \colon S \to S/G$ is the quotient map. We have

$$u \cdot \pi_G(x) = u \cdot [X/G \to S/G] = [U \times_{S/G} X/G \to X/G \to S/G],$$

and this is the same motive as $\pi_G(u \cdot x) = \pi_G[U \times_{S/G} X \to X \to S]$, since G does not act on U. □

By Lemma 1.5, the map (1.3) sends the ideal $\widetilde{\mathsf{J}}^G$ onto the ideal $\mathsf{J}_{S/G} \subset K_0(\mathrm{Var}_{S/G})$ defined in (1.1), therefore it descends to a $K(\mathrm{Var}_{S/G})$-linear map

$$\pi_G \colon \widetilde{K}^G(\mathrm{Var}_S) \to K(\mathrm{Var}_{S/G}).$$

This map extends to a map $\widetilde{\mathcal{M}}_S^G \to \mathcal{M}_{S/G}$, still denoted π_G, by setting $\pi_G(\mathbb{L}^{n/2} \cdot [X \to S]) = \mathbb{L}^{n/2} \cdot \pi_G([X \to S])$.

Furthermore, by [**4**, Lemma 3.2], if the G-action on S is free,[†] π_G descends to a $K(\mathrm{Var}_{S/G})$-linear map

$$\pi_G \colon K^G(\mathrm{Var}_S) \to K(\mathrm{Var}_{S/G}),$$

which again extends to a morphism $\pi_G \colon \mathcal{M}_S^G \to \mathcal{M}_{S/G}$.

Let $\mathfrak{S}_n$ be the symmetric group on n elements.

LEMMA 1.6 [**3**, Lemma 2.4]. *For every $n > 0$, there exists an nth power map*

$$(\,\cdot\,)^{\otimes n} \colon \mathcal{M}_S \to \widetilde{\mathcal{M}}_{S^n}^{\mathfrak{S}_n},$$

where S^n carries the natural $\mathfrak{S}_n$-action, defined by the property that for

$$T = (-\mathbb{L}^{\frac{1}{2}})^\alpha \cdot [A \xrightarrow{f} S] + \mathbb{L}^\beta \cdot [B \xrightarrow{g} S] - (-\mathbb{L}^{\frac{1}{2}})^\gamma \cdot [C \xrightarrow{h} S] - \mathbb{L}^\delta \cdot [D \xrightarrow{i} S] \in \mathcal{M}_S,$$

we have

$$T^{\otimes n} := \sum_{a+b+c+d=n} (-1)^{c+d} (-\mathbb{L}^{\frac{1}{2}})^{a\alpha/2 + b\beta + c\gamma/2 + d\delta} [X_{a,b,c,d} \to S^n],$$

where $X_{a,b,c,d}$ is the space of homomorphisms of schemes

$$s : \{1, \dots, n\} \to A \cup B \cup C \cup D$$

with the domain considered as a scheme with n points, a of which are sent to A, b of which are sent to B and so on. We consider this variety as a $\mathfrak{S}_n$-equivariant variety over S^n, sending s to the point $(j(s(1)), \dots, j(s(n)))$, where $j \colon A \cup B \cup C \cup D \to S$ is the union of the maps f, g, h, i.

The above lemma is proved in [**3**] in the case $S = \mathrm{Spec}\,\mathbb{C}$, but the proof for the general case is the same. We remark that, by definition, there is an identity

$$(-[A \to S])^{\otimes n} = (-1)^n [A \to S]^{\otimes n}. \tag{1.4}$$

2.2.1. *The monodromic motivic ring.* Let $\mu_n = \mathrm{Spec}\,\mathbb{C}[x]/(x^n - 1)$ be the group of nth roots of unity. We define a good action of the procyclic group

$$\hat{\mu} = \varprojlim \mu_n$$

as an action that factors through a good μ_n-action for some n. The additive group

$$\mathcal{M}_S^{\hat{\mu}}$$

carries a commutative bilinear associative product '$\star$' called the *convolution product*. See [**12**, Section 5] or [**19**, Section 7] for its definition. The product '$\star$' provides an alternative ring structure on the group of $\hat{\mu}$-equivariant motivic weights, and it restricts to the usual product '$\cdot$' on the subring

$$\mathcal{M}_S \subset \mathcal{M}_S^{\hat{\mu}}$$

[†]Without freeness, the naive quotient map may fail to respect the relation identifying G-bundles with trivial G-bundles.

of classes with trivial $\hat{\mu}$-action. The main role of '$\star$' will be played through the motivic Thom–Sebastiani theorem (cf. Theorem 1.19).

2.3. Lambda ring structures

Let $A \in \mathcal{M}_S$. We define

$$\widetilde{\sigma}^n(A) = \pi_{\mathfrak{S}_n}(A^{\otimes n}) \in \mathcal{M}_{S^n/\mathfrak{S}_n}.$$

The *lambda ring* operations on $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1/2}]$ are defined by setting $\sigma^n(A) = \widetilde{\sigma}^n(A)$ for A effective, and then taking the unique extension to a lambda ring on $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1/2}]$, determined by the relation

$$\sum_{i=0}^{n} \sigma^i([X] - [Y])\sigma^{n-i}([Y]) = \sigma^n([X]). \tag{1.5}$$

Note that $\sigma^n(-\mathbb{L}^{1/2}) = (-\mathbb{L}^{1/2})^n$. By [8, Remark 3.5 (4)], these operations induce a lambda ring structure on the localisation $K_0(\mathrm{St}_{\mathbb{C}}^{\mathrm{aff}})[\mathbb{L}^{-1/2}]$, and thus a lambda ring structure on $\mathcal{M}_{\mathbb{C}}$.

REMARK 1.7.   Note that by definition, $\widetilde{\sigma}^n(A) = \sigma^n(A)$ for A effective. The logical structure of the paper is such that we will often end up proving relations involving $\widetilde{\sigma}^n$ first, and then using them to prove that the motives we consider are effective, so that we can state those same relations in terms of the more well-behaved operations σ^n.

If S comes with a commutative associative map $\nu \colon S \times S \to S$, and $A \in \mathcal{M}_S$, we likewise define

$$\widetilde{\sigma}^n_\nu(A) = \nu_!(\widetilde{\sigma}^n(A)) = \nu_!\big(\pi_{\mathfrak{S}_n}(A^{\otimes n})\big) \in \mathcal{M}_S,$$

where we abuse notation by denoting by ν the map $S^n/\mathfrak{S}_n \to S$. As above, using the analogue of the relation (1.5) there is a unique set of lambda ring operators σ^n_ν agreeing with $\widetilde{\sigma}_\nu$ on effective motives.

As a special case, we obtain operations $\widetilde{\sigma}^n$ and σ^n on $\mathcal{M}_{\mathbb{C}}\llbracket t \rrbracket$ via the isomorphism

$$\mathcal{M}_{\mathbb{C}}\llbracket t \rrbracket \xrightarrow{\sim} \mathcal{M}_{\mathbb{N}} \tag{1.6}$$

defined by

$$\sum_{n \geqslant 0} [X_n]t^n \mapsto \left[\coprod_{n \in \mathbb{N}} X_n \to \{n\}\right] \tag{1.7}$$

for $X_0, X_1, \ldots$ varieties, and then extending by linearity. Here, $\mathbb{N}$ is a considered as a scheme by identifying each natural number with a distinct closed point, and this scheme is considered as a commutative monoid under the addition map.

2.4. Motivic measures

Ring homomorphisms with source $K(\mathrm{Var}_{\mathbb{C}})$ or $\mathcal{M}_{\mathbb{C}}$ are frequently called *motivic measures*, realisations, or generalised Euler characteristics. We recall some of them here.

Let $K_0(\mathrm{HS})$ be the Grothendieck ring of the abelian category HS of Hodge structures. For a complex variety X, taking its *Hodge characteristic*

$$\chi_{\mathrm{h}}(X) = \sum_{i \geqslant 0} (-1)^i \big[\mathrm{H}^i_c(X, \mathbb{Q})\big] \in K_0(\mathrm{HS})$$

defines a motivic measure. The *E-polynomial* is the specialisation

$$E(X) = \sum_{p,q \geqslant 0} (-1)^{p+q} h^{p,q} \big(\mathrm{H}_c^{p+q}(X, \mathbb{Q}) \big) u^p v^q \in \mathbb{Z}[u, v].$$

As $E(\mathbb{A}_{\mathbb{C}}^1) = uv$, the E-polynomial can be extended to a motivic measure

$$E \colon \mathcal{M}_{\mathbb{C}} \to \mathbb{Z} \left[u, v, (uv)^{-\frac{1}{2}} \right]$$

satisfying $E(\mathbb{L}^{1/2}) = -(uv)^{1/2}$. The further specialisations $u = v = q^{1/2}$, $(uv)^{1/2} = q^{1/2}$ define the *weight polynomial* $W \colon \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}[q^{\pm 1/2}]$ and one has $W(\mathbb{L}) = q$. Finally, specialising to $q^{1/2} = 1$ recovers the Euler characteristic $\chi \colon K(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$, extending to $\chi \colon \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}$ after setting

$$\chi(\mathbb{L}^{-\frac{1}{2}}) = -1.$$

See [**12**, Section 2] for a natural extension to a ring homomorphism

$$\chi \colon \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \to \mathbb{Z}.$$

REMARK 1.8. Our sign conventions differ slightly from [**3**]. We have chosen them so that all specialisations are homomorphisms of pre λ-rings. Note that, putting all the changes together, our convention that $\chi(\mathbb{L}^{-1/2}) = -1$ is the same as theirs.

2.5. *Power structures and motivic exponentials*

We recall the notion of a *power structure* on a commutative ring R, mainly following [**14, 15**].

DEFINITION 1.9. A *power structure* on a ring R is a map

$$(1 + tR \llbracket t \rrbracket) \times R \to 1 + tR \llbracket t \rrbracket$$

$$(A(t), X) \mapsto A(t)^X,$$

satisfying the following conditions:

(1) $A(t)^0 = 1$;
(2) $A(t)^1 = A(t)$;
(3) $(A(t) \cdot B(t))^X = A(t)^X \cdot B(t)^X$;
(4) $A(t)^{X+Y} = A(t)^X \cdot A(t)^Y$;
(5) $A(t)^{XY} = (A(t)^X)^Y$;
(6) $(1 + t)^X = 1 + Xt + O(t^2)$;
(7) $A(t)^X|_{t \to t^k} = A(t^k)^X$.

NOTATION 1.10. If α is a partition of an integer n, which we indicate $\alpha \vdash n$, by writing $\alpha = (1^{\alpha_1} \cdots i^{\alpha_i} \cdots r^{\alpha_r})$ we mean that there are α_i parts of size i, so that we recover n as the sum $|\alpha| = \sum_i i\alpha_i$. The number of distinct parts of α is denoted $l(\alpha) = \sum_i \alpha_i$. The automorphism group of α is the product of symmetric groups $G_\alpha = \prod_i \mathfrak{S}_{\alpha_i}$.

Let us focus on $R = K_0(\mathrm{Var}_{\mathbb{C}})$. If X is a variety and $A(t) = 1 + \sum_{n>0} A_n t^n$ is a power series in $K_0(\mathrm{Var}_{\mathbb{C}}) \llbracket t \rrbracket$, we define

$$(A(t))_\diamond^{[X]} = 1 + \sum_\alpha \pi_{G_\alpha} \left(\left[\prod_i X^{\alpha_i} \setminus \Delta \right] \cdot \prod_i A_i^{\otimes \alpha_i} \right) t^{|\alpha|}. \tag{1.8}$$

In the above formula, $\Delta \subset \prod_i X^{\alpha_i} = X^{l(\alpha)}$ is the 'big diagonal' (where at least two entries are equal), and the class

$$\left[\prod_i X^{\alpha_i} \setminus \Delta \right] \cdot \prod_i A_i^{\otimes \alpha_i} \in \widetilde{K}_0^{G_\alpha}(\mathrm{Var}_{\mathbb{C}})$$

is G_α-equivariant due to the 'power map' of Lemma 1.6. Gusein-Zade, Luengo and Melle-Hernández have proved [**14**, Theorem 2] that there is a unique power structure on $K_0(\mathrm{Var}_{\mathbb{C}})$ for which the restriction to the case where all A_i are effective is given by the formula (1.8), for every variety X. Moreover, by [**14**, Theorem 1], such a power structure is determined by the relation

$$(1 - t)^{-[X]} = \zeta_{[X]}(t),$$

where

$$\zeta_{[X]}(t) = \sum_{n \geqslant 0} \left[\mathrm{Sym}^n X \right] \cdot t^n \in K_0(\mathrm{Var}_{\mathbb{C}}) \llbracket t \rrbracket \qquad (1.9)$$

is the Kapranov motivic zeta function of X. Since we always consider effective exponents when taking powers, we just recall the recipe for dealing with general $A(t)$ and effective exponent $[X]$. First, note that for any such $A(t)$ there is an effective $B(t)$ such that $A(t) \cdot B(t) = C(t)$ is effective. Then we have

$$A(t)^{[X]} := (C(t))_\diamond^{[X]} \cdot ((B(t))_\diamond^{[X]})^{-1},$$

where both factors in the right-hand side are defined via (1.8).

LEMMA 1.11.   *Let* $[X] \in K_0(\mathrm{Var}_{\mathbb{C}})$ *be invertible in* $K_0(\mathrm{St}_{\mathbb{C}}^{\mathrm{aff}})$. *Then* $(-)_\diamond^{[X]}$ *and* $(-)^{[X]}$ *are injective maps.*

Proof. By [**9**, Remark 3.7], the power structure can be extended to $K_0(\mathrm{St}_{\mathbb{C}}^{\mathrm{aff}})$, and so the second statement follows from $(A(t)^{[X]})^{[X]^{-1}} = A(t)$.

Next we consider the first statement. Assume $A(t)_\diamond^{[X]} = B(t)_\diamond^{[X]}$. Write $A(t) = \sum_{i \geqslant 0} A_i t$ and $B(t) = \sum_{i \geqslant 0} B_i t$, where $A_0 = B_0 = 1$, and assume that we have shown that $A_i = B_i$ for $i < n$. Let $\alpha \vdash n$. Comparing the contributions from α in the t^n coefficients of $A(t)_\diamond^{[X]}$ and $B(t)_\diamond^{[X]}$, by assumption they agree for $\alpha \neq (n)$, since these terms only involve A_i and B_i for $i < n$. We deduce that the terms for $\alpha = (n^1)$ agree, and so $[X] \cdot A_n = [X] \cdot B_n$, and the result follows by injectivity of $[X] \cdot$. $\qquad \square$

As noted in [**3**], there is an extension of the power structure to $\mathcal{M}_{\mathbb{C}}$ uniquely determined by the substitution rules

$$A((-\mathbb{L}^{\frac{1}{2}})^n t)^{[X]} = A(t)^{(-\mathbb{L}^{\frac{1}{2}})^n [X]} = A(t)\big|_{t \mapsto (-\mathbb{L}^{\frac{1}{2}})^n t}.$$

2.5.1. *Motivic exponential.*   It is often handy to rephrase motivic identities in terms of the motivic exponential, which is a group isomorphism[†]

$$\mathrm{Exp} \colon t\mathcal{M}_{\mathbb{C}} \llbracket t \rrbracket \xrightarrow{\sim} 1 + t\mathcal{M}_{\mathbb{C}} \llbracket t \rrbracket.$$

Under (1.6), this can be seen as an inclusion of groups

$$K_0(\mathrm{Var}_{\mathbb{N} \setminus 0}) \hookrightarrow K_0(\mathrm{Var}_{\mathbb{N}})^\times.$$

[†]The group structures are the additive one on the source and the multiplicative one on the target.

First, define $^{\diamond}\mathrm{Exp} = \sum_{n \geqslant 0} \eth^n$, relative to the monoid $(\mathbb{N}, +)$. Then if A and B are effective classes, we set

$$\mathrm{Exp}(A - B) = {}^{\diamond}\mathrm{Exp}(A) \cdot {}^{\diamond}\mathrm{Exp}(B)^{-1}.$$

As in the proof of Lemma 1.11, $^{\diamond}\mathrm{Exp}$ and Exp are injective.

Now if $(S, \nu \colon S \times S \to S)$ is a commutative monoid in the category of schemes, with a submonoid S_+ such that the induced map $\coprod_{n \geqslant 1} S_+^{\times n} \to S$ is of finite type, we similarly define

$$^{\diamond}\mathrm{Exp}_\nu(A) = \sum_{n \geqslant 0} \eth^n_\nu(A),$$

and for A and B effective classes, we define

$$\mathrm{Exp}_\nu(A - B) = {}^{\diamond}\mathrm{Exp}_\nu(A) \cdot {}^{\diamond}\mathrm{Exp}_\nu(B)^{-1}.$$

The principal example will be

$$S = \mathrm{Sym}(U) = \coprod_{n \geqslant 0} \mathrm{Sym}^n(U)$$

for U a variety, and $S_+ = \coprod_{n \geqslant 1} \mathrm{Sym}^n(U)$. We define

$$\cup \colon \mathrm{Sym}(U) \times \mathrm{Sym}(U) \to \mathrm{Sym}(U)$$

to be the morphism sending a pair of sets of unordered points with multiplicity to their union.

Note that Exp_ν sends effective motives to effective motives, as the same is true of σ^n_ν for each n.

In order to recover a formal power series from a relative motive over $\mathrm{Sym}(U)$, we consider the operation

$$\#_! \left(\sum_{n \geqslant 0} \left[A_n \xrightarrow{f_n} \mathrm{Sym}^n(U) \right] \right) := \sum_{n \geqslant 0} [A_n] t^n.$$

In other words, we take the direct image along the 'tautological' map $\# \colon \mathrm{Sym}(U) \to \mathbb{N}$ which sends $\mathrm{Sym}^n(U)$ to the point n — recall that via (1.6) we consider power series in t with coefficients in $\mathcal{M}_\mathbb{C}$ as the same thing as elements of $\mathcal{M}_\mathbb{N}$.

PROPOSITION 1.12. *Let U, V be varieties. Set $S = \mathrm{Sym}(U \times V) = \coprod_{n \geqslant 0} \mathrm{Sym}^n(U \times V)$, and for $i \in \mathbb{N}$ denote by $\cup \colon \mathrm{Sym}(U \times V)^i \to \mathrm{Sym}(U \times V)$ the map taking i sets of points (with multiplicity) to their union (with multiplicity). Let*

$$\tilde{i}_n \colon U \times \mathrm{Sym}^n(V) \to \mathrm{Sym}^n(U \times V)$$

be the inclusion of the n-tuples $((u_1, v_1), \ldots, (u_n, v_n))$ such that $u_1 = \cdots = u_n$. Write $B = 1 + \sum_{n > 0} B_n = {}^{\diamond}\mathrm{Exp}_\cup(\sum_{n > 0} A_n) = {}^{\diamond}\mathrm{Exp}_\cup(A)$ for some set of $A_n, B_n \in K(\mathrm{Var}_{\mathrm{Sym}^n(V)})$. Define the S-motive

$$\mathsf{Z} = \sum_{n \geqslant 0} \sum_{\alpha \vdash n} \cup_! \pi_{G_\alpha} j_\alpha^* \left(\underset{i | \alpha_i \neq 0}{\boxtimes} \tilde{i}_{i,!} \left([U \xrightarrow{\mathrm{id}} U] \boxtimes B_i \right)^{\otimes \alpha_i} \right),$$

where j_α is the G_α-equivariant inclusion from the space of points in $\prod_{i | \alpha_i \neq 0} \mathrm{Sym}^i(U \times V)^{\alpha_i}$ that are not sent to the big diagonal after projection to $\prod_{i | \alpha_i \neq 0} \mathrm{Sym}^i(U)^{\alpha_i}$. Then

$$\mathsf{Z} = {}^{\diamond}\mathrm{Exp}_\cup \left(\sum_{n > 0} \tilde{i}_{n,!} \left([U \xrightarrow{\mathrm{id}} U] \boxtimes A_n \right) \right).$$

and if A is effective, one has

$$\#_! \mathsf{Z} = (\#_! B)^{[U]}.$$

Proof. The second statement follows directly from the definition of the power structure. The first arises from the decomposition of the right-hand side according to incidence partition in the U factor. $\square$

2.6. *The virtual motive of a critical locus*

Let X be a complex scheme of finite type, and let $\nu_X \colon X(\mathbb{C}) \to \mathbb{Z}$ be the canonical constructible function introduced by Behrend [**2**]. The *weighted* (or *virtual*) *Euler characteristic* of X is defined via ν_X as

$$\widetilde{\chi}(X) = \int_X \nu_X \, \mathrm{d}\chi = \sum_{r \in \mathbb{Z}} r \cdot \chi(\nu_X^{-1}(r)).$$

When X is a proper moduli space of stable sheaves on a Calabi–Yau 3-fold, this number agrees with the Donaldson–Thomas invariant of X by the main result of [**2**]. The following definition is central to this paper.

DEFINITION 1.13 [**3**]. A *virtual motive* of a scheme X is a motivic weight $\xi \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ such that $\chi(\xi) = \widetilde{\chi}(X)$.

DEFINITION 1.14. A scheme X is a *critical locus* if it is of the form

$$X = \mathrm{crit}(f) = Z(\mathrm{d}f),$$

where $f \colon U \to \mathbb{A}^1$ is a regular function on a smooth scheme U.

The Behrend function of a critical locus $X = \mathrm{crit}(f) \subset U$ agrees with the Milnor function μ_f, the function counting the number of vanishing cycles [**27**, Corollary 2.4 (iii)]. In particular, $\nu_X(x) = (-1)^{\dim U - 1}(\chi(\mathrm{MF}_{f,x}) - 1)$, where $\mathrm{MF}_{f,x}$ is the Milnor fibre of f at x. More globally, one can write

$$\nu_X = \chi(\Phi_f[\dim U - 1]),$$

where $\Phi_f[\dim U - 1] \in \mathrm{Perv}(X)$ is the perverse sheaf of vanishing cycles, the image of the constant perverse sheaf $\mathbb{Q}_U[\dim U]$ under the vanishing cycle functor $\varphi_f[-1] \colon \mathrm{Perv}(U) \to \mathrm{Perv}(U_0)$. Here $U_0 = f^{-1}(0)$ denotes the hypersurface determined by f. The pair (U, f) also determines a canonical *relative* virtual motive

$$\mathsf{MF}_{U,f} = \mathbb{L}^{-\frac{d}{2}}[-\phi_f]_X \in \mathcal{M}_X^{\hat{\mu}} \subset \mathcal{M}_U^{\hat{\mu}}, \tag{1.10}$$

where $d = \dim U$ and $[\phi_f]_X$ is the relative motivic vanishing cycle class introduced by Denef and Loeser [**12**]. It is a class in $K_0^{\hat{\mu}}(\mathrm{Var}_{U_0})$, supported on $X = \mathrm{crit}(f)$, that we view as an element of $\mathcal{M}_X^{\hat{\mu}}$. We write $[\phi_f]$ for the pushforward of $[\phi_f]_X$ to a point. We will repeatedly use the following proposition, due to Bittner [**4**].

PROPOSITION 1.15. *Let G be a finite group acting freely on a smooth variety U, let $q \colon U \to U/G$ be the quotient map and let f be a regular function on U/G. Then:*

(1) *there is a well-defined equivariant motivic vanishing cycle $[\phi_{fq}]_{U_0}^G \in K_0^{G \times \hat{\mu}}(\mathrm{Var}_{U_0})$ such that the relative motive in $K_0^{\hat{\mu}}(\mathrm{Var}_{U_0})$ induced by forgetting the G-action is $[\phi_{fq}]_{U_0}$;*
(2) *there is an equality of motives*

$$\pi_G\big([\phi_{fq}]_{U_0}^G\big) = [\phi_f]_{U_0/G} \in K_0^{\hat{\mu}}(\mathrm{Var}_{U_0/G}).$$

NOTATION 1.16. If $X = \mathrm{crit}(f) \to Y$ is a morphism of varieties, we denote by

$$[\mathrm{crit}(f) \to Y]_{\mathrm{vir}} = (\mathrm{crit}(f) \to Y)_! \mathsf{MF}_{U,f} \in \mathcal{M}_Y^{\hat{\mu}}$$

the induced *relative* virtual motive. More generally, if $\iota\colon Z \hookrightarrow \mathrm{crit}(f)$ is a locally closed subscheme and $Z \to Y$ is a morphism, we let

$$[Z \to Y]_{\mathrm{vir}} = (Z \to Y)_! \iota^* \mathsf{MF}_{U,f} \in \mathcal{M}_Y^{\hat{\mu}}.$$

When $Y = \mathrm{Spec}\,\mathbb{C}$, we denote $[Z \to \mathrm{Spec}\,\mathbb{C}]_{\mathrm{vir}}$ simply by $[Z]_{\mathrm{vir}}$.

Since the fibrewise Euler characteristic of $\mathsf{MF}_{U,f}$ equals ν_X as a function on X [**3**, Proposition 2.16], the absolute class

$$[X]_{\mathrm{vir}} = \int_X \mathsf{MF}_{U,f} = \mathbb{L}^{-\frac{d}{2}} \cdot [-\phi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \tag{1.11}$$

is a virtual motive for X in the sense of Definition 1.13.

REMARK 1.17. A critical locus $X = \mathrm{crit}(f) \subset U$ has a canonical virtual fundamental class $[X]^{\mathrm{vir}} \in A_0 X$, attached to the symmetric perfect obstruction theory determined by the Hessian of f. When X is proper, Behrend's theorem [**2**] can be phrased as

$$\chi[X]_{\mathrm{vir}} = \int_{[X]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

REMARK 1.18. If X is a smooth scheme, it can be considered as a critical locus via the zero function $f = 0 \in \Gamma(\mathscr{O}_X)$. The associated virtual motive is

$$[X]_{\mathrm{vir}} = \mathbb{L}^{-\frac{\dim X}{2}} \cdot [X] \in \mathcal{M}_{\mathbb{C}}.$$

Via the stated sign conventions, we see that $\chi[X]_{\mathrm{vir}}$ recovers the virtual Euler characteristic of the smooth scheme X, namely $\widetilde{\chi}(X) = (-1)^{\dim X} \chi(X)$.

We end this subsection with two results that are of crucial importance in calculations involving motivic vanishing cycles. Recall from Section 1.2.1 that the groups $\mathcal{M}_S^{\hat{\mu}}$ carry the convolution product '$\star$' besides the ordinary product.

THEOREM 1.19 [**11, 19**]. *Let $f\colon U \to \mathbb{A}^1$ and $g\colon V \to \mathbb{A}^1$ be regular functions on smooth varieties U and V. Consider the function $f \oplus g\colon U \times V \to \mathbb{A}^1$ given by $(x,y) \mapsto f(x) + g(y)$. Let $i\colon U_0 \times V_0 \to (U \times V)_0$ be the inclusion of the zero fibres, and let p_U and p_V be the projections from $U_0 \times V_0$. Then one has*

$$i^* \big[\phi_{f \oplus g}\big]_{(U \times V)_0} = p_U^* \big[\phi_f\big]_{U_0} \star p_V^* \big[\phi_g\big]_{V_0} \in \mathcal{M}_{U_0 \times V_0}^{\hat{\mu}}.$$

The following result will be used in Propositions 2.9 and 2.11.

THEOREM 1.20 [**3**, Theorem B.1]. *Let $f\colon U \to \mathbb{A}^1$ be a regular function on a smooth complex quasi-projective variety, with critical locus X. Assume U is acted on by a connected complex torus $\mathbb{T}$ in such a way that f is $\mathbb{T}$-equivariant with respect to a primitive character $\chi\colon \mathbb{T} \to \mathbb{G}_m$.*

(i) *If there is a one parameter subgroup $\mathbb{G}_m \subset \mathbb{T}$ such that the induced action is circle compact, then*

$$\big[\phi_f\big] = \big[f^{-1}(1)\big] - \big[f^{-1}(0)\big] \in \mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$
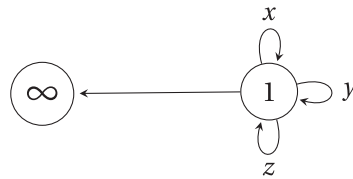
FIGURE 1. *The framed 3-loop quiver Q_{BBS}.*

(ii) *Let $\tau\colon X \to Y$ be a map to an affine variety. If, in addition to the assumption in* (i), *the hypersurface $f^{-1}(0) \subset U$ is reduced, then the relative class $[\phi_f]_Y = \tau_! [\phi_f]_X$ lies in the subring $\mathcal{M}_Y \subset \mathcal{M}_Y^{\hat{\mu}}$ of classes with trivial monodromy.*

REMARK 1.21. The original statement of this theorem in [**3**] fixed Y to be the affinisation of X — the statement above then follows from the fact that τ must factor through the affinisation, and the direct image of a monodromy-free motive is monodromy-free.

2.7. *The virtual motive of the Hilbert scheme of points*

Quivers with potentials provide a large class of examples of critical loci. For instance, consider the framed 3-loop quiver Q_{BBS} (studied by Behrend–Bryan–Szendrői) depicted in Figure 1. The arrow $1 \to \infty$ is called a *framing*, and ∞ is the *framing vertex*. Throughout the paper, the vertices of a framed quiver are ordered so that the framing vertex is last.

The space of $(n,1)$-dimensional right $\mathbb{C}Q_{\mathrm{BBS}}$-modules is the affine space $\mathrm{End}(\mathbb{C}^n)^3 \times \mathbb{C}^n$ parameterising triples of $n \times n$ matrices (A, B, C) and vectors $v \in \mathbb{C}^n$. Consider the potential $W = x[y, z]$, viewed as an element of the path algebra $\mathbb{C}\langle x, y, z \rangle$ of the (unframed) 3-loop quiver. Then by [**3**, Proposition 3.1], one has, as schemes,

$$\mathrm{Hilb}^n(\mathbb{A}^3) = \mathrm{crit}(\mathrm{Tr}\, W) \subset \mathrm{NCHilb}^n,$$

where NCHilb^n is the *non-commutative Hilbert scheme*, defined as follows. The open subscheme

$$V_n \subset \mathrm{End}(\mathbb{C}^n)^3 \times \mathbb{C}^n = \mathrm{rMod}_{(n,1)}(\mathbb{C}Q_{\mathrm{BBS}}),$$

parameterising tuples (A, B, C, v) such that $v \in \mathbb{C}^n$ generates the $\mathbb{C}\langle x, y, z \rangle$-module defined by the triple (A, B, C), carries a free GL_n-action, and $\mathrm{NCHilb}^n = V_n / \mathrm{GL}_n$ is a smooth quasi-projective variety of dimension $2n^2 + n$. The generating function $\mathsf{Z}_{\mathbb{A}^3}(t)$ for which the t^n coefficient is the virtual motive

$$\left[\mathrm{Hilb}^n(\mathbb{A}^3)\right]_{\mathrm{vir}} = \mathbb{L}^{-\frac{2n^2+n}{2}}[-\phi_{\mathrm{Tr}\, W}] \in \mathcal{M}_{\mathbb{C}}$$

was computed in [**3**, Theorem 3.7]. The result is the equation

$$\mathsf{Z}_{\mathbb{A}^3}(t) = \prod_{m \geqslant 1} \prod_{k=0}^{m-1} \left(1 - \mathbb{L}^{k+2-\frac{m}{2}} t^m\right)^{-1} \in \mathcal{M}_{\mathbb{C}}[\![t]\!].$$

Let

$$\left[\mathrm{Hilb}^n(\mathbb{A}^3)_0\right]_{\mathrm{vir}} = \int_{\mathrm{Hilb}^n(\mathbb{A}^3)_0} \iota^* \mathsf{MF}_{\mathrm{NCHilb}^n, \mathrm{Tr}\, W} \in \mathcal{M}_{\mathbb{C}} \qquad (1.12)$$

be the virtual motive of the *punctual Hilbert scheme* $\iota\colon \mathrm{Hilb}^n(\mathbb{A}^3)_0 \hookrightarrow \mathrm{Hilb}^n(\mathbb{A}^3)$ (cf. Notation 1.16), the closed subscheme parameterising subschemes entirely supported at the origin $0 \in \mathbb{A}^3$. Then the generating series

$$\mathsf{Z}_0(t) = \sum_{n \geqslant 0} \left[\mathrm{Hilb}^n(\mathbb{A}^3)_0\right]_{\mathrm{vir}} \cdot t^n$$

satisfies the relation

$$\mathsf{Z}_0(-t) = \mathrm{Exp}\left(\frac{-\mathbb{L}^{-\frac{3}{2}}t}{\left(1 + \mathbb{L}^{-\frac{1}{2}}t\right)\left(1 + \mathbb{L}^{\frac{1}{2}}t\right)}\right). \tag{1.13}$$

REMARK 1.22. As a corollary of Formula (1.13), the motive $(-1)^n[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}$ is effective, that is, it belongs to the sub semigroup $S_0(\mathrm{Var}_\mathbb{C}) \subset \mathcal{M}_\mathbb{C}$.

Behrend, Bryan and Szendrői also define a virtual motive $[\mathrm{Hilb}^n X]_{\mathrm{vir}}$ for arbitrary smooth 3-folds. The motivic partition function

$$\mathsf{Z}_X(t) = \sum_{n \geqslant 0} [\mathrm{Hilb}^n X]_{\mathrm{vir}} \cdot t^n \in \mathcal{M}_\mathbb{C}[\![t]\!]$$

is again fully determined by the punctual contributions, that is, by [3, Proposition 4.2], one has

$$\mathsf{Z}_X(-t) = \mathsf{Z}_0(-t)^{[X]}. \tag{1.14}$$

REMARK 1.23 (Related work on Quot schemes). The identity (1.14), as well as its reformulation in terms of the motivic exponential, has been generalised in [34] to the case of Quot schemes $\mathrm{Quot}_Y(F, n)$ where F is an arbitrary locally free sheaf on a smooth 3-fold. See also [33] for the 'non-virtual' setup. In higher rank, the starting point of motivic DT theory is the observation that $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$ is a global critical locus [1, Theorem 2.6]. This has also been exploited to prove a plethystic formula (the Awata–Kanno conjecture in String Theory) for the partition function of higher rank K-theoretic DT invariants [13].

3. *The local model as a critical locus*

For a smooth curve C embedded in a smooth 3-fold Y with ideal sheaf $\mathscr{I}_C \subset \mathscr{O}_Y$, we let

$$Q_C^n = \mathrm{Quot}_Y(\mathscr{I}_C, n) = \{\mathscr{I}_C \twoheadrightarrow \mathscr{F} \,|\, \dim(\mathrm{Supp}\,\mathscr{F}) = 0,\ \chi(\mathscr{F}) = n\}$$

denote the Quot scheme of n points of $\mathscr{I}_C$. Given a surjection $\theta\colon \mathscr{I}_C \twoheadrightarrow \mathscr{F}$, we can consider its kernel $\mathscr{I}_Z \subset \mathscr{I}_C$, and thus think of $[\theta] \in Q_C^n$ as a closed 1-dimensional subscheme $Z \subset Y$ containing C as its maximal purely 1-dimensional subscheme. We will switch between these two interpretations of Q_C^n without further comment. Note that we generally suppress the ambient 3-fold Y from the notation.

REMARK 2.1. When Y is projective, the association $[\theta] \mapsto \ker\theta$ defines a closed immersion into the moduli space of ideal sheaves

$$Q_C^n \hookrightarrow I_{\chi(\mathscr{O}_C)+n}(Y, [C]),$$

as proved in [32, Lemma 5.1]. This closed immersion can be generalised to higher rank sheaves, see [1, Proposition 5.1].

Let now $L \subset \mathbb{A}^3$ be a line; for concreteness set $\mathscr{I}_L = (x, y) \subset \mathbb{C}[x, y, z]$. The scheme Q_L^n parameterises surjections $\mathscr{I}_L \twoheadrightarrow N$ of $\mathbb{C}[x, y, z]$-modules, where N is n-dimensional as a

$\mathbb{C}$-vector space. As such there is a forgetful map from Q_L^n to the stack of n-dimensional $\mathbb{C}[x, y, z]$-modules, and postcomposing with the affinisation map for this stack, a 'Hilbert-to-Chow' morphism

$$\mathbf{hc}_{\mathbb{A}^3} \colon Q_L^n \to \mathrm{Sym}^n(\mathbb{A}^3). \tag{2.1}$$

This map is a special case of [**35**, Corollary 7.15].

The goal of this section is to prove the following result, which is a 1-dimensional counterpart of the analogous statement for $\mathrm{Hilb}^n(\mathbb{A}^3)$, considered in [**3**]. Part (A) is Theorem A from the introduction.

THEOREM 2.2. *Consider the embedding $L \subset \mathbb{A}^3$ from above. Then:*

(A) *the Quot scheme Q_L^n is a global critical locus, that is, there is a smooth variety U and a function $f \colon U \to \mathbb{A}^1$ such that $Q_L^n \cong \mathrm{crit}(f)$;*

(B) *the relative motive $[\phi_f]_{\mathrm{Sym}^n(\mathbb{A}^3)}$ is an element of the subring $\mathcal{M}_{\mathrm{Sym}^n(\mathbb{A}^3)} \subset \mathcal{M}_{\mathrm{Sym}^n(\mathbb{A}^3)}^{\hat{\mu}}$.*

Part (A) is proved at the end of Section 2.1, and part (B) is proved in Proposition 2.11. The main step in the proof consists of realising Q_L^n as a suitable open subscheme of the Quot scheme $Q_{C_0}^n$, where $C_0 \subset X$ is the exceptional curve in the resolved conifold X. In the next subsection, we review the critical locus structure on $Q_{C_0}^n$, and more generally, the noncommutative Donaldson–Thomas theory of the conifold as introduced in [**39**].

3.1. *Conifold Geometry*

Given a quiver $Q = (Q_0, Q_1)$ with potential W and a field K, we define $K(Q, W)$ to be the associated Jacobi algebra, that is, the quotient of the free path algebra KQ by the non-commutative derivatives $\partial W / \partial a$ for $a \in Q_1$ ranging over the arrows of Q.

Given a dimension vector $\mathbf{d} = (\mathbf{d}(i))_{i \in Q_0} \in \mathbb{N}^{Q_0}$, we define

$$\mathrm{rMod}_{\mathbf{d}}(\mathbb{C}Q) = \prod_{(i \to j) \in Q_1} \mathrm{Hom}(\mathbb{C}^{\mathbf{d}(j)}, \mathbb{C}^{\mathbf{d}(i)})$$

to be the affine space of $\mathbf{d}$-dimensional right $\mathbb{C}Q$-modules. Given a King stability condition $\zeta \in \mathbb{Q}^{Q_0}$, we denote by

$$\mathrm{rMod}_{\mathbf{d}}^{\zeta}(\mathbb{C}Q) \subset \mathrm{rMod}_{\mathbf{d}}(\mathbb{C}Q),$$

the open subscheme of ζ-semistable $\mathbb{C}Q$-modules. Both schemes are acted on by the gauge group

$$\mathrm{GL}_{\mathbf{d}} \coloneqq \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{d}(i)}$$

by change of basis.

Throughout the paper, we let X denote the *resolved conifold*, namely the total space of the rank 2 locally free sheaf $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. We denote by $C_0 \subset X$ the zero section of this vector bundle, so $C_0 \cong \mathbb{P}^1$. Let

$$\pi \colon X \to C_0$$

denote the projection onto the zero section. The local Calabi–Yau 3-fold X is the crepant resolution of the conifold singularity

$$\mathrm{Spec}\, \mathbb{C}[x, y, z, w]/(xy - zw) \subset \mathbb{A}^4,$$

and C_0 is the exceptional curve, the only strictly positive dimensional proper subvariety of X. Since C_0 is a *rigid* smooth rational curve in X, the Hilbert scheme $I_{n+1}(X, [C_0])$ of
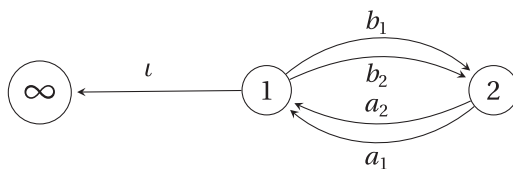
FIGURE 2. *The framed conifold quiver* $\tilde{Q}_{con}$.

1-dimensional subschemes $Z \subset X$ with homology class $[C_0]$ and $\chi(\mathscr{O}_Z) = n + 1$ agree (as a scheme) with the Quot scheme $Q^n_{C_0}$. This space was realised as a global critical locus by Nagao and Nakajima [**24**], by studying representations of the Jacobi algebra $\mathbb{C}(\tilde{Q}_{con}, W_{con})$. Here, $\tilde{Q}_{con}$ is the (framed) *conifold quiver* depicted in Figure 2, and

$$W_{con} = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1 \tag{2.2}$$

is the Klebanov–Witten potential [**16**]. We denote by Q_{con} the quiver obtained by removing from $\tilde{Q}_{con}$ the framing vertex ∞ and the arrow ι. Since we consider right modules, a $K\tilde{Q}_{con}$-module $\tilde{\rho}$ is defined by the following data.

(1) A right KQ_{con}-module ρ, with KQ_{con}-action defined via the $K\tilde{Q}_{con}$-action on $\tilde{\rho}$ and the inclusion $KQ_{con} \hookrightarrow K\tilde{Q}_{con}$.
(2) A linear map $V \to \rho_1$, with $V = \tilde{\rho}_\infty$.

REMARK 2.3. The orientation of $\tilde{Q}_{con}$ differs from [**24**, Figure 4], but our notion of framing is the same as *loc. cit.* (see (1) and (2) above) — note that we are considering *right* $\mathbb{C}\tilde{Q}_{con}$-modules throughout — see Remark 2.4.

We identify quasicoherent $\mathscr{O}_X$-modules with triples $(\mathscr{F}, \alpha_1, \alpha_2)$ where $\mathscr{F} \in \mathrm{QCoh}(\mathbb{P}^1)$ and $\alpha_1, \alpha_2 \in \mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^1}}(\mathscr{F}, \mathscr{F}(-1))$ commute in the sense that $\alpha_1(-1) \circ \alpha_2 = \alpha_2(-1) \circ \alpha_1$. Then the above noncommutative conifold is the natural enhancement of Beilinson's noncommutative $\mathbb{P}^1$: given a complex of quasicoherent sheaves $\mathscr{F}$ on $\mathbb{P}^1$, we set ρ_1 and ρ_2 to be the complexes of vector spaces $\mathcal{R}\mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^1}}(\mathscr{O}_{\mathbb{P}^1}, \mathscr{F})$ and $\mathcal{R}\mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^1}}(\mathscr{O}_{\mathbb{P}^1}(1), \mathscr{F})$. We let

$$\rho(b_1), \rho(b_2) \colon \mathcal{R}\mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^1}}(\mathscr{O}_{\mathbb{P}^1}(1), \mathscr{F}) \to \mathcal{R}\mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^1}}(\mathscr{O}_{\mathbb{P}^1}, \mathscr{F})$$

be the maps induced by the two sections x and y of $\mathscr{O}_{\mathbb{P}^1}(1)$. We set $\rho(a_i) = \alpha_i$ for $i = 1, 2$, and it is easy to check that the commutativity conditions are given precisely by the superpotential relations for W_{con}, so that we obtain in this way a right $\mathbb{C}(Q_{con}, W_{con})$-module.

This description of the non-commutative conifold makes the translation of various geometrically defined functors rather transparent. For instance, the direct image along the projection map $\pi_* \colon \mathrm{D}^b(\mathrm{QCoh}(X)) \to \mathrm{D}^b(\mathrm{QCoh}(\mathbb{P}^1))$ becomes the forgetful map from the non-commutative conifold to Beilinson's non-commutative $\mathbb{P}^1$, forgetting the action of the arrows a_1, a_2, and the direct image along the inclusion $C_0 \hookrightarrow X$ becomes the extension by zero functor.

Let

$$\mathscr{E} = \pi^*(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1))$$

and let

$$A_{con} = \mathrm{End}_{\mathscr{O}_X}(\mathscr{E})$$
$$\cong \mathbb{C}(Q_{con}, W_{con}).$$

We denote by

$$\Phi = \mathcal{R}\mathrm{Hom}(\mathscr{E}, \bullet) \colon \mathrm{D}^b(\mathrm{Coh}\, X) \overset{\sim}{\to} \mathrm{D}^b(A_{con}\text{-mod})$$

the equivalence of derived categories, where on the right-hand side of the equivalence we have the derived category of right A_{con}-modules with finitely generated total cohomology. We denote by

$$\Psi\colon M \to M \overset{\mathbf{L}}{\otimes}_{A_{\mathrm{con}}} \mathscr{E},$$

the quasi-inverse.

REMARK 2.4. As one sees from the above, the general setup leads us to consider *right A_{con}*-modules. On the other hand, for an arbitrary quiver Q, there is an equivalence of categories between K-linear Q-representations and *left KQ*-modules. Since we consider right modules (as in [**24**]), if one wants to think of modules over algebras such as A_{con} as quiver representations, one should reverse the orientation of the underlying quiver.

The chamber decomposition of the space of stability parameters for $\tilde{Q}_{\mathrm{con}}$ was worked out by Nagao and Nakajima in [**24**], where the DT and PT chambers were precisely characterised. For a generic stability condition

$$\zeta = (\zeta_1, \zeta_2, -(\zeta_1(n+1) + \zeta_2 n)) \in \mathbb{R}^3$$

in the 'DT region' for X, defined by the conditions $\zeta_1 < \zeta_2$ and $\zeta_1 + \zeta_2 < 0$, we consider the moduli space

$$\mathcal{N}_n \coloneqq \mathrm{rMod}^{\zeta}_{(n+1,n,1)}(\tilde{Q}_{\mathrm{con}})/\mathrm{GL}_{n+1} \times \mathrm{GL}_n \tag{2.3}$$

of ζ-stable framed representations of Q_{con}, having dimension vector $(n+1, n, 1)$. Here the dimension vector $(n+1, n, 1)$ refers to the vertices ordered as $(1, 2, \infty)$.

The work of Nagao–Nakajima then implies that $Q^n_{C_0}$ is isomorphic to the subscheme of $\mathcal{N}_n$ defined by the defining relations of A_{con}. Since these are exactly the non-commutative derivatives of the Klebanov–Witten potential (2.2), it follows from [**38**, Proposition 3.8] that $Q^n_{C_0}$ is identified with the critical locus of the function

$$g_n\colon \mathcal{N}_n \to \mathbb{A}^1$$

given by taking the trace of (2.2).

Consider the open subset

$$\mathcal{N}^{\circ}_n \subset \mathcal{N}_n$$

parameterising stable representations ρ such that the linear map $\rho(b_2)\colon \mathbb{C}^n \to \mathbb{C}^{n+1}$ is injective, and let

$$f_n\colon \mathcal{N}^{\circ}_n \to \mathbb{A}^1$$

be the restriction of the function g_n. We now prove that

$$Q^n_L \cong \mathrm{crit}(f_n) \subset \mathcal{N}^{\circ}_n.$$

Proof of Theorem 2.2 (A). For skyscraper sheaves of points $\mathscr{O}_x$, corresponding to representations of dimension vector $(1, 1)$ under Φ, injectivity of $\rho(b_2)$ corresponds to the condition that $\pi(x) \neq \infty$. We identify $\mathbb{A}^3$ with $X \setminus \pi^{-1}(\infty)$, and we have the following Cartesian diagram

$$
\begin{array}{ccc}
L & \lhook\joinrel\longrightarrow & \mathbb{A}^3 \\
\Big\uparrow & \square & \Big\uparrow{\scriptstyle u} \\
C_0 & \lhook\joinrel\longrightarrow & X
\end{array}
$$

where the horizontal maps are closed inclusions, and vertical maps open inclusions.

Let $\mathscr{L}$ be the tautological $\mathbb{C}(Q_{\mathrm{con}}, W_{\mathrm{con}}) \otimes \mathscr{O}_{\mathrm{crit}(g_n)}$-module. Let $\mathscr{G}$ be the submodule generated by

$$\rho(b_1)\mathscr{L} + \rho(b_2)\mathscr{L}.$$

Consider the exact sequence

$$0 \to \mathscr{G} \to \mathscr{L} \to V \to 0$$

of $\mathbb{C}(Q_{\mathrm{con}}, W_{\mathrm{con}}) \otimes \mathscr{O}_{\mathrm{crit}(g_n)}$-modules. Let $\mathrm{Spec}\,(K) \hookrightarrow \mathcal{N}_n$ be a geometric point of $\mathrm{crit}(g_n)$, corresponding to a $K(Q_{\mathrm{con}}, W_{\mathrm{con}})$-module ρ. By the above-mentioned result of Nagao–Nakajima, $\Psi(\rho)$ corresponds to a K-point of Q_C^n, and so admits a unique (up to scalar) surjective map to $\mathscr{O}_{C_0} \otimes K$, with kernel a coherent sheaf $\mathscr{F}$ with 0-dimensional support. It follows that ρ admits a unique (up to scalar) non-zero map to the nilpotent simple module at vertex 1, with kernel isomorphic to $\Phi(\mathscr{F})$. In particular, the space spanned by the image of $\rho(b_1)$ and $\rho(b_2)$ is n-dimensional. It follows that $\dim(V_K) = 1$ for all K-points, and so V is a locally free $\mathscr{O}_{\mathrm{crit}(g_n)}$-module of rank 1, and thus $\mathscr{G}$ is locally free of rank (n, n).

Let $\mathrm{Spec}\,(K) \hookrightarrow \mathrm{crit}(g_n)$ be the inclusion of a point, and let $i\colon X \times \mathrm{Spec}\,(K) \hookrightarrow X \times \mathrm{crit}(g_n)$ be the induced inclusion. The $\mathscr{O}_{X \times \mathrm{crit}(g_n)}$-module $\Psi(\mathscr{G})$ is a coherent sheaf after pullback along i, and so is a coherent sheaf in [**5**, Lemma 4.3]. The coherent sheaf $\Psi(\mathscr{L})$ is equipped with a tautological map from $\mathscr{O}_{X \times \mathrm{crit}(g_n)}$, surjective by stability, inducing the tautological surjective map

$$\mathtt{fr}\colon \mathscr{I}_{C_0} \boxtimes \mathscr{O}_{\mathrm{crit}(g_n)} \twoheadrightarrow \mathscr{G}.$$

The condition on $\rho(b_2)$ defining $\mathcal{N}_n^\circ$ implies that each $i^*\Psi(\mathscr{G})$ has support away from $\pi^{-1}(\infty)$. Since $\mathrm{crit}(f_n) = \mathrm{crit}(g_n) \cap \mathcal{N}_n^\circ$, the inverse image sheaf of $\Psi(\mathscr{G})$ on $\mathrm{crit}(f_n) \times \mathbb{A}^3$ is a vector bundle on $\mathrm{crit}(f_n)$, equipped with a surjection from $\mathscr{I}_L \boxtimes \mathscr{O}_{\mathrm{crit}(f_n)}$. This defines the map $\mathrm{crit}(f_n) \to Q_L^n$. The inverse is defined similarly: given a family of surjections $\mathscr{I}_L \twoheadrightarrow \mathscr{F}$, we obtain a family of surjections $\mathscr{I}_{C_0} \to u_*\mathscr{I}_L \to u_*\mathscr{F}$. $\qquad\square$

This completes the proof of Theorem A from the Introduction.

3.2. *Relative virtual motives*

Via the coherent sheaf $\Psi(\mathscr{G})$ of $\mathscr{O}_{X \times \mathrm{crit}(g_n)}$-modules constructed in the proof, we obtain the map

$$\mathbf{hc}_X\colon Q_{C_0}^n \to \mathrm{Sym}^n(X) \tag{2.4}$$

extending the map (2.1). It is again a special case of the Quot-to-Chow map [**35**, Corollary 7.15]. In particular, we can write

$$Q_L^n \cong Q_{C_0}^n \times_{\mathrm{Sym}^n(X)} \mathrm{Sym}^n(\mathbb{A}^3) \tag{2.5}$$

via the map $\mathbf{hc}_X$ and the inclusion $u\colon \mathbb{A}^3 \to X$. Via projection to the $\mathrm{Sym}^n(\mathbb{A}^3)$-factor, we recover the map

$$\mathbf{hc}_{\mathbb{A}^3}\colon Q_L^n \to \mathrm{Sym}^n(\mathbb{A}^3).$$

Where it is clear which of the two Hilbert–Chow maps we mean, we will drop the subscript.

REMARK 2.5. In a little more detail, at a K-valued point of $Q_{C_0}^n$, the corresponding $K(Q_{\mathrm{con}}, W_{\mathrm{con}})$-module admits a filtration by $(1, 1)$-dimensional (ζ_1, ζ_2)-stable $K(Q_{\mathrm{con}}, W_{\mathrm{con}})$-modules, that is, quadruples $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \mathbb{C}^4$ such that $\beta_1 \neq 0$ or $\beta_2 \neq 0$, modulo the equivalence relation

$$(\alpha_1, \alpha_2, \beta_1, \beta_2) \sim (z \cdot \alpha_1, z \cdot \alpha_2, z^{-1} \cdot \beta_1, z^{-1} \cdot \beta_2)$$
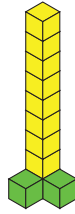
FIGURE 3 (colour online). *A singular point of the Quot scheme Q_L^2.*

for $z \in \mathbb{C}^*$. This is the fine moduli space of point sheaves on X, and we identify it with X. Then taking the union of the supports of the subquotients in the filtration gives the map (2.4) to $\mathrm{Sym}^n(X)$.

Theorem 2.2 (A) has the following immediate consequence.

COROLLARY 2.6. *The function $f_n \colon \mathcal{N}_n^\circ \to \mathbb{A}^1$ induces a relative virtual motive*

$$\mathbb{L}^{-\frac{2n^2+3n}{2}}[-\phi_{f_n}]_{Q_L^n} \in \mathcal{M}_{Q_L^n}^{\hat{\mu}},$$

that is the pullback under the inclusion $Q_L^n \hookrightarrow Q_{C_0}^n$ of the relative virtual motive

$$\mathbb{L}^{-\frac{2n^2+3n}{2}}[-\phi_{g_n}]_{Q_{C_0}^n} \in \mathcal{M}_{Q_{C_0}^n}^{\hat{\mu}}.$$

Proof. The space $\mathcal{N}_n$ is a smooth scheme of dimension $2n^2 + 3n$, so the machinery recalled in Section 1.6 gives the relative motive

$$\mathsf{MF}_{\mathcal{N}_n, g_n} = \mathbb{L}^{-\frac{2n^2+3n}{2}}[-\phi_{g_n}]_{Q_{C_0}^n}$$

according to (1.10). The statement follows since f_n is defined as the restriction of g_n to the smooth open subscheme $\mathcal{N}_n^\circ \subset \mathcal{N}_n$. $\qquad\square$

According to Notation 1.16, we can write down the absolute motives

$$[Q_{C_0}^n]_{\mathrm{vir}} = \int_{Q_{C_0}^n} \mathbb{L}^{-\frac{2n^2+3n}{2}}[-\phi_{g_n}]_{Q_{C_0}^n}$$

$$[Q_L^n]_{\mathrm{vir}} = \int_{Q_L^n} \mathbb{L}^{-\frac{2n^2+3n}{2}}[-\phi_{f_n}]_{Q_L^n}. \tag{2.6}$$

We end this subsection with two examples of the geometric and motivic behaviour of Q_L^n for low n.

EXAMPLE 2.7. The scaling action of the torus $\mathbb{G}_m^3$ on $\mathbb{A}^3$ lifts to an action on Q_L^n. Let us consider the Quot scheme Q_L^2. We will exhibit a singular point belonging to the torus fixed locus. First of all, we have $\dim Q_L^2 = 6$. Consider the point $p \in Q_L^2$ corresponding to

$$\mathscr{I}_Z = (x^2, y^2, xy, xz, yz) \subset \mathscr{I}_L \subset \mathbb{C}[x, y, z].$$

This is depicted in Figure 3. It is easy to check that $\mathrm{Hom}_{\mathbb{A}^3}(\mathscr{I}_Z, \mathscr{I}_L/\mathscr{I}_Z)$, the tangent space of Q_L^2 at p, is 10-dimensional, so that p is a singular point.

EXAMPLE 2.8. The Hilbert scheme $\mathrm{Hilb}^n \mathbb{A}^3$ is non-singular if $n \leqslant 3$ and singular otherwise, whereas Q_L^n is already singular if $n = 2$, by Example 2.7. Let us fix $n = 1$, so that both trace

functions (the ones giving rise to $\mathrm{Hilb}^1 \mathbb{A}^3$ and to Q_L^1, respectively) vanish, and the virtual motives are a shift of the naive motives by $\mathbb{L}^{-3/2}$. On the Hilbert scheme side we have, using Remark 1.18,

$$\left[\mathrm{Hilb}^1 \mathbb{A}^3\right]_{\mathrm{vir}} = \mathbb{L}^{-\frac{3}{2}} \cdot \mathbb{L}^3 = \mathbb{L}^{\frac{3}{2}},$$

while on the Quot scheme side we find, using that $Q_L^1 = \mathrm{Bl}_L \mathbb{A}^3$,

$$\left[Q_L^1\right]_{\mathrm{vir}} = \mathbb{L}^{-\frac{3}{2}} \cdot \left[\mathrm{Bl}_L \mathbb{A}^3\right]$$

$$= \mathbb{L}^{-\frac{3}{2}} \cdot \left(\left[\mathbb{A}^3 \setminus L\right] + \left[L \times \mathbb{P}^1\right]\right)$$

$$= \mathbb{L}^{\frac{3}{2}} + \mathbb{L}^{\frac{1}{2}}.$$

This is the first instance of formula (0.7).

3.3. Equivariance of the family

Let $\mathcal{R}_n = \mathrm{rMod}_{(n+1,n,1)}(\mathbb{C}\tilde{Q}_{\mathrm{con}})$ be the affine space parameterising framed $\tilde{Q}_{\mathrm{con}}$-modules of dimension vector $(n+1, n, 1)$. Let us set $G = \mathrm{GL}_{n+1} \times \mathrm{GL}_n$ and let $S \subset \Gamma(\mathscr{O}_{\mathcal{R}_n})$ be the subalgebra of functions scaling, under the G-action, as a positive power of the given character realising framed stability. Then by general GIT we have $\mathcal{N}_n = \mathrm{Proj}\, S$, and the natural inclusion $\Gamma(\mathscr{O}_{\mathcal{R}_n})^G \subset S$ induces a *projective* morphism

$$\mathsf{p}_n \colon \mathcal{N}_n \to Y_0 = \mathrm{Spec}\, \Gamma(\mathscr{O}_{\mathcal{R}_n})^G, \tag{2.7}$$

where the affine scheme Y_0 can be viewed as the GIT quotient $\mathcal{R}_n /\!\!/_0 G$ at the trivial character.

PROPOSITION 2.9. *There is an identity*

$$\left[\phi_{g_n}\right] = \left[g_n^{-1}(1)\right] - \left[g_n^{-1}(0)\right] \in \mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$

In particular, $[Q_{C_0}^n]_{\mathrm{vir}}$ *lies in* $\mathcal{M}_{\mathbb{C}}$.

Proof. The 4-dimensional torus $T = \mathbb{G}_m^4$ acts on $\mathcal{N}_n$ by

$$t \cdot (A_1, A_2, B_1, B_2, v) = (t_1 A_1, t_2 A_2, t_3 B_1, t_4 B_2, t_1 t_2 t_3 t_4 v).$$

Moreover, the trace function $g_n \colon \mathcal{N}_n \to \mathbb{A}^1$ is T-equivariant with respect to the primitive character $\chi(t) = t_1 t_2 t_3 t_4$. This means that for all $P \in \mathcal{N}_n$, we have $g_n(t \cdot P) = \chi(t) g_n(P)$. We claim that the induced action on $\mathcal{N}_n$ by the diagonal torus $\mathbb{G}_m \subset T$ is circle compact, that is, it has compact fixed locus and the limits $\lim_{t \to 0} t \cdot P$ exist in $\mathcal{N}_n$ for all $P \in \mathcal{N}_n$. Following the proof of [3, Lemma 3.4], we note that (2.7) is a projective $\mathbb{G}_m$-equivariant map, where Y_0 has a unique $\mathbb{G}_m$-fixed point, and moreover all orbits have this point in their closure. In other words, limits exist in Y_0. Therefore, by properness of p_n, we conclude that the $\mathbb{G}_m$-fixed locus in $\mathcal{N}_n$ is compact and limits exist. This proves the claim.

Then the equation involving g_n follows by part (i) of Theorem 1.20, proved by Behrend, Bryan and Szendrői. In particular, the absolute virtual motive of $Q_{C_0}^n$ carries no monodromy,

$$\left[Q_{C_0}^n\right]_{\mathrm{vir}} = \mathbb{L}^{-\frac{2n^2+3n}{2}}\left[-\phi_{g_n}\right] \in \mathcal{M}_{\mathbb{C}}. \qquad \square$$

3.4. A direct critical locus description

There is a way of writing down the above critical locus description of Q_L^n that does not involve pulling back from a moduli space of representations for the non-commutative conifold.[†]

[†]Although, to be precise, for the proof that the description really does recover Q_L^n, the only method we offer will rely on the geometry of the non-commutative conifold.
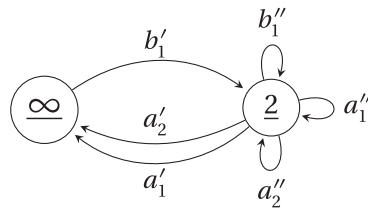
FIGURE 4. *The quiver Q_r.*

Consider again the space $\mathcal{N}_n^\circ$. In the definition, we have imposed an open condition on representations in $\mathcal{N}_n$, namely

(1) $\rho(b_2)$ is injective.

We have seen above that the points of $\mathrm{crit}(f_n)$ correspond to $\tilde{Q}_{\mathrm{con}}$-representations ρ satisfying the extra condition that the short exact sequence of vector spaces

$$0 \to \mathrm{Image}(\rho(b_2)) \to \rho_1 \to \mathrm{Coker}(\rho(b_2)) \to 0$$

is induced by a short exact sequence of $\mathbb{C}(Q_{\mathrm{con}}, W_{\mathrm{con}})$-modules, and in particular, $\mathrm{Image}(\rho(b_1)) \subset \mathrm{Image}(\rho(b_2))$. Stability then imposes the extra open condition

(2) $\mathrm{Span}(\mathrm{Image}(\rho(\iota)), \mathrm{Image}(\rho(b_2))) = \rho_1$.

Let $\mathcal{N}_n^{\circ\circ} \subset \mathcal{N}_n$ be the open substack defined by the open conditions (1) and (2) above. For a $\mathbb{C}\tilde{Q}_{\mathrm{con}}$-module parameterised by a point in $\mathcal{N}_n^{\circ\circ}$, there is a canonical direct sum decomposition $\rho_1 \cong \mathrm{Image}(\rho(b_2)) \oplus \mathrm{Image}(\rho(\iota))$ and moreover an identification between $\mathrm{Image}(\rho(b_2))$ and ρ_2 via the action of $\rho(b_2)$, and an identification of $\mathrm{Image}(\rho(\iota))$ and ρ_∞ via $\rho(\iota)$. It follows that there is an isomorphism

$$\Gamma \colon \mathcal{N}_n^{\circ\circ} \xrightarrow{\sim} \mathrm{rMod}_{(n,1)}^{\zeta'}(Q_r)/\mathrm{GL}_n \tag{2.8}$$

with the fine moduli space of ζ'-semistable $(n,1)$-dimensional $\mathbb{C}Q_r$-representations of the 'reduced' quiver Q_r depicted in Figure 4.

In (2.8), we have put $\zeta' = (-1, n)$. In a little more detail, given ρ a $\mathbb{C}\tilde{Q}_{\mathrm{con}}$-module corresponding to a point in $\mathcal{N}_n^{\circ\circ}$, we set

$$\rho_{\underline{\infty}} = \mathrm{Image}(\rho(\iota))$$

$$\rho_{\underline{2}} = \rho_2 \quad (\cong \mathrm{Image}(\rho(b_2))).$$

Then for $s = 1, 2$, we set

$$\rho(a_s') = \rho(a_s)|_{\mathrm{Image}(\rho(\iota))}$$

$$\rho(b_1') = \pi_{\mathrm{Image}(\rho(\iota))} \circ \rho(b_1)$$

$$\rho(b_1'') = \pi_{\mathrm{Image}(\rho(b_2))} \circ \rho(b_1)$$

$$\rho(a_s'') = \rho(a_s)|_{\mathrm{Image}(\rho(b_2))},$$

where π denotes the projection maps. Then

$$h_n := f_n\big|_{\mathcal{N}_n^{\circ\circ}} = \mathrm{Tr}(W_r) \circ \Gamma, \tag{2.9}$$

for W_r the potential

$$W_r = a_1'' b_1' a_2' - a_2'' b_1' a_1' + a_1'' b_1'' a_2'' - a_2'' b_1'' a_1''.$$

PROPOSITION 2.10. *There is an isomorphism of schemes*

$$Q_L^n \cong \mathrm{crit}(h_n).$$

The result follows directly from our analysis of the non-commutative conifold. From that description, we see that in the stable locus, we have $\rho(b_1') = 0$. Then the superpotential relations become

$$a_2' a_1'' = a_1' a_2''$$
$$a_1'' a_2'' = a_2'' a_1''$$
$$a_1'' b_1'' = b_1'' a_1''$$
$$a_2'' b_1'' = b_1'' a_2''$$

and the stable locus is identified with the moduli space of triples (v_x, v_y, N) where N is a $\mathbb{C}[x, y, z]$-module, $v_x, v_y \in N$ generate under the action of $\mathbb{C}[x, y, z]$, and $x v_y = y v_x$. This is the same data as a $\mathbb{C}[x, y, z]$-linear surjection $\theta \colon (x, y) \twoheadrightarrow N$, by setting $v_x = \theta(x)$ and $v_y = \theta(y)$.

The next result contains the statement of Theorem 2.2 (B).

PROPOSITION 2.11. *There is an identity*

$$\left[\phi_{f_n}\right] = \left[h_n^{-1}(1)\right] - \left[h_n^{-1}(0)\right] \in \mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$

In particular, $[Q_L^n]_{\mathrm{vir}}$ *lies in* $\mathcal{M}_{\mathbb{C}}$. *Moreover, the relative motive* $[\phi_{f_n}]_{\mathrm{Sym}^n(\mathbb{A}^3)}$ *belongs to the subring* $\mathcal{M}_{\mathrm{Sym}^n(\mathbb{A}^3)} \subset \mathcal{M}_{\mathrm{Sym}^n(\mathbb{A}^3)}^{\hat{\mu}}$ *of monodromy-free motives.*

Proof. The first statement is proved the same way as Proposition 2.9, noting that by construction, $[\phi_{f_n}] = [\phi_{h_n}]$, and by (2.8), h_n is a potential on a GIT moduli space of quiver representations admitting a contracting $\mathbb{C}^*$-action for which $\mathrm{Tr}(W_r)$ has weight three.

Next, we claim that the hypersurface $h_n^{-1}(0) \subset \mathcal{N}_n^{\circ\circ}$ is reduced. This will follow from the claim that the variety cut out by the function $\mathrm{Tr}(W_{\mathrm{con}})$ on $\mathrm{rMod}_{(n+1,n,1)}^{\zeta}(\tilde{Q}_{\mathrm{con}})$ is reduced, which is in turn weaker than the claim that the variety cut out by $\mathrm{Tr}(W_{\mathrm{con}})$ on $\mathrm{rMod}_{(n+1,n,1)}(\tilde{Q}_{\mathrm{con}})$ is reduced. This variety is affine, and so it is enough to show that the function $\mathrm{Tr}(W_{\mathrm{con}})$ is reduced. With respect to the $\mathbb{T}$-grading this function has degree $(1, 1, 1, 1)$, and so it cannot be factorised with a repeated factor.

The statement regarding $[\phi_{f_n}]_{\mathrm{Sym}^n(\mathbb{A}^3)}$ then follows from part (ii) of Theorem 1.20. $\qquad\square$

The proof of Theorem 2.2 is complete.

4. *Relative DT theory of the conifold*

The goal of this section is to express the motive $[Q_{C_0}^n]_{\mathrm{vir}}$ defined in (2.6) in terms of motivic contributions coming from the 'punctual loci' inside $Q_{C_0}^n$. For $n \in \mathbb{N}$, we define:

(i) $\mathcal{P}_{\mathrm{pt}}^n \subset Q_{C_0}^n$, the subvariety parameterising quotients $\mathscr{I}_{C_0} \twoheadrightarrow \mathscr{F}$ such that $\mathscr{F}$ is set-theoretically supported at a *fixed* single point, away from C_0; and

(ii) $\mathcal{P}_{\mathrm{crv}}^n \subset Q_{C_0}^n$, the subvariety parameterising quotients $\mathscr{I}_{C_0} \twoheadrightarrow \mathscr{F}$ such that $\mathscr{F}$ is set-theoretically supported at a *fixed* single point on the curve C_0.

Items (i) and (ii), respectively, will lead (cf. Definition 3.16) to the definition of motivic weights

$$\Omega_{\mathrm{pt}}^n, \ \Omega_{\mathrm{crv}}^n \ \in \mathcal{M}_{\mathbb{C}},$$

which will be the basic building blocks for the construction of a virtual motive of the Quot scheme Q_C^n for an arbitrary curve $C \subset Y$ in a 3-fold.

4.1. Motivic stratifications I: partially punctual strata

In what follows, we streamline proofs by deducing results for the embedded curve $C_0 \subset X$ from the embedded curve $L \subset \mathbb{A}^3$. In order to achieve this, we introduce two automorphisms of $\mathcal{N}_n$ (see (2.3)):

$$\overline{\sigma}_\lambda \cdot (\rho(\iota), \rho(a_1), \rho(a_2), \rho(b_1), \rho(b_2)) = (\rho(\iota), \rho(a_1), \rho(a_2), \rho(b_1) + \lambda\rho(b_2), \rho(b_2)),$$

$$\overline{\xi}_\lambda \cdot (\rho(\iota), \rho(a_1), \rho(a_2), \rho(b_1), \rho(b_2)) = (\rho(\iota), \rho(a_1), \rho(a_2), \rho(b_1), \rho(b_2) + \lambda\rho(b_1)).$$

Note that both of these automorphisms preserve $\operatorname{Tr}(W_{\operatorname{con}})$ and the ζ-stable locus, and so they induce automorphisms of $Q_{C_0}^n$ preserving $[\phi_{g_n}]_{Q_{C_0}^n}$. The two automorphisms of $\mathbb{C}^4$ defined by $\sigma_\lambda(x, y, z, w) = (x, y, z + \lambda w, w)$ and $\xi_\lambda(x, y, z, w) = (x, y, z, w + \lambda z)$ induce automorphisms of X commuting with $\mathbf{hc}_X$ in the sense that $\operatorname{Sym}^n(\sigma_\lambda) \circ \mathbf{hc}_X = \mathbf{hc}_X \circ \overline{\sigma}_\lambda$ and $\operatorname{Sym}^n(\xi_\lambda) \circ \mathbf{hc}_X = \mathbf{hc}_X \circ \overline{\xi}_\lambda$ (see Remark 2.5).

LEMMA 3.1. Let $\alpha \in \mathcal{M}_{\operatorname{Sym}^n(X)}^{\hat{\mu}}$ satisfy:

(1) $(\operatorname{Sym}^n(\mathbb{A}^3) \hookrightarrow \operatorname{Sym}^n(X))^* \alpha = 0$;
(2) $\sigma_\lambda^* \alpha = \alpha$ for all $\lambda \in \mathbb{C}$;
(3) $\xi_\lambda^* \alpha = \alpha$ for all $\lambda \in \mathbb{C}$.

Then $\alpha = 0$.

Proof. The space $\operatorname{Sym}^n(X)$ has an open cover by open subsets of the form $U_t = \operatorname{Sym}^n(X \setminus \pi^{-1}(t))$ for $t \in \mathbb{P}^1$: if $\gamma \in \operatorname{Sym}^n(X)$ is in the complement of $\cup_{t \in T} U_t$ for $|T| > n$, then the support of γ is spread across more than n fibres of the projection $\pi \colon X \to \mathbb{P}^1$. By our assumptions, the restriction of α to U_t is zero. By the scissor relations, it follows that $\alpha = 0$. $\square$

COROLLARY 3.2. Let $\alpha, \beta \in \mathcal{M}_{\operatorname{Sym}^n(X)}^{\hat{\mu}}$ satisfy conditions (2) and (3) from Lemma 3.1, and $(\operatorname{Sym}^n(\mathbb{A}^3) \hookrightarrow \operatorname{Sym}^n(X))^* \alpha = (\operatorname{Sym}^n(\mathbb{A}^3) \hookrightarrow \operatorname{Sym}^n(X))^* \beta$. Then $\alpha = \beta$.

The next lemma is an incarnation of the fact that taking box sum with a quadratic function (locally) does not change the vanishing cycle complex, while for global triviality one has to be mindful of monodromy. The implication is that we can pass to a 'minimal' potential at the expense of keeping track of some extra monodromy data, which in the Kontsevich–Soibelman framework, and then elsewhere, is called orientation data. In the language of potentials on 3-Calabi–Yau categories, one can think of the proof of part (2) of Lemma 3.7 as working by proceeding to a 'partially minimised' potential.

LEMMA 3.3. Let $\pi \colon \operatorname{Tot}(V) \to X$ be the projection from the total space of a vector bundle on a smooth connected variety X, and let $f \colon \operatorname{Tot}(V) \to \mathbb{C}$ be a function that is quadratic in the fibres, that is, $f(z \cdot v) = z^2 f(v)$ for $z \in \mathbb{C}$, where we have given $\operatorname{Tot}(V)$ the scaling action of $\mathbb{C}^*$. Assume $X = \operatorname{crit}(f)$, where we have identified X with the zero section of $\operatorname{Tot}(V)$.

(1) For $x \in X$, there is an equality

$$[x]_{\operatorname{vir}} = \mathbb{L}^{-\frac{\dim(X)}{2}} \in \mathcal{M}_\mathbb{C} \subset \mathcal{M}_\mathbb{C}^{\hat{\mu}}. \tag{3.1}$$

(2) Assume that $V \cong V_- \oplus V_+$ where $f|_{\operatorname{Tot}(V_-)} = f|_{\operatorname{Tot}(V_+)} = 0$. Then there is an equality

$$\left[-\phi_f\right]_X = \left[\operatorname{id} \colon X \to X\right] \in \mathcal{M}_X \subset \mathcal{M}_X^{\hat{\mu}}. \tag{3.2}$$

(3) *Under the conditions of* (2), *let G be a finite group acting freely on X, let $V \cong V_- \oplus V_+$ be a direct sum decomposition of G-equivariant vector bundles, and assume that f is G-invariant. Then*

$$\left[-\phi_f\right]_X^G = \left[\mathrm{id}\colon X \to X\right] \in \mathcal{M}_X^G \subset \mathcal{M}_X^{G \times \hat{\mu}}.$$

In part (3), we include the assumption that G acts freely so that we may apply Proposition 1.15.

Proof. Part (1): Zariski locally, we can write $\mathrm{Tot}(V) = X \times \mathbb{A}^m$, and $f = \sum_{1 \leqslant i,j \leqslant m} f_{ij} t_i t_j$ for $F = (f_{ij})_{1 \leqslant i,j \leqslant m}$ a matrix of functions on X, and $t_1, \ldots, t_m$ coordinates on $\mathbb{A}^m$. We can, and will, assume that $f_{ij} = f_{ji}$ for all i and j. For every closed point $x \in X$, the functions $(\partial/\partial t_i)f$ and $(\partial/\partial s)f$, for s local coordinates at x, generate

$$(t_1, \ldots, t_m)\mathscr{O}_{X,x}[t_1, \ldots, t_m]/(t_1, \ldots, t_m)^2 \mathscr{O}_{X,x}[t_1, \ldots, t_m].$$

On the other hand, $(\partial/\partial s)f = 0$ in this quotient. It follows that $\det(f_{ij})_{1 \leqslant i,j \leqslant m}$ is an invertible function on X. In an analytic open neighborhood of x, we may thus apply a change of coordinates and obtain $f_{ij} = \delta_{i-j}$, and then the first result follows from direct calculation, or the explicit formula of Denef and Loeser [**12**, Theorem 4.3.1].

Part (2+3): By nondegeneracy of F, if $V \cong V_- \oplus V_+$ is a decomposition as in the statement of the lemma, then $\dim(V_-) = m/2$, and the symmetric bilinear form F establishes an isomorphism of vector bundles $V_+ \cong (V_-)^*$. Replacing V_+ by $(V_-)^*$ we obtain

$$F = \begin{pmatrix} 0 & \mathrm{Id}_{m/2 \times m/2} \\ \mathrm{Id}_{m/2 \times m/2} & 0 \end{pmatrix}.$$

Letting $\mathbb{C}^*$ act by scaling V_+, the function f is equivariant of weight one. The proof of [**8**, Theorem 5.9] shows that if we have a smooth variety Y, an integer r and a function $g = \sum g_i t_i$ on $Y \times \mathbb{A}^r$, then $\pi_{Y,!}([\phi_g]_{g^{-1}(0)}) = [V(g_1, \ldots, g_r)]$. Moreover the proof generalises without modification to the case in which $\pi_Y \colon Y \times \mathbb{A}^r \to Y$ is a projection from the total space of a G-equivariant vector bundle. The second and the third parts follow, putting $Y = X \times \mathrm{Tot}(V_-)$. □

REMARK 3.4. If we relax the condition on f, part (2) may fail. For instance, consider the function $f = zx^2$ on $\mathrm{Spec}\,\mathbb{C}[z^{\pm 1}, x] \cong \mathbb{C}^* \times \mathbb{A}^1_{\mathbb{C}}$. It is easy to check that the associated virtual motive satisfies $[\mathbb{C}^*]_{\mathrm{vir}} = 0$.

REMARK 3.5. Likewise, if we relax the condition on the G-action on V, part (3) may fail. For instance, let G be the cyclic group of order 2, let $\mathrm{Tot}(V) = \mathrm{Spec}\,\mathbb{C}[x,y] \cong \mathbb{A}^2$, with $f = xy$ and G swapping x and y. Then it is easy to check that for the associated equivariant critical structure, we have $[\mathrm{pt}]_{\mathrm{vir}} = [G] - [\mathrm{pt}]$, where on the right-hand side $[G]$ is a pair of points, permuted by the G-action, and pt carries the trivial G-action. This is a consequence of the fact that the G-action on $H(\mathbb{A}^2, \Phi_{xy})$ is the sign representation, see, for instance, [**29**, Lemma 4.1].

The ultimate goal of this section is to show that as a relative motive over $\mathrm{Sym}(X)$, the virtual motive

$$\sum_{n \geqslant 0} (-1)^n \left[Q_{C_0}^n \xrightarrow{\mathbf{hc}_X} \mathrm{Sym}^n(X) \right]_{\mathrm{vir}} \tag{3.3}$$

is generated under $\mathrm{Exp}_{\cup}$ by motives that are supported on the punctual locus, and constant away from C_0, as well as constant on C_0. To get to this point will require some work, and we break the proof up by showing first that (3.3) is at least generated by motives supported

on a 'partially punctual' locus. By Lemma 3.1, it is enough to prove the analogous result for $\sum_n (-1)^n [Q_L^n \xrightarrow{\mathbf{hc}} \mathrm{Sym}^n(\mathbb{A}^3)]_{\mathrm{vir}}$.

We explain here what we mean by the partially punctual locus. Consider again the map

$$\mathbf{hc} \colon Q_L^n \to \mathrm{Sym}^n(\mathbb{A}^3).$$

The embedding of the xy-plane in $\mathbb{A}^3$ induces an embedding of varieties

$$\iota_n \colon \mathrm{Sym}^n(\mathbb{A}^2) \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^3) \tag{3.4}$$

and we denote by $Q_L^{\bullet,\bullet,\mathrm{nilp},n}$ the preimage of $\mathrm{Sym}^n(\mathbb{A}^2)$ under $\mathbf{hc}$, that is,

$$Q_L^{\bullet,\bullet,\mathrm{nilp},n} = \mathbf{hc}^{-1}(\mathrm{Sym}^n(\mathbb{A}^2)) \subset Q_L^n. \tag{3.5}$$

The map ι_n is the inclusion of the subspace of configurations of points which all have z-coordinate 0, which explains the notation $(\bullet, \bullet, \mathrm{nilp})$ — the scheme-theoretic version of this condition is that the operator corresponding to the action of z is nilpotent. So, ordering the operators corresponding to x, y, z alphabetically, the first two are unconstrained, and the third is nilpotent.

NOTATION 3.6. More generally, for $\#_x, \#_y, \#_z \in \{\mathrm{uni}, \mathrm{nilp}, \bullet\}$ we define $Q_L^{\#_x,\#_y,\#_z,n} \subset Q_L^n$ by imposing the closed conditions that for $w \in \{x, y, z\}$ the operator $\cdot w$ is nilpotent if $\#_w = \mathrm{nilp}$, or unipotent if $\#_w = \mathrm{uni}$ — so, for instance, $Q_L^n = Q_L^{\bullet,\bullet,\bullet,n}$.

There is an action of $\mathbb{A}^1$ on $\mathrm{Sym}^n(\mathbb{A}^3)$ via simultaneous addition on the z coordinate of all points in a configuration, and we let

$$\tilde{\iota}_n \colon \mathrm{Sym}^n(\mathbb{A}^2) \times \mathbb{A}^1 \to \mathrm{Sym}^n(\mathbb{A}^3) \tag{3.6}$$

be the restriction of this action. It is again an embedding, this time of the subvariety of n-tuples of points which all have the same z-coordinate (not necessarily zero). It is this locus that we call 'partially punctual'. Finally, consider the morphism

$$q_z \colon \mathrm{Sym}^n(\mathbb{A}^3) \to \mathrm{Sym}^n(\mathbb{A}^1)$$

obtained by projecting onto the z coordinate. We define $\mathbf{hc}^z = q_z \circ \mathbf{hc}$. More generally, for $a_1, \ldots, a_r$ distinct elements of $\{x, y, z\}$ and $T \subset Q_L^n$, we denote by

$$\mathbf{hc}^{a_1 \cdots a_r} \colon T \to \mathrm{Sym}^n(\mathbb{A}^r)$$

the map given by composing the restriction of $\mathbf{hc}$ to T with the projection $\mathrm{Sym}(\mathbb{A}^3) \to \mathrm{Sym}(\mathbb{A}^r)$ induced by the projection $\mathbb{A}^3 \to \mathbb{A}^r$ defined by forgetting the coordinates not contained in $a_1, \ldots, a_r$.

The space $\mathrm{Sym}^n(\mathbb{A}^1)$ is stratified according to partitions of n, and for $\alpha \vdash n$ we denote by $Q_L^{\bullet,\bullet,\alpha} \subset Q_L^n$ the corresponding stratum of the stratification of Q_L^n given by pulling back along $\mathbf{hc}^z \colon Q_L^n \to \mathrm{Sym}^n(\mathbb{A}^1)$. So, for instance,

$$Q_L^{\bullet,\bullet,(n)} = (\mathrm{Sym}^n(\mathbb{A}^2) \times \mathbb{A}^1) \times_{\mathrm{Sym}^n(\mathbb{A}^3)} Q_L^n \subset Q_L^n$$

is the fibre product of $\tilde{\iota}_n$ and $\mathbf{hc}$.

LEMMA 3.7. (1) *There is an equality*

$$\tilde{\iota}_n^* \big[Q_L^{\bullet,\bullet,(n)} \xrightarrow{\mathbf{hc}} \mathrm{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}} = \iota_n^* \big[Q_L^n \xrightarrow{\mathbf{hc}} \mathrm{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}} \boxtimes \big[\mathbb{A}^1 \xrightarrow{\mathrm{id}_{\mathbb{A}^1}} \mathbb{A}^1 \big]$$

in $\mathcal{M}_{\mathrm{Sym}^n(\mathbb{A}^2) \times \mathbb{A}^1}$. *In other words, the motive on the left-hand side is constant in the* $\mathbb{A}^1$*-factor.*

(2) *Moreover, for $\alpha \vdash n$ there is an equality*

$$\left[Q_L^{\bullet,\bullet,\alpha} \to \mathrm{Sym}^n(\mathbb{A}^3) \right]_{\mathrm{vir}} = \cup_! \pi_{G_\alpha} j_\alpha^* \left(\underset{i|\alpha_i \neq 0}{\boxtimes} \left[Q_L^{\bullet,\bullet,(i)} \to \mathrm{Sym}^i(\mathbb{A}^2) \times \mathbb{A}^1 \right]_{\mathrm{vir}}^{\otimes \alpha_i} \right) \quad (3.7)$$

where j_α is the G_α-equivariant inclusion of the complement of the pullback of the big diagonal under the G_α-equivariant projection onto z-coordinates:

$$\prod_{i|\alpha_i \neq 0} (\mathrm{Sym}^i(\mathbb{A}^2) \times \mathbb{A}^1)^{\alpha_i} \to \mathbb{A}^{\sum \alpha_i}.$$

Here G_α is the automorphism group for the partition type α, and $\cup$ is the union of points map on $\mathrm{Sym}(\mathbb{A}^3)$.

Before we begin the proof, we give some guidance for how to read the right-hand side of (3.7). Firstly, recall from Section 1.5 that the infinite union of algebraic varieties $\mathrm{Sym}(\mathbb{A}^3)$ has a symmetric monoidal structure $\cup$, given by taking unions of unordered points with multiplicity. We consider $\mathrm{Sym}^i(\mathbb{A}^2) \times \mathbb{A}^1$ as a subvariety of $\mathrm{Sym}^i(\mathbb{A}^3)$ via $\tilde{\iota}_i$. We abuse notation by writing $\cup$ again for the map $\mathrm{Sym}(\mathbb{A}^3)^m \to \mathrm{Sym}(\mathbb{A}^3)$ taking an m-tuple of sets of unordered points with multiplicity to their union. The term in big round brackets on the right is a G_α-equivariant motive via Lemma 1.6.

Proof. Consider again the space $\mathcal{N}_n^{\circ\circ} \subset \mathcal{N}_n$ from Section 2.4. We define the subspace $\mathcal{T}_n \subset \mathcal{N}_n^{\circ\circ}$ by the condition that $\mathrm{Tr}(\rho(b_1'')) = 0$. Then there is an isomorphism

$$\mathcal{T}_n \times \mathbb{A}^1 \overset{\sim}{\to} \mathcal{N}_n^{\circ\circ}$$

given by

$$(\rho(a_1''), \rho(a_2''), \rho(b_1''), \rho(a_1'), \rho(a_2'), \rho(b_1'), t) \mapsto (\rho(a_1''), \rho(a_2''), \rho(b_1'') + t \cdot \mathrm{Id}_{n \times n}, \rho(a_1'), \rho(a_2'), \rho(b_1')),$$

and the function $h_n = f_n|_{\mathcal{N}_n^{\circ\circ}}$ (cf. (2.9) and Proposition 2.10) is pulled back from a function $\overline{h}_n$ on $\mathcal{T}_n$. The stratification by partition type of $\rho(b_1'')$ for $Q_L^n \subset \mathcal{N}_n^{\circ\circ}$ is induced by a stratification of $\mathcal{T}_n$: for α a partition of n, define $\mathcal{T}_\alpha \subset \mathcal{T}_n$ to be the locally closed subvariety whose $\mathbb{C}$-points correspond to Q_r-representations for which the partition type of the generalised eigenvalues of $\rho(b_1'')$ are given by α. Then $\mathrm{crit}(\overline{h}_n) \cap \mathcal{T}_{(n)} = Q_L^{\bullet,\bullet,\mathrm{nilp},n}$, that is, the isomorphism $\mathcal{T}_n \times \mathbb{A}^1 \overset{\sim}{\to} \mathcal{N}_n^{\circ\circ}$ sends $(\mathrm{crit}(\overline{h}_n) \cap \mathcal{T}_{(n)}) \times \{0\}$ onto $Q_L^{\bullet,\bullet,\mathrm{nilp},n}$.

Let $\mathrm{Sym}_0^n(\mathbb{A}^1) \subset \mathrm{Sym}^n(\mathbb{A}^1)$ be the closed subvariety of n-tuples summing to zero. Let

$$\lambda : \mathcal{N}_n^{\circ\circ} \to \mathrm{Sym}^n(\mathbb{A}^1)$$

be the morphism taking a module ρ to the eigenvalues (with multiplicity) of $\rho(b_1'')$. Note that $\lambda|_{Q_L^n} = \mathbf{hc}^z$. Then the first equality follows from the commutativity of

$$\begin{array}{ccc}
\mathcal{T}_n \times \mathbb{A}^1 & \xrightarrow{\;\cong\;} & \mathcal{N}_n^{\circ\circ} \\
{\scriptstyle \lambda|_{\mathcal{T}_n} \times \mathrm{id}_{\mathbb{A}^1}} \downarrow & & \downarrow {\scriptstyle \lambda} \\
\mathrm{Sym}_0^n(\mathbb{A}^1) \times \mathbb{A}^1 & \xrightarrow{((t_1,\ldots,t_n),t) \mapsto (t_1+t,\ldots,t_n+t)} & \mathrm{Sym}^n(\mathbb{A}^1)
\end{array}$$

and the fact that, pulling back along the top isomorphism, the function h_n becomes $\overline{h}_n \oplus 0$.

Let $\alpha = (1^{\alpha_1} \cdots r^{\alpha_r}) \vdash n$ be a partition. We write $l(\alpha) = \sum_{i \leq r} \alpha_i$ for the total number of parts of α. For the proof of the second part of the lemma, we adapt the proof of [3,
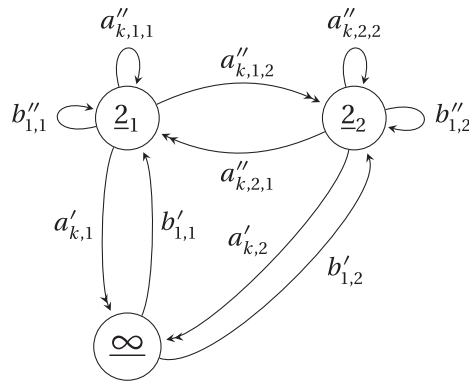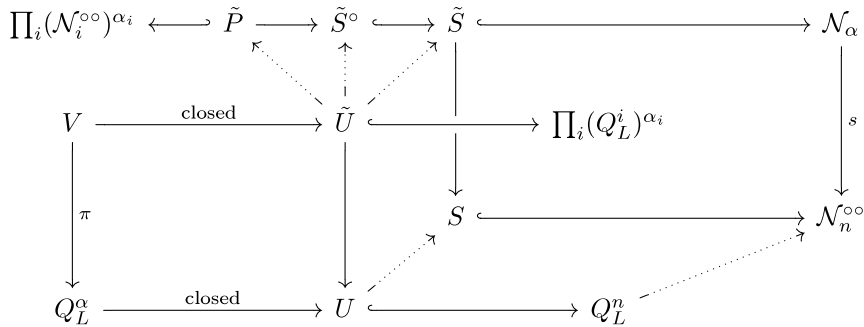
FIGURE 5. *The quiver Q_α, for the case in which $l(\alpha) = 2$. The index k varies in $\{1, 2\}$.*

Proposition 2.6], via the following commutative diagram, which will take some time to define and describe.

All of the dotted arrows correspond to the inclusion of the critical locus of a function. All hooked arrows denote *open* inclusions.

The open subspace $\tilde{U} \subset \prod_i (Q_L^i)^{\alpha_i}$ is defined by the condition that the set of eigenvalues of $\rho(b_1'')$ in each of the $l(\alpha)$ factors of the product are distinct from the set of eigenvalues for $\rho(b_1'')$ in any other factor in the product. The open subset $U \subset Q_L^n$ is the image of the étale map $\tilde{U} \to Q_L^n$ given by sending the $l(\alpha)$-tuple $\{\mathscr{I}_L \twoheadrightarrow \mathscr{F}_j\}_{j \leqslant l(\alpha)}$ to

$$\mathscr{I}_L \twoheadrightarrow \bigoplus_{j \leqslant l(\alpha)} \mathscr{F}_j.$$

The closed subset $V \subset \tilde{U}$ is defined by the condition that in each of the $l(\alpha)$ factors in the product decomposition there is only one eigenvalue. It is in fact the preimage of $Q_L^\alpha = \mathbf{hc}^{-1}(\mathrm{Sym}^\alpha(\mathbb{A}^3)) \subset U$ under the map $\tilde{U} \to U$.

We form the quiver Q_α (Figure 5) and dimension vector $\mathbf{d}_\alpha$ as follows. Set

$$(Q_\alpha)_0 = \left\{ \underline{2}_1, \ldots, \underline{2}_{l(\alpha)}, \underline{\infty} \right\}$$

$$(Q_\alpha)_1 = \left\{ a_{1,i}', a_{2,i}', b_{1,i}', b_{1,i}'' \right\}_{1 \leqslant i \leqslant l(\alpha)} \cup \left\{ a_{1,i,j}'', a_{2,i,j}'' \right\}_{1 \leqslant i,j \leqslant l(\alpha)}.$$

We set

$$t(a_{1,i}') = t(a_{2,i}') = s(b_{1,i}') = \underline{\infty}$$

$$s(a_{1,i}') = s(a_{2,i}') = t(b_{1,i}') = t(b_{1,i}'') = s(b_{1,i}'') = \underline{2}_i$$

$$t(a''_{1,i,j}) = t(a''_{2,i,j}) = \underline{2}_j$$

$$s(a''_{1,i,j}) = s(a''_{2,i,j}) = \underline{2}_i.$$

We give Q_α the dimension vector $\mathbf{d}_\alpha = (1^{\alpha_1}, \ldots, r^{\alpha_r}, 1)$, that is, we always assign the vertex $\underline{\infty}$ dimension 1, and then the rest of the dimension vector is given by the ordered set of numbers α. We give this quiver the stability condition

$$\zeta_\alpha = (-1, \ldots, -1, n).$$

A (ζ_α-stable) $\mathbb{C}Q_\alpha$-module is the same data as a (ζ'-stable) $\mathbb{C}Q_r$-module ρ, along with an ordered vector space decomposition of the vector space $\rho_{\underline{2}}$ that is preserved by the operation $\rho(b''_1)$. As such, there is a forgetful functor from $\mathbb{C}Q_\alpha$-modules to $\mathbb{C}Q_r$-modules, inducing a generically finite morphism

$$s \colon \mathcal{N}_\alpha = \mathrm{rMod}^{\zeta_\alpha}_{\mathbf{d}_\alpha}(Q_\alpha)/\mathrm{GL}_\alpha \to \mathcal{N}_n^{\circ\circ},$$

which at the level of points amounts to setting

$$s(\rho_{\underline{2}}) = \bigoplus_{i \leqslant l(\alpha)} \rho_{\underline{2}_i}.$$

The scheme is G_α-equivariant, via the obvious G_α-action on the quiver Q_α.

There is a unique potential W_α on Q_α such that $\mathrm{Tr}(W_\alpha)$ is the pullback of $\mathrm{Tr}(W_r)$ under this forgetful map. Precisely, we define

$$W_\alpha = \sum_{1 \leqslant i,j \leqslant l(\alpha)} \left(a''_{1,j,i} b'_{1,j} a'_{2,i} - a''_{2,j,i} b'_{1,j} a'_{1,i} \right) + \sum_{1 \leqslant i,j \leqslant l(\alpha)} \left(a''_{1,j,i} b''_{1,j} a''_{2,i,j} - a''_{2,j,i} b''_{1,j} a''_{1,i,j} \right).$$

We define the open subscheme $\tilde{S} \subset \mathcal{N}_\alpha$ by the condition that for every $i, j \leqslant l(\alpha)$ the endomorphisms $\rho(b''_{1,i})$ and $\rho(b''_{1,j})$ share no eigenvalues. The map

$$s \colon \tilde{S} \to \mathcal{N}_n^{\circ\circ}$$

is quasi-finite, and factors through a finite morphism $\tilde{S} \to S$ where $S \subset \mathcal{N}_n^{\circ\circ}$ is an open subscheme.

Let $\rho \in \tilde{S} \cap \mathrm{crit}(\mathrm{Tr}(W_\alpha))$. Then $s(\rho)(b'_1) = 0$ by stability, and so from the superpotential relations we deduce that $a''_{1,i,j} b''_{1,i} = b''_{1,j} a''_{1,i,j}$ and $a''_{2,i,j} b''_{1,i} = b''_{1,j} a''_{2,i,j}$ for all i, j and so from our condition on the eigenvalues of the $b''_{1,i}$ we deduce that for all $1 \leqslant i \neq j \leqslant l(\alpha)$ we have $\rho(a''_{1,i,j}) = \rho(a''_{2,i,j}) = 0$. As such, in calculating the relative vanishing cycle $[\phi_{\mathrm{Tr}(W_\alpha)}]_{\tilde{S}}$ we can restrict to the set $\tilde{S}^\circ \subset \tilde{S}$ defined by the condition that ρ remains stable after setting all $\rho(a''_{1,i,j}) = \rho(a''_{2,i,j}) = 0$ for $i \neq j$. So both the inclusions $\tilde{U} \to \tilde{S}$ and $\tilde{U} \to \tilde{S}^\circ$ are the inclusions of the critical locus of the function $\mathrm{Tr}(W_\alpha)$ on the respective targets.

The space $\tilde{S}^\circ$ is a vector bundle over

$$\tilde{P} \subset \prod_i (\mathcal{N}_i^{\circ\circ})^{\alpha_i}, \tag{3.8}$$

the open subset defined by the condition that the generalised eigenvalues of the $\rho(b''_1)$-operators from different factors are distinct. The projection from $\tilde{S}^\circ$ to $\tilde{P}$ is given by forgetting the values of $\rho(a''_{k,i,j})$ for $k = 1, 2$ and $i \neq j$.

The map π is a Galois cover with Galois group G_α, and the map $\tilde{S}^\circ \to S$ is a Galois cover in a formal neighbourhood of the morphism π.

The space $\tilde{P}$ carries the *free* G_α-action inherited from $\mathcal{N}_\alpha$, and furthermore the vector bundle $\tilde{S}^\circ$ has a direct sum decomposition $V_- \oplus V_+$, where V_- keeps track of the entries of $\rho(a''_{1,i,j})$ for $i \neq j$ and V_+ keeps track of the entries of $\rho(a''_{2,i,j})$ for $i \neq j$, and so the decomposition is preserved by the G_α-action.

Let $m = \mathrm{rk}(V_+) = \mathrm{rk}(V_-)$. If we let $G_1 = \mathbb{C}^*$ act on $\tilde{S}^\circ$ by scaling V_+ with weight 1 and V_- with weight -1, then $\mathrm{Tr}(W_\alpha)$ is $\mathbb{C}^*$-invariant. Let $G_2 = \mathbb{C}^*$ act by scaling both V_+ and V_- with weight one. Then since each term in the potential contains at most two instances of $a''_{k,i,j}$, for $k = 1, 2$, for every $\rho \in \tilde{S}^\circ$, there is a fixed constant C for which $|\mathrm{Tr}(W_\alpha)(z \cdot_2 \rho)| \leqslant C|z|^2$ where the action is via the G_2-action. We deduce that on $\tilde{S}^\circ$ we can write

$$\mathrm{Tr}(W_\alpha) = g_0 + g_1,$$

where g_0 is a function pulled back from the projection to $\tilde{P}$ and g_1 is a G_1-invariant function on $\tilde{S}^\circ$, quadratic in the fibres. After passing to a G_α-invariant Zariski open subset, we can trivialise the vector bundle $\tilde{S}^\circ$ and write

$$g_1 = \sum_{1 \leqslant i,j \leqslant m} g_{i,j} t_i s_j,$$

where t_i and s_j are coordinates on the fibre of $\mathrm{Tot}(V_+)$ and $\mathrm{Tot}(V_-)$, respectively. From the equality $\mathrm{crit}(\mathrm{Tr}(W_\alpha)) = \mathrm{crit}(g_0)$, arguing as in the proof of the second part of Lemma 3.3, the matrix $\{g_{ij}\}_{1 \leqslant i,j \leqslant m}$ is invertible, and after a change of coordinates on $\mathrm{Tot}(V_+)$ we may assume that $g_{ij} = \delta_{i-j}$. By the Thom–Sebastiani isomorphism, and the third part of Lemma 3.3 we deduce that $[\phi_{\mathrm{Tr}(W_\alpha)}]^{G_\alpha}_{\tilde{S}^\circ} = [\phi_{g_0}]^{G_\alpha}_{\tilde{P}} \in \mathcal{M}^{G_\alpha \times \hat{\mu}}_{\tilde{P}}$. Finally, we note that g_0 is the sum of the potentials on the factors of $\prod_i (\mathcal{N}_i^{\circ\circ})^{\alpha_i}$, and the result follows from Proposition 1.15 and the Thom–Sebastiani theorem. $\qquad\square$

Recall from (3.5) the subvarieties $Q_L^{\bullet,\bullet,\mathrm{nilp},n} \subset Q_L^n$, relative over $\mathrm{Sym}^n(\mathbb{A}^2)$ via the map $\mathbf{hc}^{xy}$. The next corollary follows from Proposition 1.12.

COROLLARY 3.8. *Define classes* $\Phi_n \in \mathcal{M}_{\mathrm{Sym}^n(\mathbb{A}^2)}$ *via*

$$\sum_{n \geqslant 0} (-1)^n \big[Q_L^{\bullet,\bullet,\mathrm{nilp},n} \xrightarrow{\mathbf{hc}^{xy}} \mathrm{Sym}^n(\mathbb{A}^2) \big]_{\mathrm{vir}} = {}^\diamond\mathrm{Exp}_\cup \left(\sum_{n \geqslant 1} \Phi_n \right).$$

Then

$$\sum_{n \geqslant 0} (-1)^n \big[Q_L^n \xrightarrow{\mathbf{hc}} \mathrm{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}}$$

$$= {}^\diamond\mathrm{Exp}_\cup \left(\sum_{n \geqslant 1} \Big(\mathrm{Sym}^n(\mathbb{A}^2) \times \mathbb{A}^1 \xrightarrow{\tilde{\iota}_n} \mathrm{Sym}^n(\mathbb{A}^3) \Big)_! \big(\Phi_n \boxtimes \big[\mathbb{A}^1 \xrightarrow{\mathrm{id}} \mathbb{A}^1 \big] \big) \right).$$

Proof. By Lemma 3.7 (2) and (1.4), we deduce that for $\alpha \vdash n$

$$(-1)^n \big[Q_L^{\bullet,\bullet,\alpha} \to \mathrm{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}} = \cup_! \pi_{G_\alpha} j_\alpha^* \left(\underset{i | \alpha_i \neq 0}{\boxtimes} (-1)^i \big[Q_L^{\bullet,\bullet,(i)} \to \mathrm{Sym}^i(\mathbb{A}^2) \times \mathbb{A}^1 \big]_{\mathrm{vir}}^{\otimes \alpha_i} \right).$$

The result then follows from Proposition 1.12 and

$$\big[Q_L^n \to \mathrm{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}} = \sum_{\alpha \vdash n} \big[Q_L^{\bullet,\bullet,\alpha} \to \mathrm{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}}. \qquad\square$$

4.2. Motivic stratifications II: fully punctual strata

Corollary 3.8 says that, considered as relative motives over $\mathrm{Sym}(\mathbb{A}^1)$, via projection to the z coordinate, the DT invariants are generated by classes on the small diagonals, which are moreover *constant* as relative motives over $\mathbb{A}^1$. In other words, there are relative motives over

$\mathrm{Sym}^n(\mathbb{A}^2)$ that generate the motivic DT partition function under taking the external product with the constant motive on $\mathbb{A}^1$ and taking exponentials.

We next show that these relative motives are themselves supported on the small diagonal (although they are non-constant, with a 'jump' at the intersection $\mathbb{A}^2 \cap L$) so that considered as a relative motive over $\mathrm{Sym}(\mathbb{A}^3)$, the virtual motive of Q^n_L is generated on the small diagonal.

For $n' \leqslant n$ and α a partition of $n - n'$, we define

$$\mathcal{Q}^{n',\alpha} = Q_L^{\bullet,\bullet,\mathrm{nilp},n} \cap (\mathbf{hc}^y)^{-1}(A_{n',\alpha}),$$

where $A_{n',\alpha}$ is the subvariety of $\mathrm{Sym}^n(\mathbb{A}^1)$ defined by the condition that 0 occurs n' times, and the partition defined by the $(n - n')$-tuple of points away from 0 is α. We denote by

$$\kappa^1_n \colon \mathrm{Sym}^n(\mathbb{A}^1) \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^2) \qquad\qquad \kappa^0_n \colon \mathrm{Sym}^n(\mathbb{A}^1) \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^2)$$

$$(z_1, \dots, z_n) \mapsto ((z_1,1), \dots, (z_n,1)) \qquad\qquad (z_1, \dots, z_n) \mapsto ((z_1,0), \dots, (z_n,0))$$

and

$$\tilde{\kappa}_n \colon \mathrm{Sym}^n(\mathbb{A}^1) \times \mathbb{A}^1 \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^2)$$

$$((z_1, \dots, z_n), z) \mapsto ((z_1, z), \dots, (z_n, z))$$

the inclusions analogous to ι_n and $\tilde{\iota}_n$ defined in (3.4) and (3.6). Let $l_n \colon \mathrm{Sym}^n(\mathbb{A}^1) \times \mathbb{G}_m \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^1) \times \mathbb{A}^1$ be the inclusion.

LEMMA 3.9. (1) *There is an equality*

$$l_n^* \tilde{\kappa}_n^* \big[Q_L^{\bullet,\bullet,\mathrm{nilp},n} \xrightarrow{\mathbf{hc}^{xy}} \mathrm{Sym}^n(\mathbb{A}^2) \big]_{\mathrm{vir}} = \kappa_n^{1,*} \big[Q_L^{\bullet,\bullet,\mathrm{nilp},n} \xrightarrow{\mathbf{hc}^{xy}} \mathrm{Sym}^n(\mathbb{A}^2) \big]_{\mathrm{vir}} \boxtimes \big[\mathbb{G}_m \xrightarrow{\mathrm{id}} \mathbb{G}_m \big]$$

in $\mathcal{M}_{\mathrm{Sym}^n(\mathbb{A}^1) \times \mathbb{G}_m}$. *In other words, away from* 0, *the motive*

$$\tilde{\kappa}_n^* \big[Q_L^{\bullet,\bullet,\mathrm{nilp},n} \xrightarrow{\mathbf{hc}^{xy}} \mathrm{Sym}^n(\mathbb{A}^2) \big]_{\mathrm{vir}}$$

is constant along the $\mathbb{A}^1$*-factor.*

(2) *More generally, there is an equality*

$$\Big[\mathcal{Q}^{n',\alpha} \xrightarrow{\mathbf{hc}^{xy}} \mathrm{Sym}^n(\mathbb{A}^2) \Big]_{\mathrm{vir}} = \Big[\mathcal{Q}^{n',\emptyset} \xrightarrow{\mathbf{hc}^{xy}} \mathrm{Sym}^{n'}(\mathbb{A}^2) \Big]_{\mathrm{vir}} \boxtimes_\cup$$

$$\cup_! \, \pi_{G_\alpha} j_\alpha^* \bigg(\underset{i | \alpha_i \neq 0}{\boxtimes} \Big[\mathcal{Q}^{0,(i)} \xrightarrow{\mathbf{hc}^{xy}} \mathrm{Sym}^i(\mathbb{A}^1) \times \mathbb{A}^1 \Big]_{\mathrm{vir}}^{\otimes \alpha_i} \bigg),$$

where j_α *is the inclusion of the complement to the preimage of the big diagonal in* $\mathbb{A}^{\sum_i \alpha_i}$ *under the projection* $\prod_i (\mathrm{Sym}^i(\mathbb{A}^1) \times \mathbb{A}^1)^{\alpha_i} \to \mathbb{A}^{\sum_i \alpha_i}$.

See the remarks following the statement of Lemma 3.7, as well as equation (1.2), for some guidance on how to read the right-hand side of the second equation.

Proof. Let $\mathbb{G}_m$ act on $\mathcal{N}_n^{\circ\circ}$ via

$$z \cdot (\rho(a_1'), \rho(a_2'), \rho(b_1'), \rho(a_1''), \rho(a_2''), \rho(b_1'')) = (z\rho(a_1'), z^{-1}\rho(a_2'), \rho(b_1'), z\rho(a_1''), z^{-1}\rho(a_2''), \rho(b_1'')).$$

Then $\mathrm{Tr}(W_r)$ is invariant under the $\mathbb{G}_m$-action. We define the space $\mathcal{U}_n \subset \mathcal{N}_n^{\circ\circ}$ by the condition that $\det(\rho(a_2'')) = 1$. Consider the morphism

$$J \colon \mathcal{U}_n \times \mathbb{G}_m \to \mathcal{N}_n^{\circ\circ}$$

given by restricting the $\mathbb{G}_m$-action. This is a Galois cover of the open subscheme $\mathcal{Y} \subset \mathcal{N}_n^{\circ\circ}$ defined by the condition that $\rho(a_2'')$ is invertible. Let $\mathcal{V}_n \subset \mathcal{U}_n$ be the subvariety defined by

the condition that $\rho(a_2'')$ has only one eigenvalue. Then as a μ_n-equivariant variety, there is an isomorphism

$$\mathcal{V}_n \cong \mathcal{W}_n \times \mu_n,$$

where the action on the first factor is trivial, and is the action of group multiplication on the second. Here $\mathcal{W}_n \subset \mathcal{U}_n$ is the subvariety defined by the condition that $\rho(a_2'')$ is unipotent.

Recall from Section 2.4 the function h_n on $\mathcal{N}_n^{\circ\circ}$. The function $h_n \circ J$ factors through a function $\tilde{h}$ on $\mathcal{U}_n$, since h is $\mathbb{G}_m$-invariant. By Proposition 1.15,

$$(\mathcal{Y} \to \mathcal{N}_n^{\circ\circ})^*\big[\phi_{h_n}\big]_{\mathcal{N}_n^{\circ\circ}} = (\mathcal{Y} \to \mathcal{N}_n^{\circ\circ})^*\pi_{\mu_n}J_!([\phi_{\tilde{h}}]_{\mathcal{U}_n} \boxtimes [\mathbb{G}_m \xrightarrow{\mathrm{id}_{\mathbb{G}_m}} \mathbb{G}_m]).$$

It follows that

$$l_n^*\tilde{\kappa}_n^*\iota_n^*\,\mathbf{hc}_![\phi_{h_n}]_{\mathcal{N}_n^{\circ\circ}} = l_n^*\tilde{\kappa}_n^*\iota_n^*\,\mathbf{hc}_!\,\pi_{\mu_n}J_!([\phi_{\tilde{h}}]_{\mathcal{U}_n} \boxtimes [\mathbb{G}_m \xrightarrow{\mathrm{id}_{\mathbb{G}_m}} \mathbb{G}_m])$$

$$= l_n^*\tilde{\kappa}_n^*\iota_n^*\,\mathbf{hc}_!\,\pi_{\mu_n}J_!([\phi_{\tilde{h}}]_{\mathcal{V}_n} \boxtimes [\mathbb{G}_m \xrightarrow{\mathrm{id}_{\mathbb{G}_m}} \mathbb{G}_m])$$

$$= l_n^*\tilde{\kappa}_n^*\iota_n^*\,\mathbf{hc}_!\,J_!([\phi_{\tilde{h}}]_{\mathcal{W}_n} \boxtimes [\mathbb{G}_m \xrightarrow{\mathrm{id}_{\mathbb{G}_m}} \mathbb{G}_m]),$$

implying the triviality along the $\mathbb{G}_m$-factor required for part (1). The second equality follows from the fact that only the part of the motive over $\mathcal{V}_n$ can contribute to the pullback along $\tilde{\kappa}_n$, since by definition this is the pullback along the locus where $\rho(a_2'')$ has only one eigenvalue.

For part (2), consider the quiver Q_α' defined as follows. Set

$$(Q_\alpha')_0 = \Big\{\underline{2}_0, \ldots, \underline{2}_{l(\alpha)}, \underline{\infty}\Big\}$$

$$(Q_\alpha')_1 = \big\{a_{1,i}', a_{2,i}', b_{1,i}', a_{1,i}''\big\}_{0 \leqslant i \leqslant l(\alpha)} \cup \big\{b_{1,i,j}'', a_{2,i,j}''\big\}_{0 \leqslant i,j \leqslant l(\alpha)}$$

with

$$t(a_{1,i}') = t(a_{2,i}') = s(b_{1,i}') = \underline{\infty}$$

$$s(a_{1,i}') = s(a_{2,i}') = t(b_{1,i}') = t(a_{2,i}'') = s(a_{2,i}'') = \underline{2}_i$$

$$t(a_{1,i,j}'') = t(b_{1,i,j}'') = \underline{2}_j$$

$$s(a_{1,i,j}'') = s(b_{1,i,j}'') = \underline{2}_i.$$

We give this quiver the dimension vector $\underline{d} = (n', 1^{\alpha_1}, \ldots, r^{\alpha_r}, 1)$ and the stability condition $(-1, \ldots, -1, n)$. As in the proof of Lemma 3.7, a (stable) Q_α'-representation corresponds to a (ζ'-stable) Q_r-representation ρ along with a direct sum decomposition of ρ_2 that is respected by the operator $\rho(a_2'')$. A point in $\mathcal{Q}^{n',\alpha}$ gives rise to a $\underline{d}$-dimensional Q_α'-representation, in a formal neighbourhood of which the forgetful map is a Galois covering — from this point, the proof proceeds exactly as in the proof of Lemma 3.7. $\qquad\square$

The following is proved in the same way as Corollary 3.8.

COROLLARY 3.10. *Define classes* $\Psi_n^\# \in \mathcal{M}_{\mathrm{Sym}^n(\mathbb{A}^1)}$ *for* $\# \in \{\mathrm{nilp}, \mathrm{uni}\}$, *via*

$$\sum_{n \geqslant 0}(-1)^n\big[Q_L^{\bullet,\#,\mathrm{nilp},n} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^n(\mathbb{A}^1)\big]_{\mathrm{vir}} = {}^\diamond\mathrm{Exp}_\cup\left(\sum_{n \geqslant 1}\Psi_n^\#\right).$$

Then

$$\sum_{n \geqslant 0} (-1)^n \big[Q_L^{\bullet,\bullet,\mathrm{nilp},n} \xrightarrow{\mathbf{hc}^{xy}} \mathrm{Sym}^n(\mathbb{A}^2) \big]_{\mathrm{vir}}$$

$$= {}^{\diamond}\mathrm{Exp}_{\cup} \left(\sum_{n \geqslant 1} \Big(\mathrm{Sym}^n(\mathbb{A}^1) \times \mathbb{G}_m \xrightarrow{\tilde{\kappa}_n \circ l_n} \mathrm{Sym}^n(\mathbb{A}^2) \Big)_! \big(\Psi_n^{\mathrm{uni}} \boxtimes [\mathbb{G}_m \xrightarrow{\mathrm{id}} \mathbb{G}_m] \big) \right.$$

$$\left. + \Big(\mathrm{Sym}^n(\mathbb{A}^1) \xrightarrow{\kappa_n^0} \mathrm{Sym}^n(\mathbb{A}^2) \Big)_! \Psi_n^{\mathrm{nilp}} \right).$$

We are now two-thirds of the way toward showing that the relative DT invariants are fully punctual over $\mathrm{Sym}(\mathbb{A}^3)$. Since the proof of Lemma 3.11 is almost identical to that of Lemma 3.9, while the proof of Lemma 3.12 is strictly simpler, we omit them. Before stating them, we introduce one last stratification; for $n' \leqslant n$, and $\alpha \vdash n - n'$, let

$$\mathcal{P}^{n',\alpha} \subset Q_L^{\bullet,\mathrm{nilp},\mathrm{nilp},n}$$

be the preimage under the map

$$Q_L^{\bullet,\mathrm{nilp},\mathrm{nilp},n} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^n(\mathbb{A}^1)$$

of the space of tuples of length n' at the origin, and for which the partition type of the tuple away from the origin is α. Likewise, for $\alpha \vdash n$ we define

$$\mathcal{U}^{\alpha} \subset Q_L^{\bullet,\mathrm{uni},\mathrm{nilp},n}$$

to be the preimage of $\mathrm{Sym}^{\alpha}(\mathbb{A}^1)$ under $Q_L^{\bullet,\mathrm{uni},\mathrm{nilp},n} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^n(\mathbb{A}^1)$. Let also $\Delta_n \colon \mathbb{A}^1 \to \mathrm{Sym}^n(\mathbb{A}^1)$ be the diagonal.

LEMMA 3.11. (1) *There is an equality*

$$(\mathbb{G}_m \xrightarrow{\Delta_n|_{\mathbb{G}_m}} \mathrm{Sym}^n(\mathbb{A}^1))^* \big[Q_L^{\bullet,\mathrm{nilp},\mathrm{nilp},n} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^n(\mathbb{A}^1) \big]_{\mathrm{vir}}$$

$$= (\{1\} \xrightarrow{\Delta_n|_{\{1\}}} \mathrm{Sym}^n(\mathbb{A}^1))^* \big[Q_L^{\bullet,\mathrm{nilp},\mathrm{nilp},n} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^n(\mathbb{A}^1) \big]_{\mathrm{vir}} \boxtimes [\mathbb{G}_m \xrightarrow{\mathrm{id}} \mathbb{G}_m]$$

in $\mathcal{M}_{\mathbb{G}_m}$. *In other words, away from* 0, *the motive* $\Delta_n^*[Q_L^{\bullet,\mathrm{nilp},\mathrm{nilp},n} \to \mathrm{Sym}^n(\mathbb{A}^1)]_{\mathrm{vir}}$ *is constant.*
(2) *Moreover, there is an equality*

$$\Big[\mathcal{P}^{n',\alpha} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^n(\mathbb{A}^1) \Big]_{\mathrm{vir}}$$

$$= \Big[\mathcal{P}^{n',\emptyset} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^{n'}(\mathbb{A}^1) \Big]_{\mathrm{vir}} \boxtimes_{\cup} \cup_! \pi_{G_\alpha} \left(\boxtimes_{i|\alpha_i \neq 0} \Big[\mathcal{P}^{0,(i)} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^{(i)}(\mathbb{A}^1) \Big]_{\mathrm{vir}}^{\otimes \alpha_i} \right).$$

LEMMA 3.12. (1) *There is an equality*

$$(\mathbb{A}^1 \xrightarrow{\Delta_n} \mathrm{Sym}^n(\mathbb{A}^1))^* \Big[Q_L^{\bullet,\mathrm{uni},\mathrm{nilp},n} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^n(\mathbb{A}^1) \Big]_{\mathrm{vir}}$$

$$= (\{0\} \xrightarrow{\Delta_n|_{\{0\}}} \mathrm{Sym}^n(\mathbb{A}^1))^* \Big[Q_L^{\bullet,\mathrm{uni},\mathrm{nilp},n} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^n(\mathbb{A}^1) \Big]_{\mathrm{vir}} \boxtimes \big[\mathbb{A}^1 \xrightarrow{\mathrm{id}} \mathbb{A}^1 \big]$$

in $\mathcal{M}_{\mathbb{A}^1}$. *In other words, the motive* $\Delta_n^*[Q_L^{\bullet,\mathrm{uni},\mathrm{nilp},n} \to \mathrm{Sym}^n(\mathbb{A}^1)]_{\mathrm{vir}}$ *is constant.*
(2) *Moreover, there is an equality*

$$\Big[\mathcal{U}^{\alpha} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^n(\mathbb{A}^1) \Big]_{\mathrm{vir}} = \cup_! \pi_{G_\alpha} \left(\boxtimes_{i|\alpha_i \neq 0} \Big[\mathcal{U}^{(i)} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^{(i)}(\mathbb{A}^1) \Big]_{\mathrm{vir}}^{\otimes \alpha_i} \right).$$

By the same argument as Corollary 3.8, we deduce the following.

COROLLARY 3.13. *Define classes $\Lambda_n^\#$ for $\# \in \{\mathrm{nilp}, \mathrm{uni}\}$ in $\mathcal{M}_{\mathbb{C}}$ via*

$$\sum_{n \geqslant 0} (-1)^n \big[Q_L^{\#, \mathrm{nilp}, \mathrm{nilp}, n}\big]_{\mathrm{vir}} = {}^\diamond\mathrm{Exp}\left(\sum_{n \geqslant 1} \Lambda_n^\#\right). \tag{3.9}$$

Then

$$\sum_{n \geqslant 0} (-1)^n \big[Q_L^{\bullet, \mathrm{nilp}, \mathrm{nilp}, n} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^n(\mathbb{A}^1)\big]_{\mathrm{vir}}$$

$$= {}^\diamond\mathrm{Exp}_\cup \left(\sum_{n \geqslant 1} \Lambda_n^{\mathrm{uni}} \boxtimes \big[\mathbb{G}_m \xrightarrow{\Delta_n|_{\mathbb{G}_m}} \mathrm{Sym}^n(\mathbb{A}^1)\big] + \Lambda_n^{\mathrm{nilp}} \boxtimes \big[0 \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^1)\big]\right).$$

COROLLARY 3.14. *Define $\Lambda_n \in \mathcal{M}_{\mathbb{C}}$ via*

$$\sum_{n \geqslant 0} (-1)^n \big[Q_L^{\mathrm{nilp}, \mathrm{uni}, \mathrm{nilp}, n}\big]_{\mathrm{vir}} = {}^\diamond\mathrm{Exp}\left(\sum_{n \geqslant 1} \Lambda_n\right). \tag{3.10}$$

Then

$$\sum_{n \geqslant 0} (-1)^n \big[Q_L^{\bullet, \mathrm{uni}, \mathrm{nilp}, n} \xrightarrow{\mathbf{hc}^x} \mathrm{Sym}^n(\mathbb{A}^1)\big]_{\mathrm{vir}} = {}^\diamond\mathrm{Exp}_\cup \left(\sum_{n \geqslant 1} \Lambda_n \boxtimes \big[\mathbb{A}^1 \xrightarrow{\Delta_n} \mathrm{Sym}^n(\mathbb{A}^1)\big]\right).$$

LEMMA 3.15. *In the notation of Corollaries 3.13 and 3.14, there is an equality*

$$\Lambda_n^{\mathrm{uni}} = \Lambda_n$$

for all n.

Proof. There is an automorphism S_n of $\mathcal{N}_n^{\circ\circ}$, for all n, defined by

$$\rho(a_1'') \mapsto \rho(a_2'')$$
$$\rho(a_2'') \mapsto -\rho(a_1'')$$
$$\rho(a_1') \mapsto \rho(a_2')$$
$$\rho(a_2') \mapsto -\rho(a_1'),$$

which leaves $\mathrm{Tr}(W_r)$ invariant, and so preserves $[Q_L^n \to Q_L^n]_{\mathrm{vir}}$. Now

$$\mathsf{S}_{n,!}\Big[Q_L^{\mathrm{uni}, \mathrm{nilp}, \mathrm{nilp}, n} \to Q_L\Big]_{\mathrm{vir}} = \Big[Q_L^{\mathrm{nilp}, \mathrm{uni}, \mathrm{nilp}, n} \to Q_L\Big]_{\mathrm{vir}}$$

and so $\big[Q_L^{\mathrm{uni}, \mathrm{nilp}, \mathrm{nilp}, n}\big]_{\mathrm{vir}} = \big[Q_L^{\mathrm{nilp}, \mathrm{uni}, \mathrm{nilp}, n}\big]_{\mathrm{vir}}$. The lemma then follows from the defining equations (3.9) and (3.10). $\qquad\square$

DEFINITION 3.16. Let us set

$$\Omega_{\mathrm{pt}}^n = \Lambda_n^{\mathrm{uni}}, \quad \Omega_{\mathrm{crv}}^n = \Lambda_n^{\mathrm{nilp}}.$$

These fully punctual motives express the contribution of points away from the curve and embedded on the curve, respectively.

Let

$$\iota_L \colon L \hookrightarrow \mathbb{A}^3, \quad \iota_{C_0} \colon C_0 \hookrightarrow X$$

denote the closed inclusions, and let

$$u_L \colon \mathbb{A}^3 \setminus L \hookrightarrow \mathbb{A}^3, \quad u_{C_0} \colon X \setminus C_0 \hookrightarrow X$$

denote the inclusions of the open complements.

THEOREM 3.17. *There is an equality in $\mathcal{M}_{\mathrm{Sym}(\mathbb{A}^3)}$:*

$$\sum_{n \geqslant 0} (-1)^n \big[Q_L^n \xrightarrow{\mathbf{hc}_{\mathbb{A}^3}} \mathrm{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}}$$

$$= {}^\diamond\mathrm{Exp}_\cup \left(\sum_{n \geqslant 1} \left(\Omega_{pt}^n \boxtimes \big[\mathbb{A}^3 \setminus L \xrightarrow{\Delta_n u_L} \mathrm{Sym}^n(\mathbb{A}^3) \big] + \Omega_{\mathrm{crv}}^n \boxtimes \big[L \xrightarrow{\Delta_n \iota_L} \mathrm{Sym}^n(\mathbb{A}^3) \big] \right) \right),$$

$$(3.11)$$

and an equality in $\mathcal{M}_{\mathrm{Sym}(X)}$:

$$\sum_{n \geqslant 0} (-1)^n \big[Q_{C_0}^n \xrightarrow{\mathbf{hc}_X} \mathrm{Sym}^n(X) \big]_{\mathrm{vir}}$$

$$= {}^\diamond\mathrm{Exp}_\cup \left(\sum_{n \geqslant 1} \left(\Omega_{pt}^n \boxtimes \big[X \setminus C_0 \xrightarrow{\Delta_n u_{C_0}} \mathrm{Sym}^n(X) \big] + \Omega_{\mathrm{crv}}^n \boxtimes \big[C_0 \xrightarrow{\Delta_n \iota_{C_0}} \mathrm{Sym}^n(X) \big] \right) \right).$$

$$(3.12)$$

Proof. By Corollary 3.8, we have

$$\sum_{n \geqslant 0} (-1)^n \big[Q_L^n \xrightarrow{\mathbf{hc}_{\mathbb{A}^3}} \mathrm{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}}$$

$$= {}^\diamond\mathrm{Exp}_\cup \left(\sum_{n \geqslant 1} \left(\mathrm{Sym}^n(\mathbb{A}^2) \times \mathbb{A}^1 \xrightarrow{\tilde{\iota}_n} \mathrm{Sym}^n(\mathbb{A}^3) \right)_! \left(\Phi_n \boxtimes \big[\mathbb{A}^1 \xrightarrow{\mathrm{id}} \mathbb{A}^1 \big] \right) \right).$$

Then by Corollary 3.10 and Proposition 1.12, we deduce that

$$\sum_{n \geqslant 0} (-1)^n \big[Q_L^n \xrightarrow{\mathbf{hc}_{\mathbb{A}^3}} \mathrm{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}}$$

$$= {}^\diamond\mathrm{Exp}_\cup \left(\sum_{n \geqslant 1} \left(\mathrm{Sym}^n(\mathbb{A}^1) \times \mathbb{G}_m \times \mathbb{A}^1 \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^3) \right)_! \Psi_n^{\mathrm{uni}} \boxtimes \big[\mathbb{G}_m \times \mathbb{A}^1 \xrightarrow{\mathrm{id}} \mathbb{G}_m \times \mathbb{A}^1 \big] \right.$$

$$\left. + \left(\mathrm{Sym}^n(\mathbb{A}^1) \times \{0\} \times \mathbb{A}^1 \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^3) \right)_! \Psi_n^{\mathrm{nilp}} \boxtimes \big[\{0\} \times \mathbb{A}^1 \xrightarrow{\mathrm{id}} \{0\} \times \mathbb{A}^1 \big] \right).$$

Then by Corollaries 3.13 and 3.14 and Proposition 1.12 again, we deduce that

$$\sum_{n \geqslant 0} (-1)^n \big[Q_L^n \xrightarrow{\mathbf{hc}_{\mathbb{A}^3}} \mathrm{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}}$$

$$= {}^\diamond\mathrm{Exp}_\cup \Bigg(\sum_{n \geqslant 1} \big(\mathbb{A}^1 \times \mathbb{G}_m \times \mathbb{A}^1 \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^3) \big)_! \Omega_{\mathrm{pt}}^n \boxtimes \big[\mathbb{A}^1 \times \mathbb{G}_m \times \mathbb{A}^1 \xrightarrow{\mathrm{id}} \mathbb{A}^1 \times \mathbb{G}_m \times \mathbb{A}^1 \big]$$

$$+ \big(\mathbb{G}_m \times \{0\} \times \mathbb{A}^1 \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^3) \big)_! \Omega_{\mathrm{pt}}^n \boxtimes \big[\mathbb{G}_m \times \{0\} \times \mathbb{A}^1 \xrightarrow{\mathrm{id}} \mathbb{G}_m \times \{0\} \times \mathbb{A}^1 \big]$$

$$+ \big(\{0\} \times \{0\} \times \mathbb{A}^1 \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^3) \big)_! \Omega_{\mathrm{crv}}^n \boxtimes \big[\{0\} \times \{0\} \times \mathbb{A}^1 \xrightarrow{\mathrm{id}} \{0\} \times \{0\} \times \mathbb{A}^1 \big] \Bigg),$$

and the first statement follows. For the second statement, we observe that the right-hand side of (3.12) satisfies conditions (2) and (3) of Lemma 3.1, and is obviously equal to the right-hand side of (3.11) after restriction to $\mathrm{Sym}(\mathbb{A}^3)$. Then the second statement follows from Corollary 3.2. □

5.   The virtual motive of the Quot scheme

In this section, we define a virtual motive

$$\big[Q_C^n \big]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}$$

for an arbitrary smooth curve $C \subset Y$ in a smooth 3-fold Y. Before getting to this point, we give an explicit formula for the generating series

$$\mathsf{Q}_{L/\mathbb{A}^3}(t) = \sum_{n \geqslant 0} \big[Q_L^n \big]_{\mathrm{vir}} \cdot t^n$$

$$\mathsf{Q}_{C_0/X}(t) = \sum_{n \geqslant 0} \big[Q_{C_0}^n \big]_{\mathrm{vir}} \cdot t^n$$

encoding the local absolute virtual motives attached to $L \subset \mathbb{A}^3$ and $C_0 \subset X$. We shall also prove Theorems B and C from the introduction.

5.1.   The local absolute virtual motives

Let X be, as usual, the resolved conifold. Then Theorem 3.17 implies that for $\iota \colon x \hookrightarrow X$ the inclusion of a point, the absolute motive

$$(\mathrm{Sym}^n \iota)^* \big[Q_{C_0}^n \xrightarrow{\mathbf{hc}_X} \mathrm{Sym}^n(X) \big]_{\mathrm{vir}}$$

only depends on whether x is in C_0 or not. We define:

- $\big[\mathcal{P}_{\mathrm{pt}}^n \big]_{\mathrm{vir}} = (\mathrm{Sym}^n \iota)^* \big[Q_{C_0}^n \xrightarrow{\mathbf{hc}_X} \mathrm{Sym}^n(X) \big]_{\mathrm{vir}}$ for $x \notin C_0$;
- $\big[\mathcal{P}_{\mathrm{crv}}^n \big]_{\mathrm{vir}} = (\mathrm{Sym}^n \iota)^* \big[Q_{C_0}^n \xrightarrow{\mathbf{hc}_X} \mathrm{Sym}^n(X) \big]_{\mathrm{vir}}$ for $x \in C_0$.

Explicitly, these motives are determined by the identities

$$\sum_{n \geqslant 0} (-1)^n \big[\mathcal{P}_{\#}^n \big]_{\mathrm{vir}} \cdot t^n = {}^\diamond\mathrm{Exp} \Bigg(\sum_{n > 0} \Omega_{\#}^n \cdot t^n \Bigg)$$

for $\# \in \{\mathrm{pt}, \mathrm{crv}\}$. We define the generating functions

$$\mathsf{F}_{\#}(t) = \sum_{n \geqslant 0} \big[\mathcal{P}_{\#}^n \big]_{\mathrm{vir}} \cdot t^n.$$

Via the morphisms defining power structures for varieties, Lemmas 3.7, 3.9, 3.11, 3.12 and 3.15 can be neatly summed up in the following corollary.

COROLLARY 4.1. *There are equalities in $\mathcal{M}_{\mathbb{C}}[\![t]\!]$*

$$\mathsf{Q}_{L/\mathbb{A}^3}(-t) = \mathsf{F}_{pt}(-t)_{\diamond}^{\mathbb{L}^3-\mathbb{L}} \cdot \mathsf{F}_{\mathrm{crv}}(-t)_{\diamond}^{\mathbb{L}}$$

$$\mathsf{Q}_{C_0/X}(-t) = \mathsf{F}_{pt}(-t)_{\diamond}^{\mathbb{L}^3+\mathbb{L}^2-\mathbb{L}-1} \cdot \mathsf{F}_{\mathrm{crv}}(-t)_{\diamond}^{\mathbb{L}+1}.$$

REMARK 4.2. Note that the exponents in the statement of the corollary are effective (despite the minus signs). Indeed, $\mathbb{L}^3 - \mathbb{L}$ is the class of $\mathbb{A}^3 \setminus L$, and $\mathbb{L}^3 + \mathbb{L}^2 - \mathbb{L} - 1$ is the class of $X \setminus C_0$. The '$\diamond$' subscripts are in the statement because we are yet to prove that the classes $(-1)^n[\mathcal{P}^n_{\mathrm{pt}}]_{\mathrm{vir}}$ and $(-1)^n[\mathcal{P}^n_{\mathrm{crv}}]_{\mathrm{vir}}$ are effective. As a result of Corollaries 4.4 and 4.6, we will be able to remove them and the statements remain true, now expressed properly in terms of the power structures on the Grothendieck rings of varieties.

The rest of this subsection is devoted to removing the decorations '$\diamond$' from the formulas in Corollary 4.1. We start with $\# = \mathrm{pt}$.

PROPOSITION 4.3. *For all $n \geqslant 0$, there is an identity*

$$\left[\mathcal{P}^n_{pt}\right]_{\mathrm{vir}} = \left[\mathrm{Hilb}^n(\mathbb{A}^3)_0\right]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}.$$

Proof. By definition

$$\left[\mathcal{P}^n_{\mathrm{pt}}\right]_{\mathrm{vir}} = j_n^*\left[Q_L^n \xrightarrow{\mathbf{hc}} \mathrm{Sym}^n(\mathbb{A}^3)\right]_{\mathrm{vir}},$$

where $j_n \colon p \hookrightarrow \mathrm{Sym}^n(\mathbb{A}^3)$ is the inclusion of the point $p = ((1,1,1),\ldots,(1,1,1))$. Consider the open subvariety $\overline{U}_n \subset \mathrm{rMod}_{(n,1)}(Q_r)$ defined by the condition that $\rho(a_1'')$ and $\rho(a_2'')$ are invertible, and the image of $\rho(a_1')$ generates $\rho_{\underline{2}}$ under the action of $\rho(a_1'')$, $\rho(a_2'')$ and $\rho(b_1'')$. Note that this is a *stronger* notion of stability than ζ'-stability. However, after restricting to the critical locus of $\mathrm{Tr}(W_r)$, using our invertibility assumptions, we have that for $\rho \in \overline{U}_n$ the relation

$$\rho(a_1'')^{-1}\rho(a_2'')\rho(a_1') = \rho(a_2')$$

holds, and so we deduce that ρ satisfies our stronger notion of stability if it is ζ'-stable, lies in $\mathrm{crit}(\mathrm{Tr}(W_r))$, and satisfies the above invertibility assumptions. It follows that, after setting $U_n = \overline{U}_n/\mathrm{GL}_n$, one has

$$\left[\mathcal{P}^n_{\mathrm{pt}}\right]_{\mathrm{vir}} = j_n^*\left[U_n \cap \mathrm{crit}(\mathrm{Tr}(W_r)) \xrightarrow{\mathbf{hc}} \mathrm{Sym}^n(\mathbb{A}^3)\right]_{\mathrm{vir}}.$$

Let Q_{BBS} denote the quiver obtained from Q_r by removing a_2' and b_1', in other words the framed 3-loop quiver of Figure 1. Then $\mathrm{NCHilb}^n \coloneqq \mathrm{rMod}_{(n,1)}^{\zeta'}(Q_{\mathrm{BBS}})/\mathrm{GL}_n$ is the noncommutative Hilbert scheme considered by Behrend, Bryan and Szendrői in [3] (cf. Section 1.7), and $U_n = \overline{U}_n/\mathrm{GL}_n$ is a vector bundle over $\mathrm{NCHilb}_{\circ}^n$, the open subscheme of NCHilb^n defined by invertibility of $\rho(a_1'')$ and $\rho(a_2'')$.

We introduce a new set of matrix coordinates on $\overline{U}_n$ by considering the matrices

$$\mathbb{A}_1'' = \rho(a_1'') \qquad\qquad \mathbb{A}_2'' = \rho(a_2'') \qquad\qquad \mathbb{B}_1'' = \rho(b_1'')$$

$$\mathbb{A}_2' = \rho(a_2'')\rho(a_1') - \rho(a_1'')\rho(a_2') \qquad \mathbb{A}_1' = \rho(a_1'')\rho(a_2') \qquad \mathbb{B}_1' = \rho(b_1').$$

With respect to these coordinates, we can write

$$\mathrm{Tr}(W_r)|_{U_n} = F + \mathrm{Tr}(\mathbb{B}_1'\mathbb{A}_2'),$$

where

$$F = \operatorname{Tr}(\mathbb{A}_1'') \operatorname{Tr}(\mathbb{B}_1'') \operatorname{Tr}(\mathbb{A}_2'') - \operatorname{Tr}(\mathbb{A}_2'') \operatorname{Tr}(\mathbb{B}_1'') \operatorname{Tr}(\mathbb{A}_1'')$$

is a function on NCHilb^n considered as a function on U_n by composition with the vector bundle projection $U_n \to \operatorname{NCHilb}^n$. It follows from Lemma 3.3 (2) and the motivic Thom–Sebastiani theorem that

$$j_n^* \big[U_n \cap \operatorname{crit}(\operatorname{Tr}(W_r)) \xrightarrow{\mathbf{hc}} \operatorname{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}} = j_n^* \big[\operatorname{NCHilb}^n \cap \operatorname{crit}(\operatorname{Tr}(F)) \xrightarrow{\mathbf{hc}} \operatorname{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}}$$

$$= j_n^* \big[\operatorname{Hilb}^n(\mathbb{A}^3) \xrightarrow{\mathbf{hc}} \operatorname{Sym}^n(\mathbb{A}^3) \big]_{\mathrm{vir}}$$

$$= \big[\operatorname{Hilb}^n(\mathbb{A}^3)_0 \big]_{\mathrm{vir}}$$

as required.   □

COROLLARY 4.4. *For all $n \geqslant 0$, there is an identity*

$$\Omega_{pt}^n = (-1)^n \mathbb{L}^{-\frac{3}{2}} \frac{\mathbb{L}^{\frac{n}{2}} - \mathbb{L}^{-\frac{n}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \in \mathcal{M}_{\mathbb{C}}.$$

In particular, Ω_{pt}^n is effective.

Proof. By definition, we have

$$\sum_{n \geqslant 0} (-1)^n \big[\mathcal{P}_{\mathrm{pt}}^n \big]_{\mathrm{vir}} t^n = {}^\diamond\!\operatorname{Exp}\left(\sum_{n > 0} \Omega_{\mathrm{pt}}^n t^n \right).$$

Since $[\mathcal{P}_{\mathrm{pt}}^n]_{\mathrm{vir}} = [\operatorname{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}$, we deduce (see Remark 1.22) that $(-1)^n [\mathcal{P}_{\mathrm{pt}}^n]_{\mathrm{vir}}$ is effective, and by Formula (1.13) we deduce also that

$$\sum_{n \geqslant 0} (-1)^n \big[\mathcal{P}_{\mathrm{pt}}^n \big]_{\mathrm{vir}} t^n = \operatorname{Exp}\left(\sum_{n > 0} (-1)^n \mathbb{L}^{-\frac{3}{2}} \frac{\mathbb{L}^{\frac{n}{2}} - \mathbb{L}^{-\frac{n}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} t^n \right).$$

The result follows since $^\diamond\!\operatorname{Exp}$ is injective, and agrees with Exp on effective motives.   □

The corollary implies that in the right-hand side of the formulas in Corollary 4.1, we can remove the decoration '$\diamond$' from the first factor. In other words, for all smooth quasi-projective 3-folds U, one has

$$\mathsf{F}_{\mathrm{pt}}(-t)_\diamond^{[U]} = \mathsf{F}_{\mathrm{pt}}(-t)^{[U]} = \mathsf{Z}_0(-t)^{[U]} = \mathsf{Z}_U(-t). \tag{4.1}$$

Next we deal with $\# = \mathrm{crv}$.

In [**20**, Proposition 4.3], the full motivic DT and PT theories of the resolved conifold X are computed. The sign conventions in *loc. cit.* are different from ours, but the discrepancy amounts to the substitution

$$\mathbb{L}^{\frac{1}{2}} \to -\mathbb{L}^{\frac{1}{2}}.$$

After this change is done, the motivic partition function of the stable pair theory of X reads

$$\mathsf{Z}_{\mathrm{PT}}(-s, T) = \prod_{m \geqslant 1} \prod_{j=0}^{m-1} \left(1 + \mathbb{L}^{-\frac{m}{2} + \frac{1}{2} + j} (-s)^m T \right).$$

On the other hand, the DT partition Z_{DT} function satisfies

$$\mathsf{Z}_{\mathrm{DT}}(-s, T) = \mathsf{Z}_X(-s) \cdot \mathsf{Z}_{\mathrm{PT}}(-s, T),$$

where s is the point class and T is the curve class. Again, we have adjusted the sign in the point contribution Z_X. Extracting the coefficient of T (which corresponds to picking out the contribution of the reduced curve class) and multiplying by $-s^{-1}$ (so that the $(-s)^n$ coefficient is $I_{n+1}(X, [C_0]) = Q_{C_0}^n$ on the DT side) yields an identity

$$\mathsf{DT}_{C_0/X}(-s) = \mathsf{Z}_X(-s) \cdot \mathsf{PT}_{C_0/X}(-s), \tag{4.2}$$

where

$$\mathsf{PT}_{C_0/X}(-s) = \sum_{n \geqslant 0} (-1)^n \big[\mathrm{Sym}^n C_0\big]_{\mathrm{vir}} \cdot s^n$$

$$= (1 + \mathbb{L}^{-\frac{1}{2}} s)^{-(\mathbb{L}+1)}$$

$$= (1 - \mathbb{L}^{-\frac{1}{2}} s + \mathbb{L}^{-1} s^2 - \cdots)^{\mathbb{L}+1}.$$

On the other hand,

$$\mathsf{DT}_{C_0/X}(-s) = \sum_{n \geqslant 0} (-1)^n \big[Q_{C_0}^n\big]_{\mathrm{vir}} \cdot s^n = \mathsf{Q}_{C_0/X}(-s),$$

and the latter equals

$$\mathsf{Z}_{X \setminus C_0}(-s) \cdot \mathsf{F}_{\mathrm{crv}}(-s)_\diamond^{\mathbb{L}+1} = \mathsf{Z}_X(-s) \cdot \frac{\mathsf{F}_{\mathrm{crv}}(-s)_\diamond^{\mathbb{L}+1}}{\mathsf{Z}_0(-s)^{\mathbb{L}+1}}$$

by Corollary 4.1 and Proposition 4.3. It follows that

$$\mathsf{F}_{\mathrm{crv}}(-s)_\diamond^{\mathbb{L}+1} = \mathsf{Z}_0(-s)^{\mathbb{L}+1} \cdot (1 - \mathbb{L}^{-\frac{1}{2}} s + \mathbb{L}^{-1} s^2 - \cdots)^{\mathbb{L}+1}, \tag{4.3}$$

$$= \Big(\mathsf{Z}_0(-s) \cdot (1 - \mathbb{L}^{-\frac{1}{2}} s + \mathbb{L}^{-1} s^2 - \cdots)\Big)^{\mathbb{L}+1}.$$

We shall need the following:

LEMMA 4.5. *There is an identity*

$$\mathsf{F}_{\mathrm{crv}}(-s) = \mathsf{Z}_0(-s) \cdot (1 + \mathbb{L}^{-\frac{1}{2}} s)^{-1} \in \mathcal{M}_{\mathbb{C}}\llbracket s \rrbracket.$$

In particular, $\mathsf{F}_{\mathrm{crv}}(-s)$ *is effective.*

Proof. Since $\mathsf{Z}_0(-s) \cdot (1 - \mathbb{L}^{-\frac{1}{2}} s + \mathbb{L}^{-1} s^2 - \cdots)$ is effective, we have

$$\Big(\mathsf{Z}_0(-s) \cdot (1 - \mathbb{L}^{-\frac{1}{2}} s + \mathbb{L}^{-1} s^2 - \cdots)\Big)^{\mathbb{L}+1} = \Big(\mathsf{Z}_0(-s) \cdot (1 - \mathbb{L}^{-\frac{1}{2}} s + \mathbb{L}^{-1} s^2 - \cdots)\Big)_\diamond^{\mathbb{L}+1}.$$

Since $\mathbb{L} + 1$ is invertible in $K(\mathrm{St}_{\mathbb{C}}^{\mathrm{aff}})$, by Lemma 1.11 and (4.3) we deduce the lemma. $\square$

COROLLARY 4.6. *There is an equality of motives*

$$\Omega_{\mathrm{crv}}^n = \begin{cases} -\mathbb{L}^{-\frac{1}{2}} - \mathbb{L}^{-\frac{3}{2}} & \text{if } n = 1 \\ (-1)^n \mathbb{L}^{-\frac{3}{2}} \dfrac{\mathbb{L}^{\frac{n}{2}} - \mathbb{L}^{-\frac{n}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} & \text{otherwise.} \end{cases}$$

In particular, Ω_{crv}^n *is effective.*

Proof. From Lemma 4.5 and the equations

$$(1 + \mathbb{L}^{-\frac{1}{2}} s)^{-1} = \mathrm{Exp}(-\mathbb{L}^{-\frac{1}{2}} s)$$

$$Z_0(-s) = \mathrm{Exp}\left(\sum_{n\geqslant 1} \mathbb{L}^{-\frac{3}{2}} \frac{\mathbb{L}^{\frac{n}{2}} - \mathbb{L}^{-\frac{n}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}(-s)^n\right),$$

we deduce that

$$\mathsf{F}_{\mathrm{crv}}(-s) = \mathrm{Exp}\,(P),$$

where

$$P = \left(\sum_{n\geqslant 1} \mathbb{L}^{-\frac{3}{2}} \frac{\mathbb{L}^{\frac{n}{2}} - \mathbb{L}^{-\frac{n}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}(-s)^n\right) - \mathbb{L}^{-\frac{1}{2}}s.$$

Since P is effective, $^{\diamond}\mathrm{Exp}(P) = \mathrm{Exp}(P)$. On the other hand, by definition we have the equality $\mathsf{F}_{\mathrm{crv}}(-s) = {}^{\diamond}\mathrm{Exp}(\sum \Omega_{\mathrm{crv}}^n s^n)$. The result then follows by injectivity of $^{\diamond}\mathrm{Exp}$. $\square$

Corollary 4.1 can now be restated as follows:

THEOREM 4.7.   *The absolute virtual motives of $L \subset \mathbb{A}^3$ and $C_0 \subset X$ are given by*

$$\mathsf{Q}_{L/\mathbb{A}^3}(-t) = \mathsf{Z}_{\mathbb{A}^3\setminus L}(-t) \cdot \mathsf{F}_{\mathrm{crv}}(-t)^{\mathbb{L}}$$

$$\mathsf{Q}_{C_0/X}(-t) = \mathsf{Z}_{X\setminus C_0}(-t) \cdot \mathsf{F}_{\mathrm{crv}}(-t)^{\mathbb{L}+1}.$$

Proof. Combining (4.1) with Lemma 4.5, we get $\mathsf{F}_{\mathrm{crv}}(-t)^{\mathbb{L}+1} = \mathsf{F}_{\mathrm{crv}}(-t)_{\diamond}^{\mathbb{L}+1}$. $\square$

5.2. *Proof of Theorems B and C*

Proof of Theorem B. By (3.11), there is an equality

$$\sum_{n\geqslant 0} (-1)^n \left[Q_L^n \xrightarrow{\mathbf{hc}_{\mathbb{A}^3}} \mathrm{Sym}^n(\mathbb{A}^3)\right]_{\mathrm{vir}}$$

$$= {}^{\diamond}\mathrm{Exp}_{\cup}\left(\sum_{n\geqslant 1} \left(\Omega_{\mathrm{pt}}^n \boxtimes \left[\mathbb{A}^3\setminus L \xrightarrow{\Delta_n u_L} \mathrm{Sym}^n(\mathbb{A}^3)\right] + \Omega_{\mathrm{crv}}^n \boxtimes \left[L \xrightarrow{\Delta_n \iota_L} \mathrm{Sym}^n(\mathbb{A}^3)\right]\right)\right).$$

Plugging in the results of Corollaries 4.6 and 4.4, we deduce

$$\sum_{n\geqslant 0} (-1)^n \left[Q_L^n \xrightarrow{\mathbf{hc}_{\mathbb{A}^3}} \mathrm{Sym}^n(\mathbb{A}^3)\right]_{\mathrm{vir}}$$

$$= {}^{\diamond}\mathrm{Exp}_{\cup}\left(\sum_{n\geqslant 1}\left(\Omega_{\mathrm{BBS}}^n \boxtimes \left[\mathbb{A}^3\setminus L \xrightarrow{\Delta_n u_L} \mathrm{Sym}^n(\mathbb{A}^3)\right]\right.\right.$$

$$\left. + \Omega_{\mathrm{BBS}}^n \boxtimes \left[L \xrightarrow{\Delta_n \iota_L} \mathrm{Sym}^n(\mathbb{A}^3)\right]\right) - \mathbb{L}^{-\frac{1}{2}}\left[L \xrightarrow{\Delta_1 \iota_L} \mathrm{Sym}^1(\mathbb{A}^3)\right]\right)$$

$$= \mathrm{Exp}_{\cup}\left(\sum_{n\geqslant 2}\left(\Omega_{\mathrm{BBS}}^n \boxtimes \left[\mathbb{A}^3 \xrightarrow{\Delta_n} \mathrm{Sym}^n(\mathbb{A}^3)\right]\right)\right.$$

$$\left. - \mathbb{L}^{-\frac{1}{2}}\left[L \xrightarrow{\Delta_1 \iota_L} \mathrm{Sym}^1(\mathbb{A}^3)\right] - \mathbb{L}^{-\frac{3}{2}}\left[\mathbb{A}^3 \xrightarrow{\Delta_1} \mathrm{Sym}^1(\mathbb{A}^3)\right]\right),$$

where the removal of the '$\diamond$' comes from the effectiveness statements in Corollaries 4.6 and 4.4. $\qquad\square$

Guided by Theorem 4.7, we can now define a virtual motive for an arbitrary Quot scheme Q_C^n.

DEFINITION 4.8. Let Y be a smooth 3-fold. For a smooth curve $C \subset Y$, we define classes $[Q_C^n]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}$ by the identity

$$\sum_{n \geqslant 0} \left[Q_C^n\right]_{\mathrm{vir}}(-t)^n = \mathsf{Z}_{Y \setminus C}(-t) \cdot \mathsf{F}_{\mathrm{crv}}(-t)^{[C]}.$$

We also define

$$\mathsf{Q}_{C/Y}(t) = \sum_{n \geqslant 0} \left[Q_C^n\right]_{\mathrm{vir}} \cdot t^n.$$

At this point, it is not yet clear that $[Q_C^n]_{\mathrm{vir}}$ is a virtual motive for Q_C^n, but we incorporate this in the proof of Theorem C from the introduction:

THEOREM 4.9. *For all $n \geqslant 0$, there is an equality of motives*

$$\left[Q_C^n\right]_{\mathrm{vir}} = \sum_{j=0}^{n} \left[\mathrm{Hilb}^{n-j} Y\right]_{\mathrm{vir}} \cdot \left[\mathrm{Sym}^j C\right]_{\mathrm{vir}}.$$

In other words, we have a product decomposition

$$\mathsf{Q}_{C/Y}(t) = \mathsf{Z}_Y(t) \cdot \mathsf{Z}_C(t),$$

where for a smooth variety X of dimension at most 3, $\mathsf{Z}_X(t)$ denotes the motivic partition function of the Hilbert scheme of points of X.

Proof. Combining the power structure with Lemma 4.5, we find

$$\begin{aligned}
\mathsf{Q}_{C/Y}(-t) &= \mathsf{Z}_{Y \setminus C}(-t) \cdot \mathsf{F}_{\mathrm{crv}}(-t)^{[C]} \\
&= \mathsf{Z}_0(-t)^{[Y]-[C]} \cdot \mathsf{Z}_0(-t)^{[C]} \cdot \left(1 + \mathbb{L}^{-\frac{1}{2}}t\right)^{-[C]} \\
&= \mathsf{Z}_Y(-t) \cdot \mathsf{Z}_C(-t).
\end{aligned}$$

In particular, the classes $[Q_C^n]_{\mathrm{vir}}$ are virtual motives, because

$$\chi \mathsf{Q}_{C/Y}(t) = \chi \mathsf{Z}_Y(t) \cdot \chi \mathsf{Z}_C(t) = M(-t)^{\chi(Y)} \cdot (1+t)^{-\chi(C)}$$

and this equals $\sum_n \widetilde{\chi}(Q_C^n)t^n$ by [**32**, Proposition 5.1]. $\qquad\square$

The proof of Theorem C is complete.

5.2.1. *Equivalent formulations.* Using the power structure on $\mathcal{M}_{\mathbb{C}}$ and the explicit formulas for $\mathsf{Z}_X(t)$ available from [**3**, Section 4], we find an identity

$$\mathsf{Q}_{C/Y}(t) = \left(\prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - \mathbb{L}^{k-1-\frac{m}{2}}t^m\right)^{-[Y]}\right) \cdot \left(1 - \mathbb{L}^{-\frac{1}{2}}t\right)^{-[C]}.$$

Another equivalent way to express the same identity is via motivic exponentials. If t is the variable used in the definition of motivic exponential (cf. Section 1.5.1), then

$$\mathsf{Q}_{C/Y}(-t) = \mathrm{Exp}\left(\frac{-t[Y]_{\mathrm{vir}}}{\left(1 + \mathbb{L}^{\frac{1}{2}}t\right)\left(1 + \mathbb{L}^{-\frac{1}{2}}t\right)} - t[C]_{\mathrm{vir}}\right)$$

$$= \mathrm{Exp}\left(-t[Y]_{\mathrm{vir}}\,\mathrm{Exp}(-t[\mathbb{P}^1]_{\mathrm{vir}}) - t[C]_{\mathrm{vir}}\right).$$

5.3. *Local Donaldson–Thomas invariants*

Let Y be a projective Calabi–Yau 3-fold, and let $C \subset Y$ be a smooth curve of genus g. Recall the C-local DT invariants

$$\mathsf{DT}_C^n = \int_{Q_C^n} \nu_I \, \mathrm{d}\chi$$

defined by restricting the Behrend function of the Hilbert scheme $I = I_{1-g+n}(Y, [C])$ to its closed subset $|Q_C^n| \subset I$. The BPS number $n_{g,C}$ of $C \subset Y$ is the integer

$$n_{g,C} = \nu_{I_{1-g}(Y,[C])}(\mathscr{I}_C) \in \mathbb{Z}.$$

Theorem 4.9 immediately implies the following:

COROLLARY 4.10. *Let Y be a projective Calabi–Yau 3-fold, $C \subset Y$ a smooth curve with BPS number $n_{g,C} = 1$. Then*

$$\chi[Q_C^n]_{\mathrm{vir}} = \mathsf{DT}_C^n.$$

Proof. The main result of [31] proves that

$$\mathsf{DT}_C^n = n_{g,C} \cdot \widetilde{\chi}(Q_C^n).$$

By the proof of Theorem 4.9, we know that $[Q_C^n]_{\mathrm{vir}}$ is a virtual motive, so the result follows. □

When C is infinitesimally rigid in Y, that is, $H^0(C, N_{C/Y}) = 0$, the integer DT_C^n is the degree of the virtual fundamental class

$$[Q_C^n]^{\mathrm{vir}} \in A_0(Q_C^n),$$

naturally defined (by restriction) on the *connected component*

$$Q_C^n \subset I_{1-g+n}(Y, [C]).$$

So, by Corollary 4.10, the class $[Q_C^n]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}$ can be seen as a motivic Donaldson–Thomas invariant.

REMARK 4.11. In [40, Example 5.7], one can find an example of a cohomological DT invariant in the projective case. We are not aware of other explicit examples of motivic DT invariants for *projective* Calabi–Yau 3-folds, in a setting where the moduli space parameterises *curves and points*. Without a curve in the picture, there is the virtual motive $[\mathrm{Hilb}^n Y]_{\mathrm{vir}}$ constructed in [3] for arbitrary 3-folds, and if Y is an *open* Calabi–Yau there are plenty of examples, see, for instance, [8, 9, 20, 21, 23].

REMARK 4.12. The formula $\mathsf{Q}_{L/\mathbb{A}^3} = \mathsf{Z}_{\mathbb{A}^3} \cdot \mathsf{Z}_L$ was conjectured in the second author's PhD thesis [30], where the problem was reduced to proving a motivic identity (with no 'virtualness' left to deal with) involving essentially only the stack of finite length coherent sheaves over $\mathbb{A}^2$. As it turned out, such an identity could in principle be checked by hand after performing

a complete classification of finite length modules over the polynomial ring $\mathbb{C}[x, y]$, which is known to be a *wild* problem. This classification was accomplished by Moschetti and the second author in [**22**] for modules of length $n \leqslant 4$. As a consequence, the motivic wall-crossing formula $Q_{L/\mathbb{A}^3} = Z_{\mathbb{A}^3} \cdot Z_L$ could be proven by hand up to order 4 using this classification.

6. *Categorification*

In this final section, we outline a programme for future work on a *categorified* DT/PT correspondence. Just as, since any smooth threefold analytically locally looks like $\mathbb{A}^3$, one may patch together the full cohomological understanding of the DT theory of degree 0 DT theory on $\mathbb{A}^3$ obtained in [**7**] to try to understand the degree 0 cohomological DT theory of *any* Calabi–Yau 3-fold (see [**42**, Section 3.1]), analytically locally, the inclusion of a smooth curve inside a smooth Calabi–Yau 3-fold can be modelled by $L \subset \mathbb{A}^3$, and so the intention is that by proving the conjectures below we can approach a cohomological version of the DT/PT correspondence.

6.1. *Further directions I: categorification*

By our Theorem A, the moduli space Q_L^n arises as the critical locus of the function $\mathrm{Tr}(W_r)$ on the moduli space $\mathcal{N}_n^{\circ\circ}$. Shifting the associated virtual motives, we define the generating function

$$Q_{L/\mathbb{A}^3}^{\mathrm{sh}} = \sum_{n \geqslant 0} \mathbb{L}^{-\frac{n}{2}} \big[Q_L^n \xrightarrow{\mathbf{hc}_n} \mathrm{Sym}^n \mathbb{A}^3 \big]_{\mathrm{vir}} \in \mathcal{M}_{\mathrm{Sym}(\mathbb{A}^3)}.$$

Then by Theorem B, we can write

$$Q_{L/\mathbb{A}^3}^{\mathrm{sh}} = \mathrm{Exp}_{\cup} \left(\sum_{n>0} \Delta_{n\,!} \Big(\Omega_n^{\mathrm{sh}} \boxtimes \big[\mathbb{A}^3 \xrightarrow{\mathrm{id}} \mathbb{A}^3 \big] \Big) \right) \boxtimes_{\cup} \mathrm{Exp}_{\cup} \big(\Delta_{1\,!} (\mathbb{L}^{-1} \boxtimes [L \hookrightarrow \mathbb{A}^3]) \big), \quad (5.1)$$

where

$$\Omega_n^{\mathrm{sh}} = \mathbb{L}^{-2}(1 + \mathbb{L}^{-1} + \cdots + \mathbb{L}^{1-n})$$

and $\Delta_n : \mathbb{A}^3 \to \mathrm{Sym}^n(\mathbb{A}^3)$ is, as ever, the inclusion of the small diagonal.

For the rest of this section, we will make free use of the language and foundational results concerning monodromic mixed Hodge modules, see [**36, 37**] for background on mixed Hodge modules, [**17**] for an introduction to monodromic mixed Hodge structures, and [**10**] for the theory of monodromic mixed Hodge modules in cohomological Donaldson–Thomas theory. In particular, we will use the same functor

$$\phi_{\mathrm{Tr}(W_r)}^{\mathrm{mon}} \colon \mathrm{MHM}(\mathcal{N}_n^{\circ\circ}) \to \mathrm{MMHM}(\mathcal{N}_n^{\circ\circ})$$

considered in [**10**, Section 2.1], following the discussion of a monodromic mixed Hodge structure on vanishing cycle cohomology in [**17**, Section 7.4].

In what follows, for a space X, we write $\underline{\mathbb{Q}}_X$ for the constant complex of mixed Hodge modules on X, which we think of as a complex of monodromic mixed Hodge modules by endowing it with the trivial monodromy operator. Where the choice of space X is clear, we may drop the subscript.

Given an element $[Y \xrightarrow{f} Z] \in \mathcal{M}_Z^{\hat{\mu}}$, where Z is a variety, pick $n \in \mathbb{N}$ such that the $\hat{\mu}$ action on Y factors through the projection $\hat{\mu} \to \mu_n$. Then we form the mapping torus

$$Y \times_{\mu_n} \mathbb{G}_m = (Y \times \mathbb{G}_m)/\mu_n,$$

where μ_n acts via $z \cdot (y, z') = (z \cdot y, z^{-1} z')$. We define

$$\tilde{f} \colon Y \times_{\mu_n} \mathbb{G}_m \to Z \times \mathbb{A}^1$$

$$(y, z') \mapsto (f(y), z'^n).$$

Set $\mathcal{F} = \tilde{f}_! \underline{\mathbb{Q}}_{Y \times_{\mu_n} \mathbb{G}_m} \in \mathrm{D}^b(\mathrm{MHM}(Z \times \mathbb{A}^1))$. By construction, the cohomology sheaves of this direct image are locally constant along the fibres of the projection $Z \times \mathbb{A}^1 \to Z$, away from the zero fibre. In particular, $\mathcal{F}$ is a bounded complex of *monodromic mixed Hodge modules*, and so $[\mathcal{F}] \in \mathrm{K}_0(\mathrm{MMHM}(Z))$. Sending $[Y \to Z] \mapsto [\mathcal{F}]$ defines a group homomorphism

$$\Psi \colon \mathcal{M}_Z^{\hat{\mu}} \to \mathrm{K}_0(\mathrm{MMHM}(Z)),$$

which is a λ-ring homomorphism in the event that Z is also a commutative monoid.

Now consider $\Psi(Q_{L/\mathbb{A}^3}^{\mathrm{sh}})$. This is the class of the complex of monodromic mixed Hodge modules

$$\mathcal{F}_{L/\mathbb{A}^3} \coloneqq \bigoplus_n \mathbf{hc}_{n!} \, \phi_{\mathrm{Tr}(W_r)}^{\mathrm{mon}} \underline{\mathbb{Q}}_{\mathcal{N}_n^{\circ\circ}} \otimes (\mathbb{T}^{1/2})^{-4n-2n^2}, \tag{5.2}$$

where $\mathbb{T}^{1/2}$ is a tensor square root of the complex of mixed Hodge structures $\mathrm{H}_c(\mathbb{A}^1, \mathbb{Q})$ — that is, a half Tate twist, concentrated in cohomological degree 1. From (5.1), we deduce that

$$\Psi(Q_{L/\mathbb{A}^3}^{\mathrm{sh}}) = [\mathcal{F}_{L/\mathbb{A}^3}]$$

$$= \left[\mathrm{Sym} \left((\Delta_{1!} \underline{\mathbb{Q}}_{\mathbb{A}^1} \otimes \mathbb{T}^{-1}) \oplus \left(\bigoplus_{n \geqslant 1} \Delta_{n!} \underline{\mathbb{Q}}_{\mathbb{A}^3} \otimes \left(\bigoplus_{i=-2}^{-1-n} \mathbb{T}^i \right) \right) \right) \right]. \tag{5.3}$$

It follows as in [**10**, Proposition 3.5] from finiteness of the map $\cup \colon \mathrm{Sym}(\mathbb{A}^3) \times \mathrm{Sym}(\mathbb{A}^3) \to \mathrm{Sym}(\mathbb{A}^3)$ that the object in the square brackets on the right-hand side is *pure*, in the sense that its ith cohomology is pure of weight i. From semisimplicity of the category of pure mixed Hodge modules (proved by Saito, see above references), the following two statements are equivalent:

- the complex of monodromic mixed Hodge modules $\mathcal{F}_{L/\mathbb{A}^3}$ is pure;
- there is an isomorphism

$$\mathcal{F}_{L/\mathbb{A}^3} \cong \mathrm{Sym} \left((\Delta_{1!} \underline{\mathbb{Q}}_{\mathbb{A}^1} \otimes \mathbb{T}^{-1}) \oplus \left(\bigoplus_{n \geqslant 1} \Delta_{n!} \underline{\mathbb{Q}}_{\mathbb{A}^3} \otimes \left(\bigoplus_{i=-2}^{-1-n} \mathbb{T}^i \right) \right) \right). \tag{5.4}$$

CONJECTURE 5.1. The above two statements are true. In particular, there is an isomorphism of $\mathbb{Z}$-graded mixed Hodge structures

$$\bigoplus_{n \geqslant 0} \mathrm{H}_c(Q_L^n, \phi_{\mathrm{Tr}(W_r)}^{\mathrm{mon}} \underline{\mathbb{Q}}_{\mathcal{N}_n^{\circ\circ}}) \otimes \mathbb{T}^{-2n-n^2} \cong \mathrm{Sym} \left(\bigoplus_{n \geqslant 1} \left(\bigoplus_{i=-2}^{-n-1} \mathbb{T}^i \right) \right) \otimes \mathrm{Sym}(V),$$

with n keeping track of the degree on both sides, and where V is a 1-dimensional pure weight zero Hodge structure placed in degree 1. The $\mathbb{Z}$-grading on the right-hand side comes from the grading on the object we are taking the symmetric algebra of (that is, we ignore the fact that Sym of any object acquires an extra $\mathbb{Z}$-grading).

The isomorphism in (5.4) would categorify our Theorem B, and would lift it from a formula to an isomorphism (that is, categorify it). Since it would take quite some time to even fill in the requisite definitions for the above discussion, we leave this conjecture to future work.

6.2. *Further directions II: A new CoHA module*

We recall some very general theory regarding critical cohomological Hall algebras and their representations.

First, we fix a quiver Q and a potential W. The Euler form for Q is defined by

$$\chi_Q \colon \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$$

$$(\gamma, \gamma') \mapsto \sum_{i \in Q_0} \gamma_i \gamma_i' - \sum_{a \in Q_1} \gamma_{s(a)} \gamma_{t(a)}'.$$

A stability condition $\zeta \in \mathbb{Q}^{Q_0}$ is called generic if for any two non-zero dimension vectors of the same slope, we have $\langle \gamma, \gamma' \rangle = 0$. Recall that a quiver is called symmetric if for all vertices $i, j \in Q_0$, we have

$$\#\{a \mid s(a) = i,\ t(a) = j\} \ = \ \#\{a \mid s(a) = j,\ t(a) = i\}.$$

Note that the genericity condition is vacuous for symmetric quivers such as the three loop quiver obtained by removing the framing from the quiver Q_{BBS} from Section 1.7, and the quiver Q_{con} from Section 2.1.

Fix a generic stability condition $\zeta \in \mathbb{Q}^{Q_0}$ and a slope $\theta \in (0, \infty)$. We set $\Lambda_\theta^\zeta \subset \mathbb{N}^{Q_0}$ to be the submonoid of dimension vectors of slope θ. Then as in [17] we endow the graded monodromic mixed Hodge structure

$$\mathcal{H}_{Q,W,\theta}^\zeta \coloneqq \bigoplus_{\mathbf{d} \in \Lambda_\theta^\zeta} \mathrm{H}(\mathrm{rMod}_{\mathbf{d}}^\zeta(Q)/\operatorname{GL}_{\mathbf{d}}, \phi_{\mathrm{Tr}(W)}^{\mathrm{mon}} \underline{\mathbb{Q}}) \otimes \mathbb{T}^{\chi(\mathbf{d}, \mathbf{d})/2} \tag{5.5}$$

with the Hall algebra product arising from the stack of short exact sequences of right $\mathbb{C}Q$-modules. The multiplication respects the monodromic mixed Hodge structure on (5.5) — in other words (5.5) is made into an algebra object in the tensor category of Λ_θ^ζ-graded monodromic mixed Hodge structures.

In fact we will only consider the cohomological Hall algebra for which Q is symmetric, and $\zeta = (0, \ldots, 0)$ is the *degenerate* stability condition, with $\theta = 0$. So we will drop ζ and θ from the notation and just write $\mathcal{H}_{Q,W}$ for the cohomological Hall algebra associated to Q and W.

Let $Q \subset Q_{\mathrm{fr}}$ be an inclusion of quivers, with Q a full subquiver. We do not assume that Q_{fr} is symmetric. Let $I \subset \mathbb{C}Q_{\mathrm{fr}}$ be the two-sided ideal generated by all paths in Q_{fr} not contained in Q. Then $\mathbb{C}Q \cong \mathbb{C}Q_{\mathrm{fr}}/I$, and we let $q \colon \mathbb{C}Q_{\mathrm{fr}} \to \mathbb{C}Q$ be the induced surjection.

Let W_{fr} be a potential extending W, in the sense that $qW_{\mathrm{fr}} = W$. Let $\zeta \in \mathbb{Q}^{Q_{\mathrm{fr},0}}$ be a stability condition extending the degenerate stability condition, that is, such that $\zeta|_{Q_0} = 0$. Let $\mathbf{f} \in \mathbb{N}^{Q_{\mathrm{fr},0} \setminus Q_0}$ be a *framing* dimension vector. Define

$$\mathcal{N}_{Q_{\mathrm{fr}}, \mathbf{f}}^\zeta \coloneqq \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathrm{H}(\mathrm{rMod}_{(\mathbf{d},\mathbf{f})}^\zeta(Q_{\mathrm{fr}})/\operatorname{GL}_{\mathbf{d}}, \phi_{\mathrm{Tr}(W_{\mathrm{fr}})}^{\mathrm{mon}} \underline{\mathbb{Q}}) \otimes \mathbb{T}^{\chi_Q(\mathbf{d},\mathbf{d})/2 - \chi_{Q_{\mathrm{fr}}}((\mathbf{d},0),(0,\mathbf{f}))}. \tag{5.6}$$

Via the usual correspondences, it is standard to check that $\mathcal{N}_{Q_{\mathrm{fr}}, \mathbf{f}}^\zeta$ is a module for $\mathcal{H}_{Q,W}$.

Now we make this setup more specific. Let Q be the three loop quiver, obtained by removing $\underline{\infty}$ and all arrows containing it from Q_r. Considering the inclusion $Q \subset Q_{\mathrm{BBS}}$, and setting $\zeta_\infty = -1$ and $\mathbf{f} = 1$, the module $\mathcal{N}_{Q_{\mathrm{fr}}, 1}^\zeta$ is precisely the vanishing cycle cohomology of $\mathrm{Hilb}(\mathbb{A}^3)$. It was shown in version one of [10] that in fact this module is cyclic over the CoHA $\mathcal{H}_{Q,W}$. For a more recent example of a geometrically motivated class of modules for this CoHA, that falls under the general construction above, the reader may consult [28], where an action on the space of spiked instantons is considered.

The purpose of this subsection is to add one more geometrically motivated example to the list, namely, we consider the inclusion $Q \subset Q_r$, the framing vector $\mathbf{f} = 1$, and the stability condition $\zeta_{\underline{\infty}} = -1$. Then by Proposition 2.10, we obtain

$$\mathcal{N}_{Q_{\mathrm{fr}}, \mathbf{f}}^\zeta = \bigoplus_{n \geqslant 0} \mathrm{H}(Q_L^n, \phi_{\mathrm{Tr}(W_r)}^{\mathrm{mon}} \underline{\mathbb{Q}}_{\mathcal{N}_n^{\circ\circ}}) \otimes \mathbb{T}^{-n-n^2}, \tag{5.7}$$

that is, the module constructed for the above data is precisely the vanishing cycle cohomologies of all of the quot schemes Q_L. Keeping track of the various Tate twists, (5.7) is the hypercohomology of the Verdier dual of the monodromic mixed Hodge module

$$\phi^{\mathrm{mon}}_{\mathrm{Tr}(W_r)}\underline{\mathbb{Q}}_{\mathcal{N}^{\circ\circ}_n} \otimes \mathbb{T}^{-2n-n^2}$$

considered in (5.2). According to the conjecture of the previous subsection, this module should itself be isomorphic to the underlying graded mixed Hodge module of a symmetric algebra. In fact by utilising the factorization sheaf structure implicit in our calculations in Section 3, it is not hard to find a cocommutative coproduct on (5.7), and a candidate for a compatible product, leading to our final conjecture:

CONJECTURE 5.2. The graded mixed Hodge structure $\mathcal{N}^{\zeta}_{Q_{\mathrm{fr}},\mathbf{f}}$ is a universal enveloping algebra.

The connection with Conjecture 5.1 is that by proving a version of Conjecture 5.2 over the base $\mathrm{Sym}(\mathbb{A}^3)$, one would deduce the purity conjecture as in [**7**, Theorem A].

REMARK 5.3. The above module $\mathcal{N}^{\zeta}_{Q_{\mathrm{fr}},1}$ is in fact the third in a natural sequence, the first two elements of which will be well known to the reader. Firstly, we can remove the arrow a'_2 from Q_r to obtain a new quiver Q'_r. The pullback of $\mathrm{Tr}(W_r)$ along the extension by zero morphism

$$\mathrm{rMod}^{\zeta}_{(n,1)}(Q'_r)/\mathrm{GL}_n \to \mathrm{rMod}^{\zeta}_{(n,1)}(Q_r)/\mathrm{GL}_n$$

is induced by the potential $W'_r = a''_1 b''_1 a''_2 - a''_2 b''_1 a''_1 - a''_2 b'_1 a'_1$, and it is easy to check that there is an isomorphism

$$\mathrm{H}\left(\mathrm{rMod}^{\zeta}_{(n,1)}(Q'_r)/\mathrm{GL}_n, \phi^{\mathrm{mon}}_{\mathrm{Tr}(W'_r)}\underline{\mathbb{Q}}\right) \otimes \mathbb{T}^{-n-n^2} \cong \mathrm{H}(\mathrm{Hilb}^n \mathbb{A}^2, \mathbb{Q}) \otimes \mathbb{T}^{-n}$$

with the *usual* cohomology of the Hilbert scheme for $\mathbb{A}^2$. Going further, we can remove the arrow b'_1, recovering the framed BBS quiver, with its usual potential, and the MacMahon module provided by the (vanishing cycle) cohomology of $\mathrm{Hilb}^n \mathbb{A}^3$. That the cohomology of $\mathrm{Hilb}^n \mathbb{A}^2$ finds itself sandwiched between the vanishing cycle cohomology of the Hilbert scheme of $\mathbb{A}^3$ and the quot scheme Q^n_L in this way is a mystery that we leave to future research to understand properly.

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