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Asymptotic stability in linear viscoelasticity with supplies ☆

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A B S T R A C T

We present some results about the asymptotic behavior of a linear viscoelastic system making use of the approach based on the concept of minimal state. This approach allows to obtain results in a larger class of solutions and data with respect to the classical one based on the histories of the deformation gradient. Recently, a lot of attention has been paid to find unified approaches which permit to study the asymptotic behavior with memory kernels presenting a temporal decay of which the exponential and polynomial decays are only special cases. Here we extend this unified approach to the dynamic problem in presence of supplies by using the minimal state and compare our results with those present in literature.

1. Introduction

The study of the asymptotic behavior of viscoelastic materials has been focused almost exclusively on the study of the stability in the topology defined on the histories space of the Graffi–Volterra free energy and with zero data both for the initial history and for the external source.

In this research we will study the stability and the asymptotic behavior of the energy in the states space, just as defined by Banfi [5] and then proposed again by Del Piero and Deseri [11] (see Appendix A). We will use the topology induced by the free energy introduced in [13] which allows to work in a wider space of states with respect to the one defined through the Graffi–Volterra free energy [6].

In presence of memory kernels of exponential type, many results concerning the exponential stability of the Graffi–Volterra free energy and of the one proposed in [13] have been obtained ([4,14,15] and references therein) starting from the pioneering paper by Dafermos [10]; while, if the memory kernel polynomially decays in time, long time behavior results have been obtained for vanishing data [20].

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Here, as done in [16] for a viscoelastic fluid, we will consider memory kernels presenting a temporal decay from which we can recover the exponential decay or the polynomial decay as special cases (see [19]) and prove that the decay of the solutions is strictly related to the behavior of the memory kernel.

Another innovative aspect of this research lies in the possibility of studying such problems in presence of non-zero past histories and external sources. We will compare our results with those obtained in a recent paper by Guesmia [18], which uses an adaptation of the so-called convexity method introduced in [1] within the context of the topology induced by the Graffi–Volterra free energy and with no supply.

This paper is organized as follows. In Section 2, the differential system is formulated as an initial boundary value problem, but also within the semigroup theory. Hence, we recall some results about its well posedness. In Section 3 we present our results on the asymptotic behavior and exponential stability. In Section 4, we give some general comments and state some open problems. Finally, in Appendix A, we recall the classical constitutive equation of the viscoelasticity and the notion of minimal state.

2. Setup of the problem

The initial boundary value problem, for a standard linear viscoelastic problem in a bounded domain $\Omega \subset \mathcal{R}^3$ with regular boundary $\partial\Omega$, is governed by the equation (see Appendix A).

$$\frac{\partial}{\partial t} v(x,t) = \nabla \cdot \left[g_\infty \mathbb{A} \nabla u(x,t) + \int_0^\infty g'(s) \mathbb{A} \nabla u_r^t(x,s)\, \mathrm{d}s \right] + f(x,t), \tag{2.1}$$

where the vector v denotes the velocity, the vector u the displacement, the function $u_r^t(\cdot, s) = u(\cdot, t-s) - u(\cdot, t)$ the relative past history of u, while the positive constant g_∞ and the function $\check{g}(s) = -\int_s^\infty g'(s)\mathrm{d}s$ satisfy the following conditions:

(h$_1$) $g_\infty > 0$, g' and $\check{g}$ belong to $L^1(\mathcal{R}^+)$,
(h$_2$) $\check{g}$ is positive in $\mathcal{R}^+$ and

$$\int_0^\infty \check{g}(s) \cos \omega s \, \mathrm{d}s > 0, \quad \forall \omega \in \mathcal{R}.$$

Moreover, the fourth order tensor $\mathbb{A}$ is symmetric and positive definite, the density ρ is assumed equal to 1 and f denotes the body force.

Together with Eq. (2.1), initial and boundary conditions are given by

$$\begin{aligned} & v(x,0) = v_0(x),\ u(x,0) = u_0(x),\ u^{t=0}(x,s) = u^0(x,s),\ s > 0, x \in \Omega, \\ & u(x,t) = 0, \quad x \in \partial\Omega,\ t \in \mathcal{R}.^+ \end{aligned} \tag{2.2}$$

We observe that the following equalities hold:

$$\int_t^\infty g'(s) \mathbb{A} \nabla u_r^t(x,s)\, \mathrm{d}s = \int_0^\infty g'(t+s) \mathbb{A} \nabla u_r^{t=0}(x,s)\, \mathrm{d}s + \check{g}(t) \mathbb{A} \nabla [u(x,t) - u(x,0)],$$

$$\int_0^t g'(s) \mathbb{A} \nabla u_r^t(x,s)\, \mathrm{d}s = [\check{g}(t) + g_\infty] \mathbb{A} \nabla [u(x,0) - u(x,t)] + \int_0^t [\check{g}(t-s) + g_\infty] \mathbb{A} \nabla v(x,s)\, \mathrm{d}s,$$

so that Eq. (2.1) becomes

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$$\frac{\partial}{\partial t}v(x,t) = \nabla \cdot \int_0^t [\breve{g}(t-s) + g_\infty]\mathbb{A}\nabla v(x,s)\,\mathrm{d}s + \nabla \cdot \left[\int_0^\infty g'(t+s)\mathbb{A}\nabla u_r^{t=0}(x,s)\,\mathrm{d}s + g_\infty \mathbb{A}\nabla u(x,0)\right] + f(x,t).$$

If we suppose the source f such that

$$\nabla \cdot \left[\int_0^\infty g'(t+s)\mathbb{A}\nabla u_r^{t=0}(x,s)\,\mathrm{d}s + g_\infty \mathbb{A}\nabla u(x,0)\right] + f(x,t) = \nabla \cdot \left[\mathbb{A}\nabla \imath^0(x,t)\right] \tag{2.3}$$

where $\imath^0$ is a known function, the initial boundary value problem becomes

$$\begin{aligned} &\frac{\partial}{\partial t}v(x,t) = \nabla \cdot \int_0^t [\breve{g}(t-s) + g_\infty]\mathbb{A}\nabla v(x,s)\,\mathrm{d}s + \nabla \cdot \left[\mathbb{A}\nabla \imath^0(x,t)\right], \\ &v(x,0) = v_0(x), \qquad x \in \Omega, \\ &v(x,t) = 0, \qquad x \in \partial\Omega,\ t \in \mathcal{R}^+. \end{aligned} \tag{2.4}$$

In [17] a natural weak formulation of problem (2.4) has been given in terms of virtual-work solutions. Moreover, under the hypotheses (h_1) and (h_2) on the memory kernel, an existence and uniqueness theorem has been obtained in the space

$$\mathcal{F}(\mathcal{R}^+,\Omega) = H^{\frac{1}{2}}(\mathcal{R}^+, L^2(\Omega)) \cap \mathcal{H}_g(\mathcal{R}^+, H_0^1(\Omega)),$$

where

$$\mathcal{H}_g(\mathcal{R}^+, H_0^1(\Omega)) = \left\{ v \in L^2_{\mathrm{loc}}(\mathcal{R}^+, H_0^1(\Omega)); \ \|v\|^2_{\mathcal{H}_g} = \int_0^\infty \int_0^\infty \int_\Omega g(\tau - s)\mathbb{A}\nabla v(x,\tau)\cdot \nabla v(x,s)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s < \infty \right\},$$

if the datum B belongs to

$$\mathcal{H}^*_g(\mathcal{R}^+, L^2(\Omega)) = \left\{ B \in L^2_{\mathrm{loc}}(\mathcal{R}^+, L^2(\Omega)); \ \int_0^\infty \int_0^\infty \int_\Omega \mathbb{A}B(x,\tau)\cdot \nabla v(x,s)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s < k_B\|v\|_{\mathcal{H}_g},\ \forall v \in \mathcal{H}_g(\mathcal{R}^+, H_0^1(\Omega)) \right\}.$$

We recall that $\mathcal{H}_g$ coincides with the space associated to the minimal free energy of Breuer and Onat [7,8] and, therefore, $\mathcal{H}^*_g$ is the widest space of data that one can expect.

In this paper we consider memory kernels which satisfy the condition (h_1) and the following more restrictive condition with respect to (h_2)

(h_2^*) $g'(s) < 0,\ g''(s) \geq 0, \quad s \in \mathcal{R}^+.$

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In the following, we put

$$\breve{\imath}^t(x,t) = -\int_0^\infty g'(s+\tau)u_r^t(x,s)\mathrm{d}s \tag{2.5}$$

and observe that the following relations hold:

$$\breve{\imath}^t(x,0) = -\int_0^\infty \breve{\imath}_\tau^t(x,\tau)\,\mathrm{d}\tau, \tag{2.6}$$

$$\breve{\imath}^t(x,\tau) = \breve{\imath}^0(x,t+\tau) - \int_0^t g'(x,\tau+s)u_r^t(x,s)\,\mathrm{d}s. \tag{2.7}$$

With this new variable the problem (2.1)–(2.2) can be rewritten as a first order integro-differential system as follows

$$\begin{aligned}
\frac{\partial}{\partial t}v(x,t) &= \nabla\cdot\left[g_\infty \mathbb{A}\nabla u(x,t) + \int_0^\infty \mathbb{A}\nabla\breve{\imath}_\tau^t(x,\tau)\,\mathrm{d}\tau\right]\\
\frac{\partial}{\partial t}\breve{\imath}_\tau^t(x,\tau) &= \breve{\imath}_{\tau\tau}^t(x,\tau) - \breve{g}(x,\tau)v(x,t)\\
v(x,0) &= v_0(x),\ u(x,0) = u_0(x),\ \breve{\imath}_\tau^{t=0}(x,\tau) = \imath_\tau^0(x,\tau),\ \tau>0, x\in\Omega\\
u(x,t) &= 0, \quad x\in\partial\Omega,\ t\in\mathcal{R}^+
\end{aligned} \tag{2.8}$$

where

$$\breve{\imath}_\tau^t(x,t) = -\int_0^\infty g''(s+\tau)u_r^t(x,s)\mathrm{d}s$$

and the second equation assigns the law governing the evolution in time of $\breve{\imath}_\tau^t$, while the initial datum i^0 is defined in (2.3).

In the sequel the dependence on the space variable x is omitted in order to simplify the notations and without misunderstanding.

In order to use the semigroup theory, we introduce the triplet

$$\chi(t) = (u(t), v(t), \breve{\imath}^t(\cdot))$$

and rewrite problem (2.8) as an abstract first order Cauchy problem

$$\begin{aligned}
\frac{d}{dt}\chi(t) &= \mathcal{A}\chi(t),\\
\chi(0) &= \chi_0.
\end{aligned} \tag{2.9}$$

The natural set in which to look for the solution of the problem (2.9) is the space

$$\mathcal{K} = H_0^1(\Omega)\times L^2(\Omega)\times L_g^2(\mathcal{R}^+, H_0^1(\Omega)),$$

where

$$L^2_g(\mathcal{R}^+, H^1_0(\Omega)) = \left\{ \imath_\tau : \mathcal{R}^+ \to H^1_0(\Omega); \int_0^\infty \frac{-1}{g'(\tau)} \int_\Omega \mathbb{A}\nabla \imath_\tau(\tau) \cdot \nabla \imath_\tau(\tau)\, \mathrm{d}x\, \mathrm{d}\tau < \infty \right\}.$$

The property (h_2^*) on the memory kernel allows us to introduce in $\mathcal{K}$ the following inner product:

$$\langle \chi_1(t), \chi_2(t)\rangle_{\mathcal{K}} = \int_\Omega [v_1(t) \cdot v_2(t) + \mathbb{A}\nabla u_1(t) \cdot \nabla u_2(t)]\, \mathrm{d}x - \int_0^\infty \frac{1}{g'(\tau)} \int_\Omega \mathbb{A}\nabla \breve{\imath}^t_{1\tau}(\tau) \cdot \nabla \breve{\imath}^t_{2\tau}(\tau)\, \mathrm{d}x\, \mathrm{d}\tau, \tag{2.10}$$

so that the total energy associated to the solution of problem (2.8) at time t is given by

$$\mathcal{E}(t) = \frac{1}{2}\langle \chi(t), \chi(t)\rangle_{\mathcal{K}} = \frac{1}{2}\|\chi(t)\|^2_{\mathcal{K}}$$

i.e.

$$\mathcal{E}(t) = \frac{1}{2}\left[\|v(t)\|^2 + g_\infty \|u(t)\|^2_{\mathbb{A}}\right] - \frac{1}{2}\int_0^\infty \frac{1}{g'(\tau)} \|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\, \mathrm{d}\tau, \tag{2.11}$$

where $\|\cdot\|$ denotes the usual L^2-norm in Ω, while

$$\|u(t)\|^2_{\mathbb{A}} = \int_\Omega \mathbb{A}\nabla u(t) \cdot \nabla u(t)\, \mathrm{d}x.$$

By using $(2.8)_2$, it is easy to obtain

$$\frac{d}{dt}\left[-\frac{1}{2}\int_0^\infty \frac{1}{g'(\tau)} \|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\, \mathrm{d}\tau\right] = -\int_\Omega \mathbb{A}\nabla \breve{\imath}^t(0) \cdot \nabla v(t) \mathrm{d}x + \frac{1}{2g'(0)} \|\breve{\imath}^t_\tau(0)\|^2_{\mathbb{A}} - \frac{1}{2}\int_0^\infty \frac{g''(\tau)}{[g'(\tau)]^2} \|\nabla \breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\, \mathrm{d}\tau, \tag{2.12}$$

then

$$\frac{d}{dt}\mathcal{E}(t) = \frac{1}{2g'(0)} \|\breve{\imath}^t_\tau(0)\|^2_{\mathbb{A}} - \frac{1}{2}\int_0^\infty \frac{g''(\tau)}{[g'(\tau)]^2} \|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\, \mathrm{d}\tau \le 0. \tag{2.13}$$

Relation (2.13) assures that the operator $\mathcal{A}$ is dissipative.

In [14] it has been proved that $\mathcal{A}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathcal{K}$. Making use of the results obtained by Da Prato and Sinestrari [9], it is possible to state the well posedness of the problem (2.9), i.e. if $\chi_0 \in \mathcal{D}(\mathcal{A})$, then the problem (2.9) admits one and only one strict solution $\chi \in C^1(\mathcal{R}^+, \mathcal{K}) \cap C(\mathcal{R}^+, \mathcal{D}(\mathcal{A}))$ while, if $\chi_0 \in \mathcal{K}$, then the problem (2.9) admits one and only one weak solution $\chi \in C(\mathcal{R}^+, \mathcal{K})$.

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3. Asymptotic behavior

Aim of this section is the study of the asymptotic behavior of problem (2.8). The minimal state approach allows to extend results obtained in the classical context based on the histories of the deformation gradient. More precisely in the first subsection we will show that the energy associated to the solutions of problem (2.8) has the same type of temporal decay of the memory kernel even in presence of non-vanishing supplies, while in the second one we will extend the results presented in [18] to a larger class of initial data.

3.1. First case

In this subsection we restrict ourselves to memory kernels satisfying (h_1), (h_2^*) and the restriction proposed in [19]

$$g''(t) \geq -\xi(t)g'(t), \qquad t \geq 0, \tag{3.1}$$

where ξ is a positive, non-increasing differentiable, measurable and non-integrable function such that it exists a positive constant κ for which

$$-\xi'(t) \leq \kappa\xi(t), \qquad t \geq 0. \tag{3.2}$$

Examples of such kernels can be found in [19]. In particular, a kernel presents an exponential or polynomial decay when ξ is a constant function or $\xi(t) = c(1+t)^{-1}$, respectively.

In order to study the asymptotic behavior of the energy E, we introduce the following functional:

$$L(t) = \mathcal{E}(t) + \delta\xi(t)\int\limits_{\Omega} v(t)\cdot\left[\breve{\imath}^t(0) + \lambda u(t)\right]\,\mathrm{d}x,$$

where δ and λ are suitable positive constants.

Lemma 3.1. *Let χ be a solution of problem* (2.9) *with $\chi_0 \in \mathcal{K}$. If g satisfies* (h_1), (h_2^*) *and* (3.1), *δ and λ are small enough, then there exists a positive constant c_0 such that*

$$\frac{d}{dt}L(t) \leq -c_0\xi(t)\mathcal{E}(t), \quad t > T_0. \tag{3.3}$$

Proof. Preliminarily, we observe that, thanks (2.6),

$$\|\breve{\imath}^t(0)\|^2_{\mathbb{A}} \leq -\int\limits_0^\infty g'(\xi)\,\mathrm{d}\xi\int\limits_0^\infty -\frac{1}{g'(\tau)}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\,\mathrm{d}\tau = -\breve{g}_0\int\limits_0^\infty \frac{1}{g'(\tau)}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\,\mathrm{d}\tau, \tag{3.4}$$

where we have pose $\breve{g}_0 = \breve{g}(0)$.

Moreover, fixed $T_0 > 0$, there exists $\alpha_{T_0} > 1$ such that

$$-\int\limits_0^\infty \frac{1}{g'(\tau)}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\,\mathrm{d}\tau \leq -\alpha_{T_0}\int\limits_0^t \frac{1}{g'(\tau)}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\,\mathrm{d}\tau, \qquad \forall t > T_0, \tag{3.5}$$

since $\breve{\imath}^t_\tau \in L^2_g(\mathcal{R}^+, H^1_0(\Omega))$.

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The time derivative of L is given by

$$\frac{d}{dt}L(t) = \frac{d}{dt}\mathcal{E}(t) + \delta\frac{d}{dt}\xi(t)\int\limits_{\Omega} v(t)\cdot\left[\breve{\imath}^t(0) + \lambda u(t)\right]\,\mathrm{dx}$$
$$+\,\delta\xi(t)\int\limits_{\Omega}\left\{\frac{\partial}{\partial t}v(t)\cdot\left[\breve{\imath}^t(0) + \lambda u(t)\right] + v(t)\cdot\left[\frac{\partial}{\partial t}\breve{\imath}^t(0) + \lambda v(t)\right]\right\}\mathrm{dx}. \tag{3.6}$$

Let us evaluate each term of the right-hand side of (3.6).

As a consequence of (2.13) and (3.1)

$$\frac{d}{dt}\mathcal{E}(t) \le \frac{1}{2g'(0)}\|\breve{\imath}^t_\tau(0)\|^2_{\mathbb{A}} + \frac{1}{2}\int\limits_0^t \frac{\xi(\tau)}{g'(\tau)}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\mathrm{d}\tau.$$

The properties of ξ and the inequality (3.5) yield

$$\frac{d}{dt}\mathcal{E}(t) \le \frac{1}{2g'(0)}\|\breve{\imath}^t_\tau(0)\|^2_{\mathbb{A}} + \frac{\xi(t)}{2\alpha_{T_0}}\int\limits_0^\infty \frac{1}{g'(\tau)}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\mathrm{d}\tau, \quad \forall t > T_0. \tag{3.7}$$

By using (3.2), (3.4) and Poincaré and Korn inequalities we have

$$\left|\delta\frac{d}{dt}\xi(t)\int\limits_{\Omega} v(t)\cdot\left[\breve{\imath}^t(0) + \lambda u(t)\right]\,\mathrm{dx}\right|$$
$$\le \frac{\delta k}{2}\xi(t)\left[(\varepsilon_0 + \varepsilon_1)\|v(t)\|^2 + \frac{\lambda^2 C_\Omega}{\varepsilon_0}\|u(t)\|^2_{\mathbb{A}} - \frac{\breve{g}_0 C_\Omega}{\varepsilon_1}\int\limits_0^\infty \frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)}\mathrm{d}\tau\right], \tag{3.8}$$

where the arithmetic-geometric mean inequality has been employed, with ε_i, $i = 0, 1$ arbitrary positive constants, while C_Ω is a suitable positive constant depending on the domain Ω and $\mathbb{A}$.

Recalling that χ is solution of problem (2.9) and taking into account (3.4), we get

$$\delta\xi(t)\int\limits_{\Omega}\frac{\partial}{\partial t}v(t)\cdot\left[\breve{\imath}^t(0) + \lambda u(t)\right]\,\mathrm{dx}$$
$$= \delta\xi(t)\int\limits_{\Omega}\nabla\cdot\mathbb{A}\nabla\left[g_\infty u(t) - \breve{\imath}^t(0)\right]\cdot\left[\breve{\imath}^t(0) + \lambda u(t)\right]\,\mathrm{d}x$$
$$\le \delta\xi(t)\left[\frac{\varepsilon_2(g_\infty - \lambda)^2 - 2g_\infty\lambda}{2}\|u(t)\|^2_{\mathbb{A}} - \frac{\breve{g}_0(2\varepsilon_2 + 1)}{2\varepsilon_2}\int\limits_0^\infty \frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)}\mathrm{d}\tau\right]; \tag{3.9}$$

moreover, $(2.8)_2$ yields

$$\delta\xi(t)\int\limits_{\Omega} v(t)\cdot\left[\frac{\partial}{\partial t}\breve{\imath}^t(0) + \lambda v(t)\right]\,\mathrm{dx} \le \delta\xi(t)\left[\left(\lambda + \frac{\varepsilon_3}{2} - \breve{g}_0\right)\|v(t)\|^2 + \frac{C_\Omega}{2\varepsilon_3}\|\breve{\imath}^t_\tau(0)\|^2_{\mathbb{A}}\right]. \tag{3.10}$$

Substituting (3.7)–(3.10) in (3.6) we obtain

$$\begin{aligned}\frac{d}{dt}L(t) \leq &\left[\frac{\delta C_\Omega}{2\varepsilon_3}\xi(t) + \frac{1}{2g'(0)}\right] \|\breve{\imath}^t_\tau(0)\|^2_{\mathbb{A}} \\ &- \delta\xi(t)\left[\breve{g}_0 - \lambda - \frac{k(\varepsilon_0+\varepsilon_1)+\varepsilon_3}{2}\right]\|v(t)\|^2 \\ &- \delta\xi(t)\left[\lambda - \frac{kC_\Omega\lambda^2}{2\varepsilon_0 g_\infty} - \frac{\varepsilon_2(g_\infty-\lambda)^2}{2g_\infty}\right] g_\infty\|u(t)\|^2_{\mathbb{A}} \\ &+ \xi(t)\left[\frac{1}{2\alpha_{T_0}} - \frac{\delta k C_\Omega \breve{g}_0}{2\varepsilon_1} - \frac{\delta\breve{g}_0(2\varepsilon_2+1)}{2\varepsilon_2}\right]\int_0^\infty \frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)}\mathrm{d}\tau \\ = &\ \mathcal{C}_0\|\breve{\imath}^t_\tau(0)\|^2_{\mathbb{A}} - \delta\xi(t)\left[\mathcal{C}_v\|v(t)\|^2 + \mathcal{C}_u g_\infty\|u(t)\|^2_{\mathbb{A}}\right] - \xi(t)\mathcal{C}_{\mathbf{i}}\int_0^\infty \frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)}\mathrm{d}\tau.\end{aligned}$$

Inequality (3.3) is proved if we find λ and δ such that $\mathcal{C}_0$ is not positive, while $\mathcal{C}_v$, $\mathcal{C}_u$ and $\mathcal{C}_{\mathbf{i}}$ are positive.

Choosing

$$\varepsilon_3 = -\delta C_\Omega g'(0)\xi(0),$$

$\mathcal{C}_0$ is not positive, while posing

$$\varepsilon_1 = \frac{2kC_\Omega\delta\breve{g}_0\alpha_{T_0}}{1-2\delta\breve{g}_0\alpha_{T_0}}\gamma_1 \quad \text{and} \quad \varepsilon_2 = \frac{2\delta\breve{g}_0\alpha_{T_0}}{1-2\delta\breve{g}_0\alpha_{T_0}}\gamma_2,$$

we obtain

$$\mathcal{C}_{\mathbf{i}} = \frac{1-2\delta\breve{g}_0\alpha_{T_0}}{4\alpha_{T_0}}\left[\frac{\gamma_1-1}{\gamma_1} + \frac{\gamma_2-1}{\gamma_2}\right]$$

that is positive if

$$\delta < \frac{1}{2\breve{g}_0\alpha_{T_0}} = \delta_1, \quad \gamma_1 > 1, \quad \gamma_2 > 1.$$

Moreover, if

$$\lambda = \frac{g_\infty\breve{g}_0}{2g_\infty + k^2C_\Omega} \quad \text{and} \quad \varepsilon_0 = \frac{kC_\Omega\breve{g}_0}{2g_\infty+k^2C_\Omega}, \tag{3.11}$$

$\mathcal{C}_u$ becomes

$$\mathcal{C}_u = \frac{g_\infty\breve{g}_0}{2(2g_\infty+k^2C_\Omega)} - \left[g_\infty - \frac{g_\infty\breve{g}_0}{2g_\infty+k^2C_\Omega}\right]^2 \frac{\alpha_{T_0}\breve{g}_0\gamma_2\delta}{g_\infty(1-2\alpha_{T_0}\breve{g}_0\delta)}$$

which is positive if

$$\delta < \frac{2g_\infty + k^2C_\Omega}{2\alpha_{T_0}[\breve{g}_0(2g_\infty+k^2C_\Omega) + \gamma_2(2g_\infty+k^2C_\Omega-\breve{g}_0)^2]} = \delta_2 \leq \delta_1.$$

To estimate $\mathcal{C}_v$ we observe that, as a consequence of (3.11),

$$\breve{g}_0 - \lambda - \frac{k}{2}\varepsilon_0 = \frac{\breve{g}_0}{2},$$

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so that

$$\begin{aligned}\mathcal{C}_v &= \frac{\check{g}_0}{2} - \frac{k\varepsilon_1 + \varepsilon_3}{2} \\ &= \frac{\check{g}_0}{2}\,\frac{2C_\Omega g'(0)\xi(0)\alpha_{T_0}\delta^2 - 2\alpha_{T_0}C_\Omega\left(\frac{\check{g}_0}{C_\Omega} - \frac{g'(0)\xi(0)}{2\alpha_{T_0}\check{g}_0} + k^2\gamma_1\right)\delta + 1}{1 - 2\delta\check{g}_0\alpha_{T_0}} \\ &= \frac{\check{g}_0}{2} f(\delta).\end{aligned}$$

Since $f(0) = 1$, there exists $\delta_3 > 0$ such that $\mathcal{C}_v > 0$ for any $\delta < \delta_3$. Finally, choosing

$$\delta = \min\{\delta_2, \delta_3\} = \delta_4,$$

we obtain (3.3). □

Lemma 3.2. *Let χ be a solution of problem* (2.9) *with $\chi_0 \in \mathcal{K}$. If g satisfies* (h_1), (h_2^*) *and* (3.1), *δ is small enough and λ is defined in* (3.11), *then L and E are equivalent, i.e. there exist two positive constants c_1 and c_2 such that*

$$c_1\mathcal{E}(t) \le L(t) \le c_2\mathcal{E}(t). \tag{3.12}$$

Proof. As a consequence of (3.4) and of Poincare and Korn inequalities, we have

$$\begin{aligned}\left|\int_\Omega v(t)\cdot\left[\check{\imath}^t(0) + \lambda u(t)\right]\,dx\right| &= \left|\int_\Omega v(t)\cdot\left[\check{\imath}^t(0) + \frac{g_\infty\check{g}_0}{2g_\infty + k^2C_\Omega}u(t)\right]\,dx\right| \\ &\le \beta\left[\frac{\varepsilon}{2}\|v(t)\|^2 + \frac{C_\Omega\check{g}_0}{2\varepsilon}\left(g_\infty\|u(t)\|_{\mathbb{A}}^2 - \int_0^\infty \frac{\|\check{\imath}_\tau^t(\tau)\|_{\mathbb{A}}^2}{g'(\tau)}\mathrm{d}\tau\right)\right],\end{aligned}$$

where

$$\beta = \max\left\{1, \frac{\check{g}_0 g_\infty}{(2g_\infty + k^2C_\Omega)^2}\right\}.$$

By posing $\varepsilon = \sqrt{C_\Omega\check{g}_0}$, we obtain

$$\left|\int_\Omega v(t)\cdot\left[\check{\imath}^t(0) + \lambda u(t)\right]\,dx\right| \le \beta\sqrt{C_\Omega\check{g}_0}\,\mathcal{E}(t). \tag{3.13}$$

As a consequence of the properties of ξ and (3.13) we have

$$\left(1 - \delta\xi(0)\beta\sqrt{C_\Omega\check{g}_0}\right)\mathcal{E}(t) \le L(t) \le \left(1 + \delta\xi(0)\beta\sqrt{C_\Omega\check{g}_0}\right)\mathcal{E}(t).$$

Choosing

$$\delta < \frac{1}{\xi(0)\beta\sqrt{C_\Omega\check{g}_0}} = \delta_0,$$

we obtain (3.12). □

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We conclude this subsection by stating the following theorem

Theorem 3.1. *Let χ be a solution of problem* (2.9) *with $\chi_0 \in \mathcal{K}$. If g satisfies* (h_1), (h_2^*) *and* (3.1), *then, for any $T_0 > 0$, there exist two positive constants λ_{T_0} and β_{T_0}. Depending on initial data, such that*

$$\mathcal{E}(t) \le \beta_{T_0} E(0) \exp\left[-\lambda_{T_0} \int_0^t \xi(s)ds\right], \quad t > T_0. \tag{3.14}$$

Proof. Let $\delta = \frac{1}{2}\min\{\delta_0, \delta_4\}$. As a consequence of Lemmas 3.1 and 3.2 we have

$$\frac{d}{dt}L(t) \le -\frac{c_0}{c_2}\xi(t)L(t), \quad t > T_0. \tag{3.15}$$

The integration of (3.15) gives

$$L(t) \le L(T_0)\exp\left[-\frac{c_0}{c_2}\int_{T_0}^t \xi(s)ds\right], \quad t > T_0$$

and inequalities (2.13) and (3.12) yield the result. □

Corollary 3.1. *Under the hypotheses of* Theorem 3.1, *the energy functional* (2.11) *exponentially (polynomially) decays if the memory kernel g exponentially (polynomially) decays in time.*

Proof. It is easy to show that if ξ is constant in time, then (3.1) assures the exponential decay of g, while (3.14) yields the exponential decay of the energy.

On the other hand, if $\xi(t) = c(1+t)^{-1}$, then $-g'(t) = O((1+t)^{-c})$ and from (3.14) we obtain

$$\mathcal{E}(t) \le \beta_{T_0} E(0)(1+t)^{-c\lambda_{T_0}}, \quad t > T_0. \qquad \square$$

Corollary 3.2. *Let $\mathcal{C}$ be a compact subset of $\mathcal{K}$, then, for any $T_0 > 0$, there exists two positive constants $\bar{\beta}_{T_0}$ and $\bar{\lambda}_{T_0}$ such that for any $\chi_0 \in \mathcal{C}$ the solution of problem* (2.9) *satisfies the inequality*

$$\mathcal{E}(t) \le \bar{\beta}_{T_0} E(0) \exp\left[-\bar{\lambda}_{T_0} \int_0^t \xi(s)ds\right], \quad t > T_0. \tag{3.16}$$

Proof. Let us suppose that it exists an instant $T_0 > 0$ for which inequality (3.16) is not satisfied. Then we can find a sequence of initial conditions χ_0^n, whose corresponding solutions $\chi^n(t)$ satisfy (3.14) with constants $\beta_{T_0}^n$ and $\lambda_{T_0}^n$ such that at least one of the following conditions hold:

$$\beta_{T_0}^n \longrightarrow \infty, \quad \lambda_{T_0}^n \longrightarrow 0. \tag{3.17}$$

Then, because $\mathcal{C}$ is compact, there exists a subsequence $\tilde{\chi}_0^n \in \mathcal{C}$, converging to $\tilde{\chi}_0 \in \mathcal{C}$, such that inequality (3.14) holds with constants $\tilde{\beta}_{T_0}^n$ and $\tilde{\lambda}_{T_0}^n$, respectively. Moreover, since $\tilde{\chi}_0^n \longrightarrow \tilde{\chi}_0$, we have

$$\tilde{\beta}_{T_0}^n \longrightarrow \tilde{\beta}_0, \quad \tilde{\lambda}_{T_0}^n \longrightarrow \tilde{\lambda}_0,$$

which is in contrast with (3.17). □

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3.2. Second case

In this subsection we restrict ourselves to memory kernels satisfying (h_1), (h_2^*) and the restriction proposed in [18], that is, there exists an increasing and strictly convex function $G : \mathcal{R}^+ \to \mathcal{R}^+$ of class $C^1(\mathcal{R}^+) \cap C^2(\mathcal{R}^+ \setminus 0)$, with

$$G(0) = G'(0) = 0, \qquad \lim_{\xi\to\infty} G(\xi) = \infty, \tag{3.18}$$

such that

$$-\left[\int_0^\infty \frac{g'(s)}{\Gamma(g''(s))}\,\mathrm{d}s + \sup_{s\in\mathcal{R}^+} \frac{g'(s)}{\Gamma(g''(s))}\right] < \infty, \tag{3.19}$$

where Γ is the inverse function of G.

Remark 3.1. It follows from the hypotheses (3.19) that

$$g''(s) > 0, \quad \forall s \in \mathcal{R}^+.$$

In fact, if there exists s_0 such that $g''(s_0) = 0$, since $\Gamma(0) = 0$, the boundedness of $\dfrac{g'(s)}{\Gamma(g''(s))}$ implies $g'(s_0) = 0$, in contrast with (h_2^*).

Before stating the main result of this subsection, we briefly recall some preliminaries that will be of use in what follows. For further details we refer to [18], where these results have been proved.

Proposition 3.1. *If G satisfies* (3.18)*, then*

$$\xi_1\xi_2 \le G(\xi_1) + \xi_2\,[G']^{-1}(\xi_2), \quad \xi_1, \xi_2 \in \mathcal{R}^+. \tag{3.20}$$

Moreover the function $K(\xi) = \dfrac{\xi}{\Gamma(\xi)}$ is monotone nondecreasing in $\mathcal{R}^+$ and $K(0) = 0$.

Proof. For any $\xi_1, \xi_2 \in \mathcal{R}^+$ the following inequality holds

$$\xi_1\xi_2 \le G(\xi_1) + \sup_{\xi_1>0}\,[\xi_1\xi_2 - G(\xi_1)]$$

and the assumptions of regularity on G yields

$$\sup_{\xi_1>0}\,[\xi_1\xi_2 - G(\xi_1)] = \xi_2\,[G']^{-1}(\xi_2) - G([G']^{-1}(\xi_2)) \le \xi_2\,[G']^{-1}(\xi_2).$$

Moreover, using the concavity of Γ and observing that $\Gamma(0) = 0$, we obtain

$$K(\xi_1) \le K(\xi_2) \quad \text{if} \quad \xi_1 < \xi_2,$$

while $K(0) = 0$ because $G'(0) = 0$. □

Let us introduce the following auxiliary functional

$$F_\alpha(t) = \alpha\mathcal{E}(t) + \int_\Omega v(t)\cdot\left[\check{i}^t(0) + \frac{\check{g}_0}{2}u(t)\right]\mathrm{d}x,$$

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where α is a suitable positive constant. By using the same approach used in previous Lemmas we prove the following result

Lemma 3.3. *Let χ be a solution of problem* (2.9) *with $\chi_0 \in \mathcal{K}$. If g satisfies* (h_1), (h_2^*), (3.19) *and α is large enough, there exist two positive constants κ_1 and κ_2, such that*

$$\frac{d}{dt}F_\alpha(t) \leq -\kappa_1\mathcal{E}(t) - \kappa_2 \int_0^\infty \frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)}\mathrm{d}\tau \tag{3.21}$$

and F_α is equivalent to E, i.e. there exist M_1 and M_2 such that

$$M_1\mathcal{E}(t) \leq F_\alpha(t) \leq M_2\mathcal{E}(t). \tag{3.22}$$

Proof. The techniques in the proof of Lemma 3.1 yield

$$\begin{aligned}\frac{d}{dt}F_\alpha(t) \leq &-\breve{g}_0\left[1 + \frac{g_\infty}{2\epsilon_1} + \frac{\breve{g}_0^2}{8\epsilon_3}\right]\int_0^\infty \frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)} - \frac{1}{2}\left[\breve{g}_0 - \epsilon_2\right]\|v(t)\|^2 \\ &- \frac{g_\infty}{2}\left(\breve{g}_0 - \frac{\epsilon_3}{g_\infty} - \epsilon_1\right)\|u(t)\|^2_{\mathbb{A}} + \left[\frac{C_\Omega}{2\epsilon_2} + \frac{\alpha}{2g'(0)}\right]\|\breve{\imath}^t_\tau(0)\|^2_{\mathbb{A}}.\end{aligned}$$

Choosing

$$\epsilon_1 = \epsilon_2 = \frac{\breve{g}_0}{2}, \quad \epsilon_3 = \frac{\breve{g}_0 g_\infty}{4}, \quad \alpha > -\frac{2C_\Omega g'(0)}{\breve{g}_0},$$

we obtain (3.21). Reasoning as in Lemma 3.2, it is possible to find a positive constant M such that

$$\left|\int_\Omega v(t)\cdot\left[\breve{\imath}^t(0) + \frac{\breve{g}_0}{2}u(t)\right]\mathrm{d}x\right| \leq M\mathcal{E}(t),$$

then, if α large enough, we obtain (3.22). □

We conclude these preliminary results by proving a relation between the time derivative of the energy and $\int_0^\infty \frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)}\mathrm{d}\tau$.

Lemma 3.4. *Let χ be a solution of problem* (2.9) *with $\chi_0 \in \mathcal{K}$ and*

$$\sup_{\tau>0}\frac{\|\breve{\imath}^0_\tau(\tau)\|^2_{\mathbb{A}}}{|g'(\tau)|^2} < \infty. \tag{3.23}$$

If g satisfies (h_1), (h_2^*) *and* (3.19), *then there exists a positive constant μ such that*

$$G'(\delta\mathcal{E}(t))\int_0^\infty \frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)}\mathrm{d}\tau \geq \mu\left[\frac{d}{dt}\mathcal{E}(t) - \delta\mathcal{E}(t)G'(\delta\mathcal{E}(t))\right] \tag{3.24}$$

for any $\delta > 0$ and $t \in \mathcal{R}^+$.

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Proof. By posing

$$\xi_1 = \Gamma\left(\gamma_2 \frac{g''(\tau)}{|g'(\tau)|^2}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\right), \quad \xi_2 = -\frac{\gamma_1}{\xi_1} G'(\delta\mathcal{E}(t))\frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)}$$

and recalling (2.13), inequality (3.20) yields

$$\begin{aligned}
&G'(\delta\mathcal{E}(t))\int_0^\infty \frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)}\mathrm{d}\tau \\
&\geq -\frac{\gamma_2}{\gamma_1}\int_0^\infty \frac{g''(\tau)\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{|g'(\tau)|^2}\mathrm{d}\tau + \int_0^\infty \frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)\Gamma\left(\gamma_2\frac{g''(\tau)}{|g'(\tau)|^2}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\right)}[G']^{-1}\left(\frac{-\gamma_1 G'(\delta\mathcal{E}(t))\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)\Gamma\left(\gamma_2\frac{g''(\tau)}{|g'(\tau)|^2}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\right)}\right)\mathrm{d}\tau \\
&\geq -\frac{2\gamma_2}{\gamma_1}\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}(t) + \int_0^\infty \frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)\Gamma\left(\gamma_2\frac{g''(\tau)}{|g'(\tau)|^2}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\right)}[G']^{-1}\left(\frac{-\gamma_1 G'(\delta\mathcal{E}(t))\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)\Gamma\left(\gamma_2\frac{g''(\tau)}{|g'(\tau)|^2}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\right)}\right)\mathrm{d}\tau. \qquad (3.25)
\end{aligned}$$

Therefore, in order to prove (3.24), we need to estimate the latest integral in (3.25). Firstly we observe that, if the initial data satisfy (3.23), as a consequence of (2.7) we have

$$\begin{aligned}
\frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{|g'(\tau)|^2} &\leq c\left[\frac{\|\breve{\imath}^0_\tau(t+\tau)\|^2_{\mathbb{A}}}{|g'(\tau)|^2} + \int_0^t \frac{g''(\tau+s)}{|g'(\tau)|^2}\,\mathrm{d}s\int_0^t g''(t+\tau-s)\|u(t)-u(s)\|^2_{\mathbb{A}}\,\mathrm{d}s\right] \\
&\leq c\left[\frac{\|\breve{\imath}^0_\tau(t+\tau)\|^2_{\mathbb{A}}}{|g'(\tau)|^2} + \frac{4E(0)}{g_\infty}\,\frac{|g'(t+\tau)-g'(\tau)|^2}{|g'(\tau)|^2}\right] \\
&\leq c\left[\frac{\|\breve{\imath}^0_\tau(t+\tau)\|^2_{\mathbb{A}}}{|g'(t+\tau)|^2} + \frac{8E(0)}{g_\infty}\right] \leq N_1,
\end{aligned}$$

where N_1 is a constant depending on the initial data. Then, to estimate

$$[G']^{-1}\left(-\frac{\gamma_1 G'(\delta\mathcal{E}(t))\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)\Gamma\left(\gamma_2\frac{g''(\tau)}{|g'(\tau)|^2}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\right)}\right),$$

we choose $\gamma_2 = \frac{1}{N_1}$ and recall that the function $\frac{\xi}{\Gamma(\xi)}$ is nondecreasing (see Proposition 3.1), so that

$$\begin{aligned}
\frac{-\gamma_1 G'(\delta\mathcal{E}(t))\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)\Gamma\left(\gamma_2\frac{g''(\tau)}{|g'(\tau)|^2}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\right)} &= -\frac{\gamma_1 g'(\tau)G'(\delta\mathcal{E}(t))}{\gamma_2 g''(\tau)}\frac{\gamma_2\frac{g''(\tau)}{|g'(\tau)|^2}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{\Gamma\left(\gamma_2\frac{g''(\tau)}{|g'(\tau)|^2}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\right)} \\
&\leq -N_1\gamma_1 G'(\delta\mathcal{E}(t))\frac{g'(\tau)}{\Gamma\left(g''(\tau)\right)} \leq N_1 N_2 \gamma_1 G'(\delta\mathcal{E}(t)),
\end{aligned}$$

where

$$N_2 = -\sup_{\tau>0}\frac{g'(\tau)}{\Gamma\left(g''(\tau)\right)}.$$

Recalling that G is convex and choosing $\gamma_1 = \frac{1}{N_1 N_2}$, we obtain

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$$[G']^{-1}\left(-\frac{\gamma_1 G'(\delta\mathcal{E}(t))\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)\Gamma\left(\gamma_2\frac{g''(\tau)}{|g'(\tau)|^2}\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}\right)}\right)\leq \delta\mathcal{E}(t).$$

Consequently inequality (3.25) becomes

$$\int_0^\infty \frac{\|\breve{\imath}^t_\tau(\tau)\|^2_{\mathbb{A}}}{g'(\tau)}\mathrm{d}\tau \geq \frac{2N_2}{G'(\delta\mathcal{E}(t))}\frac{d}{dt}\mathcal{E}(t)+\delta\mathcal{E}(t)\int_0^\infty \frac{N_1 g'(\tau)}{\Gamma(g''(\tau))}\mathrm{d}\tau$$

and (3.24) follows from hypothesis (3.19). □

Theorem 3.2. *Let* χ *be a solution of problem* (2.9) *with initial data* $\chi_0 \in \mathcal{K}$ *satisfying* (3.23). *If* g *satisfies* (h_1), (h_2^*) *and* (3.19), *then there exist two positive constants* δ_1 *and* δ_2 *depending on initial data such that*

$$\mathcal{E}(t)\leq \delta_1[G_1]^{-1}(\delta_2 t),$$

where

$$G_1(\xi)=\int_\xi^1 \frac{1}{sG'(\delta s)}\,\mathrm{d}s,\quad 0<\xi\leq 1,\quad \delta>0. \tag{3.26}$$

Proof. As a consequence of Lemmas 3.3 and 3.4 we obtain

$$G'(\delta\mathcal{E}(t))\frac{\mathrm{d}}{\mathrm{dt}}F_\alpha(t)+\kappa_2\mu\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}(t)\leq -(\kappa_1-\kappa_2\mu\delta)G'(\delta\mathcal{E}(t))\mathcal{E}(t).$$

Tanks to the properties of G

$$\frac{\mathrm{d}}{\mathrm{dt}}G'(\delta\mathcal{E}(t))=\delta G''(\delta\mathcal{E}(t))\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}(t)\leq 0 \tag{3.27}$$

and, if α is large enough, we get

$$\frac{\mathrm{d}}{\mathrm{dt}}G'(\delta\mathcal{E}(t))F_\alpha(t)\leq 0.$$

Choosing $\delta<\frac{\kappa_1}{\kappa_2\mu}$, so that $\kappa_3=\kappa_1-\kappa_2\mu\delta>0$ and introducing

$$H_\lambda(t)=\lambda[G'(\delta\mathcal{E}(t))F_\alpha(t)+\kappa_2\mu\mathcal{E}(t)],\quad \lambda>0,$$

we have

$$\frac{\mathrm{d}}{\mathrm{dt}}H_\lambda(t)\leq -\lambda\kappa_3 G'(\delta\mathcal{E}(t))\mathcal{E}(t).$$

Moreover, thanks to inequalities (3.22) and (3.27), we can choose λ such that

$$H_\lambda(t)\leq\mathcal{E}(t),\quad H_\lambda(0)\leq 1;$$

it follows that H_λ is equivalent to E and

$$0<H_\lambda(t)\leq 1,\quad \forall\, t\in\mathcal{R}^+.$$

Observing that the function $\xi \to \xi G'(\delta\xi)$ is nondecreasing, we obtain

$$\frac{1}{G'(\delta H_\lambda(t))H_\lambda(t)}\frac{\mathrm{d}}{\mathrm{dt}}H_\lambda(t) \leq -\lambda\kappa_3,$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{dt}}G_1(H_\lambda(t)) \geq -\lambda\kappa_3, \tag{3.28}$$

where G_1 is defined in (3.26). Upon integration over $[0, t]$, inequality (3.28) yields

$$G_1(H_\lambda(t)) \geq \lambda\kappa_3 t + G_1(H_\lambda(0))) \geq \lambda\kappa_3 t.$$

Therefore, since G_1 is non-decreasing and $G_1(1) = 0$, we have

$$H_\lambda(t) \leq [G_1]^{-1}(\lambda\kappa_3 t).$$

The equivalence between H_λ and E gives the result. □

4. Conclusions and open problems

In this paper, by using the approach based on the notion of minimal state we have extended the results obtained by Messaoudi in [19] to evolutive problems presenting non-vanishing initial data and/or non-vanishing supplies and we have revisited the method proposed by Guesmia [18] in the context of the classical histories approach, enlarging the space of the admissible initial data.

It should be observed that the first method allows to state that the energy has the same type of decay of the memory kernel, but, since it is not possible to give an explicit estimate for the constant λ_{T_0} in (3.14), it gives no information on the rate of decay.

On the other hand, with the second method, it is possible give explicit estimates on the rate of decay of the energy provided that the initial data satisfy restriction (3.23), that is for initial data belonging to a proper subset of $\mathcal{K}$. For example, as observed by Guesmia by choosing

$$G(t) = t^{\frac{1+q}{q}} \quad q \in \left]0, \frac{p-1}{2}\right[,$$

if the memory kernel has a polynomial behavior, i.e.

$$-g_1'(t) = \frac{c}{(1+t)^p}, \quad p > 1,$$

then it exists a positive constant C such that

$$\mathcal{E}(t) \leq \frac{C}{(1+t)^q}, \quad \forall t \in \mathbb{R}^+,\ \forall q \in \left]0, \frac{p-1}{2}\right[.$$

It is evident from this example that we are not able to get an optimal decay rate and the reason lies in the difficulty in finding a function G which can improve the previous estimate.

Therefore, getting the optimal decay rate (as done in [3] and [2] for vanishing sources) is a very interesting open problem. Finally, since for the second method we have restricted the space of admissible data for technical reasons, proving Theorem 3.2 for initial data belonging to $\mathcal{K}$ is another interesting open problem.

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Appendix A

The constitutive equation of the stress tensor T for a standard linear viscoelastic solid is given by

$$T(x,t) = \mathbb{G}_\infty(x)E(x,t) + \int_0^\infty \mathbb{G}'(x,s)E_r^t(x,s)\,\mathrm{d}s, \tag{A.1}$$

where $E = \mathrm{sym}\nabla u$ is the infinitesimal strain tensor and $E_r^t(x,s) = E(x,t-s) - E(x,t) = E^t(x,s) - E(x,t)$ its relative past history, while $\mathbb{G}_\infty(x)$ and $\mathbb{G}'(x,\cdot)$ are fourth order symmetric tensors with $\mathbb{G}'(x,\cdot) \in L^1(\mathcal{R}^+)$ and $\mathbb{G}_\infty$ and is positive definite, moreover the tensor

$$\check{\mathbb{G}}(x,\tau) = -\int_t^\infty \mathbb{G}'(x,s)\,\mathrm{d}s$$

belongs to $L^1(\mathcal{R}^+)$, is positive definite in $\mathcal{R}^+$ and

$$\mathbb{G}_c(x,\omega) = \int_0^\infty \check{\mathbb{G}}(x,\tau)\cos\omega s\,\mathrm{d}s$$

is positive definite for any $\omega \in \mathcal{R}$.

For this system the differential system is given by

$$\rho\frac{\partial}{\partial t}v(x,t) = \nabla\cdot T(x,t) + \rho f(x,t).$$

It has been observed (see [12,17]) that the linear constitutive equation (A.1) allows us to introduce the following equivalence relation among histories: we say that $E_1^t(x,s)$ is equivalent to $E_2^t(x,s)$ if, for any $\tau \in \mathcal{R}^+$,

$$\mathbb{G}_\infty(x)E_1(x,t) + \int_0^\infty \mathbb{G}'(x,s+\tau)E_{1r}^t(x,s)\mathrm{d}s = \mathbb{G}_\infty(x)E_2(x,t) + \int_0^\infty \mathbb{G}'(x,s+\tau)E_{2r}^t(x,s)\mathrm{d}s \tag{A.2}$$

and, by using equality (A.2), to introduce the notion of minimal state (see [5,11]) by means of the pair $(E(x,t), \breve{I}^t(x,\cdot))$, where

$$\breve{I}^t(x,\tau) = -\int_0^\infty \mathbb{G}'(x,s+\tau)E_r^t(x,s)\mathrm{d}s. \tag{A.3}$$

As a consequence of (A.3) we obtain the following results:

$$\begin{aligned}
&T(x,t) = \mathbb{G}_\infty(x)E(x,t) - \breve{I}^t(x,0),\\
&\lim_{\tau\to\infty}\breve{I}^t(x,\tau) = 0,\\
&\frac{\partial}{\partial t}\breve{I}^t(x,\tau) = \breve{I}_\tau^t(x,\tau) - \check{\mathbb{G}}(x,\tau)\dot{E}(x,t),\\
&\breve{I}^t(x,\tau) = \breve{I}^0(x,t+\tau) - \int_0^t \mathbb{G}'(x,\tau+s)E_r^t(x,s)\,\mathrm{d}s,
\end{aligned}$$

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where

$$\check{\mathbb{G}}(x,\tau) = -\int_t^\infty \mathbb{G}'(x,s)\,\mathrm{d}s$$

and the index τ denotes the derivative with respect to τ.

In this paper we consider homogeneous materials and memory kernels of the following type

$$\mathbb{G}(s) = g(s)\mathbb{A},$$

where $\mathbb{A}$ is a constant fourth order symmetric tensor positive definite on Sym and g is a scalar function which satisfies the properties (h_1) and(h_2).

Moreover, by posing

$$\check{i}^t(x,t) = -\int_0^\infty g'(s+\tau)u_r^t(x,s)\mathrm{d}s,$$

we have

$$\check{I}^t(x,t) = \mathbb{A}\nabla\check{i}^t(x,t).$$

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