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Uniformly expanding Markov maps of the real line: exactness and infinite mixing

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Abstract

We give a fairly complete characterization of the exact components of a large class of uniformly expanding Markov maps of $\mathbb{R}$. Using this result, for a class of $\mathbb{Z}$-invariant maps and finite modifications thereof, we prove certain properties of infinite mixing recently introduced by the author.

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Contents

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1 Introduction

Uniformly expanding Markov maps of the interval represent a paradigm for chaotic dynamical systems. They make a fairly large class of non-trivial maps, and possess the standard ingredient for chaos, namely, hyperbolicity—insofar as expansivity can be understood as the one-dimensional version of hyperbolicity. On the other hand, they are simple enough to be more or less fully understood via the techniques of the modern theory of dynamical systems; see the excellent textbook by Boyarsky and Góra [BG].

In infinite ergodic theory, the analogues of such maps are the uniformly expanding Markov maps of $\mathbb{R}$ (see, e.g., Fig. 1 further down). Not much is known about them, at least to this author. We are especially interested in "translation indifferent" maps, that is, maps whose local properties are uniformly bounded throughout $\mathbb{R}$. By way of counterexample, we are not interested in Boole's transformation [AW], which is very close to the identity outside of a compact set, or in Bugiel's maps [B1, B2], which are designed to preserve a finite measure.

In this note we are concerned with the mixing properties of a very general class of uniformly expanding Markov maps of the real line.

Initially, we consider the exactness property, which is a strong notion of mixing that has the advantage of being defined in the same way in both finite and infinite ergodic theory. We prove a series of results that characterize the ergodic and exact components of a map in terms of its combinatorics relative the Markov partition. The characterization is rather precise outside of the invariant set where the orbits escape to $\pm\infty$. Understandably, the ways in which an orbit can escape are many and not easily classifiable. With a few extra assumptions, however, we are able to give a comprehensive description of the exact components of this set as well. A byproduct of all these results is a number of easily checkable sufficient conditions for the exactness of a uniformly expanding Markov map.

Later, we apply the notions of mixing for infinite-measure-preserving dynamical systems (for short, *infinite mixing*) recently introduced by the author in [L4]. We present these notions, within the present scope, in Section 4 below and refer the reader to [L4, L5, L7] for a more thorough discussion. (The last reference, in particular, uses a more intuitive notation and contains several results that are used in this article.)

For this part, we specialize to a much narrower but still nontrivial class of maps. We consider both *quasi-lifts* of expanding circle maps, i.e., piecewise smooth, translation invariant maps $\mathbb{R} \longrightarrow \mathbb{R}$, whose quotient on a fundamental domain is an expanding map of the circle (see Fig. 2), and *finite modifications* thereof, namely, maps that coincide with a quasi-lift of an expanding circle map outside a bounded domain (see Fig. 3). In both cases, we prove versions of *global-local mixing* and *global-global mixing*. Very loosely, global-local mixing means that any *global observable* (roughly, a bounded function) and any *local observable* (an integrable function) decorrelate in time. Global-global mixing means that the same happens for any two global observables.

Of course, we have stronger results for the more specialized class of systems, that is, the quasi-lifts. In particular, we prove a property, denoted **(GLM2)**, that can be recast like this: For any global observable $F : \mathbb{R} \longrightarrow \mathbb{C}$ and any probability measure ν, absolutely continuous w.r.t. a reference infinite measure μ, if $T_*^n \nu$ denotes the push-forward of ν via the map T^n, then

$$\lim_{n\to\infty} T_*^n \nu(F) = \overline{\mu}(F), \tag{1.1}$$

where $\overline{\mu}(F)$ represents, in a sense that is specified below, the average of F over $\mathbb{R}$, relative to μ. Thus (1.1) can be regarded as a sort of weak convergence of $T_*^n \nu$, the statistical state of the system at time n, to the "equilibrium state" $\overline{\mu}$, which is independent of the initial condition ν. The global observables play the role of test functions.

We are unable to prove this strong property for all finite modifications of quasi-lifts of circle maps, but we certainly believe it to be true for a large class of such systems. For this reason, we give an example for which a very strong version of **(GLM2)** can be indeed be proved. In a sense which will be explained below, cf. Section 3.3, this example represents a random walk in $\mathbb{Z}$.

This is how the paper is organized. In Section 2 we introduce our maps and present several results on their exact components, from the more general statements to the ones that require extra assumptions. In Section 3 we focus on three subclasses of maps: the quasi-lifts of expanding circle maps, their finite modifications and the random walks. In Section 4 we give our definitions of infinite mixing and apply them to the systems of Section 3. The proofs of all the main results are found in Section 5. The Appendix comprises two sections: in the first we discuss the importance of some of our assumptions and in the second we place a standard distortion argument used in Section 5.

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2 Setup and exactness

In this section we give the precise definition of our maps of interest and give a number or results about their exactness properties. We start with a characterization of the exact components which intersect the conservative part of the phase space. Their complement, the 'escape part', has a more complicated dynamics: the results for this part will require more assumptions, cf. (A5)-(A7) in Section 2.2.

Let $(a_j)_{j \in \mathbb{Z}}$ be a collection of real numbers such that $\lim_{j \to \pm\infty} a_j = \pm\infty$ and

(A1) $\exists \theta > 0$ such that $0 < a_{j+1} - a_j \le \theta$, $\forall j \in \mathbb{Z}$.

Let $I_j := [a_j, a_{j+1}]$. We call $\{I_j\}_{j \in \mathbb{Z}}$ a partition of $\mathbb{R}$ even though formally it is not—the substance of what we discuss in this paper would not change if we made the cleaner yet more cumbersome choice $I_j := [a_j, a_{j+1})$. Let us denote by $\mathscr{M}$ the σ-algebra generated by the I_j.

We consider $T : \mathbb{R} \longrightarrow \mathbb{R}$, a surjective Markov map relative to $\{I_j\}$. More precisely,

(A2) $T|_{(a_j, a_{j+1})}$ has a unique extension $\tau_j : I_j \longrightarrow J_j$, which is twice differentiable and bijective onto $J_j \in \mathscr{M}$. Equivalently, $J_j := \bigsqcup_{k=k_{1j}}^{k_{2j}} I_k$, for some $k_{1j} \le k_{2j}$.

Notice that the above implies that T is two-sided non-singular w.r.t. the Lebesgue measure m. This means that, for all Borel sets A, $m(T^{-1}A) = 0 \Leftrightarrow m(A) = 0$.

Let τ_j' and τ_j'' denote, respectively, the first and second derivatives of τ_j. Then:

(A3) $\exists \lambda > 1$ such that $|\tau_j'| \ge \lambda$, $\forall j \in \mathbb{Z}$;

(A4) $\exists \eta > 0$ such that $\left| \dfrac{\tau_j''}{(\tau_j')^2} \right| \le \eta$, $\forall j \in \mathbb{Z}$.

An example of a map satisfying (A1)-(A4) is shown in Fig. 1.

2.1 Classification of Markov intervals

Throughout the paper we use the following

Convention. All equalities/inclusions of subsets of $\mathbb{R}$ are intended mod m within $\mathscr{B}$, the Borel σ-algebra of $\mathbb{R}$. In particular, the strict inclusion $A \subset B$ means $m(A \cap B^c) = 0$ and $m(B \setminus A) > 0$.

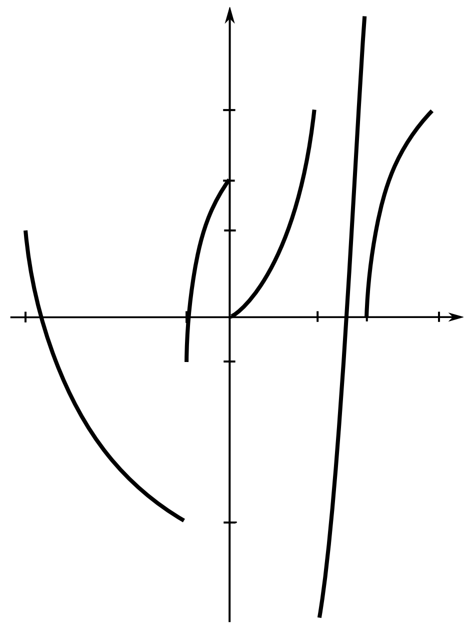

Figure 1: A uniformly expanding Markov map $\mathbb{R} \longrightarrow \mathbb{R}$.

For a Markov map, the elements of its Markov partition—in our case, the intervals I_j—can be classified in analogy with the states of a Markov chain; cf., e.g., [S, Chap. VIII] or [G, Chap. 4]. In recalling the definitions below, we will say indifferently that the interval I_j possesses a certain property or that the state j possesses that property.

The *transition matrix associated to* T is the stochastic matrix $\mathcal{P} = \mathcal{P}_T := (p_{jk})_{j,k \in \mathbb{Z}}$, where $p_{jk} := m(T^{-1} I_k \,|\, I_j)$. The surjectivity of T implies that, for every k, there exists j such that $p_{jk} > 0$. We denote by $p_{jk}^{(n)}$ the entries of $\mathcal{P}^n$. The Markov property (A2) implies that

$$p_{jk}^{(n)} > 0 \iff m(T^{-n} I_k \,|\, I_j) > 0 \iff T^n I_j \supset I_k. \tag{2.1}$$

When the above occurs for some $n \in \mathbb{Z}^+$, we say that the interval I_k is *accessible* from I_j, or that I_j *feeds* I_k.

The intervals I_j, I_k (or the states j, k) are called *communicating* if each one is accessible form the other. By convention, we declare that I_j communicates with itself. This establishes an equivalence relation on $\mathbb{Z}$. The corresponding equivalence classes are referred to as the *(communicating) classes* of T, and are denoted $\mathbb{Z}_\alpha$, with $\alpha \in \aleph$, some countable set. We also call

$$M_\alpha := \bigsqcup_{j \in \mathbb{Z}_\alpha} I_j \tag{2.2}$$

the *set associated to* $\mathbb{Z}_\alpha$. If $\mathbb{Z}$ is one (hence the only) communicating class, T is called *irreducible*.

If I_j feeds some I_k, but the viceversa does not hold, we say that I_j is *inessential*. It is easy to see that the property of feeding a state, or being accessible from a state, carries over within a communicating class. So we say, for example, that a certain class is accessible from I_j, or it is inessential, etc. The states or classes that are not inessential are obviously called *essential*.

An essential class $\mathbb{Z}_\alpha$ that is not accessible from any external states, that is, such that $j \in \mathbb{Z}_\alpha$ and $k \notin \mathbb{Z}_\alpha$ imply $p_{jk}^{(n)} = p_{kj}^{(n_1)} = 0$, $\forall n, n_1 \in \mathbb{Z}^+$, is called *isolated*. The index set of all isolated classes is denoted $\aleph_{\mathrm{iso}}$. An essential class that is fed by at least one external state is called *terminal*. For example, $\mathbb{Z}_\alpha$ can be a terminal class of the state k, or of the inessential class $\mathbb{Z}_\beta$, etc. Notice that an inessential state can have more than one terminal class, or none—the latter possibility can occur because the set of states is infinite. The index set of all terminal classes is denoted $\aleph_{\mathrm{ter}}$.

The integer

$$d_j := \text{g.c.d.} \left\{ n \in \mathbb{Z}^+ \;\middle|\; p_{jj}^{(n)} > 0 \right\} \tag{2.3}$$

is called the *period* of I_j (if the r.h.s. of (2.3) is empty, set $d_j := 0$). Since this definition is the same as for the Markov chain generated by $\mathcal{P}$, we know [G, Thm. 4.2.2] that two intervals in the same class have the same period. This will be henceforth called the period of the class $\mathbb{Z}_\alpha$, denoted d_α. We say that the *period of* T is d if $d_j = d$, $\forall j \in \mathbb{Z}$. We say that T is *aperiodic* if it has period 1.

Finally, let us endow $\mathbb{Z}$ with a graph structure by declaring that an edge exists between j and k if and only if $p_{jk} + p_{kj} > 0$, that is, if $TI_j \supset I_k$ or $TI_k \supset I_j$. This is not the usual graphical representation of the transition probabilities, which is a directed graph: it is its undirected version. It is easy to see that a communicating class of T and all the states feeding it are all contained in one connected component of this graph. In fact, a connected component may contain more that one terminal class, but only one isolated class (which, in that case, coincides with the connected component). If there is only one connected component, we say that T is *Markov-indecomposable*. Obviously, an irreducible T is Markov-indecomposable.

2.2 Exactness properties

We denote by $\mathcal{C}$ and $\mathcal{D}$ the conservative and dissipative parts of T, respectively. It is known that $T^{-1}\mathcal{C} \supseteq \mathcal{C}$ and $T^{-1}\mathcal{D} \subseteq \mathcal{D}$ (mod m, which is implicit by our convention) [A, Chap. 1]. As is customary in the field of non-singular dynamical systems, a set A is called *invariant* relative to T, or T-*invariant*, if $T^{-1}A = A$. The set

$$\mathcal{I}(A) = \mathcal{I}_T(A) := \bigcup_{k \in \mathbb{Z}^+} \bigcup_{k \in \mathbb{N}} T^{-k} T^n A \tag{2.4}$$

is called the *invariant hull* of A w.r.t. T. It is the smallest T-invariant set containing A. Let us define the following invariant sets:

$$\mathcal{I}_{\mathcal{CD}} := \mathcal{I}(\mathcal{C}) \cap \mathcal{I}(\mathcal{D}); \tag{2.5}$$

$$\mathcal{I}_{\mathcal{C}} := \mathcal{I}(\mathcal{C}) \setminus \mathcal{I}_{\mathcal{CD}}; \tag{2.6}$$

$$\mathcal{I}_{\mathcal{D}} := \mathcal{I}(\mathcal{D}) \setminus \mathcal{I}_{\mathcal{CD}}; \tag{2.7}$$

Since $\mathcal{C}$ and $\mathcal{D}$ are complementary, $\mathbb{R} = \mathcal{I}_{\mathcal{C}} \sqcup \mathcal{I}_{\mathcal{D}} \sqcup \mathcal{I}_{\mathcal{CD}}$. We may call $\mathcal{I}_{\mathcal{C}}$ the conservative-invariant part, $\mathcal{I}_{\mathcal{D}}$ the dissipative-invariant part, and $\mathcal{I}_{\mathcal{CD}}$ the mixed part of $\mathbb{R}$. We will see below how $\mathcal{I}_{\mathcal{D}}$ also deserves the name of 'escape part'.

Definition 2.1 *In the present context, a set $A \subseteq \mathbb{R}$ is called an* ergodic component *of T if $m(A) > 0$, A is T-invariant and A has no T-invariant subset of strictly smaller measure. It is called an* exact component *of T if it is an ergodic component and $T|_A$ is exact.*

Observe that the above is a rather stringent definition of ergodic component, not allowing for zero-measure ergodic components, which can be defined via the Ergodic Decomposition Theorem [A, §2,2]. On the other hand, the next proposition shows that the union of all null invariant sets of $\mathcal{I}(\mathcal{C}) = \mathcal{I}_{\mathcal{C}} \sqcup \mathcal{I}_{\mathcal{CD}}$ is negligible.

Given $x \in \mathbb{R}$, recall the definition of $\omega(x)$, the ω-limit set of x: it is the set of all the accumulation points of $(T^n(x))_{n\in\mathbb{N}}$ [W, Chap. 5]. Call Ω the set of all $x \in \mathbb{R}$ with a non-empty $\omega(x)$. Evidently, Ω is measurable and invariant.

Proposition 2.2 $\Omega = \mathcal{I}(\mathcal{C}) = \mathcal{I}_{\mathcal{C}} \sqcup \mathcal{I}_{\mathcal{CD}}$. *Furthermore, Ω is decomposed mod m into a countable number of (positive-measure) ergodic components.*

PROOF. Section 5.

The following results concern the exactness properties of $T|_{\mathcal{I}_{\mathcal{C}}}$ and $T|_{\mathcal{I}_{\mathcal{CD}}}$.

Theorem 2.3 *Under the assumptions (A1)-(A4), $\mathcal{I}_{\mathcal{C}}$ is made up of at most countably many ergodic components of T, denoted E_α. (Here α is just a generic index; see however Proposition 2.5.) The periods of all $I_j \subset E_\alpha$ are the same: we denote them d_α. Also, E_α splits into d_α exact components of T^{d_α}, denoted $E_{\alpha,i}$, with $0 \le i \le d_\alpha - 1$. Each $E_{\alpha,i} \in \mathscr{M}$. These are 'cyclic components' of T in that $TE_{\alpha,i} = E_{\alpha,i+1}$, for $i \in \{0, 1, \ldots, d_\alpha - 2\}$, and $TE_{\alpha,d_\alpha-1} = E_{\alpha,0}$.*

PROOF. Section 5.

Corollary 2.4 *The following holds:*

(a) $\mathcal{I}_{\mathcal{C}} \in \mathscr{M}$.

(b) Every $E_\alpha \in \mathscr{M}$.

(c) Every $I_j \subset \mathcal{I}_{\mathcal{C}}$ belongs to an $\mathscr{M}$-measurable exact component of T^{d_j}.

(d) If T is Markov-indecomposable and aperiodic with a non-null $\mathcal{I}_{\mathcal{C}}$, then it is conservative, irreducible and exact.

PROOF. Assertion *(b)* follows trivially from Theorem 2.3. So does *(a)* from *(b)*. Assertion *(d)* is also easy: if $m(\mathcal{I}_{\mathcal{C}}) > 0$ then $\mathcal{I}_{\mathcal{C}}$ contains at least one ergodic component E_α, which contains at least one interval I_j. The Markov-indecomposability of T implies that $E_\alpha = \mathcal{I}_T(I_j)$ intersects I_k, $\forall k \in \mathbb{Z}$. But $E_\alpha \in \mathscr{M}$, whence $E_\alpha = \mathbb{R}$, which cannot be split in smaller cyclic components by aperiodicity. Therefore, T is irreducible and exact, and $\mathbb{R} = \mathcal{C}$. Finally, not only does *(c)* follow from the theorem, as is apparent, but the viceversa holds as well. This will be shown in the proof of Theorem 2.3 in Section 5. Q.E.D.

As intuition suggests, the ergodic components of T have much to do with the communicating classes introduced in Section 2.1. In the remainder of this section we will establish relations between the two. For the moment, let us remark that the set of all states that either belong to or feed an essential class does not necessarily equal $\mathbb{Z}$, for there might be inessential states which have no terminal class. The collection of all the latter states will be denoted $\mathbb{Z}_\infty$.

The next two propositions assume (A1)-(A4) and use the notation of Section 2.1. They will be proved in Section 5.

Proposition 2.5 *Each ergodic component $E_\alpha \subseteq \mathcal{C}$ equals M_α, cf. (2.2), for some $\alpha \in \aleph_{\mathrm{iso}}$. (Hence the integer d_α of Theorem 2.3 is the period of $\mathbb{Z}_\alpha$.) Viceversa, if $\alpha \in \aleph_{\mathrm{iso}}$ and $\#\mathbb{Z}_\alpha < \infty$, then M_α is an ergodic component $E_\alpha \subseteq \mathcal{C}$. If $\alpha \in \aleph_{\mathrm{iso}}$ and $\#\mathbb{Z}_\alpha = \infty$, then M_α is either an ergodic component $E_\alpha \subseteq \mathcal{C}$, or a T-invariant subset of $\mathcal{I}_{\mathcal{D}}$.*

For $\alpha \in \aleph_{\mathrm{ter}}$, set

$$E_\alpha := \bigcup_{n \in \mathbb{N}} T^{-n} M_\alpha; \tag{2.8}$$

$$T_\alpha := T|_{M_\alpha} : M_\alpha \longrightarrow M_\alpha. \tag{2.9}$$

Observe that the definition (2.8) is consistent with the statements of Proposition 2.5, that is, with the case $\alpha \in \aleph_{\mathrm{iso}}$. In such case, in fact, (2.8) reduces to $E_\alpha = M_\alpha$.

Proposition 2.6 *Each ergodic component within $\mathcal{I}_{\mathcal{C}\mathcal{D}}$ is of the form E_α, cf. (2.8), for some $\alpha \in \aleph_{\mathrm{ter}}$. Also, $M_\alpha \subseteq \mathcal{C}$, $E_\alpha \setminus M_\alpha \subseteq \mathcal{D}$, and, for a.e. $x \in E_\alpha$, $\omega(x) = M_\alpha$. The map T_α defined in (2.9) is conservative and ergodic, and M_α splits into d_α exact components of $T_\alpha^{d_\alpha}$, which are cyclic in the sense of Theorem 2.3 (again, d_α is the period of $\mathbb{Z}_\alpha$).*

Viceversa, if $\alpha \in \aleph_{\mathrm{ter}}$ and $\#\mathbb{Z}_\alpha < \infty$, then E_α is an ergodic component contained in $\mathcal{I}_{\mathcal{C}\mathcal{D}}$, with the above properties. If $\alpha \in \aleph_{\mathrm{ter}}$ and $\#\mathbb{Z}_\alpha = \infty$, then E_α is either an ergodic component contained in $\mathcal{I}_{\mathcal{C}\mathcal{D}}$, with the above properties, or a T-invariant subset of $\mathcal{I}_{\mathcal{D}}$.

Corollary 2.7 $\mathcal{C} \in \mathscr{M}$.

PROOF. Propositions 2.6 and 2.2 show that $\mathcal{I}_{\mathcal{CD}} \cap \mathcal{C}$ is the union of a countable number of $M_\alpha \in \mathscr{M}$, with α ranging in a subset of $\aleph_{\mathrm{ter}}$. But $\mathcal{C} = \mathcal{I}_\mathcal{C} \sqcup (\mathcal{I}_{\mathcal{CD}} \cap \mathcal{C})$. Corollary 2.4*(a)* completes the proof. Q.E.D.

Notice that the 'mixed ergodic components' $E_\alpha \subseteq \mathcal{I}_{\mathcal{CD}}$ need not be $\mathscr{M}$-measurable. For example, if j feeds both $\mathbb{Z}_\alpha$ and $\mathbb{Z}_\beta$, then part of I_j will belong to E_α and part to E_β.

Also observe that characterizing the exact components of $T_\alpha^{d_\alpha}$ on the ω-limit set M_α gives complete information about the exactness properties of T on E_α. In fact, if $M_{\alpha,i}$ $(0 \le i \le d_\alpha - 1)$ denote the exact components of $T_\alpha^{d_\alpha}$ inside M_α, then

$$E_{\alpha,i} := \bigcup_{n \in \mathbb{N}} T^{-nd_\alpha} M_{\alpha,i} \tag{2.10}$$

are cyclic sets for $T|_{E_\alpha}$, on each of which the $(d_\alpha)^{\mathrm{th}}$ power of the map is exact. This can be seen as follows. If A is a positive-measure subset of $E_{\alpha,i}$, there exist $B \subseteq A$, $m(B) > 0$, and $N \in \mathbb{N}$ such that, $\forall n \ge N$, $T^{nd_\alpha} B \subseteq M_{\alpha,i}$. For all such n, however, since T^{d_α} is exact on $M_{\alpha,i}$,

$$\bigcup_{k \in \mathbb{N}} T^{-kd_\alpha} T^{kd_\alpha} T^{nd_\alpha} B = M_{\alpha,i}, \tag{2.11}$$

whence

$$\bigcup_{j \in \mathbb{N}} T^{-jd_\alpha} T^{jd_\alpha} B = E_{\alpha,i}. \tag{2.12}$$

Since $B \subseteq A$, (2.12) holds as well with A in the place of B. This proves that any positive-measure subset of $E_{\alpha,i}$, in the tail σ-algebra of T^{d_α}, is the entire $E_{\alpha,i}$.

Understandably, if the dynamical system preserves the Lebesgue or a similar measure, the mixed part of the reference space, whose dynamics is dissipative in the basin of attraction of a conservative set, must be null:

Proposition 2.8 *If T preserves a measure μ equivalent to m (this means, $\mu \ll m$ and $m \ll \mu$), then $\Omega = \mathcal{C}$. Equivalently: $\mathcal{I}_\mathcal{C} = \mathcal{C}$, $\mathcal{I}_\mathcal{D} = \mathcal{D}$, $\mathcal{I}_{\mathcal{CD}} = \varnothing$.*

PROOF. Pick a function $\zeta \in L^1(\mathbb{R}, \mu)$ such that $\zeta(x) > 0$, $\forall x \in \mathbb{R}$. By definition of Ω and Prop. 1.1.6 of [A],

$$\Omega \subseteq \left\{ x \in \mathbb{R} \;\middle|\; \sum_{n=0}^{\infty} \zeta \circ T^n(x) = \infty \right\} = \mathcal{C} \bmod \mu, \tag{2.13}$$

i.e., $\Omega \subseteq \mathcal{C} \bmod m$. The reverse inclusion is obvious, and in any case implied by Proposition 2.2, which also gives the other claims. Q.E.D.

We now come to $\mathcal{I}_\mathcal{D}$.

Proposition 2.9 $\mathcal{I}_\mathcal{D} \supseteq \bigsqcup_{j \in \mathbb{Z}_\infty} I_j$.

PROOF. Every I_j with $j \in \mathbb{Z}_\infty$ cannot intersect (hence be contained in) $\mathcal{I}_\mathcal{C}$, otherwise, by Proposition 2.5, it would be essential; and cannot intersect $\mathcal{I}_{\mathcal{CD}}$, otherwise, by Proposition 2.6, it would intersect E_α, for some $\alpha \in \aleph_{\mathrm{ter}}$, implying that $\mathbb{Z}_\alpha$ is a terminal class for j. Q.E.D.

A thorough description of the ergodic and exact components of $\mathcal{I}_\mathcal{D}$, like we have for $\mathcal{C}$ and $\mathcal{I}_{\mathcal{CD}}$ via Propositions 2.5 and 2.6, is an arduous task. Counterexamples 1 and 2 of Section A.1 of the Appendix corroborate this intuition. But if we agree to a few, simple, extra assumptions on $\mathcal{I}_\mathcal{D}$, the situation improves a great deal:

(A5) $\exists \rho > 0$ such that, $\forall x \in \mathcal{I}_\mathcal{D}$, $|T(x) - x| \le \rho$.

(A6) $\exists \theta_o \in (0, \theta)$ such that, $\forall j \in \mathbb{Z}$ with $m(I_j \cap \mathcal{I}_\mathcal{D}) > 0$, $\theta_o \le a_{j+1} - a_j$; cf. (A1).

(A7) $\forall j \in \mathbb{Z}$ with $m(I_j \cap \mathcal{I}_\mathcal{D}) > 0$, $TI_j \supset I_j$.

We describe (A5) by saying that T has a bounded action on $\mathcal{I}_\mathcal{D}$. In view of (A1), (A6) and (A2), this amounts to the existence of

$$\kappa := \max_{\substack{j \in \mathbb{Z} \\ m(I_j \cap \mathcal{I}_\mathcal{D}) > 0}} (k_{2j} - k_{1j} + 1). \tag{2.14}$$

In other words, $\kappa \in \mathbb{Z}^+$ is the maximum number of Markov intervals in $J_j = TI_j$, for all j such that I_j has a non-negligible intersection with $\mathcal{I}_\mathcal{D}$.

For all such j, (A7) assumes in addition that $k_{1j} \le j \le k_{2j}$. In this case notice that also $k_{2j} - k_{1j} \ge 1$, otherwise $TI_j = I_j$, which is prohibited by (A3).

Remark 2.10 Of course, one does not know the set $\mathcal{I}_\mathcal{D}$ *a priori*. One can however ensure that (A5)-(A7) hold if, for instance, T has a bounded action on the whole of $\mathbb{R}$, or on $\mathcal{D}$; and if the conditions of (A6)-(A7) are verified for all $j \in \mathbb{Z}$, or at least all j that do not belong to no essential class $\mathbb{Z}_\alpha$, with $\#\mathbb{Z}_\alpha < \infty$ (cf. Propositions 2.5 and 2.6).

Theorem 2.11 *Under the assumptions (A1)-(A5), $\mathcal{I}_\mathcal{D} = \mathcal{D}_{+\infty} \sqcup \mathcal{D}_{-\infty}$, where*

$$\mathcal{D}_{\pm\infty} := \left\{ x \in \mathbb{R} \;\middle|\; \lim_{n \to \infty} T^n(x) = \pm\infty \right\}.$$

Both sets are clearly T-invariant. If (A6) holds, then $m(\mathcal{D}_{\pm\infty}) \in \{0, \infty\}$. If (A7) also holds, then each of the two sets is either null or an exact component of T.

PROOF. Section 5.

3 Examples

We can use the results of Section 2.2 to find many examples of exact maps. We first focus on the simplest cases that are not piecewise linear. This is done for the purposes of Section 4, in which certain properties of infinite mixing are verified, for the first time, for truly non-linear maps. After that, we discuss an important class of piecewise linear maps, the random walks. They motivate the definitions of Section 2.1 and provide examples for the relevance of certain assumptions of Section 2.2.

3.1 Quasi-lifts of expanding circle maps

Let us consider the case where the elements of the Markov partition have the same size, e.g., $I_j := [aj, a(j+1)]$, for some $a > 0$, and T acts in the same way on each of them, that is, for $x \in I_j$, $\tau_j(x) = \tau_0(x - aj) + aj$. This is equivalent to

$$T \circ \sigma = \sigma \circ T, \tag{3.1}$$

where $\sigma(x) = \sigma_a(x) = x + a$. In other words, T is translation invariant (by the quantity a). If $\tau_0 : I_0 \longrightarrow J_0$ is bijective, twice differentiable with bounded second derivative, expanding and such that $J_0 = [ak_{1,0}, a(k_{2,0} + 1)]$, with $k_{1,0} \leq 0$ and $k_{2,0} \geq 0$—cf. (A2)—it is easy to see that all the conditions (A1)-(A7) are verified. See Fig. 2.

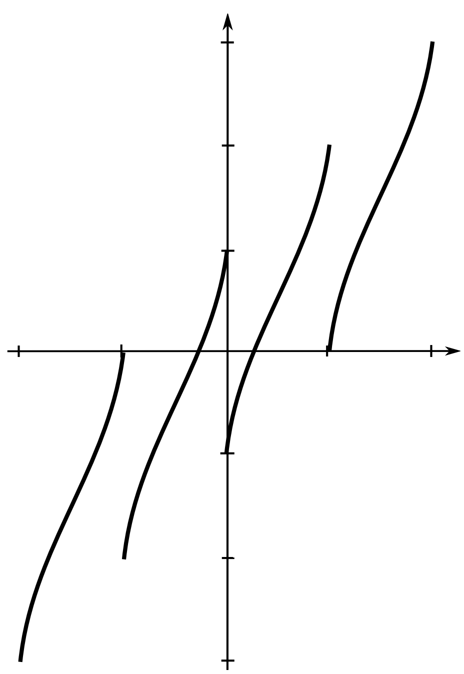

Figure 2: An example of a quasi-lift of an expanding circle map.

If $\mathbb{S}_a$ is the circle constructed by identifying the endpoints of $[0, a]$, and $T_a : \mathbb{S}_a \longrightarrow \mathbb{S}_a$ is defined by $T_a(x) := T(x) \bmod a$, we observe that T_a is a uniformly expanding map of the circle, with bounded distortion and at most one non-regular point, which happens to be a fixed point. It thus possesses a number of strong stochastic properties. In particular, there exists an invariant measure μ_a, equivalent to m_a, the Lebesgue measure on $\mathbb{S}_a$, which makes $(\mathbb{S}_a, \mu_a, T_a)$ Bernoulli, with exponential decay or correlations for a large class of observables, etc. These results are proved, e.g., in [BG, Chaps. 5 & 8]. It may be worth remarking that T_a is irreducible and aperiodic on $\mathbb{S}_a$, so it is exact by the same arguments proving Theorem 2.3.

T is a sort of lift of $\mathbb{S}_a$ to $\mathbb{R}$, which we might call *quasi-lift*. It is apparent that T preserves μ, the σ-invariant measure whose restriction to $[0, a) \cong \mathbb{S}_a$ is μ_a. Of course, μ is equivalent to m. If we set $h_\mu := d\mu/dm$, the above statements read

$$h_\mu(x) = \sum_{y \in T^{-1}(x)} \frac{h_\mu(y)}{|T'(y)|}, \tag{3.2}$$

for all $x \neq aj$ ($j \in \mathbb{Z}$); and $h_\mu(x) = h_\mu(x+a)$, $h_\mu(x) > 0$, for all $x \in \mathbb{R}$. An important consequence of the invariance of μ is that $\mathcal{C} = \mathcal{I}_\mathcal{C}$ and $\mathcal{D} = \mathcal{I}_\mathcal{D}$ (Proposition 2.8).

A quasi-lift can be thought of as a $\mathbb{Z}$-*extension* of T_a, that is, a self-map T_ϕ of $[0, a) \times \mathbb{Z}$ of the form $T_\phi(y, j) = (T_a(y), j + \phi(y))$, where $\phi : [0, a) \longrightarrow \mathbb{Z}$ [A, Chap. 8]. ϕ is called (by some) *discrete displacement*. In fact, set $\phi(x) := k$ for all $x \in I_0 \cap T^{-1}(I_k)$ (these sets are intervals and partition $I_0 \cong [0, a)$, for $k_{1,0} \leq k \leq k_{2,0}$). The map $\Psi(y, j) := aj + y$ defines an isomorphism between the measure spaces $([0, a) \times \mathbb{Z}\, , \mu_a \otimes \mathbb{Z})$ and $(\mathbb{R}, \mu)$, and one readily verifies that $T_\phi = \Psi^{-1} \circ T \circ \Psi$. Therefore, T_ϕ preserves $\mu_a \otimes \mathbb{Z}$.

The quantity

$$\mathbb{E}_{\mu_a}(\phi) := \frac{1}{\mu_a([0, a))} \int_0^a \phi \, d\mu_a \tag{3.3}$$

will be called the *drift* of T. Notice that there is no harm in using μ instead of μ_a in (3.3). We will do so throughout the paper.

The assumption (A7) imposes stringent conditions on ϕ, making the present systems special examples of $\mathbb{Z}$-extensions of expanding circle maps. Quasi-lifts and similar maps have often been used in nonlinear physics as toy models for normal and anomalous diffusion; see, e.g., [AC1, AC2], [K, KHK], [SJ, Sect. 3.3], and references therein.

Proposition 3.1 *A quasi-lift of an expanding circle map, as defined above, is exact. Furthermore, up to null sets, $\mathbb{R}$ coincides with $\mathcal{C}$, $\mathcal{D}_{+\infty}$, or $\mathcal{D}_{-\infty}$, depending on the drift $\mathbb{E}_\mu(\phi)$ being, respectively, zero, positive, or negative.*

Proof. This result is a corollary of the main theorem of [AD]. For a much simpler proof, which is self-contained within the present paper, see Section 5.

3.2 Finite modifications of quasi-lifts

Slightly more complex examples than the quasi-lifts of circle maps are the so-called finite modifications of quasi-lifts of circle maps. If T_o is a quasi-lift as defined earlier, $T : \mathbb{R} \longrightarrow \mathbb{R}$ is a *finite modification* of T_o if there exists $k_o \in \mathbb{Z}^+$ such that $T(x) = T_o(x)$, for all $x \notin \bigsqcup_{j=-k_o}^{k_o} I_j$. Finite, or *local*, modifications of translation-invariant dynamical systems have been studied in more complicated contexts as well, such as billiards [L1, L2, DSzV].

We assume that T verifies (A1)-(A7). Observe that if T is a finite modification of T_o, then T^n is a finite modification of T_o^n, which is a $\mathbb{Z}$-extension by (3.1). In fact, (A5) *et seq.* show that, if $x \in I_j$, $T(x)$ can land at most $\kappa - 1$ intervals away form I_j, hence, for all $x \neq \bigsqcup_{j=-k_o-n(\kappa-1)}^{k_o+n(\kappa-1)} I_j$, $T^n(x) = T_o^n(x)$.

Under certain conditions, a finite modification of a quasi-lift is also exact.

Proposition 3.2 *Let T be a finite modification of a quasi-lift of an expanding circle map. It T verifies (A1)-(A7), is Markov-indecomposable and preserves a measure μ equivalent to m, then it is exact.*

Proof. See Section 5.

Remark 3.3 The significance of the hypotheses of Proposition 3.2 is clarified by Counterexamples 3 and 4 of Section A.1 of the Appendix.

Under slightly stronger conditions, T verifies a trichotomy similar to that of Proposition 3.1 for quasi-lifts. The statement of this result is relatively cumbersome and requires terminology that is more appropriately introduced later. For this reason we present it in Section 5, under the reference Proposition 5.7.

We emphasize that the measure μ of Proposition 3.2 need not be the same measure preserved by T_o, which we henceforth call μ_o. However, by way of example and in view of later application (cf. Proposition 4.5), we now present a method to construct finite modifications of quasi-lifts of circle maps which preserve the original measure.

To simplify things further, we assume that μ_o is the Lebesgue measure. The general case can be worked out in a similar fashion. In view of the notation of Section 2, indicate with τ_{oj} the extension of $T_o|_{(a_j,a_{j+1})}$ to I_j, and set $J_{oj} := \tau_{oj}(I_j)$. Define also $\mathcal{J} := \{j \in \mathbb{Z} \mid J_{oj} \supset I_0\}$; notice that, by (A2), either $J_{oj} \supset I_0$ or $J_{oj} \cap I_0 = \varnothing$, mod m. Now, pick a C^2 function $\psi : \mathbb{R} \longrightarrow \mathbb{R}_0^+$ that is compactly supported in $(a_0, a_1) = (0, a)$. For $j \in \mathcal{J}$, denote by $\varphi_{oj} : J_{oj} \longrightarrow I_j$ the inverse function of τ_{oj}, and set $\varphi_j := \varphi_{oj} + \delta_j \psi$, which defines a function on J_{oj}. Here $(\delta_j)_{j\in\mathcal{J}}$ is a collection of numbers so small, in absolute value, that φ_j is a monotonic, hence bijective, function $J_{oj} \longrightarrow I_j$. (Recall that, by (A2), φ_{oj} is monotonic on $J_{oj} \supset I_0$.) Also, they satisfy

$$\sum_{j\in\mathcal{J}} \mathrm{sign}(\varphi'_{oj})\, \delta_j = 0. \tag{3.4}$$

Finally, define $\tau_j := \varphi_j^{-1}$, for $j \in \mathcal{J}$, and $\tau_j := \tau_{oj}$, otherwise. This determines the map T, except at the points $a_j = aj$, which are negligible. An example of this construction is shown in Fig. 3.

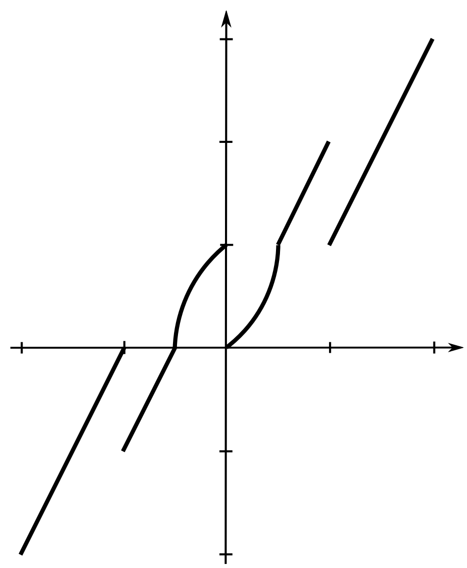

Figure 3: A finite modification of a quasi-lift of a circle map, constructed with the procedure given in Section 3.2, for the case $\mu_o = m$.

T verifies (A1)-(A2) and (A5)-(A7) by construction. If the δ_j are sufficiently small, (A3)-(A4) are verified as well. As for the invariance of $\mu_o = m$, the reader can check that

$$\sum_{y \in T^{-1}\{x\}} \frac{1}{|T'(y)|} = \sum_{y \in T_o^{-1}\{x\}} \frac{1}{|T_o'(y)|}, \tag{3.5}$$

at least for all $x \notin a\mathbb{Z}$. This means that T preserves the Lebesgue measure if and only if T_o does, which was assumed.

3.3 Random walks

A very special family of uniformly expanding Markov maps $\mathbb{R} \longrightarrow \mathbb{R}$ is given by those representing random walks in $\mathbb{Z}$.

Let $\mathcal{Q} = (q_{jk})_{j,k \in \mathbb{Z}}$ be the transition matrix of a random walk in $\mathbb{Z}$, namely, $q_{jk} \in [0, 1]$ is the probability that the walker jumps from the site j to the site k. In line with (A5), we assume that the walk only admits bounded jumps, i.e., there exists $\bar{\kappa} \in \mathbb{Z}^+$ such that $q_{jk} = 0$, for all $|k - j| > \bar{\kappa}$ (although the construction we give below can be easily adapted to the case of unbounded jumps, cf. [L6]).

The map $T = T_Q$ associated to the above random walk is defined as follows. For $j \in \mathbb{Z}$ and $k \in \{j - \bar{\kappa}, j - \bar{\kappa} + 1, \ldots, j + \bar{\kappa}\}$, set

$$I_{jk} := \left[j + \sum_{i=j-\bar{\kappa}}^{k-1} q_{ji} \, , \, j + \sum_{i=j-\bar{\kappa}}^{k} q_{ji} \right], \tag{3.6}$$

with the understanding that, when $k = j - \bar{\kappa}$, the first of the above sums is zero. So $\{I_{jk}\}_{k=j-\bar{\kappa}}^{j+\bar{\kappa}}$ is a partition of $[j, j+1]$ into intervals of length, respectively, $\{q_{jk}\}_{k=j-\bar{\kappa}}^{j+\bar{\kappa}}$. Notice that, if $q_{jk} = 0$, I_{jk} reduces to a point. We exclude such degenerate intervals. The complete collection $\{I_{jk}\}_{j,k\in\mathbb{Z}}$ is the Markov partition for our map. For x in the interior of I_{jk} define

$$T(x) := \frac{1}{q_{jk}} \left(x - j - \sum_{i=j-\bar{\kappa}}^{k-1} q_{ji} \right) + k. \tag{3.7}$$

For all other x, the definition of $T(x)$ is irrelevant. In other words, T maps I_{jk} affinely onto $[k, k+1]$, see Fig. 4.

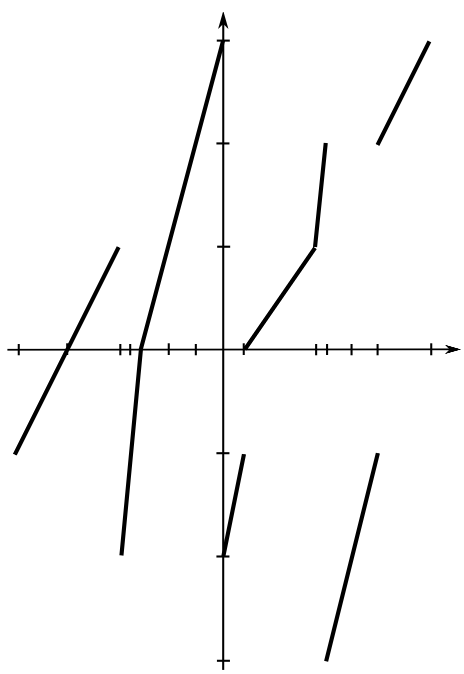

Figure 4: A map T associated a random walk. The marks on the abscissa indicate the Markov intervals I_{jk}, while those on the ordinate represent the intervals $[k, k+1]$.

A little thinking shows that, if we take a uniformly random $x \in (k_0, k_0 + 1) \setminus \bigcup_{n\in\mathbb{N}} T^{-n}\mathbb{Z}$ and look at its itinerary w.r.t. the partition $\{[k, k+1]\}_{k\in\mathbb{Z}}$, calling $k_n =$

$k_n(x)$ the unique integer such that $T^n(x) \in (k_n, k_n+1)$, then $(k_n)_{n\in\mathbb{N}}$ is the random walk on $\mathbb{Z}$ determined by the transition matrix $\mathcal{Q}$ and the initial state k_0.

Let us observe that $\mathcal{Q}$ is not the same as $\mathcal{P}_T$, the transition matrix associated to the map T, as presented in Section 2.1. They are however closely related. Denoting by δ_{jk} the Kronecker delta, and using that $I_{jk} = [j, j+1] \cap T^{-1}[k, k+1]$, we get

$$\begin{aligned} p_{jk\,,\,j_1k_1} &:= m(T^{-1} I_{j_1k_1} \,|\, I_{jk}) \\ &= \delta_{j_1,k}\, \frac{m([j,j+1] \cap T^{-1}[k,k+1] \cap T^{-2}[k_1,k_1+1])}{m([j,j+1] \cap T^{-1}[k,k+1])} \\ &= \delta_{j_1,k}\, q_{k,k_1}. \end{aligned} \tag{3.8}$$

The map T always verifies the assumptions (A1), (A2), (A4) and (A5). If $\sup_{j,k} q_{jk} < 1$, (A3) and (A6) are also verified.

On the other hand, (A7) cannot hold in great generality. In fact, it is verified only if it amounts to the null condition.

Proposition 3.4 *For a map T associated to a random walk such that $\sup_{j,k} q_{jk} < 1$, (A7) can only hold if $\mathcal{I}_\mathcal{D}$ is null.*

PROOF. The Markov partition of T is $\{I_{jk}\}_{j,k}$. Suppose by absurd that $m(\mathcal{I}_\mathcal{D}) > 0$. There exist $j, k \in \mathbb{Z}$ such that $m(I_{jk} \cap \mathcal{I}_\mathcal{D}) > 0$. If (A7) holds, $T I_{jk} \supset I_{jk}$. However, by construction, $T I_{jk} \cap I_{jk} = \varnothing \pmod m$, $\forall j \neq k$, whence $j = k$. Since $\mathcal{I}_\mathcal{D} = \mathcal{D}_{+\infty} \sqcup \mathcal{D}_{-\infty}$ and both sets are invariant (Theorem 2.11), the trajectories of all $x \in I_{jj} \cap \mathcal{I}_\mathcal{D}$ must eventually leave I_{jj}, implying that $m(I_{jk} \cap \mathcal{I}_\mathcal{D}) > 0$, for some $k \neq j$. This contradicts what we have just shown. Q.E.D.

Remark 3.5 This proposition does not imply that—say—random walks with a non-zero drift cannot be represented by maps satisfying (A7). They can, only not w.r.t. the Markov partition $\{I_{jk}\}_{j,k}$. For example, consider the quasi-lift determined by $\tau_0(x) := 3x$, where τ_0 is the branch of T defined on $[0, 1] =: I_0$, cf. Section 3.1. This map represents the homogeneous random walk $q_{j,j} = q_{j,j+1} = q_{j,j+2} = 1/3$, $\forall j \in \mathbb{Z}$. Clearly $\mathbb{R} = \mathcal{I}_\mathcal{D} = \mathcal{D}_{+\infty}$. Nonetheless, as discussed in Section 3.1, T verifies all (A1)-(A7), relative to the Markov partition $\{I_j = [j, j+1]\}_{j\in\mathbb{Z}}$.

The following simple result will be useful in the remainder.

Proposition 3.6 *The map T associated to the random walk determined by $\mathcal{Q}$ preserves the Lebesgue measure m if and only if $\mathcal{Q}$ is doubly stochastic, i.e., $\sum_{j\in\mathbb{Z}} q_{jk} = 1$, $\forall k \in \mathbb{Z}$.*

PROOF. We prove this simple proposition by means of the Perron-Frobenius operator $P = P_T$, which is the operator $L^1(\mathbb{R}, m) \longrightarrow L^1(\mathbb{R}, m)$ uniquely determined by the identity

$$\int_\mathbb{R} (F \circ T) g \, dm = \int_\mathbb{R} F (Pg) \, dm, \tag{3.9}$$

with $F \in L^\infty(\mathbb{R}, m)$ and $g \in L^1(\mathbb{R}, m)$. It is well known [BG] that, for a.e. $x \in \mathbb{R}$,

$$(Pg)(x) = \sum_{y \in T^{-1}\{x\}} \frac{g(y)}{|T'(y)|}. \tag{3.10}$$

For T as in the statement of the proposition, this reads: for all $k \in \mathbb{Z}$ and $x \in (k, k+1)$,

$$(Pg)(x) = \sum_{\substack{j \in \mathbb{Z} \\ q_{jk} > 0}} q_{jk}\, g(\tau_{jk}^{-1}(x)) = \sum_{j \in \mathbb{Z}} q_{jk}\, g(\tau_{jk}^{-1}(x)), \tag{3.11}$$

where, in accordance with the notation of (A2), τ_{jk} is the branch of T defined on I_{jk}, cf. (3.7).

If we allow (3.10)-(3.11) to act on $g \in L^\infty$ as well, it is clear that T preserves m if and only if $P1 = 1$, with $1(x) \equiv 1$ (see also (3.2)); that is, if and only if $\sum_j q_{jk} = 1$ for all k. Q.E.D.

Markov maps representing random walks are also useful in this paper for they provide examples which clarify the importance of some of our earlier assumptions. The reader is referred to Section A.1 of the Appendix.

4 Infinite mixing

In this section we consider the notions of infinite mixing introduced in [L4] and further developed in [L7]. We first formalize them for the case of uniformly expanding maps of the real line and then apply them to the examples of Sections 3.1 and 3.2.

4.1 Generalities

Consider a "translation-indifferent" $T : \mathbb{R} \longrightarrow \mathbb{R}$. With this imprecise term we mean that the relevant properties of T—e.g., expansivity, distortion—are uniform throughout $\mathbb{R}$. In other words, the map does not single out any special region of $\mathbb{R}$. In this vague sense, all the examples of Section 3 are translation-indifferent, even the finite modifications of quasi-lifts, because the modification does not alter the nature of the map there. Suppose that T preserves a Lebesgue-absolutely continuous measure μ, which we assume infinite due to translation-indifference.

We call *global observable* any complex-valued function $F \in L^\infty(\mathbb{R}, \mu)$ such that

$$\overline{\mu}(F) := \lim_{r \to \infty} \frac{1}{\mu([x_0 - r, x_0 + r])} \int_{x_0 - r}^{x_0 + r} F \, d\mu \tag{4.1}$$

exists uniformly in x_0 and independently of it, as the notation suggests. Clearly, the class of all global observables forms a linear space, containing, for example, the constant functions, all functions that differ from a constant by a bounded integrable function, or all bounded F with $\lim_{|x| \to \infty} F(x) < \infty$, etc. Naturally, one is interested in more complicated observables, such as periodic, quasi-periodic and generally oscillating functions: to determine whether they are global observables, one should know μ.

If we restrict T to be a Markov map verifying (A5), we can view this definition within the general framework presented in [L4, L5, L7]. We especially refer the reader to [L7], which uses the same notation as the present paper and contains several results needed here.

We first assume that $\exists \theta_1, \theta_2 > 0$ such that, $\forall j \in \mathbb{Z}$,

$$\theta_1 \leq \mu(I_j) \leq \theta_2. \tag{4.2}$$

This makes sense for translation-indifferent systems, cf. (A1) and (A6). The collection of sets

$$\mathscr{V} := \left\{ \bigsqcup_{j=k}^{\ell} I_j \;\middle|\; k \leq \ell \right\}. \tag{4.3}$$

is called the *exhaustive family* relative to the Markov partition of T: its elements play the role of "large boxes" in phase space. Since global observables are bounded, it is easy to see that F verifies (4.1) if and only if

$$\lim_{M \to \infty} \sup_{\substack{V \in \mathscr{V} \\ \mu(V) \geq M}} \left| \frac{1}{\mu(V)} \int_V F \, d\mu - \overline{\mu}(F) \right| = 0. \tag{4.4}$$

We describe this situation by saying that the average of F over $V \in \mathscr{V}$, also denoted $\mu_V(F) := \mu(V)^{-1} \int_V F \, d\mu$, converges in the *infinite-volume limit* to $\overline{\mu}(F)$. The notation

$$\lim_{V \nearrow \mathbb{R}} \mu_V(F) = \overline{\mu}(F) \tag{4.5}$$

is short for (4.4). $\overline{\mu}(F)$ is called the *infinite-volume average* of F.

We also call *local observable* any complex-valued $g \in L^1(\mathbb{R}, \mu)$. For any such g we use the customary notation $\mu(g) := \int_{\mathbb{R}} g \, d\mu$.

Let us consider two (sub)classes $\mathcal{G}$ and $\mathcal{L}$ of global and local observables, respectively. Relative to $\mathcal{G}$ and $\mathcal{L}$, one says that the dynamical system $(\mathbb{R}, \mu, T)$ is mixing of type **(GLM1)** if, for all $F \in \mathcal{G}$ and $g \in \mathcal{L}$, with $\mu(g) = 0$,

$$\lim_{n \to \infty} \mu((F \circ T^n) g) = 0. \tag{GLM1}$$

It is mixing of type **(GLM2)** if, for all $F \in \mathcal{G}$ and $g \in \mathcal{L}$,

$$\lim_{n \to \infty} \mu((F \circ T^n) g) = \overline{\mu}(F) \, \mu(g). \tag{GLM2}$$

It is immediate to see that **(GLM2)** is equivalent to (1.1) and implies **(GLM1)**. As they involve the pairing of a global and a local observable, we say that these are two definitions of *global-local mixing*. (There exists another definition of global-local mixing, which is a uniform version of **(GLM2)** and is denoted **(GLM3)** in [L7]. We do not consider it here.)

The following is a trivial consequence of a well-known theorem of Lin [Li].

Proposition 4.1 *An exact dynamical system is* **(GLM1)***-mixing for any choice of* $\mathcal{G} \subseteq L^\infty$ *and for* $\mathcal{L} = L^1$ *(viz. any choice of* $\mathcal{L} \subseteq L^1$*).*

PROOF. See [L7, Thm. 3.5*(a)*].

When we consider the "decorrelation" between two global observables, we study the so-called *global-global mixing*. We have two definitions for it. The system is called mixing of type **(GGM1)** if, for all $F, G \in \mathcal{G}$, $\overline{\mu}((F \circ T^n)G)$ exists for all sufficiently large n, and

$$\lim_{n\to\infty} \overline{\mu}((F \circ T^n)G) = \overline{\mu}(F)\,\overline{\mu}(G). \tag{GGM1}$$

It is called mixing of type **(GGM2)** if, for all $F, G \in \mathcal{G}$,

$$\lim_{\substack{V\nearrow\mathbb{R}\\ n\to\infty}} \mu_V((F \circ T^n)G) = \overline{\mu}(F)\,\overline{\mu}(G). \tag{GGM2}$$

The above limit, which we call *joint infinite-volume and time limit*, means

$$\lim_{M\to\infty} \sup_{\substack{V\in\mathscr{V}\\ \mu(V)\geq M\\ n\geq M}} \left| \frac{1}{\mu(V)} \int_V (F \circ T^n) G \, d\mu - \overline{\mu}(F)\,\overline{\mu}(G) \right| = 0. \tag{4.6}$$

The second definition is in essence stronger than the first, as the following proposition shows.

Proposition 4.2 *If* $F, G \in \mathcal{G}$ *are such that* $\overline{\mu}((F \circ T^n)G)$ *exists for all* n *large enough (depending on* F, G*), then*

$$\lim_{\substack{V\nearrow\mathbb{R}\\ n\to\infty}} \mu_V((F \circ T^n)G) = b \quad\Longrightarrow\quad \lim_{n\to\infty} \overline{\mu}((F \circ T^n)G) = b. \tag{4.7}$$

In particular, if the above hypothesis holds $\forall F, G \in \mathcal{G}$*, then* **(GGM2)** *implies* **(GGM1)***.*

PROOF. See [L7, Prop. 2.3].

Some of the maps considered in this paper give a good sense of the relative strength of **(GGM2)** and **(GGM1)**, as the latter property will be trivially verified while the former will remain an open question; cf. Proposition 4.5.

For an in-depth discussion on the meaning and relevance of the above definitions we refer the reader to [L4]. Here we just point out that, in order for them to make sense as indicators of decorrelation, it must be that, for all $F \in \mathcal{G}$ and $n \in \mathbb{N}$,

$$\overline{\mu}(F \circ T^n) = \overline{\mu}(F). \tag{4.8}$$

As shown in [L4], this is guaranteed by the following hypothesis: for all $n \in \mathbb{N}$,

$$\lim_{V \nearrow \mathbb{R}} \frac{\mu(T^{-n}V \triangle V)}{\mu(V)} = 0, \tag{4.9}$$

in the sense of the infinite-volume limit, as in (4.4). The above is easily verified for all dynamical systems which verify (A1)-(A2), (A5)-(A6). In fact, recalling the definition (2.14), if $V = \bigsqcup_{j=k}^{\ell} I_j$, with $\ell - k$ sufficiently large, it is easy to see that

$$\bigsqcup_{j=k+n(\kappa-1)}^{\ell-n(\kappa-1)} I_j \ \subset \ T^{-n}V \ \subset \bigsqcup_{j=k-n(\kappa-1)}^{\ell+n(\kappa-1)} I_j, \tag{4.10}$$

whence

$$T^{-n}V \triangle V \ \subset \bigsqcup_{j=k-n(\kappa-1)}^{k+n(\kappa-1)} I_j \ \sqcup \bigsqcup_{j=\ell-n(\kappa-1)}^{\ell+n(\kappa-1)} I_j. \tag{4.11}$$

Using (4.11) and (4.2), we see that the numerator of (4.9) is bounded above by $(4n(\kappa-1)+2)\theta_2$, while the denominator is bounded below by $(\ell-k+1)\theta_1$. But the infinite-volume limit here corresponds precisely to the limit $\ell - k \to +\infty$ (uniformly in k, ℓ), whence the assertion.

4.2 Results for quasi-lifts and their finite modifications

Now, let T be a quasi-lift of an expanding circle maps, as in Section 3.1. We are going to show that all the definitions of infinite mixing, both global-local and global-global, are verified for suitable choices of the global observables.

If ψ is either a global or a local observable, and $k \in \mathbb{Z}^+$, set

$$\mathcal{A}_k \psi := \frac{1}{k} \sum_{j=0}^{k-1} \psi \circ \sigma^j. \tag{4.12}$$

By (3.1), this operator commutes with the dynamics, namely $\mathcal{A}_k(\psi \circ T) = (\mathcal{A}_k \psi) \circ T$. Now define

$$\mathcal{G}_1 := \left\{ F \in L^\infty \;\middle|\; \exists F_a = F_a \circ \sigma : \lim_{k\to\infty} \| \mathcal{A}_k F - F_a \|_\infty = 0 \right\}; \tag{4.13}$$

$$\mathcal{G}_2 := \overline{\mathrm{span}_{\mathbb{C}} \left\{ F \in L^\infty \mid \exists \beta \in \mathbb{R} : F \circ \sigma = e^{\imath a \beta} F \right\}}, \tag{4.14}$$

where the bar denotes closure in the L^∞-norm. In other words, $\mathcal{G}_1$ is the space of all essentially bounded functions whose $(a\mathbb{Z})$-average converges uniformly to a periodic function (of period a); $\mathcal{G}_2$ is the space generated by the quasiperiodic functions w.r.t. $a\mathbb{Z}$. Clearly, $\mathcal{G}_2 \subset \mathcal{G}_1$ (observe that $\mathcal{G}_1$ is closed). To see that all these functions are global observables, we need to verify that every $F \in \mathcal{G}_1$ possesses an infinite-volume average $\overline{\mu}(F)$, in the sense of (4.4). In this case, V is of the form $[ak, a(\ell+1)]$, which gives:

$$\frac{1}{\mu(V)} \int_V F \, d\mu = \frac{1}{(\ell - k + 1)\mu(I_0)} \int_{ak}^{a(\ell+1)} F \, d\mu = \int_{I_0} \mathcal{A}_{\ell-k+1} F \circ \sigma^k \, d\mu_{I_0} \tag{4.15}$$

which, by the hypotheses on F, converges to

$$\int_{I_0} F_a \, d\mu_{I_0} =: \overline{\mu}(F), \tag{4.16}$$

as $\ell - k \to \infty$ (that is, as $\mu(V) \to \infty$, uniformly in $V \in \mathscr{V}$).

Examples of elements of $\mathcal{G}_2$ are the functions $E_\gamma(x) := e^{\imath\gamma x}$, $\gamma \in \mathbb{R}$. An example of $F \in \mathcal{G}_1 \setminus \mathcal{G}_2$ is given by $F := \sum_{j\in\mathbb{Z}} b_j 1_{I_j}$, with $(b_j)_{j\in\mathbb{Z}}$ a non-periodic sequence such that $\{b_{2k}, b_{2k+1}\} = \{0, 1\}$, for all $k \in \mathbb{Z}$.

We have:

Theorem 4.3 *A quasi-lift of an expanding circle map, as defined in Section 3.1, is:*

(a) mixing of type **(GLM1)** *for any $\mathcal{G} \subset L^\infty$ and $\mathcal{L} = L^1$;*

(b) mixing of type **(GLM2)** *w.r.t. $\mathcal{G}_1$ and L^1;*

(c) mixing of type **(GGM1)** *and* **(GGM2)** *w.r.t. $\mathcal{G}_2$.*

Proof. See Section 5.

The results we have for finite modifications of quasi-lifts are less satisfactory.

Definition 4.4 *Given μ, μ_o, two σ-finite, infinite, measures on $\mathbb{R}$, we write $\overline{\mu} = \overline{\mu}_o$ when they admit the same global observables and coincide on them. This means, for all bounded $F : \mathbb{R} \longrightarrow \mathbb{C}$, $\overline{\mu}(F)$ exists if and only if $\overline{\mu}_o(F)$ does, and they are equal.*

The above situation can occur, for example, when $h_\mu - h_{\mu_o} \in L^1(\mathbb{R}, m)$, where h_μ, h_{μ_o} are the densities of μ, μ_o, respectively.

Proposition 4.5 *Let T be a finite modification of a quasi-lift T_o which verifies (A1)-(A7), and call μ_o the measure preserved by T_o (cf. Section 3.2). If T is Markov-indecomposable and preserves a Lebesgue-equivalent measure μ such that $\overline{\mu} = \overline{\mu}_o$, then T is:*

(a) mixing of type **(GLM1)** *for any $\mathcal{G} \subset L^\infty$ and $\mathcal{L} = L^1$;*

(b) mixing of type **(GGM1)** *w.r.t. $\mathcal{G}_2$.*

Proof. See Section 5.

Remark 4.6 The proof of Proposition 4.5 will show that, if one drops the hypothesis $\overline{\mu} = \overline{\mu}_o$, statement *(a)* still holds. As to *(b)*, one still has that $\lim_{n\to\infty} \overline{\mu}_o((F \circ T^n)G) = \overline{\mu}_o(F)\overline{\mu}_o(G)$, for all $F, G \in \mathcal{G}_2$.

4.3 Example: finite modification of a homogeneous random walk

The results of the previous section convince one that the mixing properties of finite modifications of quasi-lifts are harder to prove, in general, than those of quasi-lifts. In particular this holds for the important notion that we have called **(GLM2)**. However, one expects **(GLM2)** to hold true for a large class of maps. In this section we present one such case. Even though we pick a specific example, the technique generalizes easily to other maps of the same kind.

Let $T = T_{\mathcal{Q}}$ be the map associated to the random walk given by

$$\mathcal{Q} := \frac{1}{9} \left(\begin{array}{ccccccccccccc} \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & & & \\ & 1 & 2 & 3 & 2 & 1 & & & & & & & \\ & & 1 & 2 & 3 & 2 & 1 & & & & & & \\ \hline & & & 1 & 2 & 5 & 1 & 0 & & & & & \\ & & & & 1 & 1 & 5 & 1 & 1 & & & & \\ & & & & & 0 & 1 & 5 & 2 & 1 & & & \\ \hline & & & & & & 1 & 2 & 3 & 2 & 1 & & \\ & & & & & & & 1 & 2 & 3 & 2 & 1 & \\ & & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{array} \right), \tag{4.17}$$

with the convention that the entries of $\mathcal{Q}$ are null outside of the shown diagonal strip. This matrix is doubly stochastic, so T preserves m (Proposition 3.6). Also, as indicated in (4.17), its rows $(q_{jk})_k$ fail to be translations of one another only for $j \in \{-1, 0, 1\}$, hence T is a finite modification of a map T_o representing a homogeneous random walk. More examples of suitable $\mathcal{Q}$ can be constructed using the ideas of [L3, App. A]. There is a sizable literature about finite modification of translation-invariant random walks. Some recent references include [PSz, N, PP, IP].

We will show that the dynamical system $(\mathbb{R}, m, T)$ verifies a very strong instance of **(GLM2)**. Define $\mathscr{V}' := \{[-k, k] \subset \mathbb{R} \mid k \in \mathbb{Z}^+\}$. This exhaustive family is smaller than the one we have introduced in (4.3), which in this specific case reads $\mathscr{V} = \{[k, \ell+1] \subset \mathbb{R} \mid k \le \ell \in \mathbb{Z}\}$. In a sense, up to inessential variations, $\mathscr{V}'$ is the smallest collection of sets that make sense as an exhaustive family, because it contains only one increasing sequence of sets that covers the phase space $\mathbb{R}$. Therefore, the class of functions

$$\mathcal{G}' := \left\{ F \in L^\infty(\mathbb{R}, \mu) \;\middle|\; \exists \overline{m}'(F) := \lim_{k \to \infty} \frac{1}{2k+1} \int_{-k}^{k} F \, dm \right\} \tag{4.18}$$

is essentially the largest class of global observables one can imagine for the dynamical system at hand, because $\overline{m}'$ is the infinite-volume average w.r.t. $\mathscr{V}'$; cf. (4.4)-(4.5).

Remark 4.7 It is worthwhile to point out that $\mathcal{G}'$ is not simply a larger class of global observables than $\mathcal{G}_1$ and $\mathcal{G}_2$. By using $\mathscr{V}'$ in lieu of $\mathscr{V}$ here, we have changed the notion of infinite-volume average from $\overline{m}$, cf. (4.3)-(4.5), to $\overline{m}'$, cf. (4.18), and therefore extended the very concept of global observable. Notice that, if $\overline{m}(F)$ exists, $\overline{m}'(F) = \overline{m}(F)$.

Proposition 4.8 *The map T defined above is irreducible, conservative and exact. Also, in addition to the statements of Proposition 4.5, it is* **(GLM2)** *relative to the exhaustive family $\mathscr{V}'$, the class of global observables $\mathcal{G}'$ and the class of local observables L^1.*

PROOF. See Section 5.

Observe that no form of global-global mixing can hold for T w.r.t. $\mathcal{G}'$. This is an instance of a general phenomenon that in [L4, Sect. 3] we have called *surface effect*. Very briefly, a counterexample is constructed by choosing, e.g., $F = G = \Theta$, the Heaviside function, which belongs in $\mathcal{G}'$, with $\overline{m}'(\Theta) = 1/2$. Since T has a bounded action, in the sense of (A5), it is clear that $\Theta \circ T^n(x) = \Theta(x)$, for all $|x| > \rho n$ (in this particular case one can take $\rho = 2$). So, for all $n \in \mathbb{N}$,

$$\lim_{V \nearrow \mathbb{R}} m_V((\Theta \circ T^n)\Theta) = \overline{m}'(\Theta^2) = \frac{1}{2} \neq \overline{m}'(\Theta)^2,$$

contradicting both **(GGM1)** and **(GGM2)**. (Recall that, in the latter, the convergence in n is uniform w.r.t. the one in V, and viceversa.)

The above says no more and no less than: $\mathscr{V}'$ is the wrong exhaustive family for the global-global mixing of quasi-lifts, or finite modifications thereof; cf. [L4, Sect. 3]. We still expect **(GGM1-2)** to hold for general classes of global observables, relative to the exhaustive family $\mathscr{V}$.

5 Proofs

In this section we prove all the results stated in the previous sections (except for the simplest ones, whose proofs have already been given). We start by laying out the technical tools that will be needed in the proofs of all the theorems and propositions of Section 2.2.

For $\boldsymbol{j} = (j_0, j_1, \ldots, j_{n-1}) \in \mathbb{Z}^n$, set

$$I_{\boldsymbol{j}}^{(n)} := I_{j_0} \cap T^{-1} I_{j_1} \cdots \cap T^{-n+1} I_{j_{n-1}}. \tag{5.1}$$

By (A1)-(A3), $\{I_{\boldsymbol{j}}^{(n)}\}_{\boldsymbol{j} \in \mathbb{Z}^n}$ is a Markov partition for T^n (as always, modulo the endpoints of the intervals) and $m(I_{\boldsymbol{j}}^{(n)}) \leq \theta \lambda^n$. We call $\mathscr{M}^n$ the generated σ-algebra. For $n \geq 1$, let $\mathscr{M}^n[x]$ denote the only element of $\{I_{\boldsymbol{j}}^{(n)}\}$ such that $x \in \mathscr{M}^n[x]$ (in case x belongs to two such elements, being an endpoint of both, we make the convention that $\mathscr{M}^n[x]$ is the interval on the right). $\mathscr{M}[x]$ will be short for $\mathscr{M}^1[x]$.

Lemma 5.1 *If x is a density point of A, with $m(A) > 0$, then*

$$\lim_{n\to\infty} m(T^n A \,|\, \mathscr{M}[T^n(x)]) = 1.$$

PROOF. By the hypothesis on x,

$$\lim_{n\to\infty} m(A \,|\, \mathscr{M}^n[x]) = 1. \tag{5.2}$$

Setting $B_{x,n} := \mathscr{M}^{n+1}[x] \setminus A$, (5.2) is equivalent to $\lim_{n\to\infty} m(B_{x,n} \,|\, \mathscr{M}^{n+1}[x]) = 0$, whence, by Corollary A.3 (Section A.2 of the Appendix),

$$\lim_{n\to\infty} m(T^n B_{x,n} \,|\, T^n \mathscr{M}^{n+1}[x]) = 0. \tag{5.3}$$

Since T^n is a bijection $\mathscr{M}^{n+1}[x] \longrightarrow \mathscr{M}[T^n(x)]$ and $\mathscr{M}[T^n(x)] \setminus T^n A \subseteq T^n B_{x,n}$, (5.3) implies the lemma. Q.E.D.

Corollary 5.2 *For $d \in \mathbb{Z}^+$, suppose that A is T^d-invariant, with $m(A) > 0$; x is a density point of A; and y an accumulation point of $(T^{dn}(x))_{n\in\mathbb{N}}$. If $y \in (a_j, a_{j+1})$, for some $j \in \mathbb{Z}$, then $I_j \subseteq A$ (this means, as always,* mod m*). If $y = a_j$ and $(T^{dn}(x))_n$ accumulates to y from the right (respectively, left), then $I_j \subseteq A$ (respectively $I_{j-1} \subseteq A$).*

PROOF. Evaluate the limit of Lemma 5.1 on any subsequence $(dn_i)_{i\in\mathbb{N}}$ such that $T^{dn_i}(x) \to y$ from the right/left, respectively. Q.E.D.

PROOF OF PROPOSITION 2.2. Clearly $\mathcal{C} \subseteq \Omega$, and so $\mathcal{I}(\mathcal{C}) \subseteq \Omega$. The first assertion of the proposition will be proved once we show the reverse inclusion. This amounts to show that the invariant set $\mathcal{I}_{\mathcal{D}\Omega} := \mathcal{I}_{\mathcal{D}} \cap \Omega$ is null.

Suppose not. For $j \in \mathbb{Z}$, let Ω_j be the set of all $x \in \Omega$ whose ω-limit satisfies at least one of the following conditions:

1. $\omega(x) \cap (a_j, a_{j+1})$ is not empty;
2. $a_j \in \omega(x)$ and $(T^n(x))_{n\in\mathbb{N}}$ accumulates to a_j from the right;
3. $a_{j+1} \in \omega(x)$ and $(T^n(x))_{n\in\mathbb{N}}$ accumulates to a_{j+1} from the left.

Clearly, Ω_j is T-invariant, and so is $\mathcal{I}_{\mathcal{D}\Omega} \cap \Omega_j$. Since $\mathcal{I}_{\mathcal{D}\Omega}$ has positive measure and the orbits of all its elements accumulate somewhere, there exists $j \in \mathbb{Z}$ such that $m(\mathcal{I}_{\mathcal{D}\Omega} \cap \Omega_j) > 0$. We claim that the orbit of a.e. $x \in I_j$ returns to I_j infinitely many times in the future. If not, there would exist $B \subseteq I_j$, with $m(B) > 0$, and $N \in \mathbb{N}$, such that $I_j \cap \bigcup_{n\ge N} T^n B$ is null. One the other hand, the typical $x \in \mathcal{I}_{\mathcal{D}\Omega} \cap \Omega_j$ is a density point of the same set. Applying Corollary 5.2 (with $d = 1$) we get $I_j \subseteq \mathcal{I}_{\mathcal{D}\Omega} \cap \Omega_j$. Thus $B \subseteq \mathcal{I}_{\mathcal{D}\Omega} \cap \Omega_j$, which is absurd because, by construction, no point of B can belong to Ω_j.

Observe that, in the terminology of Section 2.1, we have just proved that I_j is essential.

Now take any positive-measure $A \subseteq I_j$. A.e. $x \in A$ is both a density point of A and a recurrent point to I_j. Choose one such x. The proof of Lemma 5.1 shows that there exists a return time n such that $T^n(A \cap \mathscr{M}^{n+1}[x])$ is so large within $\mathscr{M}[T^n(x)] = I_j$ to have a non-null intersection with A. Since T^n acts as a two-sided non-singular bijection $\mathscr{M}^{n+1}[x] \longrightarrow \mathscr{M}[T^n(x)]$, we obtain

$$m(A \cap T^{-n}A) \geq m(A \cap \mathscr{M}^{n+1}[x] \cap T^{-n}A) > 0. \tag{5.4}$$

We have thus proved that I_j cannot contain wandering sets, which is a contradiction because $I_j \subseteq \mathcal{I}_{\mathcal{D}\Omega} \cap \Omega_j \subseteq \mathcal{D}$. This concludes the proof of the first assertion of Proposition 2.2.

For the second assertion it suffices to prove that every invariant $A \subseteq \Omega$, with $m(A) > 0$, contains a positive-measure, invariant subset B which cannot be further decomposed in invariant subsets of strictly smaller measure. So, consider one such A. The previous arguments show that, for some $j \in \mathbb{Z}$, $m(A \cap \Omega_j) > 0$ and $I_j \subseteq A \cap \Omega_j$. Therefore $B := \mathcal{I}_T(I_j) \subseteq A$ cannot be further decomposed in smaller invariant subsets, ending the proof of Proposition 2.2.

We add a few remarks. The above conclusion states that $\mathcal{I}_T(I_j)$ is an ergodic component of T. If we only take the forward images of I_j, we see that $\bigcup_{n\in\mathbb{N}} T^n I_j \subseteq \mathcal{C}$, because $I_j \subset \mathcal{C}$, as shown earlier, and $T\mathcal{C} \subseteq \mathcal{C}$. With reference to the definitions of Section 2.1—see in particular (2.2) and (2.9)—let α be the unique index in $\aleph$ such that $j \in \mathbb{Z}_\alpha$. Then $M_\alpha = \mathcal{I}_{T_\alpha}(I_j) = \bigcup_{n\in\mathbb{N}} T^n I_j \subseteq \mathcal{C}$, and T_α is conservative and ergodic. Q.E.D.

5.1 Exactness

Recall that $\mathscr{B}$ denotes the Borel σ-algebra of $\mathbb{R}$. Let us introduce the other σ-algebras that we are concerned with. For $n \in \mathbb{N}$,

$$\mathscr{I}^n := \left\{ A \in \mathscr{B} \,\middle|\, T^{-n}A = A \bmod m \right\} \tag{5.5}$$

is the T^n*-invariant* σ*-algebra*. (From now on, as declared in Section 2, we will always imply 'mod m'.) Clearly, if n is a multiple of k, $\mathscr{I}^k \subseteq \mathscr{I}^n$. $\mathscr{I}$ will be short for $\mathscr{I}^1$. The *tail* σ*-algebra* is defined to be:

$$\mathscr{T} := \bigcap_{n=0}^{\infty} T^{-n}\mathscr{B}. \tag{5.6}$$

Of course, $\mathscr{I}^n \subseteq \mathscr{T}$, for all $n \in \mathbb{N}$.

Given a σ-algebra $\mathscr{A}$ and a Borel B, we will denote by $\mathscr{A} \cap B := \{A \cap B \mid A \in \mathscr{A}\}$ the *trace* of $\mathscr{A}$ in B.

At the core of all exactness proofs will be the following generalization of a criterion by Miernowski and Nogueira [MN]:

Proposition 5.3 *Consider the dynamical system $(X, \mathscr{A}, \nu, S)$, where $(X, \mathscr{A}, \nu)$ is a σ-finite measure space and S a non-singular endomorphism on it (i.e., $\nu(A) = 0 \Rightarrow \nu(S^{-1}A) = 0$). Denote by $\mathscr{I} := \{A \in \mathscr{A} \mid S^{-1}A = A \bmod \nu\}$ and $\mathscr{T} := \bigcap_{n=0}^{\infty} S^{-n}\mathscr{A}$, respectively, the invariant and tail σ-algebras. Clearly, $\mathscr{I} \subseteq \mathscr{T}$. If, $\forall A \in \mathscr{T}$ with $\nu(A) > 0$, $\exists n = n(A)$ such that $\nu(S^{n+1}A \cap S^n A) > 0$, then $\mathscr{I} = \mathscr{T}$.*

In other words, under the above hypotheses, the non-null ergodic components of S are also exact components. The proof of Proposition 5.3, together with a converse statement, can be found in [L8, Prop. A.2].

PROOF OF THEOREM 2.3. We start by proving that, for all $d \geq 1$, the ergodic components of T^d within $\mathcal{I}_\mathcal{C}$ (equivalently, the ergodic components of $(T|_{\mathcal{I}_\mathcal{C}})^d$) are $\mathscr{M}$-measurable. In fact, given a T^d-invariant $A \subseteq \mathcal{I}_\mathcal{C}$, consider $j \in \mathbb{Z}$ such that $m(A \cap I_j) > 0$. Since A is in the conservative part of T (or T^d, which is the same) the typical $x \in A \cap I_j$ is a density point of A and is recurrent to the interior of I_j, w.r.t. T^d. Corollary 5.2 shows that $I_j \subseteq A$. Thus, $\mathscr{I}^d \cap \mathcal{I}_\mathcal{C} \subseteq \mathscr{M} \cap \mathcal{I}_\mathcal{C}$ and the claim is proved.

Consider an ergodic component $E_\alpha \subseteq \mathcal{I}_\mathcal{C}$. Since $E_\alpha \in \mathscr{M}$, it contains whole Markov intervals. Set $\mathbb{Z}_o := \{j \in \mathbb{Z} \mid I_j \subseteq E_\alpha\}$. For all $j \in \mathbb{Z}_o$, $\mathcal{I}_T(I_j) = E_\alpha$ and I_j is essential (by conservativity). So all these I_j communicate with each other. Therefore $\mathbb{Z}_o$ is a communicating class $\mathbb{Z}_\alpha$ and E_α is the corresponding M_α, cf. (2.2). In particular, the periods of all intervals $I_j \subset E_\alpha$ are the same. Let $d_\alpha \geq 1$ denote their common value. For the sake of simplicity, in the remainder of this proof we write d for d_α.

Now fix $A \subseteq E_\alpha$ with $m(A) > 0$. We are going to show that $\exists n \in \mathbb{Z}^+$ such that

$$m(T^{(n+1)d}A \cap T^{nd}A) > 0. \tag{5.7}$$

Therefore Proposition 5.3 can by applied to $T^d|_{E_\alpha}$, implying that the ergodic components of T^d within E_α are exact.

We start with a simple lemma concerning the return times of a state in a countable-state Markov chain:

Lemma 5.4 *Denoting $R_j := \{n \in \mathbb{Z}^+ \mid p_{jj}^{(n)} > 0\}$, and recalling that $d_j :=$ g.c.d.(R_j), there exists $n \in R_j$ such that $n + d_j \in R_j$.*

PROOF. By (2.1), R_j is an additive set, therefore $R_j - R_j$ is a $\mathbb{Z}$-ideal. By principality, $R_j - R_j = \mathbb{Z}d_j$, therefore $\exists n, n_1 \in R_j$ such that $d_j = n_1 - n$. Q.E.D.

Choose j such that $m(A \cap I_j) > 0$. By Lemma 5.4,

$$T^{n_1} I_j \supset I_j; \tag{5.8}$$

$$T^{n_1+d} I_j \supset I_j, \tag{5.9}$$

for some $n_1 \in \mathbb{Z}^+$. Thus, $T^d T^{n_1} I_j \supset I_j$. The Markov property of T^d implies that $\exists B \subset T^{n_1} I_j$ such that $T^d|_B : B \longrightarrow I_j$ is a bijection. (Incidentally, $B \in \mathscr{M}^d$.)

In view of (5.8), $\exists \delta > 0$ such that every $A' \subset \mathbb{R}$ with $m(A'|I_j) > 1-\delta$ verifies both

$$m(T^{n_1}A'|I_j) > \frac{1}{2}; \tag{5.10}$$

$$m(T^{n_1}A'|B) > 1 - \frac{1}{2D}, \tag{5.11}$$

where D is the distortion constant that appears in Corollary A.3 of Section A.2. That corollary, together with the definition of B and (5.11), gives

$$m(T^{n_1+d}A'|I_j) > \frac{1}{2}. \tag{5.12}$$

Therefore, by (5.10) and (5.12),

$$m(T^{n_1+d}A' \cap T^{n_1}A') > 0. \tag{5.13}$$

At this point, observe that a.a. $x \in A \cap I_j$ are both density points of A and recurrent to the interior of I_j, that is, $\mathscr{M}[T^n(x)] = I_j$ for infinitely many n. Choose any such x. By Lemma 5.1, there exists a large enough n_2 such that $m(T^{n_2}A|I_j) > 1-\delta$. This means that (5.13) can be applied with $T^{n_2}A$ in the place of A'. More precisely, $m(T^{n_1+n_2+d}A \cap T^{n_1+n_2}A) > 0$. By the non-singularity of T, this inequality holds as well if n_1+n_2 is replaced with any $nd \geq n_1+n_2$, which proves (5.7).

So, the ergodic components of T^d inside E_α are also exact components of T^d. Let us study them. This part of the proof uses a standard argument from the classification of states for Markov chains [S, Sect. VIII.2].

Choose j such that $I_j \subset E_\alpha$. Define $\mathbb{Z}'_0 := \{j\}$ and, for $\ell \geq 1$,

$$\mathbb{Z}'_\ell := \left\{ k \in \mathbb{Z} \;\middle|\; p^{(\ell)}_{jk} > 0 \right\}. \tag{5.14}$$

In other words, using also the Markov property of T, $\bigsqcup_{k\in\mathbb{Z}'_\ell} I_k = T^\ell I_j$. Given $k \in \mathbb{Z}'_\ell$, for all n such that $p^{(n)}_{kj} > 0$ (and there are infinitely many of them, because all these Markov intervals are contained in a conservative ergodic component of T) we have $p^{(\ell+n)}_{jj} > 0$, hence $n \equiv -\ell \pmod d$ (because $d_j = d$). This implies that the sets

$$\mathbb{Z}''_i := \bigcup_{\substack{\ell\in\mathbb{N} \\ \ell\equiv i \,(\mathrm{mod}\, d)}} \mathbb{Z}'_\ell, \tag{5.15}$$

defined for $i \in \{0,1,\ldots,d-1\}$, are pairwise disjoint. It is clear that their union is the communicating class $\mathbb{Z}_\alpha$ that contains j, and so $E_\alpha = M_\alpha$, as observed earlier. For $i \in \{0,1,\ldots,d-1\}$, set $E_{\alpha,i} := \bigsqcup_{k\in\mathbb{Z}''_i} I_k$. By the definitions (5.14)-(5.15),

$$TE_{\alpha,i} = E_{\alpha,i+1 \,(\mathrm{mod}\ d)}, \tag{5.16}$$

which is one of the assertions of Theorem 2.3.

It remains to prove that each $E_{\alpha,i}$ is an ergodic component of T^d. First off, since $E_\alpha = \bigsqcup_{i=0}^{d-1} E_{\alpha,i}$ is T-invariant, and by (5.16), $T^{-d} E_{\alpha,i} = E_{\alpha,i}$. Suppose by absurd that $E_{\alpha,i}$ could be split in two non-trivial T^d-invariant sets A, B. By the two-sided non-singularity of the map, $T^{-i}A, T^{-i}B$ would be non-trivial T^d-invariant subsets of $E_{\alpha,0}$. They would also belong to $\mathscr{M}$, as we have shown at the start of this proof. So one of them, say $T^{-i}A$, must contain I_j. However, by definition of $E_{\alpha,0}$, that is, by definition of $\mathbb{Z}_0''$, $\bigcup_{n\in\mathbb{N}} T^{nd} I_j = E_{\alpha,0}$, giving that $T^{-i}A = E_{\alpha,0}$, a contradiction. This ends the proof of Theorem 2.3. Q.E.D.

Proof of Proposition 2.5. The first assertion was all but shown in the previous proof: a conservative ergodic component must be of the form M_α, for some $\alpha \in \aleph$. Moreover, $\mathbb{Z}_\alpha$ is an isolated class because $T^{-1}M_\alpha = M_\alpha$. In other words, $\alpha \in \aleph_{\mathrm{iso}}$.

Viceversa, given $\alpha \in \aleph_{\mathrm{iso}}$, consider the T-invariant set $A := M_\alpha \cap \Omega$: its Lebesgue measure can be either positive or zero.

If $m(A) > 0$, Corollary 5.2 (with $d = 1$) entails that $I_j \subseteq A$, for some $j \in \mathbb{Z}_\alpha$. By definition of communicating class, $\forall k \in \mathbb{Z}_\alpha$, $\exists n \geq 1$ such that $T^n I_j \supset I_k$. This shows that $A \supseteq M_\alpha$, whence $A = M_\alpha$. It also shows that I_j has infinitely many Markov returns. The same arguments as in the proof of Proposition 2.2 prove that there are no wandering sets in I_j. Since j is arbitrary, $M_\alpha \subseteq \mathcal{C}$. But M_α is T-invariant, whence $M_\alpha \subseteq \mathcal{I}_{\mathcal{C}}$.

In the case $m(A) = 0$, $M_\alpha \subseteq \mathcal{I}_{\mathcal{D}}$ by Proposition 2.2.

Lastly, observe that $\#\mathbb{Z}_\alpha < \infty$ entails that M_α is compact, which implies the first of the two cases above. Q.E.D.

Proof of Proposition 2.6. Let E_o be an ergodic component of $\mathcal{I}_{\mathcal{CD}}$, whence $m(E_o) > 0$. Recall the definition of Ω_j from the proof of Proposition 2.2. For at least for one j, the invariant set $E_o \cap \Omega_j$ has positive measure. The same arguments as in the aforementioned proof show that I_j is essential and it is contained in E_o. It follows that $\mathbb{Z}_\alpha$, the communicating class that contains j, is essential, and $M_\alpha \subseteq E_o$. Then $TM_\alpha = M_\alpha$ and $E_\alpha = E_o$, for both are the ergodic component containing I_j.

Lemma 5.5 *A positive-measure $W \subseteq M_\alpha$ is a wandering set for T if and only if it is a wandering set for T_α, namely, the conservative and dissipative parts of T and T_α coincide within M_α.*

Proof of Lemma 5.5. Given W as in the statement of the lemma, set

$$W_1' := T^{-1}W \cap M_\alpha = T_\alpha^{-1}W; \tag{5.17}$$

$$W_1'' := T^{-1}W \setminus M_\alpha. \tag{5.18}$$

Of course, $T^{-1}W \cap W \neq \varnothing \;\Leftrightarrow\; T_\alpha^{-1}W \cap W \neq \varnothing$. Also, since $T^{-1}M_\alpha \supseteq M_\alpha$, one has that $T^{-k}W_1'' \cap M_\alpha = \varnothing$, $\forall k \geq 0$. Applying the same reasoning with $T_\alpha^{-1}W$ in the place of W and so on, recursively, we establish that, for all $n \geq 1$,

$$T^{-n}W \cap W \neq \varnothing \iff T_\alpha^{-n}W \cap W \neq \varnothing, \tag{5.19}$$

which was to be proved. Q.E.D.

By definition of T_α, cf. (2.9), $\mathbb{Z}_\alpha$ is an isolated class for T_α, which is irreducible. Proposition 2.5, applied to T_α instead of T, proves that one, and only one, of the following occurs:

1. T_α is conservative and enjoys all the ergodic properties listed in the statement of Theorem 2.3. By Lemma 5.5, $M_\alpha \subseteq \mathcal{C}$;
2. T_α is dissipative. By Lemma 5.5, $M_\alpha \subseteq \mathcal{D}$.

In view of (2.8) and the inclusion $T^{-1}\mathcal{D} \subseteq \mathcal{D}$, the latter case implies $E_\alpha \subseteq \mathcal{D}$, which is impossible because E_α is a mixed ergodic component by hypothesis. So the statement about the ergodic properties is proved.

Now consider $W := T^{-1}M_\alpha \setminus M_\alpha$. Certainly $m(W) > 0$, otherwise $T^{-1}M_\alpha = M_\alpha$ and $E_\alpha = M_\alpha \subseteq \mathcal{C}$, which is false, again because $E_\alpha \subseteq \mathcal{I}_{\mathcal{CD}}$. This proves in particular that $\mathbb{Z}_\alpha$ is not an isolated class, hence $\alpha \in \aleph_{\text{ter}}$. One readily checks that the sets $\{T^{-n}W\}_{n\in\mathbb{N}}$ are pairwise disjoint, so W is T-wandering. Moreover, $\bigcup_{n\in\mathbb{N}} T^{-n}W = E_\alpha \setminus M_\alpha \subseteq \mathcal{D}$.

For the first part of Proposition 2.6 it remains to show that $\omega(x) = M_\alpha$, for a.e. $x \in E_\alpha$. But this follows trivially from (2.8)-(2.9) and the fact that T_α is conservative and ergodic.

Viceversa, suppose $\alpha \in \aleph_{\text{ter}}$. This implies that $T^{-1}M_\alpha \supset M_\alpha$ (strictly mod m, according to our convention). The arguments used in the paragraph before the last one prove that $m(E_\alpha \setminus M_\alpha) > 0$ and $E_\alpha \setminus M_\alpha \subseteq \mathcal{D}$.

Now, set $A := M_\alpha \cap \Omega$. This is a T_α-invariant set, so the proof of Proposition 2.5 applies, with T_α in lieu of T. There are two cases:

1. $m(A) > 0$. In this case, $A = M_\alpha$ is a conservative ergodic component of T_α—the unique component, in fact. By Lemma 5.5, $M_\alpha \subseteq \mathcal{C}$;
2. $m(A) = 0$. In this case, a.e. point of M_α has an empty ω-limit w.r.t. T_α, equivalently, w.r.t. T. By Proposition 2.2, $M_\alpha \subseteq \mathcal{I}_{\mathcal{D}}$.

Observe that the two cases above correspond to the two cases described in the first part of this proof. In any event, the first one gives $E_\alpha \subseteq \mathcal{I}_{\mathcal{CD}}$, and the second one gives $E_\alpha \subseteq \mathcal{I}_{\mathcal{D}}$. As in the proof of Proposition 2.5, $\#\mathbb{Z}_\alpha < \infty$ implies the first case. Q.E.D.

Proof of Theorem 2.11. For $x \in \mathcal{I}_{\mathcal{D}} = \mathbb{R} \setminus \Omega$ (cf. Proposition 2.2), one and only one of the following occurs:

1. $\lim_{n\to\infty} T^n(x) = +\infty$;
2. $\lim_{n\to\infty} T^n(x) = -\infty$;

3. $\limsup_{n\to\infty} T^n(x) = +\infty$ and $\liminf_{n\to\infty} T^n(x) = -\infty$.

In the third case, (A5) implies that the orbit of x intersects $[0, \rho]$ infinitely many times, thus having an accumulation point there. This is a contradiction, and so $\mathcal{I}_{\mathcal{D}} = \mathcal{D}_{+\infty} \sqcup \mathcal{D}_{-\infty}$.

The remaining assertions will be proved only for $\mathcal{D}_{+\infty}$, the arguments for $\mathcal{D}_{-\infty}$ being completely analogous.

Assume (A6) and $m(\mathcal{D}_{+\infty}) > 0$. By definition of the invariant set $\mathcal{D}_{+\infty}$, the orbit of a.e. $x \in \mathcal{D}_{+\infty}$ visits an infinite number of distinct intervals I_{j_n}, with $m(I_{j_n} \cap \mathcal{I}_{\mathcal{D}}) > 0$. Here $(j_n)_n$ is a subsequence of $\mathbb{Z}$ which depends on x. By (A6),

$$\sum_n m(I_{j_n}) = \infty. \tag{5.20}$$

On the other hand, a.e. $x \in \mathcal{D}_{+\infty}$ is also a density point of $\mathcal{D}_{+\infty}$. Hence, by Lemma 5.1 and (5.20), $m(\mathcal{D}_{+\infty}) = \infty$.

Lastly, we assume (A7) too and prove that $\mathcal{D}_{+\infty}$ is an exact component of T. We need the following distortion lemma.

Lemma 5.6 *Under the assumptions (A1)-(A7), there exists $D_1 > 1$ such that, for all measurable $A' \subseteq \mathbb{R}$ and all $j \in \mathbb{Z}$, $m(A'|I_j) > 1-\delta$ implies $m(TA'|I_k) > 1 - D_1\delta$, for all $k_{1j} \le k \le k_{2j}$ (equivalently, for all k such that $I_k \subset TI_j =: J_j$).*

PROOF OF LEMMA 5.6. Recall the meaning of the constants θ, θ_o, κ; cf. (A1), (A5)-(A7) and following remarks.

For A', j, k as in the statement of the lemma, set $B := I_j \setminus A'$. By (A1), $m(B) < \delta\theta$. By Corollary A.3, T expands B by a rate that is at most D times the average expansion rate of I_j:

$$\frac{m(TB)}{m(B)} \le D\, \frac{m(TI_j)}{m(I_j)} \le D\, \frac{\kappa\theta}{\theta_o}. \tag{5.21}$$

In the worst case, TB lands entirely in I_k, whence $m(TB|I_k) < D\kappa(\theta/\theta_o)^2\delta$. Setting $D_1 := D\kappa(\theta/\theta_o)^2$ and noticing that $TA' \cap I_k \supseteq I_k \setminus TB$ yields the desired result. Q.E.D.

Back to the proof of Theorem 2.11: given $A \subset \mathbb{R}$, we say that a set of the type $C = \bigsqcup_{k=i}^{\ell} I_k$ is *A-prevalent* if $\ell - i + 1 \ge \kappa$ and $m(A|I_k) > 1/2$, $\forall k \in \{i, i+1, \ldots, \ell\}$. In other words, C is made up of at least κ Markov intervals, in each of which the relative measure of A is bigger than half; κ is the positive integer defined in (2.14).

Suppose, by absurd, that $A, B \subset \mathcal{D}_{+\infty}$ are disjoint, T-invariant and of positive measure. We prove that, for a typical $x \in A$, $x_n := T^n(x)$ belongs to an A-prevalent set, for all n large enough.

In fact, set $\delta := D_1^{-\kappa+1}/2$, where D_1 is the universal constant provided by Lemma 5.6. By Lemma 5.1 and the invariance of A, $\exists n_1 = n_1(x)$ such that, $\forall n \ge n_1$, $m(A|\mathscr{M}[x_n]) > 1 - \delta$. Applying Lemma 5.6 recursively $\kappa - 1$ times gives $m(A|I_k) >$

$1/2$, for all $I_k \subset T^{\kappa-1}\mathscr{M}[x_n]$. Observe that, by (A2) and (A7), $T^{\kappa-1}\mathscr{M}[x_n] = \bigsqcup_{k=i}^{\ell} I_k$, for some $i < \ell$ that depend on x and n. If we show that

$$\ell - i + 1 \geq \kappa, \tag{5.22}$$

we have proved that $T^{\kappa-1}\mathscr{M}[x_n]$ is an A-prevalent set that contains $x_{n+\kappa-1}$. This would give the assertion made in the previous paragraph, because the above argument holds for *all* $n \geq n_1(x)$. But (5.22) is easily verified: by (A2) and (A7), $(T^j\mathscr{M}[x_n])_{j\geq 0}$ is an increasing sequence of sets that are unions of adjacent Markov intervals. The sequence must be strictly increasing, otherwise, for some j, $T^j\mathscr{M}[x_n]$ would be forward-invariant, contradicting that $x_{n+j} \to +\infty$, as $j \to \infty$. This implies that $T^{\kappa-1}\mathscr{M}[x_n]$ is made up of at least κ intervals, which is precisely (5.22).

So there are infinitely many A-prevalent sets in any right half-line of $\mathbb{R}$. On the other hand, applying the above to a typical $y \in B$, we have that $y_n := T^n(y)$ belongs to a B-prevalent set, for all large n. But $y_n \to +\infty$, and the distance between y_n and y_{n-1}, in terms of intervals, is at most $\kappa - 1$. This means that, for n big enough, a point y_n must fall in an A-prevalent set. But this is a contradiction, since A-prevalent sets and B-prevalent sets cannot overlap. Hence, A and B cannot be disjoint and $\mathcal{D}_{+\infty}$ is an ergodic component.

To prove that it is also an exact component we apply Proposition 5.3 to $T|_{\mathcal{D}_{+\infty}}$. In fact, given $A \subseteq \mathcal{D}_{+\infty}$, the arguments used earlier show that, for a.e. $x \in A$ and all large n, depending on x, $m(T^nA|\mathscr{M}[x_n]) > 1 - 1/2D_1 > 1/2$. By Lemma 5.6 and (A7), $m(T^{n+1}A|\mathscr{M}[x_n]) > 1/2$, whence $m(T^{n+1}A \cap T^nA) > 0$, which is the hypothesis of Proposition 5.3. Q.E.D.

5.2 Quasi-lifts and finite modifications

Proof of Proposition 3.1. Because of (A7) and (3.1), T is aperiodic and Markov-indecomposable. Recall that for this map $\mathcal{C} = \mathcal{I}_{\mathcal{C}}$. If T is not dissipative, then it is conservative, irreducible and exact by Corollary 2.4*(d)*.

If T is dissipative, Proposition 2.8 and Theorem 2.11 entail that there are at most two exact components: $\mathcal{D}_{+\infty}$ and $\mathcal{D}_{-\infty}$. Suppose that neither has measure zero. By (3.1), both components are invariant for the action of σ, thus $m(\mathcal{D}_{\pm\infty}|I_j)$ is constant in j. But the proof of Theorem 2.11 shows that there exists a (sufficiently large) j such that $m(\mathcal{D}_{+\infty}|I_j) > 1/2$. This, then, holds for all $j \in \mathbb{Z}$. The same can be proved for $m(\mathcal{D}_{-\infty}|I_j)$. It follows that our assumption was wrong and there is only one exact component.

In order to characterize which type of exact component $\mathbb{R}$ one obtains, depending on ϕ, we look at $S_n\phi(y) := \sum_{k=0}^{n-1} \phi \circ T_a(y)$, the Birkhoff sum of ϕ for the dynamical system $([0,a), \mu_a, T_a)$. The collection of all these random variables is also referred to as the (additive) *cocycle* generated by ϕ. Any such cocycle is called *recurrent* if, for μ_a-a.e. y,

$$\liminf_{n\to\infty} |S_n\phi(y)| = 0. \tag{5.23}$$

A classical result by Atkinson [At] shows that, if T_a is ergodic and ϕ is integrable w.r.t μ_a—both holding here—then $(S_n\phi)_{n\in\mathbb{N}}$ is recurrent if and only if $\mathbb{E}_\mu(\phi)=0$.

Suppose this is the case. The iterates of the $\mathbb{Z}$-extension T_ϕ, cf. Section 3.1, are of the form

$$T_\phi^n(y,j)=(T_a^n(y),j+S_n\phi(y)). \tag{5.24}$$

By (5.23)-(5.24), the T_ϕ-orbit of m-a.e. point of $[0,a)\times\{j\}$ has infinitely many returns there. Passing to its conjugated map T, this means that a.e. $x\in I_j$ has infinitely many returns to I_j. Thus, T is conservative. (This is obvious by the invariance of μ, but would hold anyway by the Markov properties of T, cf. proof of Proposition 2.2.)

Suppose instead $\mathbb{E}_\mu(\phi)>0$. The ergodicity of T_a gives that, for a.e. $y\in[0,a)$, $\lim_{n\to\infty}S_n\phi(y)=+\infty$. Once again, (5.24) and the correspondance between T_ϕ and T prove that $\lim_{n\to\infty}T^n(x)=+\infty$ for a.e. $x\in\mathbb{R}$, namely, $\mathbb{R}=\mathcal{D}_{+\infty}$. Analogously for the third case. Q.E.D.

Proof of Proposition 3.2. T is Markov-indecomposable by hypothesis and aperiodic by (A7). If $\mathcal{C}=\mathcal{I}_\mathcal{C}$ has positive measure, then, by Corollary 2.4*(d)*, T is conservative, irreducible and exact, ending the proof of the Proposition.

Hence, let us assume that $\mathcal{D}=\mathcal{I}_\mathcal{D}=\mathbb{R}$. Using the notation of Section 3.2, let us see what implications this has on the dynamics of T_o.

We denote by $\mathcal{C}_o$ the conservative part of T_o, and by $\mathcal{D}^o_{\pm\infty}$ the sets defined in Theorem 2.11, relative to T_o. By Proposition 3.1, one of these sets is the whole $\mathbb{R}$ and T_o is exact. If $\mathbb{R}=\mathcal{C}_o$, ergodicity and conservativity entail that the forward T_o-orbit of a.e. $x\in\mathbb{R}$ intersects $B:=\bigsqcup_{j=-k_o}^{k_o}I_j$. This occurs in particular for a.e. $x\in\mathbb{R}\setminus B$, where T_o and T coincide, implying that the forward T-orbits of a.a. $x\in\mathbb{R}$ accumulate in B. This conclusion contradicts the dissipativity of T, i.e., $\mathbb{R}=\mathcal{I}_\mathcal{D}$.

Therefore, either $\mathcal{D}^o_{+\infty}$ or $\mathcal{D}^o_{-\infty}$ has full measure. Suppose, w.l.g., that it is $\mathcal{D}^o_{+\infty}$. We want to prove that the same occurs for $\mathcal{D}_{+\infty}$, equivalently, $m(\mathcal{D}_{-\infty})=0$. Assume the contrary and define, for $\ell\in\mathbb{N}$,

$$A_\ell:=\left\{x\in\mathcal{D}_{-\infty}\;\middle|\;T^n(x)\in\bigsqcup_{j<-k_o}I_j,\ \forall n\ge\ell\right\}. \tag{5.25}$$

Clearly, $A_\ell\subseteq A_{\ell+1}$ and $\bigcup_{\ell\in\mathbb{N}}A_\ell=\mathcal{D}_{-\infty}$. Thus, $\exists\ell$ such that $m(A_\ell)>0$. On the other hand, $T^\ell A_\ell\subseteq\mathcal{D}^o_{-\infty}$, because $T^\ell A_\ell\subseteq T^\ell\mathcal{D}_{-\infty}=\mathcal{D}_{-\infty}$ and $T^n(x)=T_o^n(x)$, $\forall n\ge\ell$ (as all such points lie outside of B). The non-singularity of T gives $m(\mathcal{D}^o_{-\infty})\ge m(T^\ell A_\ell)>0$, which is a contradiction. Thus, our assumption was wrong and $\mathcal{D}_{+\infty}=\mathbb{R}$ mod m. Finally, T is exact by Theorem 2.11. Q.E.D.

As promised in Section 3.2, we present here a stronger version of Proposition 3.2, together with its proof.

Proposition 5.7 *Let T be a finite modification of the quasi-lift T_o. Denote by μ_o the σ-invariant measure preserved by T_o, and by ϕ_o discrete displacement function* *for T_o, as introduced in Section 3.1. Suppose that T verifies (A1)-(A7) and preserves a Lebesgue-equivalent measure μ with the following properties:*

(i) $\exists \theta_2 > 0$ *such that* $\mu(I_j) \le \theta_2$, $\forall j \in \mathbb{Z}$;

(ii) $\overline{\mu} = \overline{\mu}_o$ *(Definition 4.4).*

Then $\mathbb{R}$ equals $\mathcal{C}$, $\mathcal{D}_{+\infty}$, or $\mathcal{D}_{-\infty}$, depending on $\mathbb{E}_{\mu_o}(\phi_o)$, the drift of T_o, being, respectively, zero, positive, or negative.

Observe that, unlike Proposition 3.1, the case $\mathbb{R} = \mathcal{C}$ does not guarantee that T is exact (cf. Countexample 3 of Appendix A.1). But, if T is also Markov-indecomposable, exactness holds by Corollary 2.4*(d)*. This shows how Proposition 3.2 is a corollary of Proposition 5.7.

Proof of Proposition 5.7. We use notation and several arguments from the proof of Proposition 3.2.

If $\mathbb{E}_{\mu_o}(\phi_o) = 0$, that is, $\mathcal{C}_o = \mathbb{R}$, the part of the previous proof that shows that a.a. orbits accumulate in B still holds. Since $\mathcal{I}_{\mathcal{CD}}$ is null by the invariance of μ, it must be $\mathcal{C} = \mathbb{R}$.

If $\mathbb{E}_{\mu_o}(\phi_o) > 0$, namely, $\mathcal{D}^o_{+\infty} = \mathbb{R}$, the argument given earlier whereby $m(\mathcal{D}_{-\infty}) = 0$ continues to work. But $m(\mathcal{D}_{+\infty}) > 0$, because $\mathcal{D}_{+\infty}$ coincides with $\mathcal{D}^o_{+\infty} = \mathbb{R}$ on a large set on the "right end" of $\mathbb{R}$. In order to prove that $\mathcal{D}_{+\infty}$ has full measure, we need to verify that $m(\mathcal{C}) = 0$.

Suppose instead that $m(\mathcal{C}) > 0$. We show that $\exists k_1 \in \mathbb{Z}$ such that

$$\mathcal{C} = \bigsqcup_{j<k_1} I_j, \quad \mathcal{D}_{+\infty} = \bigsqcup_{j \ge k_1} I_j. \tag{5.26}$$

Recall that $\mathcal{C} = \mathcal{I}_{\mathcal{C}}$ is $\mathscr{M}$-measurable by Corollary 2.4*(a)*. If (5.26) does not hold, there exist $j_2 \in \mathbb{Z}$ with $I_{j_2} \subset \mathcal{C}$, and a positive-measure set of $x \in \mathcal{D}_{+\infty}$ such that x lies to the left of I_{j_2} and $T(x)$ lies to the right of I_{j_2}. Therefore, $\exists j_1 < j_2$ such that a positive-measure subset of such x belong in I_{j_1}. Since $\mathcal{D}_{+\infty} = \mathbb{R} \setminus \mathcal{C}$ is also $\mathscr{M}$-measurable, $I_{j_1} \subseteq \mathcal{D}_{+\infty}$, whence $TI_{j_1} \subseteq \mathcal{D}_{+\infty}$. But, for any $x \in I_{j_1}$ with the properties stated earlier, (A7) shows that both x and $T(x)$ belong in TI_{j_1}. Since TI_{j_1} is an interval, it must include I_{j_2} too, which is absurd, because $I_{j_2} \subset \mathcal{C}$. Thus (5.26) is established.

For all $\ell > k_1$, set $B_\ell := \bigsqcup_{j=k_1}^{\ell-1} I_j$. The invariance of μ gives

$$\mu(B_\ell \setminus T^{-1}B_\ell) = \mu(T^{-1}B_\ell \setminus B_\ell). \tag{5.27}$$

The above l.h.s. comprises all the points that leave B_ℓ in one iteration of T; the r.h.s. comprises all the points that enter B_ℓ in one iteration of T. Now, recall the meaning of $k_{1,0}, k_{2,0}$ from (A2), and choose a sufficiently large $k_2 > k_1$ so that T and T_o

coincide on $\bigsqcup_{j \ge k_2 - k_{2,0}} I_j$. From now on, we restrict to $\ell \ge k_2$. Since $\bigsqcup_{j \ge k_1} I_j = \mathcal{D}_{+\infty}$ is T-invariant, points can only leave or enter B_ℓ "through its right end". In formula:

$$B_\ell \setminus T^{-1} B_\ell = \bigsqcup_{i=1}^{k_{2,0}} \bigsqcup_{j=0}^{k_{2,0}-i} I_{\ell-i} \cap T^{-1} I_{\ell+j}; \tag{5.28}$$

$$T^{-1} B_\ell \setminus B_\ell = \bigsqcup_{i=0}^{|k_{1,0}|-1} \bigsqcup_{j=1}^{|k_{1,0}|-i} I_{\ell+i} \cap T^{-1} I_{\ell-j}. \tag{5.29}$$

We can replace T with T_o in the above r.h.sides, as already observed. As ℓ varies, the resulting intervals are translation of each other and it is relatively straightforward to evaluate their μ_o-measures: translate the r.h.sides of (5.28) and (5.29) via the maps $\sigma^{-\ell+i}$ and $\sigma^{-\ell-i}$, respectively. This yields:

$$\mu_o(B_\ell \setminus T^{-1} B_\ell) = \sum_{k=1}^{k_{2,0}} k \, \mu_o(I_0 \cap T_o^{-1} I_k); \tag{5.30}$$

$$\mu_o(T^{-1} B_\ell \setminus B_\ell) = \sum_{k=1}^{|k_{1,0}|} k \, \mu_o(I_0 \cap T_o^{-1} I_{-k}). \tag{5.31}$$

Therefore, for all $\ell \ge k_2$,

$$\begin{aligned} \mu_o(B_\ell \setminus T^{-1} B_\ell) - \mu_o(T^{-1} B_\ell \setminus B_\ell) &= \sum_{k=-k_{1,0}}^{k_{2,0}} k \, \mu_o(I_0 \cap T_o^{-1} I_k) \\ &= \mu_o(I_0) \, \mathbb{E}_{\mu_o}(\phi_o) > 0, \end{aligned} \tag{5.32}$$

as we have assumed.

Now consider the function

$$F := \sum_{j \in \mathbb{Z}} \left(1_{B_{k_2} \setminus T^{-1} B_{k_2}} - 1_{T^{-1} B_{k_2} \setminus B_{k_2}} \right) \circ \sigma^j. \tag{5.33}$$

Once again, observe that there is no harm in replacing T with T_o in the above definition. Since F is bounded and periodic, it is a global observable. For $k \ge k_2$, set $V_k := \bigsqcup_{\ell=k_2}^{k} I_\ell = [ak_2, a(k+1)]$. It is easily verified that

$$F 1_{V_k} - \sum_{\ell=k_2}^{k} \left(1_{B_\ell \setminus T^{-1} B_\ell} - 1_{T^{-1} B_\ell \setminus B_\ell} \right) \tag{5.34}$$

is a compactly supported bounded function whose L^∞-norm is constant in k and whose support, though varying with k, is always contained in at most $2(|k_{1,0}| + k_{2,0})$ Markov intervals. Therefore, by the hypothesis *(i)*, the integral of (5.34) is uniformly bounded in k. But it is a requirement of Definition 4.4 that μ be an infinite measure, whence

$$\lim_{k\to+\infty} \mu_{V_k}(F) = \lim_{k\to+\infty} \frac{1}{\mu(V_k)} \sum_{\ell=k_2}^{k} \big(\mu(B_\ell \setminus T^{-1}B_\ell) - \mu(T^{-1}B_\ell \setminus B_\ell) \big) = 0, \tag{5.35}$$

the last equality coming from (5.27). On the other hand, via the periodicity of μ_o and F, and using (5.32),

$$\lim_{k\to+\infty} (\mu_o)_{V_k}(F) = \frac{1}{\mu_o(I_0)} \big(\mu_o(B_{k_2} \setminus T^{-1}B_{k_2}) - \mu_o(T^{-1}B_{k_2} \setminus B_{k_2}) \big) > 0. \tag{5.36}$$

The hypothesis *(ii)* implies that the l.h.sides of (5.35) and (5.36) are the same, which is a contradiction. Therefore, the supposition $m(\mathcal{C}) > 0$ was wrong, proving that $\mathcal{D}_{+\infty} = \mathbb{R}$.

Analogously, $\mathbb{E}_{\mu_o}(\phi_o) < 0$ gives $\mathcal{D}_{-\infty} = \mathbb{R}$. Q.E.D.

5.3 Infinite mixing

Proof of Theorem 4.3. Assertion *(a)* comes from Proposition 4.1, because T is exact by Proposition 3.1.

As for *(b)*, fix $F \in \mathcal{G}_1$. Using definition (4.12), one verifies that, for all $k \in \mathbb{Z}^+$,

$$\frac{\mu((F \circ T^n) 1_{[0,ak]})}{\mu([0,ak])} = \int_{I_0} (\mathcal{A}_k F) \circ T^n \, d\mu_{I_0}. \tag{5.37}$$

This can be seen by writing $1_{[0,ak]} = \sum_{j=0}^{k-1} 1_{I_0} \circ \sigma^{-j}$ and using the commutativity of T and σ, which are both μ-invariant. By definition of $\mathcal{G}_1$, for every $\varepsilon > 0$, there exists a large enough k such that, for all $n \in \mathbb{N}$,

$$\left| \int_{I_0} (\mathcal{A}_k F) \circ T^n \, d\mu_{I_0} - \int_{I_0} F_a \circ T^n \, d\mu_{I_0} \right| \le \varepsilon. \tag{5.38}$$

Recalling from Section 3.1 the definition of the measure-preserving dynamical system $(\mathbb{S}_a, \mu_a, T_a)$, we see that, since F_a is a-periodic,

$$\begin{aligned} \int_{I_0} F_a \circ T^n \, d\mu_{I_0} &= \frac{1}{\mu(I_0)} \int_{\mathbb{S}_a} F_a \circ T_a^n \, d\mu_a \\ &= \frac{1}{\mu(I_0)} \int_{\mathbb{S}_a} F_a \, d\mu_a \\ &= \int_{I_0} F_a \, d\mu_{I_0} = \overline{\mu}(F), \end{aligned} \tag{5.39}$$

as per definition (4.16) (with the slight abuse of notation whereby the projection of F_a to $\mathbb{S}_a \cong I_0$ is still called F_a).

The following lemma is an easy consequence of the exactness of T.

Lemma 5.8 *If, for some $b \in \mathbb{C}$ and $\varepsilon \geq 0$, the limit*

$$\limsup_{n\to\infty} \left| \frac{\mu((F \circ T^n)g)}{\mu(g)} - b \right| \leq \varepsilon$$

holds for some $g \in L^1$, with $\mu(g) \neq 0$, then it holds for all $g \in L^1$, with $\mu(g) \neq 0$.

PROOF. See [L7, Lem. 3.6].

Equations (5.37)-(5.39) show that the lemma can be applied, for any $\varepsilon > 0$, with $b = \overline{\mu}(F)$ and $g = 1_{[0,ak]}$, with k depending on ε. Therefore, for all g with $\mu(g) \neq 0$,

$$\lim_{n\to\infty} |\mu((F \circ T^n)g) - \overline{\mu}(F)\mu(g)| = 0. \tag{5.40}$$

The case $\mu(g) = 0$ was already covered when we proved **(GLM1)**. This ends the proof of *(b)*.

By simple density arguments, **(GGM2)** will be verified once the limit (GGM2) is proved for any pair of quasiperiodic observables F, G. More precisely, assume $F \circ \sigma = e^{\imath a\beta} F$ and $G \circ \sigma = e^{\imath a\gamma} G$. Notice that, if $\beta \neq 0 \bmod 2\pi/a$, then $\overline{\mu}(F) = 0$. The analogous implication holds for G.

Set $g = G1_{I_0}/\mu(I_0)$. Since $V = [ak, a(\ell+1)] = \bigsqcup_{j=k}^{\ell} I_j$, we can write

$$G1_V = \sum_{j=k}^{\ell} G1_{I_j} = \sum_{j=k}^{\ell} ((G \circ \sigma^j)1_{I_0}) \circ \sigma^{-j} = \mu(I_0) \sum_{j=k}^{\ell} e^{\imath a\gamma j} g \circ \sigma^{-j}, \tag{5.41}$$

whence, using the quasi-periodicity of F, the commutativity of T and σ, and the σ-invariance of μ,

$$\begin{aligned} \mu_V((F \circ T^n)G) &= \frac{1}{(\ell - k + 1)\mu(I_0)} \int_{\mathbb{R}} (F \circ T^n) G1_V \, d\mu \\ &= \frac{1}{\ell - k + 1} \sum_{j=k}^{\ell} \int_{\mathbb{R}} (F \circ T^n) e^{\imath a\gamma j} (g \circ \sigma^{-j}) \, d\mu \\ &= \frac{\sum_{j=k}^{\ell} e^{\imath a(\beta+\gamma)j}}{\ell - k + 1} \int_{\mathbb{R}} (F \circ T^n) g \, d\mu. \end{aligned} \tag{5.42}$$

In the last term above, the factor in front of the integral is bounded by 1 uniformly in k, ℓ, namely, in V, while the integral does not depend on it. In fact, the latter term is $\mu((F \circ T^n)g)$ and, by **(GLM2)**, converges to $\overline{\mu}(F)\mu(g)$, as $n \to \infty$ (since $\mathcal{G}_2 \subset \mathcal{G}_1$).

We now have three cases:

1. $\underline{\beta \neq 0 \bmod 2\pi/a}$. In this case $\overline{\mu}(F) = 0$, therefore (5.42) converges to 0, as $n \to \infty$, uniformly in V. In particular, it converges to 0 in the joint infinite-volume and time limit; cf. (4.6).

2. $\beta = 0 \bmod 2\pi/a$ and $\gamma \neq 0 \bmod 2\pi/a$. In this case $\overline{\mu}(G) = 0$ and the factor in front of the integral in (5.42) vanishes when $\ell - k \to \infty$; that is, uniformly in V as $\mu(V) \to \infty$, i.e., in the infinite-volume limit. On the other hand, the integral is bounded by $\|F\|_\infty \|g\|_1$, uniformly in n. This implies that, again, in the joint infinite-volume and time limit, (5.42) converges to 0.

3. Both β and γ are 0 mod $2\pi/a$. In this case the factor in front of the integral is identically 1, (5.42) no longer depends on V and, for $n \to \infty$, tends to $\overline{\mu}(F)\mu(g) = \overline{\mu}(F)\overline{\mu}(G)$, which is the same as the joint infinite-volume and time limit, here.

In all these cases, the limit (GGM2) is verified.

In view of Proposition 4.2, **(GGM1)** will be shown once we have proved that $\overline{\mu}((F \circ T^n)G)$ exists for all $F, G \in \mathcal{G}_2$ and $n \in \mathbb{N}$. Once again, since $\overline{\mu}$ is a continuous functional in the L^∞-norm, it is enough to prove the assertion for F, G quasiperiodic. But, in that case, $F \circ T^n$ is quasiperiodic by (3.1), which implies the same for $(F \circ T^n)G$, which thus has an infinite-volume average. This completes the proof of assertion *(c)* of Theorem 4.3. Q.E.D.

Proof of Proposition 4.5. As in the previous proof, assertion *(a)* comes from Propositions 3.2 and 4.1.

Now for *(b)*. Consider $F, G \in \mathcal{G}_2$ and fix $n \in \mathbb{N}$. Since T^n is a finite modification of T_o^n, as emphasized in Section 3.2, the function $(F \circ T^n)G$ differs from $(F \circ T_o^n)G$ by a compactly supported and bounded function of $\mathbb{R}$. This shows that $\overline{\mu}_o((F \circ T^n)G)$ exists if and only if $\overline{\mu}_o((F \circ T_o^n)G)$ does, and they are equal. Theorem 4.3*(c)* then implies that

$$\lim_{n\to\infty} \overline{\mu}_o((F \circ T^n)G) = \overline{\mu}_o(F)\overline{\mu}_o(G), \tag{5.43}$$

for all $F, G \in \mathcal{G}_2$. By the hypothesis on μ, the above holds as well with $\overline{\mu}$ in place of $\overline{\mu}_o$, ending the proof of *(b)*. Q.E.D.

5.4 The example of the random walk

Proof of Proposition 4.8. The irreducibility of T is apparent from the expression of $\mathcal{Q}$, see (4.17) and (3.8). The exactness then comes from Proposition 3.2. Moreover, the proof of Proposition 3.2 (in Section 5.2) shows that, under its hypotheses, a finite modification of a conservative map is conservative. In the case at hand, T_o is clearly conservative, whereby T is as well.

Thus, Proposition 4.8 will be proved once the limit (GLM2) is proved for any $F \in \mathcal{G}'$ and some $g \in L^1$, with $m(g) \neq 0$; see Lemma 5.8 *et seq.* We take g to be of the form

$$g_\pi := \sum_{j\in\mathbb{Z}} \pi_j 1_{I_j}, \tag{5.44}$$

where $\pi := (\pi_j)_{j \in \mathbb{Z}}$ is a symmetric (i.e., $\pi_j = \pi_{-j}$, $\forall j \in \mathbb{Z}$) and half-monotonic (i.e., $\pi_j \geq \pi_{j+1}$, $\forall j \in \mathbb{N}$) stochastic vector on $\mathbb{Z}$. In particular, g_π is a density w.r.t. the Lebesgue measure, namely, $g_\pi \geq 0$ and $m(g_\pi) = 1$.

The following lemma (which extends [BCLL, Lem. 7] to the non-homogeneous random walk at hand) states in particular that the set of densities thus constructed is closed under the action of the dynamics, that is, under the action of the Perron-Frobenius operator P introduced in (3.9)-(3.11).

Lemma 5.9 *If π is a symmetric and half-monotonic stochastic vector on $\mathbb{Z}$ and g_π is its corresponding density on $\mathbb{R}$ via (5.44), then $P^n g_\pi = g_{\pi^{(n)}}$, where $\pi^{(n)} := \pi \mathcal{Q}^n$ is the evolution at time n of the initial state π for the random walk described above. Moreover, $\pi^{(n)}$ is symmetric and half-monotonic.*

Proof of Lemma 5.9. We prove all the assertions for $n = 1$ and the lemma will follow by induction. For the sake of the notation, let us denote $\pi' := \pi^{(1)} = \pi \mathcal{Q}$.

By (3.11) and (5.44), for all $x \in (k, k+1)$, $(P g_\pi)(x) = \sum_j \pi_j q_{jk} =: \pi'_k$. This means that $P g_\pi = g_{\pi'}$. Also, by the symmetry properties of π and $\mathcal{Q}$,

$$\pi'_{-k} = \sum_j \pi_j q_{j,-k} = \sum_j \pi_{-j} q_{j,-k} = \sum_j \pi_j q_{-j,-k} = \sum_j \pi_j q_{jk} = \pi'_k. \tag{5.45}$$

Finally, using both the symmetry and the half-monotonicity of π,

$$\begin{aligned} \pi'_0 - \pi'_1 &= [(\pi_{-2} + \pi_{-1} + 5\pi_0 + \pi_1 + \pi_2) - (\pi_0 + 5\pi_1 + 2\pi_2 + \pi_3)]/9 \\ &= (4\pi_0 - 3\pi_1 - \pi_3)/9 > 0; \end{aligned} \tag{5.46}$$

$$\begin{aligned} \pi'_1 - \pi'_2 &= [(\pi_0 + 5\pi_1 + 2\pi_2 + \pi_3) - (\pi_0 + 2\pi_1 + 3\pi_2 + 2\pi_3 + \pi_4)]/9 \\ &= (3\pi_1 - \pi_2 - \pi_3 - \pi_4)/9 > 0; \end{aligned} \tag{5.47}$$

and, for $k \geq 2$,

$$\begin{aligned} \pi'_k - \pi'_{k+1} &= [(\pi_{k-2} + 2\pi_{k-1} + 3\pi_k + 2\pi_{k+1} + \pi_{k+2}) \\ &\quad - (\pi_{k-1} + 2\pi_k + 3\pi_{k+1} + 2\pi_{k+2} + \pi_{k+3})]/9 \\ &= (\pi_{k-2} + \pi_{k-1} + \pi_k - \pi_{k+1} - \pi_{k+2} - \pi_{k+3})/9 > 0. \end{aligned} \tag{5.48}$$

Therefore π' is decreasing on $\mathbb{N}$ and the proposition is proved. Q.E.D.

Now for the core argument. Without loss of generality we assume that $\overline{m}'(F) = 0$ (for (GLM2) is trivial when F is a constant). Set $f_j := \int_j^{j+1} F \, dm$. The assumption implies that $\forall \varepsilon > 0$, $\exists \ell \in \mathbb{N}$ such that, $\forall k \geq \ell$,

$$\frac{1}{2k+1} \left| \sum_{j=-k}^{k} f_j \right| = \left| m_{[-k,k+1]}(F) \right| \leq \frac{\varepsilon}{2}. \tag{5.49}$$

By (3.9) and Lemma 5.9 we have

$$
\begin{aligned}
\int_{\mathbb{R}} (F \circ T^n) g_\pi \, dm &= \int_{\mathbb{R}} F g_{\pi^{(n)}} \, dm \\
&= \sum_{j \in \mathbb{Z}} f_j \, \pi_j^{(n)} \\
&= \sum_{k=0}^{\infty} \left(\pi_k^{(n)} - \pi_{k+1}^{(n)} \right) \sum_{j=-k}^{k} f_j \\
&=: S_\ell + S'_\ell,
\end{aligned}
\tag{5.50}
$$

where S_ℓ and S'_ℓ correspond to restricting the outer summation to $\sum_{k=0}^{\ell-1}$ and $\sum_{k=\ell}^{\infty}$, respectively. Observe that the third equality of (5.50) comes from disintegrating the density $(\pi_j^{(n)})_j$ in "horizontal slices" of width $2k+1$ and height $\pi_k^{(n)} - \pi_{k+1}^{(n)}$.

By (5.49) we obtain

$$
\begin{aligned}
|S'_\ell| &\le \sum_{k=\ell}^{\infty} \left(\pi_k^{(n)} - \pi_{k+1}^{(n)} \right) \left| \sum_{j=-k}^{k} f_j \right| \\
&\le \frac{\varepsilon}{2} \sum_{k=\ell}^{\infty} \left(\pi_k^{(n)} - \pi_{k+1}^{(n)} \right) (2k+1) \le \frac{\varepsilon}{2},
\end{aligned}
\tag{5.51}
$$

because $\sum_{k \in \mathbb{N}} (\pi_k^{(n)} - \pi_{k+1}^{(n)})(2k+1) = \sum_{j \in \mathbb{Z}} \pi_j^{(n)} = 1$, as in (5.50).

To estimate S_ℓ we need a property of the dynamical system which is an easy consequence of its exactness.

Lemma 5.10 *For all $f \in L^\infty \cap L^1$ and $g \in L^1$,*

$$
\lim_{n \to \infty} m((f \circ T^n) g) = \lim_{n \to \infty} m(f (P^n g)) = 0.
$$

Proof. See [L7, Thm. 3.5*(b)*]. (In our terminology, see [L7], the above notion is called *local-local mixing*, or **(LLM)**, and is easily seen to be equivalent to the zero-type property of [HK].)

Let us apply Lemma 5.10 with $f = 1_{[-\ell+1, \ell]}$ and $g = g_\pi$. There exists $N \in \mathbb{N}$ such that, $\forall n \ge N$,

$$
\sum_{j=-\ell+1}^{\ell-1} \pi_j^{(n)} = \int_{-\ell+1}^{\ell} g_{\pi^{(n)}} \, dm = \int_{-\ell+1}^{\ell} P^n g_\pi \, dm \le \frac{\varepsilon}{2 \|F\|_\infty}. \tag{5.52}
$$

Therefore

$$
\begin{aligned}
|S_\ell| &\le \sum_{k=0}^{\ell-1} \left(\pi_k^{(n)} - \pi_{k+1}^{(n)} \right) \left| \sum_{j=-k}^{k} f_j \right| \\
&\le \|F\|_\infty \sum_{k=0}^{\ell-1} \left(\pi_k^{(n)} - \pi_{k+1}^{(n)} \right) (2k+1) \qquad (5.53) \\
&\le \|F\|_\infty \sum_{j=-\ell+1}^{\ell-1} \pi_j^{(n)} \le \frac{\varepsilon}{2}.
\end{aligned}
$$

Since N is chosen depending on ℓ, which in turns depends on ε, (5.50), (5.51) and (5.53) prove the assertion. Q.E.D.

A Appendix

A.1 The importance of certain assumptions

In this section we present some examples—or rather counterexamples—of maps which clarify the role of some of our less obvious assumptions. We refer in particular to Theorem 2.11, which describes the exact components of $\mathcal{I}_{\mathcal{D}}$, and Propositions 3.2 and 5.7, which apply to finite modifications of quasi-lifts. All the maps we present are Markov maps associated to random walks.

Counterexample 1: With reference to Theorem 2.11, this example illustrates the relevance of (A7) for the exactness properties of T on $\mathcal{I}_{\mathcal{D}}$.

Let T be the map associated to the random walk with the following transition probabilities:

$$
\begin{aligned}
\forall j \le -1, &\quad q_{j,j-2} = 1/2, \ q_{j,j} = q_{j,j+2} = 1/4; \\
\forall j \in \{0,1\}, &\quad q_{j,-2} = q_{j,-1} = q_{j,0} = q_{j,1} = q_{j,2} = q_{j,3} = 1/6; \\
\forall j \ge 2, &\quad q_{j,j-2} = q_{j,j} = 1/4, \ q_{j,j+2} = 1/2.
\end{aligned}
\qquad (A.1)
$$

All other q_{jk} are necessarily null. As seen in Section 3.3, T verifies (A1)-(A6). It is also irreducible.

Denote $\mathbb{R}_{\mathrm{even}} := \bigsqcup_{j\in\mathbb{Z}} [2j, 2j+1)$ and $\mathbb{R}_{\mathrm{odd}} := \mathbb{R} \setminus \mathbb{R}_{\mathrm{even}}$. For $\epsilon \in \{\mathrm{even}, \mathrm{odd}\}$ indicate with $\mathcal{D}^{\epsilon}_{\pm\infty}$ the set of all $x \in \mathbb{R}$ such that $\lim_{n\to\infty} T^n(x) \to \pm\infty$ and $T^n(x) \in \mathbb{R}_\epsilon$ for all sufficiently large n. This defines four invariant sets. Once we prove that all of them have infinite Lebesgue measure, we have shown that $\mathcal{I}_{\mathcal{D}}$ has at least four ergodic components. This demonstrates that, if (A1)-(A6) hold but (A7) does not, the last assertion of Theorem 2.11 fails, even for an irreducible T.

So let us verify that $m(\mathcal{D}^{\mathrm{even}}_{+\infty}) = \infty$, the proof for the other sets being analogous. Consider the map T_1 corresponding to the homogeneous random walk $q_{j,j-2} = q_{j,j} =$

$1/4$, $q_{j,j+2} = 1/2$, $\forall j \in \mathbb{Z}$. Clearly $T_1^{-1}\mathbb{R}_{\text{even}} = \mathbb{R}_{\text{even}}$ and, by the transience of the random walk, $\lim_{n\to\infty} T_1^n(x) \to +\infty$ for all x. This implies that there is an infinite-measure set of $x \in \mathbb{R}_{\text{even}}$ such that $T_1^n(x) \geq 2$, $\forall n \in \mathbb{N}$. Since T and T_1 coincide on $\mathbb{R}_{\geq 2}$, all such x belong to $\mathcal{D}_{+\infty}^{\text{even}}$, which therefore has infinite Lebesgue measure. (A more refined analysis, using the arguments of Section 5.1, would show that $\mathcal{C} = \mathcal{I}_{\mathcal{CD}} = \varnothing$, and the four sets $\{\mathcal{D}_{\pm\infty}^{\epsilon}\}$ are the exact components of T.)

Counterexample 2: The assumption (A5) is even more important than (A7) in Theorem 2.11. The following example shows that if (A5) does not hold—even if all other assumptions do—the tail σ-algebra of a Markov map can be very large, making the system very far from being exact.

Consider the map T representing the following random walk:

$$\begin{array}{ll} \forall j \leq 1, & q_{j,j} = q_{j,j+1} = 1/2; \\ \forall j \geq 2, & q_{j,j} = q_{j,j+1} = \ldots = q_{j,2^{j^2}-1} = 1/(2^{j^2} - j). \end{array} \tag{A.2}$$

In other words,

$$T(x) = \begin{cases} j + 2(x - j), & x \in [j, j+1), \quad j \leq 1; \\ j + (2^{j^2} - j)(x - j), & x \in [j, j+1), \quad j \geq 2. \end{cases} \tag{A.3}$$

The intervals $I_j = [j, j+1]$ provide an alternative Markov partition for T—see Remark 3.5—relative to which the system verifies (A1)-(A4), (A6)-(A7). Evidently, it does not verify (A5).

Using the notation of (A2), let us denote by τ_j the branch of T over I_j, which is expressed by the r.h.s. of (A.3). For $j \geq 3$, set

$$X_j := \tau_j^{-1}\left([2^{(j-1)^2}, 2^{j^2}) \right) = \left[j + \frac{2^{(j-1)^2} - j}{2^{j^2} - j}, j+1 \right) \subset I_j. \tag{A.4}$$

Define also $X := \bigsqcup_{j \geq 3} X_j$ and $Y := \bigcap_{n \geq 0} T^{-n} X$. By construction, $T|_X$ is a bijection $X \longrightarrow \mathbb{R}_{\geq 16}$, hence $T|_Y$ is an invertible self-map of Y. See Fig. 5.

Lemma A.1 $m(Y) > 0$.

Proof. It will suffice to show that $m(I_3 \cap Y) > 0$. Set $X'_j = I_j \setminus X_j$ and $X' := \bigsqcup_{j \geq 3} X'_j$. By (A.4),

$$m(X'_j) < \frac{2^{(j-1)^2} - j}{2^{j^2} - j} < \frac{2^{(j-1)^2}}{2^{j^2}} = 2^{-2j+1}. \tag{A.5}$$

For every $n \geq 1$, T^n acts as a piecewise linear bijection $I_3 \cap \bigcap_{i=0}^{n-1} T^{-i} X \longrightarrow [k_n, \ell_n + 1)$, for some $k_n, \ell_n \geq 3$. Let us call this map L_n. We have:

$$I_3 \cap Y = X_3 \cap \bigcap_{n=1}^{\infty} T^{-n} X = X_3 \cap \bigcap_{n=1}^{\infty} L_n^{-1}(X \cap [k_n, \ell_n + 1)). \tag{A.6}$$

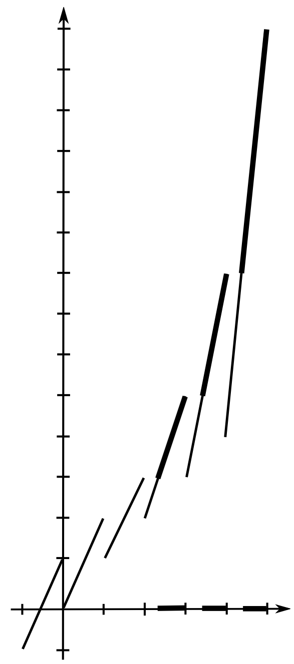

Figure 5: A rough sketch of the map T of Counterexample 2. The bold segments on the abscissa indicate the set X. The bold parts of the graph of T represent $T|_X$, which is invertible.

The integers k_n, ℓ_n can be calculated recursively from the definition of X_j. For example, $k_1 = 2^4$, $k_2 = 2^9 - 1$, $k_2 = 2^{(2^4-1)^2}$, $k_2 = 2^{(2^9-1)^2} - 1$, etc. A very generous lower bound for k_n is n. In view of (A.5)-(A.6), the complementary set of Y, w.r.t. I_3, measures

$$
\begin{aligned}
m\left(I_3 \cap \bigcup_{n=0}^{\infty} T^{-n} X' \right) &\le m(X'_3) + \sum_{n=1}^{\infty} m\big(L_n^{-1} (X' \cap [k_n, \ell_n + 1)) \big) \\
&= \frac{13}{509} + \sum_{n=1}^{\infty} \sum_{j=k_n}^{\ell_n} m(L_n^{-1} X'_j).
\end{aligned} \tag{A.7}
$$

On the other hand,

$$\begin{aligned}\sum_{j=k_n}^{\ell_n} m(L_n^{-1}X'_j) &= \sum_{j=k_n}^{\ell_n} m(L_n^{-1}X'_j \,|\, L_n^{-1}I_j)\, m(L_n^{-1}I_j) \\ &= \sum_{j=k_n}^{\ell_n} m(X'_j \,|\, I_j)\, m(L_n^{-1}I_j) \\ &< \sum_{j=k_n}^{\ell_n} 2^{-2j+1}\, m(L_n^{-1}I_j) \\ &< 2^{-2n+1}.\end{aligned} \tag{A.8}$$

In the above we have used, from top to bottom: the affinity of $L_n^{-1}|_{I_j}$, the inequality (A.5), the lower bound $n \le k_n \le j$, and the fact that $L_n^{-1}[k_n, \ell_n + 1) = I_3 \cap \bigcap_{i=0}^{n-1} T^{-i}X$, whose Lebesgue measure is less than 1.

The estimate (A.8) shows that the l.h.s. of (A.7) is less than 1, which is equivalent to $m(I_3 \cap Y) > 0$, as claimed. (Incidentally, by the same arguments as above, one proves that $m(Y \cap I_\ell) \ge c$, for some $c > 0$ and all $\ell \ge 3$, showing that, in fact, $m(Y) = \infty$.) Q.E.D.

The bijectivity of $T|_Y : Y \longrightarrow Y$ ensures that, for all $B \subseteq Y$ and all $n \in \mathbb{N}$, $B = T^{-n}T^nB$, implying that $B \in \mathscr{T}$, the tail σ-algebra of T defined in (5.6). Equivalently, using the notation introduced in Section 5.1, $\mathscr{T} \cap Y = \mathscr{B} \cap Y$. In more suggestive terms, T cannot lose memory about Y, as it is invertible there!

Counterexample 3: The next map is as much a counterexample for Proposition 3.2 as an example for Proposition 5.7. It is a finite modification of a quasi-lift which preserves the same measure as the quasi-lift, but fails to be exact because it is not Markov-indecomposable or, which is the same here, because it has more than one conservative ergodic component.

Let T correspond to the random walk given by:

$$\begin{aligned} & q_{-1,-2} = 1/3,\ q_{-1,-1} = 2/3; \\ & q_{0,0} = 2/3,\ q_{0,1} = 1/3; \\ \forall j \notin \{-1, 0\}, \quad & q_{j,j-1} = q_{j,j} = q_{j,j+1} = 1/3. \end{aligned} \tag{A.9}$$

T is a finite modification of a map T_o which is associated to a homogeneous random walk. The latter is thus a quasi-lift. By means of Proposition 3.6, one readily checks that both T and T_o preserve m. On the other hand, $T^{-1}\mathbb{R}^{\pm} = \mathbb{R}^{\pm}$, showing that T has at least two exact components. In fact, it is not hard to see that T is conservative and $E_1 := \mathbb{R}^+$, $E_2 := \mathbb{R}^-$ are the only two exact components. Since these sets are unions of Markov intervals, T is Markov-decomposable.

Counterexample 4: Finally, we show that the preservation of a measure equivalent to Lebesgue is also a crucial hypothesis for both Propositions 3.2 and 5.7.

Let T be given by this random walk:

$$\begin{aligned} & q_{-1,-2} = 1/3, \ q_{-1,-1} = 2/3; \\ & q_{1,1} = 2/3, \ q_{1,2} = 1/3; \\ \forall j \notin \{-1,1\}, \quad & q_{j,j-1} = q_{j,j} = q_{j,j+1} = 1/3. \end{aligned} \tag{A.10}$$

This is a finite modification of the same T_o as in the previous case. In this case, however, T does not preserve an m-equivalent measure, as can be seen, e.g., by the fact that $[0,1]$ has only one inverse branch, which contracts by a factor 3.

What happens is that $T|_{\mathbb{R}^-} : \mathbb{R}^- \longrightarrow \mathbb{R}^-$ and $T|_{\mathbb{R}_{\geq 1}} : \mathbb{R}_{\geq 1} \longrightarrow \mathbb{R}_{\geq 1}$ are exact. Also, the Markov interval $I_{0,-1} = [0,1/3]$ feeds $\mathbb{R}^-$; $I_{0,1} = [2/3,1]$ feeds $\mathbb{R}_{\geq 1}$; and $I_{0,0} = [1/3,2/3]$ feeds both $\mathbb{R}^-$ and $\mathbb{R}_{\geq 1}$. There are only two communicating classes, $\mathbb{Z}_1$ and $\mathbb{Z}_2$, which are terminal. The corresponding sets are, respectively, $M_1 = \mathbb{R}^-$ and $M_2 = \mathbb{R}_{\geq 1}$, with basins $E_1 = \mathbb{R}_{<1/2}$ and $E_2 = \mathbb{R}_{>1/2}$; cf. (2.2), (2.8)-(2.9). Lastly, T is Markov-indecomposable because $I_{0,0}$ feeds both M_1 and M_2.

In conclusion, T is not exact and $\mathbb{R} = \mathcal{I}_{\mathcal{CD}}$, contrary to both the statements of Proposition 3.2 and Proposition 5.7.

A.2 Distortion

In this section of the Appendix we prove a standard distortion result that is used in Section 5.1.

Lemma A.2 *Under the hypotheses of Section 2, and using the notation (5.1), there exists $D > 1$ such that, for all $n \geq 1$, $\boldsymbol{j} \in \mathbb{Z}^n$, $x, y \in I_{\boldsymbol{j}}^{(n)}$,*

$$D^{-1} \leq \frac{|(T^n)'(x)|}{|(T^n)'(y)|} \leq D.$$

Proof. For $0 \leq k \leq n$, set $x_k := T^k(x)$ and $y_k := T^k(y)$.

Convention. The notation (x_k, y_k) denotes both the interval (x_k, y_k), when $x_k < y_k$, and the interval (y_k, x_k), when $y_k < x_k$.

By definition of $I_{\boldsymbol{j}}^{(n)}$, see (5.1), we have

$$(x_k, y_k) \subset I_{(j_k, \ldots, j_{n-1})}, \tag{A.11}$$

so that T is twice differentiable on (x_k, y_k). Therefore,

$$\begin{aligned} \log \frac{|(T^n)'(x)|}{|(T^n)'(y)|} &= \sum_{k=0}^{n-1} \log \frac{|T'(x_k)|}{|T'(y_k)|} \\ &= \sum_{i=1}^{n} \left(\log |T'(\tau_{j_{i-1}}^{-1}(x_i))| - \log |T'(\tau_{j_{i-1}}^{-1}(y_i))| \right) \\ &= \sum_{i=1}^{n} \left| \frac{T''(\tau_{j_{i-1}}^{-1}(z_i))}{T'(\tau_{j_{i-1}}^{-1}(z_i))} \, \frac{1}{T'(\tau_{j_{i-1}}^{-1}(z_i))} \right| (x_i - y_i), \end{aligned} \tag{A.12}$$

where $z_i \in (x_i, y_i)$. By (A.11), using (A1) and (A3), we get $|x_i - y_i| \le \theta\lambda^{n-i}$, which, using (A4) as well, gives

$$\left| \log \frac{|(T^n)'(x)|}{|(T^n)'(y)|} \right| \le \sum_{i=1}^{n} \eta\, \theta\, \lambda^{n-i} \le \frac{\eta\, \theta}{1-\lambda}. \tag{A.13}$$

Renaming the rightmost term of (A.13) $\log D$ yields the assertion. Q.E.D.

Corollary A.3 *Let* $\boldsymbol{j} = (j_0, j_1, \ldots, j_n) \in \mathbb{Z}^{n+1}$ *be such that* $m(I_{\boldsymbol{j}}^{(n+1)}) > 0$. *If* $B \subset I_{\boldsymbol{j}}^{(n+1)}$, *then* $T^n B \subset I_{j_n}$ *and*

$$\frac{m(T^n B)}{m(I_{j_n})} \le D \, \frac{m(B)}{m(I_{\boldsymbol{j}}^{(n+1)})}.$$

Proof. By construction, $T^n I_{\boldsymbol{j}}^{(n+1)} = I_{j_n}$, giving the first assertion. As for the second,

$$\frac{m(T^n B)}{m(T^n I_{\boldsymbol{j}}^{(n+1)})} = \frac{\int_B |(T^n)'(x)|\, dx}{\int_{I_{\boldsymbol{j}}^{(n+1)}} |(T^n)'(x)|\, dx} \le \frac{\max_B |(T^n)'|}{\min_{I_{\boldsymbol{j}}^{(n+1)}} |(T^n)'|} \, \frac{m(B)}{m(I_{\boldsymbol{j}}^{(n+1)})}, \tag{A.14}$$

which, through Lemma A.2, yields the corollary. Q.E.D.

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