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This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

Published Version:
Orlandelli, E. (2021). Labelled calculi for quantified modal logics with definite descriptions. JOURNAL OF LOGIC AND COMPUTATION, 31(3), 923-946 [10.1093/logcom/exab018].

Availability:
This version is available at: https://hdl.handle.net/11585/819001 since: 2021-09-06

Published:
DOI: http://doi.org/10.1093/logcom/exab018

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(Article begins on next page)

12 October 2024

This is the final peer-reviewed accepted manuscript of:

Eugenio Orlandelli, Labelled calculi for quantified modal logics with definite descriptions, *Journal of Logic and Computation*, 2021

The final published version is available online at: https://doi.org/10.1093/logcom/exab018

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Labelled calculi for quantified modal logics with definite descriptions

Eugenio Orlandelli, *Department of Philosophy and Communication Studies, University of Bologna, via Zamboni 38, Bologna, Italy. Email: eugenio.orlandelli@unibo.it*

Abstract

We introduce labelled sequent calculi for quantified modal logics with definite descriptions. We prove that these calculi have the good structural properties of **G3**-style calculi. In particular, all rules are height-preserving invertible, weakening and contraction are height-preserving admissible and cut is syntactically admissible. Finally, we show that each calculus gives a proof-theoretic characterization of validity in the corresponding class of models.

1 Introduction

The proof-theoretic study of propositional modal logics is now a well-developed subject thanks to the introduction of generalizations of Gentzen-style sequent calculi. In particular, we have internal calculi – e.g., hypersequents [6] and nested sequents [2] – whose sequents are interpretable in the modal language, and we have external calculi – e.g., display calculi [6], labelled sequent calculi [21, 28], and annotated tableaux [3, 7] – whose sequents are not interpretable in the basic modal language. Nevertheless, with the only exception of labelled calculi [8, 19, 21, 23, 22, 28], the proof-theoretic study of quantified modal logics (QMLs) has remained rather underdeveloped, see [24] for some considerations on hypersequents and display calculi for QMLs. One interesting problem that is still open is that of presenting a satisfactory approach to the structural proof theory for QMLs with definite descriptions: the only cut-free calculi are the Gentzen-style calculi for QMLs with definite description *à la* Garson [13] that have been presented in [14].

Starting from our work in [23], we introduce labelled calculi for the QMLs with descriptions and λ-abstraction that are studied by Fitting and Mendelsohn [11]. We show that these calculi have good structural properties – all rules are height-preserving invertible, weakening and contraction are height-preserving admissible, and cut is (syntactically) admissible – and characterize validity in the appropriate semantic classes. In so doing we solve a problem left open in [14] where we read:

> [Fitting and Mendelsohn's one] is probably the most subtle theory of definite descriptions [...] As such it certainly deserves attention but it is difficult to provide a suitable sequent formalization of it. [p. 388]

This work was partially supported by the Academy of Finland, research project no. 1308664.
I am grateful to Sara Negri for valuable discussion of the ideas and results presented in this paper. Thanks are also due to two anonymous reviewers for their helpful comments.

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The rest of this introduction gives a quick survey of Fitting and Mendelsohn's QMLs with definite descriptions and explains why labelled calculi are the ideal formalism to study their structural proof theory.

As it is convincingly argued in [11], the need for non-rigid and non-denoting terms originates from problems already touched upon in the classical works of Frege [12] and Russell [25]. First, as Frege noticed, even if both 'the morning star' and 'the evening star' denote Venus and even if the ancient knew that objects are self-identical, the Babylonians did not know that 'the morning star is identical with the evening star'. Despite this, if we treat definite descriptions as genuine terms, in standard QMLs we can prove that the Babylonians knew it because terms are rigid designators. Moreover, Russell showed that the sentence 'The present king of France is not bald' is semantically ambiguous. The expression 'The present king of France' does not actually denote anyone. Hence, if we read 'The present king of France is not bald' as saying 'the object denoted by the expression "The present king of France" is such that he is bald', we have a false sentence. If, instead, we read it as saying 'it is false that there is an object that is denoted by the expression "The present king of France" and he is bald', we have a true sentence. Nevertheless, in standard QMLs we cannot express the second reading – i.e., the true one – where negation has wide scope over the definite description (unless we explain away terms expressing descriptions, as Russell did [25]).

As the two examples above show, if definite descriptions are taken as genuine terms, we must extend the language of QMLs with non-rigid and non-denoting terms, but this extension is not trivial, cf. [11]. The problem, roughly, is that if t is a non-rigid or non-denoting term, the formula $\circ Pt$ (for $\circ \in \{\Diamond, \neg\}$) becomes ambiguous. When it is evaluated in a possible world w of some model, the formula $\Diamond Pt$ might either mean that there is a world u that is accessible from w and such that Pt is true therein, or it might mean that the object denoted by t in w satisfies the unary predicate P is some world v that is accessible from w. Analogously, $\neg Pt$ might either mean that it is false that (in w) there is one and only object that is denoted by t and that satisfies P, or it might mean that the one and only object denoted by t (in w) does not satisfy P. For rigid and always denoting terms the two readings are equivalent. For non-rigid and non-denoting terms, instead, neither reading entails the other, and, therefore, we need some scoping mechanism to disambiguate the formulas $\Diamond Pt$ and $\neg Pt$. The solution adopted in [11] is that of extending the language with the operator of predicate abstraction λ. The two readings of $\circ Pt$ (for $\circ \in \{\Diamond, \neg\}$) can thus be expressed, respectively, by the (semantically independent) formulas:

$$\lambda x(\circ Px).t \qquad \text{and} \qquad \circ\,(\lambda x Px.t)$$

All in all, Russell's [25] (and Smullyan's [26]) proposal of explaining away definite descriptions by means of quantification and identity originates from the need to have a scoping mechanism for the formal representation of non-rigid and non-denoting terms. By using λ to abstract predicates from formulas we have a scoping mechanism for terms and, therefore, we don't need to explain them away. One essential feature of this approach is that non-rigid and non-denoting terms, such as definite descriptions, can occur in formulas only when their predication is mediated by the operator λ and not as one of the *relata* of an atomic formula: if t is a (possibly) non-rigid or non-denoting term, $\lambda x Px.t$ is a formula but Pt is not a formula. Otherwise, we would

have a problem in interpreting $\Diamond Pt$ (for t non-rigid) and $\neg Pt$ (for t non-denoting).

It is well-known that labelled calculi allow to give well-behaved sequent calculi for all first-order semantically definable propositional modal logic. The key idea is that of extending the language of sequent calculus in order to internalize relational semantics into the syntax: we add *world labels* (representing worlds) and *relational atoms* (representing the accessibility relation), and we replace modal formulas with *labelled modal formulas* (representing satisfaction at a world). This allows to give well-behaved rules for the modalities (that are just like rules for restricted first-order quantifiers). Moreover, thanks to the presence of relational atoms, it allows to use the method of axioms as rules [18, 20] to transform the first-order semantic conditions that define interesting modal logics into rules of the calculus. This can be done directly for coherent (aka geometric) semantic conditions (i.e., formulas of shape $\forall \bar{x}(A \supset B)$, where neither A nor B contains $\forall$ and $\supset$) and indirectly for non-coherent ones (via the method of coherentisation of arbitrary first-order formulas [10]).

As we will show, the strategy of internalizing the semantics works equally well for QMLs with definite descriptions. In particular, in order to internalize the semantics presented in [11] we will need to add to the labelled language also *denotation formulas* of the shape $D(t, x, w)$ that, when t is a definite description, express the non-trivial fact that t denotes one object in the world w. In this way we can easily define an external calculus for the QMLs with definite descriptions presented in [11]. Even if it is possible to define well-behaved internal calculi for some QMLs without definite descriptions [24] (e.g., by applying the embeddings given in [4, 5]), we believe that labelled calculi provides the best tool for the logics we are considering because it seems hard to define a calculus for them without using denotation formulas or some other extension of the language – e.g., annotated terms [3, 7] – that cannot be interpreted in the language of modal logic. In a nutshell, the problem is that something like denotation formulas are needed to cope with definite descriptions and the only way to interpret a denotation formula $D(t, x, w)$ in the object language (or to do without something like denotation formulas as in [14, 15]) is via an identity atom of shape $t = x$. But, if the modal language allows for formulas of the shape $t = x$ with t a definite description, then λ-abstraction looses its role of scoping mechanism and we run into problems with substitutivity of identicals, see [11, Chapter 10.1], and with cut-elimination for identity atoms [14, Section 5].

The paper is organized as follows. Section 2 sketches the labelled calculi for QMLs presented in [21]. In particular, the language and semantics of standard QMLs are introduced in Section 2.1, and labelled calculi for these logics are outlined in Section 2.2. In Section 3, QMLs with identity and definite descriptions are introduced. Different approaches to descriptions are briefly compared (Section 3.1) and the syntax and the semantics of the QMLs with definite descriptions presented in [11] are sketched (Section 3.2). Then, in Section 4, labelled calculi for these logics are introduced. Section 5 shows that these calculi have the good structural properties that are distinctive of **G3**-style calculi, and Section 6 shows that they are sound and complete with respect to the appropriate classes of quantified modal frames. We conclude in Section 7 by showing how the present approach can be extended to cover the quantified extensions of all first-order definable propositional modal logics and how it can simulate some other approaches to definite descriptions.

2 Quantified Modal Logics

In this section, we present QMLs based on a varying domain semantics defined, for simplicity and following [11, 21], over a signature not containing functions of any arity nor the identity symbol, and we present labelled calculi for these logics. Functions of any arity can be added without any problem (as long as they are interpreted rigidly). Identity will be added in Section 3. Apart from some minor adjustment, the semantics is as in [11, Chap. 4.7], and the calculi are as in [21, Chap. 12.1]. This section is needed to make the paper self-contained and it might be skipped by readers already familiar with QMLs and labelled calculi.

2.1 Syntax and Semantics

Let $\mathcal{S}$ be a signature containing, for every $n \in \mathbb{N}$, an at most denumerable set $REL_n^{\mathcal{S}}$ of n-ary predicate letters $P_1^n, P_2^n, \ldots$, and let VAR be a denumerable set of variables $x_1, x_2, \ldots$. The language $\mathscr{L}$ is given by the grammar:

$$A ::= P^n x_1, \ldots, x_n \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \forall x A \mid \exists x A \mid \Box A \mid \Diamond A \qquad (\mathcal{L})$$

where $P_n \in REL_n^{\mathcal{S}}$ and $x, x_1, \ldots, x_n \in VAR$. We use the following metavariables:

- P, Q, R for predicate letters;
- x, y, z for variables;
- p, q, r for atomic formulas;
- A, B, C for formulas.

We follow the standard conventions for parentheses. The formulas $\top$, $\neg A$ and $A \supset\subset B$ are defined as expected. The notions of *free* and *bound occurrences* of a variable in a formula are the usual ones. Given a formula A, we use $A[y/x]$ to denote the formula obtained by replacing each free occurrence of x in A with an occurrence of y, provided that y is free for x in A – i.e., no new occurrence of y is bound by a quantifier.

A *model* (over the signature $\mathcal{S}$) is a tuple:

$$\mathscr{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{V} \rangle$$

where

- $\mathcal{W} \neq \varnothing$ is a nonempty set of (possible) *worlds* (to be denoted by $w, v, u \ldots$);
- $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ is a binary *accessibility relation* between worlds;
- $\mathcal{D} : \mathcal{W} \longrightarrow 2^D$ is a function mapping each world to a possibly empty set of objects D_w (its *domain*), where $D_{\mathcal{W}} = \bigcup_{w \in \mathcal{W}} D_w$ is nonempty and disjoint from $\mathcal{W}$;
- $\mathcal{V} : \mathcal{S} \times \mathcal{W} \longrightarrow 2^{(D_{\mathcal{W}})^n}$ is a *valuation* function mapping, at each world w, each n-ary predicate $P (\in \mathcal{S})$ to a subset of $(D_{\mathcal{W}})^n$.

A *frame* $\mathcal{F}$ is a triple $\langle \mathcal{W}, \mathcal{R}, \mathcal{D} \rangle$ (i.e. it is a model without valuation), and a model $\mathscr{M}$ is *based on a frame* $\mathcal{F}$ if $\mathscr{M} = \langle \mathcal{F}, \mathcal{V} \rangle$. We will say that a frame $\langle \mathcal{W}, \mathcal{R}, \mathcal{D} \rangle$ has:

- *Increasing domain* if $\forall w, v \in \mathcal{W},\ w\mathcal{R}v$ implies $D_w \subseteq D_v$;
- *decreasing domain* if $\forall w, v \in \mathcal{W},\ w\mathcal{R}v$ implies $D_w \supseteq D_v$;

- *Constant domain* if $\forall w, v \in \mathcal{W},\ w\mathcal{R}v$ implies $D_w = D_v$.

Given a model $\mathscr{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{V} \rangle$, an *assignment* (over $\mathscr{M}$) is a function $\sigma : VAR \longrightarrow D_{\mathcal{W}}$ mapping each variable x to an element of the union of the domains of the model. Moreover, for $o \in D_{\mathcal{W}}$, $\sigma^{x \triangleright o}$ denotes the assignment behaving like σ except for x that is mapped to the object o.

Definition 2.1 (Satisfaction) Given a model $\mathscr{M}$, an assignment σ over it, and a world w of that model, we define the notion of *satisfaction* of an $\mathscr{L}$-formula A as follows:

$\sigma \vDash^{\mathscr{M}}_{w} Px_1, \dots x_n$	iff	$\langle \sigma(x_1), \dots, \sigma(x_n) \rangle \in \mathcal{V}(P, w)$
$\sigma \nvDash^{\mathscr{M}}_{w} \bot$		
$\sigma \vDash^{\mathscr{M}}_{w} B \wedge C$	iff	$\sigma \vDash^{\mathscr{M}}_{w} B$ and $\sigma \vDash^{\mathscr{M}}_{w} C$
$\sigma \vDash^{\mathscr{M}}_{w} B \vee C$	iff	$\sigma \vDash^{\mathscr{M}}_{w} B$ or $\sigma \vDash^{\mathscr{M}}_{w} C$
$\sigma \vDash^{\mathscr{M}}_{w} B \supset C$	iff	$\sigma \nvDash^{\mathscr{M}}_{w} B$ or $\sigma \vDash^{\mathscr{M}}_{w} C$
$\sigma \vDash^{\mathscr{M}}_{w} \forall x B$	iff	for all $o \in D_w$, $\sigma^{x \triangleright o} \vDash^{\mathscr{M}}_{w} B$
$\sigma \vDash^{\mathscr{M}}_{w} \exists x B$	iff	for some $o \in D_w$, $\sigma^{x \triangleright o} \vDash^{\mathscr{M}}_{w} B$
$\sigma \vDash^{\mathscr{M}}_{w} \Box B$	iff	for all $v \in \mathcal{W}$, $w\mathcal{R}v$ implies $\sigma \vDash^{\mathscr{M}}_{v} B$
$\sigma \vDash^{\mathscr{M}}_{w} \Diamond B$	iff	for some $v \in \mathcal{W}$, $w\mathcal{R}v$ and $\sigma \vDash^{\mathscr{M}}_{v} B$

The notions of *truth at a world w of a model* ($\vDash^{\mathscr{M}}_{w} A$), *truth in a model* ($\vDash^{\mathscr{M}} A$), and *validity in a (class of) frames* ($\mathcal{F}(\in \mathcal{C}) \vDash A$) are defined as usual.

As it is well known, some notable formulas are valid in classes of frames defined by properties of the accessibility relation and/or of the domains. In particular, Table 1 presents some well-known (coherent) propositional correspondence results, as well as correspondence results for increasing, decreasing and constant domain frames. By an *$\mathscr{L}$-logic* **Q.L** we mean the set of all $\mathscr{L}$-formulas that are valid in a class of frames. We use standard names for $\mathscr{L}$-logics – e.g., **Q.K** stands for the set of $\mathscr{L}$-formulas valid in the class of all frames, and **Q.S4⊕CBF/BF/UI** stands for the set of $\mathscr{L}$-formulas valid in the class of all reflexive and transitive frames with increasing/decreasing/constant domain. We say that *$\mathscr{M}$ is a model for* **Q.L** whenever $\mathscr{M}$ is based on a frame in the class that defines **Q.L**.

2.2 Labelled Sequent Calculi

Labelled sequent calculi for $\mathscr{L}$-logics have been considered in [21, Chapter 12.1] (see [28] for a related approach). These calculi are based on extending the modal language in order to internalize the semantics of QMLs. First of all, we introduce a set LAB of fresh variables, called *labels*. Labels will be denoted by $w, v, u, \dots$ and will be used to represent worlds. Then, we extend the set of formulas by adding atomic formulas of shape $x \in w$ – expressing that (the object assigned to) x is in the domain of quantification of (the world represented by) w – and of shape $w\mathscr{R}v$ – expressing that v is accessible from w. Lastly, we replace each $\mathscr{L}$-formula A with the labelled formula $w : A$ – expressing that A holds at w. A *labelled sequent* is an expression:

$$\Gamma \Rightarrow \Delta$$

TABLE 1: Modal axioms and corresponding semantic properties

$T := \Box A \supset A$	reflexivity:$=\forall w \in \mathcal{W}(w\mathcal{R}w)$
$D := \Box A \supset \Diamond A$	seriality:$=\forall w \in \mathcal{W}\exists u \in \mathcal{W}(w\mathcal{R}u)$
$4 := \Box A \supset \Box\Box A$	transitivity:$=\forall w, v, u \in \mathcal{W}(w\mathcal{R}v \wedge v\mathcal{R}u \supset w\mathcal{R}u)$
$5 := \Diamond A \supset \Box\Diamond A$	Euclideaness:$=\forall w, v, u \in \mathcal{W}(w\mathcal{R}v \wedge w\mathcal{R}u \supset v\mathcal{R}u)$
$B := A \supset \Box\Diamond A$	symmetry:$=\forall w, v \in \mathcal{W}(w\mathcal{R}v \supset v\mathcal{R}w)$
$CBF := \Box\forall x A \supset \forall x \Box A$	increasing domain
$BF := \forall x \Box A \supset \Box\forall x A$	decreasing domain
$UI := \forall x A \supset A[y/x]$	constant domain

where Γ is a multiset composed of labelled formulas and of atomic formulas of shape $x \in w$ or $w\mathscr{R}v$, and where Δ is a multiset of labelled formulas. Given a formula E of this extended language, $E[w/v]$ is the formula obtained by substituting each occurrence of v in E with an occurrence of w. Substitution of variables is extended to formulas of the extended language as expected, and both kinds of substitution are extended to sequents by applying them componentwise.

The rules of the calculus **G3Q.K**, for the minimal $\mathscr{L}$-logic **Q.K**, are given in Table 2. For each logic **Q.L** extending **Q.K**, the calculus **G3Q.L** is obtained by extending **G3Q.K** with the non-logical rules of Table 3 that express proof-theoretically the coherent semantic properties which define **Q.L** (cf. Table 1). Whenever a calculus contains rule *Eucl*, it contains also all its contracted instances $Eucl^c$ (see [21, p. 100]). Observe that CBF (BF) is not derivable in calculi where rule *Incr* (*Decr*) is not primitive nor admissible (given Proposition 2.2.8, this can be checked semantically).

A **G3Q.L**-*derivation* of a sequent $\Gamma \Rightarrow \Delta$ is a tree of sequents, whose leaves are initial sequents, whose root is $\Gamma \Rightarrow \Delta$, and which grows according to the rules of **G3Q.L**. As usual, we consider only derivations of *pure sequents* – i.e., sequents where no variable has both free and bound occurrences. The *height* of a **G3Q.L**-derivation is the number of nodes of its longest branch. We say that $\Gamma \Rightarrow \Delta$ is **G3Q.L**-derivable (with height n), and we write **G3Q.L** $\vdash^{(n)} \Gamma \Rightarrow \Delta$, if there is a **G3Q.L**-derivation (of height at most n) of $\Gamma \Rightarrow \Delta$ or of an alphabetic variant of $\Gamma \Rightarrow \Delta$. A rule is said to be *(height-preserving) admissible* in **G3Q.L**, if, whenever its premisses are **G3Q.L**-derivable (with height at most n), also its conclusion is **G3Q.L**-derivable (with height at most n). In each rule depicted in Tables 2 and 3, Γ and Δ are called *contexts*, the formulas occurring in the conclusion are called *principal*, and the formulas occurring in the premiss(es) only are called *active*.

The following proposition presents the main meta-theoretical properties of **G3Q.L**. The proofs can be found in [21, Chap. 12.1].

Proposition 2.2 (Properties of G3Q.L)

1. Sequents of shape $w : A, \Gamma \Rightarrow \Delta, w : A$ (with A non-atomic) are **G3Q.L**-derivable.
2. α-conversion is height-preserving admissible: if **G3Q.L** $\vdash^n \Gamma \Rightarrow \Delta$, then **G3Q.L** $\vdash^n \Gamma' \Rightarrow \Delta'$, where Γ' (Δ') is obtained from Γ (Δ) by renaming bound variables.

TABLE 2: Rules of **G3Q.K**

initial sequents:	$w : p, \Gamma \Rightarrow \Delta, w : p$, with p atomic
logical rules:	
$\dfrac{}{w : \bot, \Gamma \Rightarrow \Delta}\ L\bot$	
$\dfrac{w : A, w : B, \Gamma \Rightarrow \Delta}{w : A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge$	$\dfrac{\Gamma \Rightarrow \Delta, w : A \quad \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \wedge B}\ R\wedge$
$\dfrac{w : A, \Gamma \Rightarrow \Delta \quad w : B, \Gamma \Rightarrow \Delta}{w : A \vee B, \Gamma \Rightarrow \Delta}\ L\vee$	$\dfrac{\Gamma \Rightarrow \Delta, w : A, w : B}{\Gamma \Rightarrow \Delta, w : A \vee B}\ R\vee$
$\dfrac{\Gamma \Rightarrow \Delta, w : A \quad w : B, \Gamma \Rightarrow \Delta}{w : A \supset B, \Gamma \Rightarrow \Delta}\ L\supset$	$\dfrac{w : A, \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \supset B}\ R\supset$
$\dfrac{w : A[y/x], y \in w, w : \forall x A, \Gamma \Rightarrow \Delta}{y \in w, w : \forall x A, \Gamma \Rightarrow \Delta}\ L\forall$	$\dfrac{z \in w, \Gamma \Rightarrow \Delta, w : A[z/x]}{\Gamma \Rightarrow \Delta, w : \forall x A}\ R\forall,\ z \text{ fresh}$
$\dfrac{z \in w, w : A[y/x], \Gamma \Rightarrow \Delta}{w : \exists x A, \Gamma \Rightarrow \Delta}\ L\exists,\ z \text{ fresh}$	$\dfrac{y \in w, \Gamma \Rightarrow \Delta, w : \exists x A, w : A[y/x]}{y \in w, \Gamma \Rightarrow \Delta, w : \exists x A}\ R\exists$
$\dfrac{v : A, w\mathscr{R}v, w : \Box A, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, w : \Box A, \Gamma \Rightarrow \Delta}\ L\Box$	$\dfrac{w\mathscr{R}u, \Gamma \Rightarrow \Delta, u : A}{\Gamma \Rightarrow \Delta, w : \Box A}\ R\Box,\ u \text{ fresh}$
$\dfrac{w\mathscr{R}u, u : A, \Gamma \Rightarrow \Delta}{w : \Diamond A, \Gamma \Rightarrow \Delta}\ L\Diamond,\ u \text{ fresh}$	$\dfrac{w\mathscr{R}v, \Gamma \Rightarrow \Delta, w : \Diamond A, v : A}{w\mathscr{R}v, \Gamma \Rightarrow \Delta, w : \Diamond A}\ R\Diamond$

TABLE 3: Non-logical (aka coherent) rules

$\dfrac{w\mathscr{R}w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref_{\mathcal{W}}$	$\dfrac{v\mathscr{R}u, w\mathscr{R}v, w\mathscr{R}u, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, w\mathscr{R}u, \Gamma \Rightarrow \Delta,}\ Eucl$	$\dfrac{v\mathscr{R}v, w\mathscr{R}v, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, \Gamma \Rightarrow \Delta,}\ Eucl^c$
$\dfrac{w\mathscr{R}u, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta,}\ Ser,\ u \text{ fresh}$	$\dfrac{w\mathscr{R}u, w\mathscr{R}v, v\mathscr{R}u, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, v\mathscr{R}u, \Gamma \Rightarrow \Delta}\ Trans$	$\dfrac{v\mathscr{R}w, w\mathscr{R}v, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, \Gamma \Rightarrow \Delta}\ Sym$
$\dfrac{x \in v, x \in w, w\mathscr{R}v, \Gamma \Rightarrow \Delta}{x \in w, w\mathscr{R}v, \Gamma \Rightarrow \Delta}\ Incr$	$\dfrac{x \in w, x \in v, w\mathscr{R}v, \Gamma \Rightarrow \Delta}{x \in v, w\mathscr{R}v, \Gamma \Rightarrow \Delta}\ Decr$	$\dfrac{x \in w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Cons$

3. The following rules of substitution are height-preserving admissible in **G3Q.L**:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma[y/x] \Rightarrow \Delta[y/x]}\ [y/x] \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma[w/v] \Rightarrow \Delta[w/v]}\ [w/v]$$

   where y is free for x in each formula occurring in Γ, Δ for rule $[y/x]$.
4. The following rules of weakening are height-preserving admissible in **G3Q.L**:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \Delta'}\ RW$$

5. Each rule of **G3Q.L** is height-preserving invertible.
6. The following rules of contraction are height-preserving admissible in **G3Q.L**:

$$\frac{\Gamma', \Gamma', \Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow \Delta, \Delta', \Delta'}{\Gamma \Rightarrow \Delta, \Delta'}\ RC$$

7. The following rule of Cut is admissible in **G3Q.L**:

$$\frac{\Gamma \Rightarrow \Delta, w : A \quad w : A, \Gamma' \Rightarrow \Delta'}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ Cut$$

8. **G3Q.L** is sound and complete with respect to **Q.L**.

3 Quantified modal logics with definite descriptions

3.1 Preliminary discussion

Before introducing the formal machinery used by Fitting and Mendelshon [11] to deal with definite descriptions, we take a minute to outline some of the main semantic approaches to definite descriptions, see [14, 15] for more details. Given a formula with one free variable $A(x)$, let us consider the description:

$$\text{the } x \text{ such that } A(x) \tag{3.1}$$

First of all, we can simply deny that (3.1) has to be represented by a genuine term of the formal language. In this case, following Russell [25], we can explain it away by means of of quantification and identity as follows:

$$\exists x(A(x) \wedge \forall y(A(y) \supset y = x))$$

This is a very simple solution in that we simply get rid of the problem of giving a semantics for improper definite descriptions.

If, instead, following Frege [12] we maintain that (3.1) has to be represented by a genuine term of the formal language, we extend the language with terms of shape:

$$\imath x A(x)$$

and we have to give a satisfactory semantics for descriptions. If a description is proper, then it denotes the one and only existing object that satisfies it (in a given world). The problem is what to do when a description is improper because either it is true of no object –e.g., $\imath x(x \neq x)$ – or it is true of more than one object – e.g., $\imath x(x = x)$. If we don't want to extend the language then we cannot allow for non-denoting terms. Thus, we have to accept that improper descriptions denote some object. In a constant domain setting – i.e., when we use the quantification theory of classical logic – we can follow Montague and Kalish [17] and assume that all improper descriptions denote a chosen object. If, instead, we are using a varying domain semantics – i.e., the quantification theory of positive free logic – then we can follow Garson [13] and assume that each improper description denotes a non-existing object (without thereby assuming it is the same one for all improper descriptions).

Nevertheless, these two solutions are not satisfactory in that it is more natural to maintain that an improper description simply fail to denote. Moreover, like Russell's approach, these two approaches do not disentangle designation from existence.

The addition of λ-abstraction to the language allows Fitting and Mendelsohn [11] to avoid these shortcomings: proper descriptions denote the one and only object that satisfies them (be it an existing object or not) and improper descriptions simply fail to denote. Failure of denotation will not be a problem because definite descriptions do not occur as *relata* of atomic formulas – i.e., $P^1 \imath x A$ is not a formula – but only as terms applied via λ. Hence we can simply impose that if t is an improper description then $\lambda x A.t$ is false (for all A): this implies that $\neg(\lambda x A.t)$ is true – i.e., that the sentence At is false – without thereby implying that $\lambda x \neg A.t$ is true – i.e., that the negation of A is true of the object denoted by t.

3.2 Syntax and semantics

Let us consider the same signature $\mathcal{S}$ of Section 2.1. Functions of any arity are omitted for simplicity; they may be added with minor modifications (see footnote 1), but they are already expressible *via* definite descriptions. The sets of terms and formulas of the language $\mathscr{L}^\lambda$ are defined simultaneously as follows:

$$t ::= x \mid \imath x A$$

$$A ::= P^n x_1, \ldots, x_n \mid x_1 = x_2 \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \forall x A \mid \exists x A \mid \Box A \mid \Diamond A \mid \lambda x A.t$$

where $P_n \in REL_n^{\mathcal{S}}$ and $x, x_1, \ldots, x_n \in VAR$. Observe that definite descriptions can occur in a formula only as terms applied by the operator λ. We continue to use the conventions and notions introduced in Section 2.1 with the following additions:

- in $\lambda x A.t$ all occurrences of x (save for the displayed t in case $t \equiv x$) are bound by λx;
- t, r, s range over terms.

Frames ($\mathcal{F}$), models ($\mathcal{M}$), and assignments (σ) are defined as in Section 2.1. Because of definite descriptions, we have to define the notions of denotation and satisfaction together.

Definition 3.1 (Denotation and satisfaction) Given a model $\mathcal{M}$, an assignment σ over it, and a world w of that model, we simultaneously define the notions of *denotation* of a term t and *satisfaction* of an $\mathscr{L}$-formula A as follows:

- Denotation of a term t:
  $\mathcal{V}_w^\sigma(x) = \sigma(x)$
  $\mathcal{V}_w^\sigma(\imath x A) = o$ iff o is the one and only member of $D_{\mathcal{W}}$ such that $\sigma^{x \rhd o} \vDash_w^{\mathcal{M}} A$
- Satisfaction is defined by extending Definition 2.1 with the following clauses:
  $\sigma \vDash_w^{\mathcal{M}} x = y$ iff $\sigma(x) = \sigma(y)$
  $\sigma \vDash_w^{\mathcal{M}} \lambda x A.t$ iff $\mathcal{V}_w^\sigma(t)$ is defined and $\sigma^{x \rhd \mathcal{V}_w^\sigma(t)} \vDash_w^{\mathcal{M}} A$

Truth and validity are defined as in Section 2.1. All correspondence results of Section 2 carries over to QMLs with definite descriptions.

Finally, we show the generality of this approach to definite descriptions by showing how it can simulate the other (non-eliminative) approaches considered in Section 3 (see [15] for a presentation of Montague and Kalish-style descriptions and [14] for Garson-style ones).

Proposition 3.2

1. A Montague and Kalish-style [17] description can be defined as:

$$\imath_m x A \equiv \imath x(\exists! y(A[y/x]) \veebar Ux)$$

   where $\exists!$ is the unique existential quantifier, $\veebar$ is exclusive disjunction and U is a constant monadic predicate axiomatized by the coherent axioms $U(y) \supset \exists z(y = z)$ and $U(y) \wedge U(z) \supset y = z$ (our language does not contain constants).
2. A Garson-style [13] description can be defined as:

$$\imath_g x A \equiv \imath x(\exists! y(A[y/x]) \veebar U_{\imath x A}(x))$$

   where $U_{\imath x A}$ is a constant predicate, that is parametric on $\imath x A$ (modulo alphabetic variants), such that $\forall y(U_{\imath x A}(y) \supset \bot)$ and $U_{\imath x A}(y) \wedge U_{\imath x A}(z) \supset y = z$.

4 Labelled calculi

In order to introduce labelled sequent calculi for QMLs with definite descriptions, we extend the language of labelled calculi with *denotation formulas* of shape $D(t, x, w)$, which will be used to express that the variable x denotes the object denoted in w by the term t. From now on, a sequent $\Gamma \Rightarrow \Delta$ is an expression where Γ is a multiset of labelled $\mathcal{L}^\lambda$-formulas and of formulas of shape $D(t, x, w)$, $x \in w$ or $w\mathscr{R}v$; and Δ is a multiset of labelled $\mathcal{L}^\lambda$-formulas and of denotation formulas only. The following non-standard definition of weight will be essential in Sections 5 and 6.

Definition 4.1 (Weight of terms and formulas)

- The weight of a term t is 0 if t is a variable and, if $t \equiv \imath x A$, it is equal to the weight of $w : A$;
- The weight of a labelled $\mathcal{L}^\lambda$-formula $w : A$ is defined as the number of operators that differs from $\bot$ (and $=$) occurring in A plus the weight of each occurrence of a term in A;
- The weight of a formula $D(t, x, w)$ is equal to the weight of the term t;
- the weight of formulas of shape $x \in w$ and $w\mathscr{R}v$ is 0.

The rules of the calculus **G3Qλ.L** are the rules of **G3Q.L**, see Tables 2 and 3, plus the initial sequents and rules given in Table 4. Observe that the rules for identity contain the labelled version of the non-logical rules first introduced in [20]. When $w : y = x$ holds, by *Repl* we can replace x with y in any atomic formula that, so to say, talks about w. Rule *RigVar* implies that if x and y denote the same object in some world, they do so in each world. Thus, variables behave as rigid designators and labels could be omitted from identities. We choose to keep them in order to have a more uniform notation.

TABLE 4: Additional rules for **G3λ.L**

Initial sequents:	$D(y,x,w),\Gamma \Rightarrow \Delta, D(y,x,w)$
Rules for identity:	
$\dfrac{w : x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Ref}_{=}$	$\dfrac{v : y = z, w : y = z, \Gamma \Rightarrow \Delta}{w : y = z, \Gamma \Rightarrow \Delta}\ \mathit{RigVar}$
$\dfrac{E[z/x], E[y/x], w : y = z, \Gamma \Rightarrow \Delta}{E[y/x], w : y = z, \Gamma \Rightarrow \Delta,}\ \mathit{Repl}$	E is either $D(x_i, x_j, w)$ or $x_i \in w$ or $w : p$ for p atomic
Rules for λ:	
$\dfrac{D(t,z,w), w : B[z/x], \Gamma \Rightarrow \Delta}{w : \lambda x B.t, \Gamma \Rightarrow \Delta}\ L\lambda,\ z \text{ fresh}$	
$\dfrac{\Gamma \Rightarrow \Delta, w : \lambda x B.t, D(t,y,w) \qquad \Gamma \Rightarrow \Delta, w : \lambda x B.t, w : B[y/x]}{\Gamma \Rightarrow \Delta, w : \lambda x B.t}\ R\lambda$	
Rules for $D(\dots)$:	
$\dfrac{w : A[x_2/x_1], D(\imath x_1 A, x_2, w), \Gamma \Rightarrow \Delta}{D(\imath x_1 A, x_2, w), \Gamma \Rightarrow \Delta}\ LD_1$	
$\dfrac{D(\imath x_1 A, x_2, w), \Gamma \Rightarrow \Delta, w : A[y/x_1] \qquad w : x_2 = y, D(\imath x_1 A, x_2, w), \Gamma \Rightarrow \Delta}{D(\imath x_1 A, x_2, w), \Gamma \Rightarrow \Delta}\ LD_2$	
$\dfrac{\Gamma \Rightarrow \Delta, w : A[x_2/x_1] \qquad w : A[z/x_1], \Gamma \Rightarrow \Delta, w : x_2 = z}{\Gamma \Rightarrow \Delta, D(\imath x_1 A, x_2, w)}\ RD,\ z \text{ fresh}$	
$\dfrac{D(x,x,w), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{DenVar}$	$\dfrac{w : y = x, D(y,x,w), \Gamma \Rightarrow \Delta}{D(y,x,w), \Gamma \Rightarrow \Delta}\ \mathit{DenId}$

The satisfaction clause for $\lambda x A.t$ in a world w is similar to that for $\exists x A$, the only difference being that A has to be satisfied not by some arbitrary object of D_w, but by the one and only object of $D_{\mathcal{W}}$ that is denoted by t in that world of that model. Therefore the rules for λ are like the ones for $\exists$ in intuitionistic logic with existence predicate (ILE), see [1, 27], save that they are restricted by formulas of shape $D(t,x,w)$ instead of by atoms of shape $\mathcal{E}t$ as in ILE.

Next, we briefly explain the rules for $D(t,x,w)$. The universal rule DenId ensures that if (in w) y picks the object denoted by x, then x and y denote the same object; and the universal rule DenVar ensures that variables denote at every world. The rules LD_i and RD are obtained as meaning-explanation of the denotation clause for definite descriptions. This is done by first rewriting the denotation clause for $\imath x A$ as:

$$\mathcal{V}^{\sigma}_{w}(\imath x A) = o_1 \quad \text{iff} \quad \sigma^{x \rhd o_1} \vDash^{\mathcal{M}}_{w} A \text{ and } \forall o_2 \in D_{\mathcal{W}}\, (\sigma^{x \rhd o_2} \vDash^{\mathcal{M}}_{w} A \supset o_2 = o_1)$$

Then, from the left-to-right (right-to-left) direction of this semantic clause we easily obtain the rules LD_i (RD).

As shown in [16], for ILE it is possible to obtain a simpler calculus by replacing the

two premisses rule

$$\frac{\Gamma \Rightarrow \mathcal{E}t \quad \Gamma \Rightarrow A[t/x]}{\Gamma \Rightarrow \exists x A}\ R\exists^*$$

with the (equivalent) one premiss rule

$$\frac{\mathcal{E}t, \Gamma \Rightarrow A[t/x]}{\mathcal{E}t, \Gamma \Rightarrow \exists x A}\ R\exists$$

The same phenomenon holds for QMLs with varying domains where we can use either two-premisses versions of rules $R\exists$ and $L\forall$ [14] or the simpler one-premiss versions thereof [21]. In Table 2 we have used the one-premiss rules and the same has been done for the rule $R\lambda$ in [23]. Here, instead, we are forced to adopt the two-premisses version of the rules $R\lambda$ and LD_2 because the presence of definite description would impair the admissibility of cut with the one-premiss version of these rules. The problem, roughly, is that $D(\imath x_1 A, x_2, w)$ (or $w : A[y/x_1]$ for rule LD_2) is not atomic and, therefore, it cannot be a principal formula of the rule $R\lambda$ (LD_2, respectively) if we want to prove the cut-elimination theorem.

5 Structural properties

Lemma 5.1 (Initial sequents) Let A be an arbitrary $\mathcal{L}^\lambda$-formula. Sequents of the following shapes are **G3Qλ.L**-derivable:

1. $w : A, \Gamma \Rightarrow \Delta, w : A$
2. $D(\imath x A, z, w), \Gamma \Rightarrow \Delta, D(\imath x A, z, w)$

Proof. The two cases are proved by simultaneous induction on the weight of the principal formula. For the inductive steps it is enough to apply, root first, the rules for the outermost operator ($D(\dots)$ included) of the principal formula and then the inductive hypothesis (IH). To illustrate, for $D(\imath x A, z, w), \Gamma \Rightarrow \Delta, D(\imath x A, z, w)$ we have:

$$\dfrac{\dfrac{\dfrac{}{w : A[z/x], \dots \Rightarrow w : A[z/x]}\ IH}{D(\imath x A, z, w), \Gamma \Rightarrow \Delta, w : A[z/x]}\ LD_1 \qquad \dfrac{\dfrac{}{w : A[y/x], \dots \Rightarrow \dots, w : A[y/x]}\ IH \qquad w : z = y, \dots \Rightarrow w : z = y}{w : A[y/x], D(\imath x A, z, w), \Gamma \Rightarrow \Delta, w : z = y}\ LD_2}{D(\imath x A, z, w), \Gamma \Rightarrow \Delta, D(\imath x A, z, w)}\ RD$$

∎

Lemma 5.2 (α-conversion) **G3Qλ.L** $\vdash^n \Gamma \Rightarrow \Delta$ entails **G3Qλ.L** $\vdash^n \Gamma' \Rightarrow \Delta'$, where Γ' (Δ') is obtained from Γ (Δ) by renaming some bound variable (without capturing variables).

Proof. The proof is by induction on the height of the **G3Qλ.L**-derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. To illustrate, suppose we know that **G3Qλ.L** $\vdash^n w : \lambda x A.t, \Gamma \Rightarrow \Delta$, and we want to show that **G3Qλ.L** $\vdash^n w : \lambda y A[y/x].t, \Gamma \Rightarrow \Delta$ (with y fresh). If $w : \lambda x A.t$ is not principal in the last step of $\mathcal{D}$, the proof is straightforward. Else, we transform

$$\frac{D(t, z, w), w : A[z/x], \Gamma \Rightarrow \Delta}{w : \lambda x A.t, \Gamma \Rightarrow \Delta}\ L\lambda \qquad \text{into} \qquad \frac{\dfrac{D(t, z, w), w : A[z/x], \Gamma \Rightarrow \Delta}{D(t, z, w), w : (A[y/x])[z/y], \Gamma \Rightarrow \Delta}\ \star}{w : \lambda y A[y/x].t, \Gamma \Rightarrow \Delta}\ L\lambda$$

where the step $\star$ is height-preserving admissible since, having assumed that y is fresh, $w : (A[y/x])[z/y]$ is just a cumbersome notation for $w : A[z/x]$. ∎

Lemma 5.3 (Substitutions) The following rules of substitution are height-preserving admissible in **G3Qλ.L**:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma[y/x] \Rightarrow \Delta[y/x]}\ {\scriptstyle [y/x]} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma[w/v] \Rightarrow \Delta[w/v]}\ {\scriptstyle [w/v]}$$

where y is free for x in each formula occurring in Γ, Δ for rule $[y/x]$.

Proof. Both proofs are by induction on the height of the derivation $\mathcal{D}$ of the premiss $\Gamma \Rightarrow \Delta$. The base cases and the inductive steps where the last rule is not a rule from Table 4 are proved in [21, Lemma 12.4].

We consider explicitly only the case of rule $[y/x]$ where the last step is by $L\lambda$ and the substitution $[y/x]$ clashes with its variable condition. E.g., the last step of $\mathcal{D}$ is

$$\frac{D(t, y, w), w : A[y/z], \Gamma' \Rightarrow \Delta}{w : \lambda z A.t, \Gamma' \Rightarrow \Delta}\ {\scriptstyle L\lambda}$$

with x occurring free in $w : A[y/z], \Gamma', \Delta$ and/or $t \equiv x$. We apply IH twice to the premiss of the last step of $\mathcal{D}$, the first time to replace y with y', for some fresh variable y', and the second time to replace x with y. We finish by applying rule $L\lambda$. Thus, assuming $z \not\equiv x$, we have transformed $\mathcal{D}$ into $\mathcal{D}[y/x]$:

$$\cfrac{\cfrac{\cfrac{\cfrac{D(t, y, w), w : A[y/z], \Gamma' \Rightarrow \Delta}{D(t, y', w), w : A[y'/z], \Gamma' \Rightarrow \Delta}\ {\scriptstyle IH}}{D(t[y/x], y', w), w : (A[y'/z])[y/x], \Gamma'[y/x] \Rightarrow \Delta[y/x]}\ {\scriptstyle IH}}{D(t[y/x], y', w), w : (A[x/y])[y'/z], \Gamma'[y/x] \Rightarrow \Delta[y/x]}\ {\scriptstyle \star}}{w : \lambda z(A[y/x]).(t[y/x]), \Gamma'[y/x] \Rightarrow \Delta[y/x]}\ {\scriptstyle L\lambda}$$

which has the same height as $\mathcal{D}$ because the steps by IH are height-preserving admissible and the step by $\star$ is an height-preserving admissible rewriting that is feasible because $z \not\equiv x$ and $y' \notin \{y, x\}$. ∎

Theorem 5.4 (Weakening) The following rules are height-preserving admissible in **G3Qλ.L**:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta}\ {\scriptstyle LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \Delta'}\ {\scriptstyle RW}$$

Proof. The proofs are by induction on the height of the derivation $\mathcal{D}$ of the premiss $\Gamma \Rightarrow \Delta$. The base cases and the inductive cases where the last step of $\mathcal{D}$ is not by a rule from Table 4 are proved in [21, Thm. 12.5]. The proofs of the inductive cases when the last step of $\mathcal{D}$ is by $L\lambda$ or by RD ($R\lambda$ or LD_i) are analogous to the ones in [21, Thm. 12.5] with last step of $\mathcal{D}$ by rule $L\exists$ ($R\exists$, respectively). The remaining cases are similar to the other ones by non-logical (coherent) rules and can be omitted. ∎

Lemma 5.5 The rules for λ and for $D(\dots)$ are invertible.

Proof. Rules $R\lambda$, LD_i, *DenVar*, and *DenId* are 'Kleene-invertible' thanks to the repetition of the principal formula in the premiss(es) and to the admissiblity of Weakening.

Let's consider RD. Suppose we have a **G3Q**λ**.L**-derivation $\mathcal{D}$ of height n of the sequent $\Gamma \Rightarrow \Delta, D(\imath x A, z, w)$. If $n = 0$ or if $D(\imath x A, z, w)$ is principal in the last step of $\mathcal{D}$, the lemma holds trivially. Else we reason by cases on the last step R of $\mathcal{D}$ and we transform the derivation(s) of its premiss(es) $\Gamma_i \Rightarrow \Delta_i, D(\imath x A, z, w)$ $(i \in \{1, 2\})$ as follows: first, if R has a variable condition, we apply an hp-admissible instance of substitution to ensure it differs from y; then we apply the inductive hypothesis to obtain either $\Gamma'_i \Rightarrow \Delta'_i, w : A[z/x]$ or $w : A[y/x], \Gamma'_i \Rightarrow \Delta'_i, w : z = y$; finally we apply an instance of rule R.

Rule $L\lambda$ can be treated analogously. ∎

Corollary 5.6 (Invertibility) Each rule of **G3Q**λ**.L** is height-preserving invertible.

Theorem 5.7 (Contraction) The following rules are height-preserving admissible in **G3Q**λ**.L**:

$$\frac{\Gamma', \Gamma', \Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow \Delta, \Delta', \Delta'}{\Gamma \Rightarrow \Delta, \Delta'}\ RC$$

Proof. The proof is handled by a simultaneous induction on the height of the derivations of the premisses of LC and RC. Without loss of generality, we assume the multiset we are contracting is made of only one formula E.

The base cases obviously hold, and the proof of nductive cases depend on whether zero, one, or two instances of E are principal in the last step R of the derivation $\mathcal{D}$ of the premiss. If zero instances are principal in R, we apply IH to the premiss(es) of R and then an instance of rule R, and we are done.

If one instance is principal and R is by a propositional rule or by one of $R\forall, L\exists, R\Box, L\Diamond, L\lambda$ and RD, we proceed by first applying invertibility to that rule, then we apply IH as many times as needed, and we conclude by applying an instance of that rule. If, instead, one instance is principal and R is by a rule with repetition of the principal formula(s) in the premiss, we use the hp-admissibility of the rules of weakening, then IH and R.

If two instances are principal, R is a coherent rule (rule *Repl* included) and, if needed, we make use of the fact that R satisfies the closure condition (see [21, p. 100]). To illustrate, the case of *Euclid* is taken care by the presence of its contracted instances $Euclid^c$. For $Trans$, we have three occurences of $w\mathscr{R}w$ in the premiss of this rule instances: two principal and one active. We apply IH twice and we are done. For *Repl*, the active formula of the last rule instance must be of shape $w : x = x$, and, after having applied IH, we can get rid of it by applying $Ref_=$.[1] ∎

Theorem 5.8 (Cut) The following rules of Cut are admissible in **G3Q.L**:

$$\frac{\Gamma \Rightarrow \Delta, w : A \quad w : A, \Gamma' \Rightarrow \Delta'}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ Cut$$

$$\frac{\Gamma \Rightarrow \Delta, D(t, x, w) \quad D(t, x, w), \Gamma' \Rightarrow \Delta'}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ Cut'$$

Proof. We prove the two cases simultaneously. The proof, which extends that of [21, Thm. 12.9], considers an uppermost instance of either Cut or Cut' which is handled

[1] If the language contains functions, then we must add contracted instances of rule *Repl* to derive the (valid) sequent $w : x = f(x) \Rightarrow w : x = f(f(x))$ without applying contraction.

by a principal induction on the weight of the cut-formula with a sub-induction on the sum of the heights of the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of the two premisses of cut (*cut-height*, for shortness). The proof can be organized in four exhaustive cases: in **case 1** one of the two premisses is an initial sequent. In **case 2** the cut formula is not principal in the left premiss only and in **case 3** it is not principal in the right premiss. Finally, in **case 4**, the cut formula is principal in both premisses.

In **case 1**, the conclusion of Cut (Cut') is an initial sequent and, therefore, we can dispense with that instance of Cut (Cut').

In **case 2**, we transform the derivation as follows. First, if the last rule applied in $\mathcal{D}_1$ has a variable condition, we apply an height-preserving admissible substitution to rename its *eigenvariable* with a fresh one. Then, we apply one instance of Cut (Cut') on each premiss of $\mathcal{D}_1$ with the conclusion of $\mathcal{D}_2$. These instances of Cut (Cut') are admissible by IH because they have a lesser cut-height. We finish by applying an instance of the last rule applied in $\mathcal{D}_1$.

Case 3 is similar to case 2 (swapping $\mathcal{D}_1$ and $\mathcal{D}_2$). To illustrate, suppose the last step of $\mathcal{D}_2$ is by $L\lambda$ (with y *eigenvariable*), we transform

$$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad \dfrac{D(t,y,v), w : A, v : B[y/x], \Gamma' \Rightarrow \Delta'}{w : A, v : \lambda x B.t, \Gamma' \Rightarrow \Delta'}\,L\lambda}{v : \lambda x B.t, \Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,Cut$$

into

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, w : A \quad \dfrac{D(t,y,v), w : A, v : B[y/x], \Gamma' \Rightarrow \Delta'}{D(t,z,v), w : A, v : B[z/x], \Gamma' \Rightarrow \Delta'}\,[z/y]}{D(t,z,v), v : B[z/x], \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut}{v : \lambda x B.t, \Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,L\lambda$$

In **case 4**, we have subcases according to the principal operator of the cut-formula. We consider only the case where the cut-formula is of shape $w : \lambda y B.t$ or $D(\imath x A, z, w)$ (see [21, Thm. 12.9] for the other cases).

- Cut formula is $w : \lambda y B.t$. We transform

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\ \Gamma' \Rightarrow \Delta', w : \lambda y B.t, D(t,x,w)\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_{12}\\ \Gamma' \Rightarrow \Delta', w : \lambda y B.t, w : B[x/y]\end{array}}{\Gamma' \Rightarrow \Delta', w : \lambda y B.t}\,R\lambda \quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{21}\\ D(t,z,w), w : B[z/y], \Gamma \Rightarrow \Delta\end{array}}{w : \lambda y B.t, \Gamma \Rightarrow \Delta}\,L\lambda}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,Cut$$

into:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{12}\\ \Gamma' \Rightarrow \Delta', w : B[x/y], w : \lambda y B.t\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_{2}\\ w : \lambda y B.t, \Gamma \Rightarrow \Delta\end{array}}{\Gamma', \Gamma \Rightarrow \Delta, \Delta', w : B[x/y]}\,Cut_1 \quad \begin{array}{c}\vdots\,\mathcal{D}_{3}\\ w : B[x/y], \Gamma', \Gamma, \Gamma \Rightarrow \Delta, \Delta, \Delta'\end{array}}{\Gamma', \Gamma', \Gamma, \Gamma, \Gamma \Rightarrow \Delta, \Delta, \Delta, \Delta', \Delta'}\,Cut_2}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,LC{+}RC$$

where $\mathcal{D}_3$ is the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\ \Gamma' \Rightarrow \Delta', D(t,x,w), w : \lambda y B.t\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_{2}\\ w : \lambda y B.t, \Gamma \Rightarrow \Delta\end{array}}{\Gamma', \Gamma \Rightarrow \Delta, \Delta', D(t,x,w)}\,Cut_3 \quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{21}\\ D(t,z,w), w : B[z/y], \Gamma \Rightarrow \Delta\end{array}}{D(t,x,w), w : B[x/y], \Gamma \Rightarrow \Delta}\,[x/z]}{w : B[x/y], \Gamma', \Gamma, \Gamma \Rightarrow \Delta, \Delta, \Delta'}\,Cut'_4$$

Cut_1 and Cut_3 are admissible because they have has lesser cut-height; Cut_2 and Cut'_4 because their cut-formula has lower weight.

- Cut formula is $D(\imath xA, z, w)$. We have two subcases according to whether the right premiss is by LD_1 or LD_2.

  - If right premise is by LD_1, we have:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\ \Gamma' \Rightarrow \Delta', w : A[z/x]\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_{12}\\ w : A[y/x], \Gamma' \Rightarrow \Delta', w : z = y\end{array}}{\Gamma' \Rightarrow \Delta', D(\imath xA, z, w)}\,RD \quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{21}\\ w : A[z/x], D(\imath xA, z, w), \Gamma \Rightarrow \Delta\end{array}}{D(\imath xA, z, w), \Gamma \Rightarrow \Delta}\,LD_1}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,Cut'$$

  and we transform it into:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\ \Gamma' \Rightarrow \Delta', w : A[z/x]\end{array} \quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{1}\\ \Gamma' \Rightarrow \Delta', D(\imath xA, z, w)\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_{21}\\ D(\imath xA, z, w), w : A[z/x], \Gamma \Rightarrow \Delta\end{array}}{w : A[z/x], \Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,Cut'_1}{\Gamma', \Gamma', \Gamma \Rightarrow \Delta, \Delta', \Delta'}\,Cut_2}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,LC{+}RC$$

where Cut'_1 has lesser cut-height and Cut_2 has a cut formula of lower weight.
  - if the right premise by LD_2, we have:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{11}\\ \Gamma' \Rightarrow \Delta', w : A[z_1/x]\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_{12}\\ w : A[y/x], \Gamma' \Rightarrow \Delta', w : z_1 = y\end{array}}{\Gamma' \Rightarrow \Delta', D(\imath xA, z_1, w)}\,RD \quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{21}\\ D(\imath xA, z_1, w), \Gamma \Rightarrow \Delta, w : A[z_2/x]\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_{22}\\ w : z_1 = z_2, D(\imath xA, z_1, w), \Gamma \Rightarrow \Delta\end{array}}{D(\imath xA, z_1, w), \Gamma \Rightarrow \Delta}\,LD_2}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,Cut'$$

we transform it into:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{1}\\ \Gamma' \Rightarrow \Delta', D(\imath xA, z_1, w)\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_{21}\\ D(\imath xA, z_1, w), \Gamma \Rightarrow \Delta, w : A[z_2/x]\end{array}}{\Gamma', \Gamma \Rightarrow \Delta, \Delta', w : A[z_2/x]}\,Cut'_1 \quad \begin{array}{c}\vdots\,\mathcal{D}_{3}\\ w : A[z_2/x], \Gamma', \Gamma', \Gamma \Rightarrow \Delta, \Delta', \Delta'\end{array}}{\Gamma', \Gamma', \Gamma', \Gamma, \Gamma \Rightarrow \Delta, \Delta, \Delta', \Delta', \Delta'}\,Cut_2}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,LC{+}RC$$

where $\mathcal{D}_3$ is the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{12}\\ w : A[y/x], \Gamma' \Rightarrow \Delta', w : z_1 = y\end{array}}{w : A[z_2/x], \Gamma' \Rightarrow \Delta', w : z_1 = z_2}\,[z_2/y] \quad \dfrac{\begin{array}{c}\vdots\,\mathcal{D}_{1}\\ \Gamma' \Rightarrow \Delta', D(\imath xA, z_1, w)\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_{22}\\ D(\imath xA, z_1, w), w : z_1 = z_2, \Gamma \Rightarrow \Delta\end{array}}{w : z_1 = z_2, \Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,Cut'_3}{w : A[z_2/x], \Gamma', \Gamma', \Gamma \Rightarrow \Delta, \Delta', \Delta'}\,Cut_4$$

Cut'_1 and Cut'_3 are admissible because they have lesser cut-height; Cut_2 and Cut_4 because their cut-formula has lower weight. ∎

Lemma 5.9 (Properties of identity)

1. Identity is an equivalence relation in **G3Q**λ.**L**;
2. The following sequents are **G3Q**λ.**L**-derivable:
   (a) $\Rightarrow w : x = x$
   (b) $w : z = y, w : A[z/x] \Rightarrow w : A[y/x]$
   (c) $w : z = y, D(t, x_1, w)[z/x] \Rightarrow D(t, x_1, w)[y/x]$
3. The following rules of replacement are admissible in **G3Q**λ.**L**:

$$\frac{w : A[y/x], w : z = y, w : A[z/x], \Gamma \Rightarrow \Delta}{w : z = y, w : A[z/x], \Gamma \Rightarrow \Delta}\ \mathit{Repl}_1$$

$$\frac{D(t, x_1, w)[y/x], w : z = y, D(t, x_1, w)[z/x], \Gamma \Rightarrow \Delta}{w : z = y, D(t, x_1, w)[z/x], \Gamma \Rightarrow \Delta}\ \mathit{Repl}_2$$

Proof. The proofs of items 1 and 2(a) are left to the reader.

We prove cases 2(b) and 2(c) by simultaneous induction on the weight of the 'replacement formula'. All cases but that of 2(b) with A of shape $\lambda x_1 B.t$ and that of 2(c) with t of shape $\imath x_2 B$ are left to the reader.

In the first case, assuming, w.l.o.g., $x_1 \notin \{x, y, z\}$, we have:

$$\dfrac{\dfrac{\dfrac{}{D(t[z/x], x_2, w), w : z = y \Rightarrow D(t[y/x], x_2, w)}\ IH_{2(c)} \quad \dfrac{\dfrac{}{w : (B[x_2/x_1])[z/x], w : z = y \Rightarrow w : (B[x_2/x_1])[y/x]}\ IH_{2(b)}}{w : (B[z/x])[x_2/x_1], w : z = y \Rightarrow w : (B[y/x])[x_2/x_1]}\ \star}{D(t[z/x], x_2, w), w : B([z/x])[x_2/x_1], w : z = y \Rightarrow w : \lambda x_1 B[y/x].t[y/x]}\ R\lambda}{w : z = y, w : \lambda x_1 B[z/x].t[z/x] \Rightarrow w : \lambda x_1 B[y/x].t[y/x]}\ L\lambda$$

Where the step by $\star$ is a rewriting that is feasible because $\{x_1, x_2\} \cap \{x, y, z\} = \varnothing$, and the step by $IH_{2(c)/(b)}$ is by induction on case (c)/(b) of the lemma, respectively.

In the second case, assuming $\{x_1, x_2\} \cap \{x, y, z\} = \varnothing$, we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{w : z = y, w : (B[x_1/x_2])[z/x] \Rightarrow w : (B[x_1/x_2])[y/x]}\ IH_{2(b)}}{w : z = y, w : (B[z/x])[x_1[z/x]/x_2] \Rightarrow w : (B[y/x])[x_1[y/x]/x_2]}\ \star}{w : z = y, D(\imath x_2(B[z/x]), x_1[z/x], w) \Rightarrow w : (B[y/x])[x_1[y/x]/x_2]}\ LD_1 \quad \vdots\, \mathcal{D}_2}{w : z = y, D(\imath x_2(B[z/x]), x_1[z/x], w) \Rightarrow D(\imath x_2(B[y/x]), x_1[y/x], w)}\ RD$$

where the derivation $\mathcal{D}_2$ is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{w : (B[x_3/x_1])[y/x], w : y = z \Rightarrow w : (B[x_3/x_1])[z/x]}\ IH_{2(b)}}{w : (B[x_3/x_1])[y/x], w : z = y \Rightarrow w : (B[x_3/x_1])[z/x]}\ Sym_= }{w : B[x_3/x_1[y/x]], w : z = y \Rightarrow w : B[x_3/x_1[z/x]]}\ \star \quad \dfrac{\dfrac{}{w : (x_1 = x_3)[z/x], w : z = y \Rightarrow w : (x_1 = x_3)[y/x]}\ IH_{2(b)}}{w : x_1[z/x] = x_3, w : z = y \Rightarrow w : x_1[y/x] = x_3}\ \star}{w : B[x_3/x_1[y/x]], w : z = y, D(\imath x_2(B[z/x]), x_1[z/x], w) \Rightarrow w : x_1[y/x] = x_3}\ LD_2$$

Where all the steps marked with $\star$ are syntactic rewritings that are feasible because $\{x_1, x_2, x_3\} \cap \{x, z, y\} = \varnothing$, and the admissibility of $Sym_=$ follows from Lemma 5.9.1.

Finally, item (3) follows from item (2) thanks to the admissibility of *Cut* and *Contraction*, as it is shown by the following derivation for Repl_1 (the case of Repl_2 uses Cut' and Lemma 5.9.2(c), and the proof proceeds in the same way):

$$\dfrac{\dfrac{\dfrac{}{w : z = y, w : A[z/x] \Rightarrow w : A[y/x]}\ 5.9.2(b) \quad w : A[y/x], w : z = y, w : A[z/x], \Gamma \Rightarrow \Delta}{w : z = y, w : z = y, w : A[z/y], w : A[z/x], \Gamma \Rightarrow \Delta}\ \mathit{Cut}}{w : z = y, w : A[z/x], \Gamma \Rightarrow \Delta}\ LC$$

∎

Example 5.10 If y is new to z and to A, then the $\mathcal{L}^{\lambda}$-formula:

$$\lambda x_1(x_1 = z).\imath x A \supset (A[z/x] \wedge (A[y/x] \supset y = z\ )$$

is **G3Q**λ**.L**-derivable (see [11, Section 12.5] for a discussion of this formula).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{w : A[z/x], w : A[y/x], D(\imath x A, y, w), w : y = z \Rightarrow w : A[z/x]}\ \text{Lemma 5.1.1}}{w : A[y/x], D(\imath x A, y, w), w : y = z \Rightarrow w : A[z/x]}\ Repl_1}{D(\imath x A, y, w), w : y = z \Rightarrow w : A[z/x]}\ LD_1}{w : \lambda x_1(x_1 = z).\imath x A \Rightarrow w : A[z/x]}\ L\lambda$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{w : A[y/x], \ldots \Rightarrow w : y = z, w : A[y/x]}\ 5.1.1 \quad \dfrac{}{w : y = z, w : A[y/x], \ldots \Rightarrow w : y = z}\ 5.1.1}{w : A[y/x], D(\imath x A, a, w), w : a = z \Rightarrow w : y = z}\ LD_2}{D(\imath x A, a, w), w : a = z \Rightarrow w : A[y/x] \supset y = z}\ R\supset}{w : \lambda x_1(x_1 = z).\imath x A \Rightarrow w : A[y/x] \supset y = z}\ L\lambda$$

6 Soundness and completeness

6.1 Soundness

Definition 6.1 Given a model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{V} \rangle$, let $f : LAB \cup VAR \longrightarrow \mathcal{W} \cup D_{\mathcal{W}}$ be a function mapping labels to worlds of the model and mapping variables to objects of the union of the domains of the model. We say that:

$\mathcal{M}$ satisfies $w : A$ under f	iff	$f \vDash^{\mathcal{M}}_{f(w)} A$
$\mathcal{M}$ satisfies $x \in w$ under f	iff	$f(x) \in D_{f(w)}$
$\mathcal{M}$ satisfies $w\mathscr{R}v$ under f	iff	$f(w)\mathcal{R}f(v)$
$\mathcal{M}$ sat. $D(t, x, w)$ under f	iff	$\begin{cases} \forall o \in D_{\mathcal{W}}(f^{y \triangleright o} \vDash^{\mathcal{M}}_{f(w)} A \text{ iff } o = f(x)) & \text{if } t \equiv \imath y A; \\ f(t) = f(x) & \text{if } t \equiv y; \end{cases}$

Given a sequent $\Gamma \Rightarrow \Delta$ we say that it is **Q**λ**.L**-*valid* iff for every pair $\mathcal{M}, f$ where $\mathcal{M}$ is a model for **Q**λ**.L**, if $\mathcal{M}$ satisfies under f all formulas in Γ then $\mathcal{M}$ satisfies under f some formula in Δ.

Theorem 6.2 (Soundness) If a sequent $\Gamma \Rightarrow \Delta$ is **G3Q**λ**.L**-derivable, then it is **Q**λ**.L**-valid.

Proof. The proof is by induction on the height of the **G3Q**λ**.L**-derivation of $\Gamma \Rightarrow \Delta$. The base case holds since Γ and Δ have one formula in common, and it is easy to see that the propositional rules, the rules for $\forall$ and $\exists$, and the rules for $\Box$ and $\Diamond$ preserve validity on every model, see [21, Thm. 12.13].

For rule $L\lambda$, let the last step of $\mathcal{D}$ be:

$$\dfrac{D(t, y, w), w : A[y/x], \Gamma \Rightarrow \Delta}{w : \lambda x A.t, \Gamma \Rightarrow \Delta}\ L\lambda$$

with y not free in $w : \lambda x A.t, \Gamma, \Delta$. Suppose that $\mathcal{M}$ satisfies under f all formulas in Γ and the formula $w : \lambda x A.t$. We have to prove that $\mathcal{M}$ satisfies under f also some

formula in Δ. Since $f \vDash^{\mathcal{M}}_{f(w)} \lambda x A.t$, we know that, in $f(w)$, the term t denotes some object $o \in D_{\mathcal{W}}$ and that $f^{y \rhd o} \vDash^{\mathcal{M}}_{f(w)} A[y/x]$, where y does not occur in Γ, A. This implies that $\mathcal{M}$ satisfies under $f^{y \rhd o}$ all formulas in $D(t, y, w), w : A[y/x], \Gamma$. Thus, by IH, $\mathcal{M}$ satisfies under $f^{y \rhd o}$ some formula in Δ. Since y does not occur in Δ, we conclude that $\mathcal{M}$ satisfies under f some formula in Δ.

For rule $R\lambda$, let the last step of $\mathcal{D}$ be:

$$\frac{\Gamma \Rightarrow \Delta, w : \lambda x A.t, D(t, y, w) \qquad \Gamma \Rightarrow \Delta, w : \lambda x A.t, w : A[y/x]}{\Gamma \Rightarrow \Delta, w : \lambda x A.t} \; R\lambda$$

We consider an arbitrary pair $\mathcal{M}, f$ satisfying all formulas in Γ. By IH we know that if $\mathcal{M}, f$ does not satisfy some formula in $\Delta, w : \lambda x A.t$, it satisfies both $D(t, y, w)$ and $w : A[y/x]$. If $\mathcal{M}, f$ satisfies some formula in $\Delta, w : \lambda x A.t$ there is nothing to prove. Else, $\mathcal{M}$ satisfies under f the formulas $w : A[y/x]$ and $D(t, y, w)$, in this case it is easy to see that $\mathcal{M}$ satisfies under f also $\lambda x A.t$.

For rule LD_1, let the last step of $\mathcal{D}$ be:

$$\frac{w : A[x_2/x_1], D(\imath x_1 A, x_2, w), \Gamma \Rightarrow \Delta}{D(\imath x_1 A, x_2, w), \Gamma \Rightarrow \Delta} \; LD_1$$

Let us consider a pair $\mathcal{M}, f$ that satisfies all formulas in Γ and $D(\imath x_1 A, x_2, w)$. The latter means that

$$\forall o \in D_{\mathcal{W}} (f^{x_1 \rhd o} \vDash^{\mathcal{M}}_{f(w)} A \text{ iff } o = f(x_2)\,) \tag{6.1}$$

Hence, we know that $\mathcal{M}, f$ satisfies $w : A[x_2/x_1]$ and, by IH, we conclude that it satisfies also some formula in Δ.

For rule LD_2, we proceed as for LD_1. If the last step of $\mathcal{D}$ is:

$$\frac{D(\imath x_1 A, x_2, w), \Gamma \Rightarrow \Delta, w : A[y/x_1] \qquad w : x_2 = y, D(\imath x_1 A, x_2, w), \Gamma \Rightarrow \Delta}{D(\imath x_1 A, x_2, w), \Gamma \Rightarrow \Delta} \; LD_2$$

and if $\mathcal{M}, f$ is a pair that satisfies $D(\imath x_1 A, x_2, w)$, then we know that (6.1) holds. Assume that $\mathcal{M}, f$ satisfies also all formulas in Γ. By IH, the left premise entails that if $\mathcal{M}, f$ does not satisfy some formula in Δ then it satisfies $w : A[y/x_1]$ and, hence $f^{x \rhd f(y)} \vDash^{\mathcal{M}}_{f(w)} A$. But then (6.1) entails that $f(y) = f(x_2)$ – i.e., $\mathcal{M}, f$ satisfies also $w : x_2 = y$. By induction on the right premiss we conclude that the pair $\mathcal{M}, f$ satisfies some formula in Δ.

For rule RD, let the last step of $\mathcal{D}$ be:

$$\frac{\Gamma \Rightarrow \Delta, w : A[x_2/x_1] \qquad w : A[z/x_1], \Gamma \Rightarrow \Delta, w : x_2 = z}{\Gamma \Rightarrow \Delta, D(\imath x_1 A, x_2, w)} \; RD, z \text{ fresh}$$

Let $\mathcal{M}, f$ satisfy all formulas in Γ. If it also satisfies some formula in Δ we are done. Else, by IH, we know it satisfies $w : A[x_2/x_1]$. Let o be any object such that $f^{x_1 \rhd o} \vDash^{\mathcal{M}}_{f(w)} A$. We consider a pair $\mathcal{M}, f'$ where $\mathcal{M}$ is as before and f' is like f save that $f'(z) = o$. Thanks to the variable condition on z, this implies that $\mathcal{M}, f'$ satisfies $w : x_2 = z$. Hence, $\mathcal{M}, f'$ satisfies $D(\imath x_1 A, x_2, w)$. We conclude the same holds for $\mathcal{M}, f$.

The rules for identity preserves validity on every model: the proof is standard for rules $Ref_=$ and $Repl$. For rule $RigVar$ it depends on the fact that variables are rigid designators. Also the rules $DenVar$ and $DenId$ preserve validity on every model. For $DenVar$ this depends on the fact that variables denote in every world. For $DenId$, this holds because the fact that $\mathcal{M}$ satisfies $D(x, y, w)$ under f means that $f(x) = f(y)$. Therefore, $\mathcal{M}$ must also satisfy under f the formula $w : x = y$.

For the proof that each non-logical rule in Table 3 preserves validity over the appropriate class of frames, we refer the reader to [21, Thm. 12.13]. ∎

6.2 Completeness

Theorem 6.3 (Completeness) If a sequent $\Gamma \Rightarrow \Delta$ is **Q**λ**.L**-valid, it is **G3Q**λ**.L**-derivable.

Proof. The proof is organized in four main steps. First, in Def. 6.4, we sketch a root-first **G3Q**λ**.L**-proof-search procedure. Second, in Def. 6.5, we define the notion of saturation for a branch of a **G3Q**λ**.L**-proof-search tree and, in Proposition 6.6, we show that, for every sequent, a **G3Q**λ**.L**-proof-search either gives us a **G3Q**λ**.L**-derivation of that sequent, or it has a saturated branch. Third, in Def. 6.8, we define a model $\mathcal{M}^{\mathcal{B}}$ out of a saturated branch $\mathcal{B}$. Finally, in Lemma 6.9, we prove that $\mathcal{M}^{\mathcal{B}}$ is a model for **Q**λ**.L** that falsifies $\Gamma \Rightarrow \Delta$. ∎

Definition 6.4 A **G3Q**λ**.L**-*proof-search tree* for a sequent $\Gamma \Rightarrow \Delta$ is a tree of sequents generated according to the following inductive procedure. At **step 0** we write the one node tree $\Gamma \Rightarrow \Delta$. At **step n + 1**, if all leaves of the tree generated at step n are initial sequents, the procedure ends. Else, we continue the bottom-up construction by applying, to each leaf that is not an initial sequent, each applicable instance of a rule of **G3Q**λ**.L** (by invertibility of the rules, there is no prescribed order in which these rules need to be applied) or, if no rule instance is applicable, we copy the leaf on top of itself. For rules $Ref_=$, $Ref_{\mathcal{W}}$, *Ser, Cons* and *DenVar*, we consider applicable only instances where, save for *eigenvariables*, all terms and labels occurring in the active formula of that instance already occur in the leaf. See [21, Thm. 12.14] for the details of the inductive procedure (the reader might easily fill the missing details).

Definition 6.5 (Saturation) A branch $\mathcal{B}$ of a **G3Q**λ**.L**-proof-search tree for a sequent is **Q**λ**L**-*saturated* if it satisfies the following conditions, where $\mathbf{\Gamma}$ ($\mathbf{\Delta}$) is the union of the antecedents (succedents) occurring in that branch,

1. no $w : p$ occurs in $\mathbf{\Gamma} \cap \mathbf{\Delta}$;
2. no $D(y, x, w)$ occurs in $\mathbf{\Gamma} \cap \mathbf{\Delta}$;
3. $w : \bot$ does not occur in $\mathbf{\Gamma}$;
4. if $w : A \wedge B$ is in $\mathbf{\Gamma}$, then both $w : A$ and $w : B$ are in $\mathbf{\Gamma}$;
5. if $w : A \wedge B$ is in $\mathbf{\Delta}$, then at least one of $w : A$ and $w : B$ is in $\mathbf{\Delta}$;
6. if $w : A \vee B$ is in $\mathbf{\Gamma}$, then at least one of $w : A$ and $w : B$ is in $\mathbf{\Gamma}$;
7. if $w : A \vee B$ is in $\mathbf{\Delta}$, then both $w : A$ and $w : B$ are in $\mathbf{\Delta}$;
8. if $w : A \supset B$ is in $\mathbf{\Gamma}$, then $w : A$ is in $\mathbf{\Delta}$ or $w : B$ is in $\mathbf{\Gamma}$;
9. if $w : A \supset B$ is in $\mathbf{\Delta}$, then $w : A$ is in $\mathbf{\Gamma}$ and $w : B$ is in $\mathbf{\Delta}$;

10. if both $w : \forall x A$ and $y \in w$ are in $\mathbf{\Gamma}$, then $w : A[y/x]$ is in $\mathbf{\Gamma}$;
11. if $w : \forall x A$ is in $\mathbf{\Delta}$, then, for some z, $w : A[z/x]$ is in $\mathbf{\Delta}$ and $z \in w$ is in $\mathbf{\Gamma}$;
12. if $w : \exists x A$ is in $\mathbf{\Gamma}$, then, for some z, both $w : A[z/x]$ and $z \in w$ are in $\mathbf{\Gamma}$;
13. if $w : \exists x A$ is in $\mathbf{\Delta}$ and $y \in w$ is in $\mathbf{\Gamma}$, then $w : A[y/x]$ is in $\mathbf{\Delta}$;
14. if both $w : \Box A$ and $w\mathscr{R}v$ are in $\mathbf{\Gamma}$, then $v : A$ is in $\mathbf{\Gamma}$;
15. if $w : \Box A$ is in $\mathbf{\Delta}$, then, for some u, $u : A$ is in $\mathbf{\Delta}$ and $w\mathscr{R}u$ is in $\mathbf{\Gamma}$;
16. if $w : \Diamond A$ is in $\mathbf{\Gamma}$, then, for some u, both $u : A$ and $w\mathscr{R}u$ are in $\mathbf{\Gamma}$;
17. if $w : \Diamond A$ is in $\mathbf{\Delta}$ and $w\mathscr{R}v$ is in $\mathbf{\Gamma}$, then $v : A$ is in $\mathbf{\Delta}$;
18. if $w : \lambda x A.t$ is in $\mathbf{\Gamma}$, then, for some z, both $D(t, z, w)$ and $w : A[z/x]$ are in $\mathbf{\Gamma}$;
19. if $w : \lambda x A.t$ is in $\mathbf{\Delta}$, then, for each $y\,(\in \mathcal{B})$, at least one of $D(t, y, w)$ and $w : A[y/x]$ is in $\mathbf{\Delta}$;
20. if $D(\imath x_1 A, x_2, w)$ is in $\mathbf{\Gamma}$, then $w : A[x_2/x_1]$ is in $\mathbf{\Gamma}$ and, for each $y\,(\in \mathcal{B})$, either $w : A[y/x_1]$ is in $\mathbf{\Delta}$ or $w : x_2 = y$ is in $\mathbf{\Gamma}$;
21. if $D(\imath x_1 A, x_2, w)$ is in $\mathbf{\Delta}$, then $w : A[x_2/x_1]$ is in $\mathbf{\Delta}$ or, for some z, both $w : A[z/x_1]$ is in $\mathbf{\Gamma}$ and $w : x_2 = z$ is in $\mathbf{\Delta}$;
22. if the principal formulas of some instance of one of $Ref_=$, $Repl$, $RigVar$, $DenVar$, and $DenId$ is in $\mathbf{\Gamma}$, then also the corresponding active formulas are in $\mathbf{\Gamma}$.

23_R. if R is a non-logical rule of **G3Q**λ**.L**, then for each set of principal formulas of R that are in $\mathbf{\Gamma}$ also the corresponding active formulas are in $\mathbf{\Gamma}$ (for some *eigenvariable* of that rule, if any).

Proposition 6.6 Let us consider a **G3Q**λ**.L**-proof-search tree for a sequent $\mathcal{S}$, two cases are possible: either the tree is finite or not. If the tree is finite then all of its leaves are initial sequents and it grows by applying rules of **G3Q**λ**.L**. Hence, the tree is a **G3Q**λ**.L**-derivation of $\mathcal{S}$ and, by Theorem 6.2, $\mathcal{S}$ is **Q**λ**.L**-vaild. Else, $\mathcal{S}$ is not **G3Q**λ**.L**-derivable and, by König's Lemma, the tree has an infinite branch $\mathcal{B}$ that is **Q**λ**.L**-saturated since every applicable rule instance has been applied at some step of the construction of the tree.

Proposition 6.7 Let $\mathbf{\Gamma}$ ($\mathbf{\Delta}$) be the union of the antecedents (succedents) of a **Q**λ**.L**-saturated branch. It is immediate to notice that, by saturation under rule $Ref_=$ and $Repl$ (cf. Def. 6.5,22), the set of variables x, y such that $w : x = y$ is in $\mathbf{\Gamma}$ is an equivalence class $[x]_w$. Moreover, by saturation under $RigVar$, the same equivalence class holds with respect to each label v occurring in $\mathbf{\Gamma}, \mathbf{\Delta}$ (hence we allow ourselves to use $[x]$ instead of $[x]_w$).

Definition 6.8 Let $\mathcal{B}$ be a saturated branch of a **G3Q**λ**.L**-proof-search tree for a sequent, and let $\mathbf{\Gamma}$ be the union of its antecedents. The model $\mathcal{M}^{\mathcal{B}} = \langle \mathcal{W}^{\mathcal{B}}, \mathcal{R}^{\mathcal{B}}, \mathcal{D}^{\mathcal{B}}, \mathcal{V}^{\mathcal{B}} \rangle$ is defined from $\mathcal{B}$ as follows ($\mathcal{M}^{\mathcal{B}}$ is well-defined thanks to Definition 6.5,1–3, Lemma 5.1,1–2, and Proposition 6.7):

- $\mathcal{W}^{\mathcal{B}}$ is the set of all labels occurring in $\mathcal{B}$;
- $\mathcal{R}^{\mathcal{B}}$ is such that $w\mathcal{R}^{\mathcal{B}}v$ iff $w\mathscr{R}v$ occurs in $\mathcal{B}$;
- $\mathcal{D}^{\mathcal{B}}$ is such that, for each $w \in \mathcal{W}^{\mathcal{B}}$, D_w is the set containing, for each variable x such that $x \in w$ occurs in $\mathcal{B}$, the equivalence class $[x]$ of all x, y such that $w : x = y$ occurs in $\mathcal{B}$;

- $\mathcal{V}^{\mathcal{B}}$ is such that, for every predicate $P^n \in \mathcal{S}^{\lambda}$, $\mathcal{V}(P^n, w)$ is the set of all n-tuples of equivalence classes of variables $\langle [x_1], \ldots, [x_n] \rangle$ such that $w : Px_1, \ldots, x_n$ occurs in $\mathbf{\Gamma}$.

Lemma 6.9 If $\mathcal{M}^{\mathcal{B}}$ is the model defined from a saturated branch $\mathcal{B}$ of a **G3Qλ.L**-proof-search tree for a sequent $\Gamma \Rightarrow \Delta$ and σ is the assignment defined by $\sigma(x) = [x]$, then, for each labelled formula $w : A$ occurring in $\mathcal{B}$ and for each denotation formula $D(t, x, w)$ occurring in $\mathcal{B}$,

1. $\sigma \vDash_w^{\mathcal{M}^{\mathcal{B}}} A$ iff $w : A$ occurs in $\mathbf{\Gamma}$
2. $\mathcal{V}_w^{\mathcal{B},\sigma}(t) = [x]$ iff $D(t, x, w)$ occurs in $\mathbf{\Gamma}$
3. $\mathcal{M}^{\mathcal{B}}$ is a model for **Qλ.L**.

Proof. The proof of **claims 1 and 2** is by simultaneous induction on the weight of $w : A$ and of $D(t, x, w)$, respectively.

We start with **claim 1**. The base case holds thanks to the definition of $\mathcal{V}^{\mathcal{B}}$, and the inductive cases depends on the construction of $\mathcal{M}^{\mathcal{B}}$ and on properties 4–19 of the definition of saturated branches (Def. 6.5). To illustrate, suppose $w : A \equiv w : \lambda x B.t$.

If $w : A$ occurs in $\mathbf{\Gamma}$, then, by Def. 6.5.18, for some z, $D(t, z, w)$ and $w : B[z/x]$ are in $\mathbf{\Gamma}$. By induction on claim 2, this implies that $\mathcal{V}_w^{\mathcal{B},\sigma}(t) = [z]$ and, by induction on claim 1, it also implies that $\sigma^{x \rhd [z]} \vDash_w^{\mathcal{M}^{\mathcal{B}}} B$. Thus, $\sigma \vDash_w^{\mathcal{M}^{\mathcal{B}}} \lambda x B.t$.

Suppose that $w : \lambda x B.t$ is in $\mathbf{\Delta}$. If there is no variable y such that $D(t, y, w)$ is in $\mathbf{\Gamma}$, then, by (the latter fact and by) induction on claim 2, we immediately have that $\mathcal{V}_w^{\mathcal{B},s}(t)$ is undefined. Thus, $\sigma \nvDash_w^{\mathcal{M}^{\mathcal{B}}} \lambda x B.t$. Else, given Def. 6.5.2 and Lemma 5.1.2, Def. 6.5.19 entails that, for each y such that $D(t, y, w)$ is in $\mathbf{\Gamma}$, $w : B[y/x]$ is in $\mathbf{\Delta}$. By induction on claim 2, we have that $\mathcal{V}_w^{\mathcal{B},\sigma}(t) = [y]$ and, thanks to induction on claim 1 (and 5.1.1, 6.5.1), $\sigma^{x \rhd [y]} \nvDash_w^{\mathcal{M}^{\mathcal{B}}} B$. We conclude that $\sigma \nvDash_w^{\mathcal{M}^{\mathcal{B}}} \lambda x B.t$.

Next, we consider **Case 2**. In the base case t is a variable y and, by construction of $\mathcal{D}^{\mathcal{B}}$, we know that $\mathcal{V}_w^{\mathcal{B},\sigma}(y) = [x]$ iff $w : x = y$ occurs in $\mathbf{\Gamma}$. The right-to left implication holds thanks to saturation under rule *DenId*, and the left-to-right one thanks to saturation under rules *DenVar* and *Repl* (it is enough to consider an instance of *Repl* with principal formulas $w : x = y$ and $D(z, x, w)[x/z]$).

If, instead, $t \equiv \imath y B$, we make use of the properties 20 and 21 of the definition of saturated branch to prove that, whenever $D(\imath y B, x, w)$ is in $\mathbf{\Gamma}$, $[x]$ is the only member of $D_{\mathcal{W}^{\mathcal{B}}}$ such that $\sigma^{y \rhd [x]} \vDash_w^{\mathcal{M}^{\mathcal{B}}} B$.

If $D(\imath y B, x, w)$ occurs in $\mathbf{\Gamma}$, then Def. 6.5.20 entails that (i) $w : B[x/y]$ is in $\mathbf{\Gamma}$ and (ii) for each $z \in \mathcal{B}$, if $w : B[z/y]$ occurs in $\mathbf{\Gamma}$ then also $w : x = z$ occurs in $\mathbf{\Gamma}$. By induction on claim 1 and by construction of $D_{\mathcal{W}^{\mathcal{B}}}$, fact (i) implies that $[x]$ is such that $\sigma^{y \rhd [x]} \vDash_w^{\mathcal{M}^{\mathcal{B}}} B$. Moreover, fact (ii) implies that for each $[z] \in D_{\mathcal{W}^{\mathcal{B}}}$, $\sigma^{y \rhd [z]} \vDash_w^{\mathcal{M}^{\mathcal{B}}}$ only if $[z] = [x]$. Thus, we conclude that $\mathcal{V}_w^{\mathcal{B},\sigma}(\imath y B) = [x]$.

If $D(\imath y B, x, w)$ occurs in $\mathbf{\Delta}$, then either $w : B[x/y]$ is in $\mathbf{\Delta}$ or, for some $z \in \mathcal{B}$, $w : A[z/y]$ is in $\mathbf{\Gamma}$ and $w : x = z$ is in $\mathbf{\Delta}$. In the first case $[x]$ is such that $\sigma^{y \rhd [x]} \nvDash_w^{\mathcal{M}^{\mathcal{B}}} B$; in the second case there is $[z] \in D_{\mathcal{W}^{\mathcal{B}}}$ such that $\sigma^{y \rhd [z]} \vDash_w^{\mathcal{M}^{\mathcal{B}}} B$ and $[z] \neq [x]$. In both cases we can conclude that $\mathcal{V}_w^{\mathcal{B},\sigma}(\imath y B) \neq [x]$.

Claim 3 holds thanks to property 23_R of saturated branch: if a non-logical rule R is in **G3Qλ.L**, then we have to show that $\mathcal{M}^{\mathcal{B}}$ satisfies the semantic property corresponding to R. This holds by construction of $\mathcal{M}^{\mathcal{B}}$ since $\mathcal{B}$ is saturated with

respect to rule R. For example, for rule *Decr*, we have to prove that if $w\mathcal{R}^{\mathcal{B}}v$ then $D_v \subseteq D_w$. By construction of $\mathcal{R}^{\mathcal{B}}$, we know that $w\mathcal{R}^{\mathcal{B}}v$ implies that $w\mathscr{R}v$ occurs in $\mathbf{\Gamma}$. Let us now consider a generic $[x] \in \mathcal{D}^{\mathcal{B}}$. If $[x] \in D_v$ then $x \in v$ occurs in $\mathbf{\Gamma}$. By saturation under rule *Decr*, we have that $x \in w$ occurs in $\mathbf{\Gamma}$ and, hence, $[x] \in D_w$. We conclude that $\mathcal{M}^{\mathcal{B}}$ is based on a frame with decreasing domain. ∎

7 Conclusion

We have introduced labelled sequent calculi that characterize the QMLs with definite descriptions introduced in [11], and we have studied their structural properties. To the best of our knowledge, this is the first proof-theoretic study of these logics. In [11] prefixed tableaux for these logics have been considered, but there is no study of their structural properties. Notice that, even if we have considered only the $Q\lambda$-extensions of propositional modal logics $\mathbf{L}$ in the cube of normal modalities, the present approach can be extended, in a modular way, to the $Q\lambda$-extensions of any propositional modal logic whose class of frames is first-order definable (by applying, if needed, the *coherentisation technique* introduced in [10]). We can, e.g., introduce a calculus characterizing validity in the class of all constant domain frames satisfying *confluence*:

$$\forall w, v, u \in \mathcal{W}(w\mathcal{R}v \wedge w\mathcal{R}u \supset \exists w' \in \mathcal{W}(v\mathcal{R}w' \wedge u\mathcal{R}w'))$$

From [9], we know that confluence corresponds to Geach's axiom $2 := \Diamond\Box A \supset \Box\Diamond A$ and that the quantified modal axiomatic system $Q.2 \oplus BF$ (see [9]) is incomplete with respect to the class of all confluent constant domain frames (i.e., the logic $\mathbf{Q\lambda.K2{\oplus}UI}$). Nevertheless, confluence is a coherent property, and it can be expressed in labelled calculi by the rule:

$$\frac{v\mathscr{R}w', u\mathscr{R}w', w\mathscr{R}v, w\mathscr{R}u, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, w\mathscr{R}u, \Gamma \Rightarrow \Delta}\ {\scriptstyle Conf,\ w' \text{ fresh}}$$

It can be proved that the labelled calculus $\mathbf{G3Q\lambda.K}$+{*Conf, Cons*} is sound and complete with respect to the class of confluent constant domain frames.

If, instead, we consider the logic $\mathbf{Q\lambda.S4.M{\oplus}UI}$ – i.e. the set of $\mathcal{L}^{\lambda}$-formulas that are valid in the class of constant domain frames that are reflexive, transitive, and *final*:

$$\forall w \in \mathcal{W}\exists v \in \mathcal{W}(w\mathcal{R}v \wedge \forall u \in \mathcal{W}(v\mathcal{R}u \supset v = u)) \tag{7.1}$$

then we have the problem that finality is not a coherent property because of the universal quantifier in the scope of an existential one. Nevertheless, as it is shown in [10], we can transform it into a set of coherent conditions by extending the language with a fresh one-place predicate constant Fin that replaces $\forall u \in \mathcal{W}(v\mathcal{R}u \supset v = u))$ in (7.1) (thus making it coherent) and that is axiomatized by the following coherent condition:

$$\forall w \in \mathcal{W}\exists v \in \mathcal{W}(w\mathscr{R}v \wedge Fin(v))$$

We can now express finality in labelled calculi by means of the following two coherent rules (where $v = u$ is governed by the rules in [21, Table 11.7]):

$$\frac{w\mathscr{R}v, Fin(v), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle Fin_1,\ v \text{ fresh}} \qquad \frac{v = u, Fin(v), v\mathscr{R}u, \Gamma \Rightarrow \Delta}{Fin(v), v\mathscr{R}u, \Gamma \Rightarrow \Delta}\ {\scriptstyle Fin_2}$$

By applying this coherentisation strategy we can easily obtain a labelled calculus (with good structural properties) for the $\mathbf{Q}\lambda$-extensions of any first-order definable propositional modal logic. This is far more general than any other existing proof-theoretic characterization result for QMLs (see [4, 5] for partial translations of these results to internal calculi). Moreover, given Proposition 3.2, we can easily simulate the QMLs with definite descriptions *à la* Montague and Kalish or *à la* Garson. All we have to do is to add the following coherent rules for Montague and Kalish's predicate U (rule U_1 is dispensable over constant domains):

$$\frac{x \in w, w : U(x)\Gamma \Rightarrow \Delta}{w : U(x), \Gamma \Rightarrow \Delta}\ U_1 \qquad \frac{w : x = y, w : U(x), w : U(y), \Gamma \Rightarrow \Delta}{w : U(x), w : U(y), \Gamma \Rightarrow \Delta}\ U_2$$

and the following ones for Garson's predicate $U_{\imath xA}$:

$$\frac{}{y \in w, w : U_{\imath xA}(y), \Gamma \Rightarrow \Delta}\ U_{Ax,1} \qquad \frac{w : z = y, w : U_{\imath xA}(z), w : U_{\imath xA}(y), \Gamma \Rightarrow \Delta}{w : U_{\imath xA}(z), w : U_{\imath xA}(y), \Gamma \Rightarrow \Delta}\ U_{Ax,2}$$

Hence, we easily obtain a labelled version of the Gentzen-style calculi defined in [14] for Garson's descriptions and in [15] for Montague and Kalish's ones. One advantage of the labelled version over the existing Gentzen-style ones is that, whereas the rules of the Getzen-style calculi are not apt for proof search, see [14, Section 4], the labelled version allows for proof-search and for the construction of countermodels from a failed proof search. In particular, we bypass the problems noted in [14, Section 4] for the rules for identity because the language $\mathcal{L}^\lambda$ is such that descriptions cannot occur in identity atoms.

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Received Day Month year.