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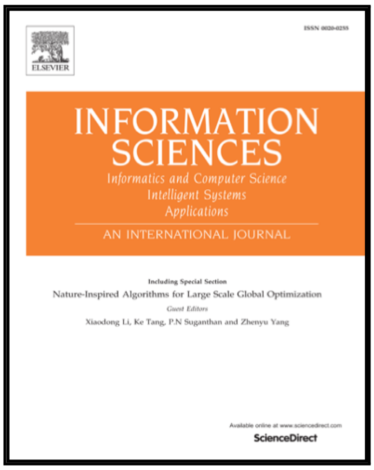

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On Possibilistic Representations of Fuzzy Intervals

Luciano Stefanini*, Maria Letizia Guerra†

Abstract

It is acknowledged that a fuzzy interval has two equivalent representations given in terms of the so called Left and Right sides of the membership function (LR-representation) or in terms of the Lower and Upper branches defining the endpoints of the α-cuts (LU-representation).

In this paper we suggest an additional representation of fuzzy intervals called ACF-representation (using an average cumulative function instead of the membership function), based on possibility theory.

We illustrate how to build the new representation and we state its basic properties. The main result is that the Average Cumulative (AC) function can be uniquely defined for any fuzzy interval and it is possible to move from one representation to the others through appropriate transformations.

An interesting link can be established between ACF-representation and quantile functions, with a possible statistical interpretation useful in real application.

We also recommend a parametric form of the AC function.

KEYWORDS: possibility distribution, parametric representations, fuzzy intervals, quantiles, average cumulative function.

1 Introduction

Representation of fuzzy intervals may take advantage of some key concepts emerging from possibility theory. Possibility theory has been widely studied; in particular, for a given normal, upper-semicontinuous and quasi-concave membership function two dual functions, called the possibility and the necessity measures have been introduced by Dubois and Prade in [12] and [13]. The relationship between membership functions and possibility distributions was primary introduced by Zadeh ([44]) in order to provide a graded semantics to natural language statements. Many other aspects have been focused by Dubois and Prade in [14] (see also the recent paper [16]) for normal fuzzy sets; Klir in

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[27] generalizes the standard fuzzy-set interpretation to non normal fuzzy sets too. Furthermore, Dubois in [11] shows that many notions in statistics are well interpreted by the numerical possibility theory and Baudrit and Dubois (details in [2]) analyze the existing relations between possibility theory, imprecise probability and belief functions. More recently, Couso and Sanchez in [7] and in [8] rephrase the possibilistic interpretation of fuzzy sets to define fuzzy random variables and confidence interval for fuzzy approximations.

In this paper we present a new representation of fuzzy intervals (called ACF-representation) based on possibility theory: associated to the membership function, we define an Average Cumulative Function (ACF), which is monotonic with values in $[0, 1]$. We illustrate how to build the representation and we analyze its basic properties. The ACF can be uniquely defined for any fuzzy interval and we show that the $\alpha-$cuts of a fuzzy interval u can be directly obtained from the ACF.

A relevant aspect motivating the use of the ACF representation stays in the fact that it can be successfully adopted to determine the membership function from experimental data. An interesting link is indeed established between AC function and quantile functions with a possibly statistical interpretation potentially useful in real applications.

The ACF-representation is crucial because a one-to-one relationship can be established between the set of ACFs and the membership functions of fuzzy intervals. In the specific case of continuous membership function, the ACF is continuous too (and monotonic) and it has the same properties of a statistical cumulative distribution function (cdf). We will shortly discuss about parametric ACF functions, based on monotonic spline functions, in order to obtain families of "basic" functions useful in computations and applications.

As well known, any fuzzy interval has two equivalent representations given in terms of the so called Left and Right side of the membership function (LR-representation introduced by Dubois Prade) or in terms of Lower and Upper branches defining the endpoints of the α-cuts (LU-representation introduced by Voxman and Goetschel - see, e.g., [3]). We will analyze the basic relationship between ACF and LU representations, in order to obtain a strict connection between the LR, LU and ACF representations; it is then possible to go from one representation to the other two, through appropriate transformations.

The paper is organized in five sections; in section 2 we define the ACF, its basic properties and its relation with the LU-representation i.e. the $\alpha-$cuts. In section 3 we show the connection between the ACF and the quantile functions and in section 4 the parametric representation of ACF is detailed with examples. The fifth section ends the paper with some comments and ideas for future research.

2 Average cumulative functions associated to a fuzzy interval

We consider real fuzzy intervals u with compact support $[a,b]$ and compact nonempty core $[c,d] \subset [a,b]$ where $a \leq c \leq d \leq b \in \mathbb{R}$; they are defined in terms of a quasi-concave, upper-semicontinuous function $u : \mathbb{R} \longrightarrow [0,1]$ such that $[a,b] = cl(\{x|u(x) > 0\})$ is the support and $[c,d] = \{x|u(x) = 1\}$ is the core (here, $cl(A)$ is the closure of set A). The space of fuzzy intervals will be denoted by $\mathbb{R}_{\mathcal{F}}$.

We first consider the so-called *proper* fuzzy intervals, such that $a < c \leq d < b$. The membership function of $u \in \mathbb{R}_{\mathcal{F}}$ can be represented in the form

$$u(x) = \begin{cases} 0 & \text{if} \quad x < a \\ u^L(x) & \text{if} \quad a \leq x < c \\ 1 & \text{if} \quad c \leq x \leq d \\ u^R(x) & \text{if} \quad d < x \leq b \\ 0 & \text{if} \quad x > b \end{cases} \tag{1}$$

where $u^L : [a,c] \longrightarrow [0,1[$ is a nondecreasing right-continuous function, $u^L(x) > 0$ for $x \in]a,c]$, called the *left side* of the fuzzy interval and $u^R : [d,b] \longrightarrow [0,1]$ is a nonincreasing left-continuous function, $u^R(x) > 0$ for $x \in [d,b[$, called the *right side* of the fuzzy interval. If $c = d$ then u is called a fuzzy number and $\{c\}$ is the core or u. Before the end of subsection 2.1, we will generalize the ACF representation to include any form of fuzzy intervals, including special and non-proper (crisp) cases.

We extend the two functions $u^L(x)$ and $u^R(x)$ to the real domain by setting

$$u^L_{ext}(x) = \begin{cases} 0 & if \quad x < a \\ u^L(x) & if \quad a \leq x < c \\ 1 & if \quad x \geq c \end{cases} \tag{2}$$

$$u^R_{ext}(x) = \begin{cases} 1 & if \quad x \leq d \\ u^R(x) & if \quad d < x \leq b \\ 0 & if \quad x > b. \end{cases} \tag{3}$$

Following Dubois-Kerre-Mesiar-Prade (see, e.g., [18], [19]), a fuzzy interval $u \in \mathbb{R}_{\mathcal{F}}$ can be viewed as a possibility distribution on the real numbers and there exists a pair of cumulative distribution functions, called the lower cdf and the upper cdf of u, respectively, based on the extended left side function $u^L_{ext}(x)$ and the extended right side function $u^R_{ext}(x)$. As described in [18], a fuzzy interval u with membership (1) can be equivalently characterized by the pair (Pos_u, Nec_u) of function $Pos_u : \mathbb{R} \longrightarrow [0,1]$ and $Nec_u : \mathbb{R} \longrightarrow [0,1]$ given by

$$Pos_u(x) = \sup\{u(t) \mid t \leq x\} = u^L_{ext}(x) \tag{4}$$

$$Nec_u(x) = 1 - \sup\{u(t) \mid t > x\} = 1 - \lim_{t \downarrow x} u^R_{ext}(t). \tag{5}$$

By construction, the two distribution functions Pos_u and Nec_u are nondecreasing and càdlàg (French "continue à droite, limite à gauche", right continuous with left limits) in all points belonging to their domain $\mathbb{R}$.

For interpretations of Pos_u and Nec_u it can useful to consider the extended literature on possibility theory, born with the papers [44], [13], [19] and recently extended with the books [4] and [15], [16].

In the present work, instead of the pair (Pos_u, Nec_u), we consider a modified pair of functions where the second component is substituted by

$$F_u^R(x) = 1 - u_{ext}^R(x) = \begin{cases} 0 & if \quad x \leq d \\ 1 - u^R(x) & if \quad d < x \leq b \\ 1 & if \quad x > b. \end{cases} \tag{6}$$

For uniformity of notation, we also denote

$$F_u^L(x) = u_{ext}^L(x). \tag{7}$$

Both functions F_u^L and F_u^R are non decreasing and, from the upper semicontinuity of u, F_u^L is right continuous while F_u^R is left continuous. Clearly, we have:

$$u\,(x) = F_u^L(x) - F_u^R(x) \quad \forall x \in \mathbb{R}. \tag{8}$$

Remark that, for continuous membership functions, we always have $\lim\limits_{t \downarrow x} u_{ext}^R(t) = u_{ext}^R(x)$ and $F_u^R(x) = Nec_u(x)$ and any weighted average (convex combination) of the two functions F_u^L and F_u^R can be used to represent a fuzzy interval.

Definition 1 *For a fixed value of $\lambda \in [0,1]$, the λ-Average Cumulative function (λ-ACF) of u is defined to be the following convex combination of F_u^L and F_u^R, for all $x \in \mathbb{R}$,*

$$\begin{aligned} F_u^{(\lambda)}(x) &= (1-\lambda)F_u^L(x) + \lambda F_u^R(x) \\ &= \begin{cases} 0 & if \quad x < a \\ (1-\lambda)u^L(x) & if \quad a \leq x < c \\ 1-\lambda & if \quad c \leq x \leq d \\ 1-\lambda u^R(x) & if \quad d < x \leq b \\ 1 & if \quad x > b. \end{cases} \end{aligned} \tag{9}$$

$F_u^{(\lambda)}$ is non-decreasing, right continuous on $]-\infty, d[$ and left continuous on $]c, +\infty[$. For the value $\lambda = \frac{1}{2}$ we denote $F_u^{(\frac{1}{2})}(x)$ simply by $F_u(x)$.

Remark 2 *The two functions F_u^L and F_u^R can be derived also in the setting of the functions of bounded variation (BV); indeed, if we define the total variation function of u as:*

$$V_u\,(x) = \sup \left\{ \sum_{j=1}^{n} \left| u\,(t_j) - u\,(t_{j-1}) \right| ; t_j \in \mathbb{P}_x \right\}$$

if $x \in a, b]$, where $\mathbb{P}_x = \{a = t_0 < t_1 < ... < t_n = x\}$ is a finite decomposition of $[a, x]$, $V_u(x) = 0$ if $x \leq a$ and $V_u(x) = V_u(b)$ if $x \geq b$, then we have (see [1]):

$$F_u^L(x) = \frac{V_u(x) + u(x)}{2} \tag{10}$$

$$F_u^R(x) = \frac{V_u(x) - u(x)}{2}. \tag{11}$$

In this context, F_u^L and F_u^R are called the positive and the negative variations of u, respectively, and equation (8) is called the Jordan decomposition of u. It is well known that F_u^L is non-decreasing and right continuous while F_u^R is non-decreasing and left continuous.

Remark 3 *In the continuous case, i.e. when the membership function of u is continuous, we have*

$$\begin{aligned} F_u^{(\lambda)}(x) &= (1-\lambda)u_{ext}^L(x) + \lambda\left(1 - u_{ext}^R(x)\right) \\ &= (1-\lambda)Pos_u(x) + \lambda Nec_u(x). \end{aligned}$$

As a consequence, for continuous u, each function $F_u^{(\lambda)} = (1-\lambda)F_u^L + \lambda F_u^R$ is nondecreasing and càdlàg for all $\lambda \in [0, 1]$; it can be considered as a cdf, as indeed $\lim_{x \longrightarrow -\infty} F_u^{(\lambda)}(x) = 0$ and $\lim_{x \longrightarrow +\infty} F_u^{(\lambda)}(x) = 1$; its generalized inverse, defined by $(F_u^{(\lambda)})^{-1}(t) = \inf\{x \in \mathbb{R} | F_u^{(\lambda)}(x) \geq t\} = \sup\{x \in \mathbb{R} | F_u^{(\lambda)}(x) < t\}$ is also called, in statistical literature, the quantile function of $F_u^{(\lambda)}$.

Remark 4 *The average of the possibility and necessity functions $\frac{1}{2}Pos_u(x) + \frac{1}{2}Nec_u(x)$ is called "credibility distribution" by Liu (see [33]); it coincides with F_u when u is a continuous fuzzy interval.*

In Figure 1 we represent an LR fuzzy interval u and the corresponding functions $F_u^{(\lambda)}$ for some values of $\lambda \in [0, 1]$.

Proposition 5 *The λ-ACF has the following translation property: for a given fuzzy interval $u \in \mathbb{R}_{\mathcal{F}}$ and a number $\rho \in \mathbb{R}$, the translated fuzzy interval $v = u + \rho$, with membership function $v(x) = u(x - \rho)$, is such that*

$$\begin{aligned} F_{u+\rho}^{(\lambda)}(x) &= (1-\lambda)v_{ext}^L(x) + \lambda v_{ext}^R(x) \\ &= (1-\lambda)u_{ext}^L(x-\rho) + \lambda u_{ext}^R(x-\rho) \\ &= F_u^{(\lambda)}(x-\rho). \end{aligned} \tag{12}$$

An interesting connection between the λ-ACF of u and the opposite fuzzy interval $-u$ can be established. We recall that, from the extension principle, the fuzzy number $-u$ can be defined by the following membership function

$$(-u)(x) := u(-x), \text{ for all } x \in \mathbb{R}.$$

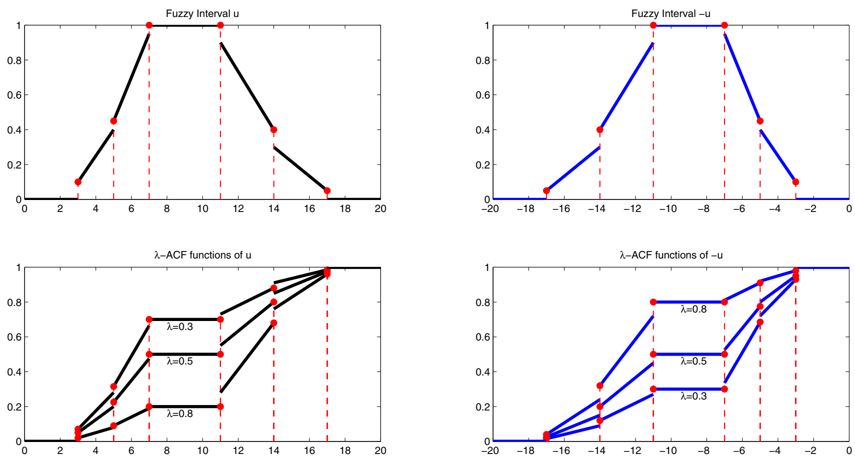

Figure 1: A fuzzy interval u (top,left) and $-u$ (top,right) and the corresponding $lambda$-ACFs, for $\lambda \in \{0.3, 0.5, 0.8\}$.

We can also write $(-u)(-x) := u(x)$, i.e., the membership value $(-u)(-x)$ of $-x \in \mathbb{R}$, with respect to the fuzzy interval $(-u)$, is the same as the membership value of x with respect to u.

We have the following property, relating the λ-ACF of u and the $(1-\lambda)$-ACF of $-u$.

Lemma 6 *Let $u \in \mathbb{R}_{\mathcal{F}}$ and let $-u \in \mathbb{R}_{\mathcal{F}}$ be its opposite interval; then, the following equality is true for all $\lambda \in [0, 1]$*

$$F_u^{(\lambda)}(-x) + F_{-u}^{(1-\lambda)}(x) = 1, \text{ for all } x \in \mathbb{R}$$

where $F_{-u}^{(1-\lambda)}$ is the $(1-\lambda)$-ACF of $-u$.

Proof. Let $[a, b]$ be the support of u and $[c, d]$ its core, with $a \leq c \leq d \leq b$, so that the support and the core of $-u$ are, respectively, $[-b, -a]$ and $[-d, -c]$, with $-b \leq -d \leq -c \leq -a \leq 0$. The membership function of $-u$ is given by

$$(-u)(x) = \begin{cases} 0 & \text{if} \quad x < -b \\ u^R(-x) & \text{if} \quad -b \leq x < -d \\ 1 & \text{if} \quad -d \leq x \leq -c \\ u^L(-x) & \text{if} \quad -c < x \leq -a \\ 0 & \text{if} \quad x > -a \end{cases}$$

so that the extended functions of $-u$ are

$$(-u)_{ext}^L(x) = u_{ext}^R(-x) \tag{13}$$

$$(-u)^R_{ext}(x) = u^L_{ext}(-x) \tag{14}$$

It follows that the two functions $F^L_{-u}(x) = (-u)^L_{ext}(x) = u^R_{ext}(-x)$ and $F^R_{-u}(x) = 1-(-u)^R_{ext}(x) = 1-u^L_{ext}(-x)$ are, respectively, nondecreasing, right continuous and nonincreasing, left continuous.
Then, from $F^{(1-\lambda)}_{-u}(x) = \lambda(-u)^L_{ext}(x) + (1-\lambda)\left(1-(-u)^R_{ext}(x)\right)$ with (13)-(14) and $F^{(\lambda)}_u(-x) = (1-\lambda)F^L_u(-x) + \lambda F^R_u(-x)$ we obtain

$$\begin{aligned} F^{(\lambda)}_u(-x) + F^{(1-\lambda)}_{-u}(x) &= (1-\lambda)\,F^L_u(-x) + \lambda F^R_u(-x) \\ &\quad + \lambda F^L_{-u}(-x) + (1-\lambda)\,F^R_{-u}(x) \\ &= (1-\lambda)\,u^L_{ext}(-x) + \lambda\left(1-u^R_{ext}(-x)\right) \\ &\quad + \lambda u^R_{ext}(-x) + (1-\lambda)\left(1-u^L_{ext}(-x)\right) \\ &= \lambda + (1-\lambda) = 1. \end{aligned}$$

■

Remark 7 *From Lemma 6 we immediately deduce the following formula for the λ-ACF of $-u$:*

$$F^{(\lambda)}_{-u}(-x) = 1 - F^{(1-\lambda)}_u(x), \text{ for all } x \in \mathbb{R}.$$

The latest lemma 6 is applied to prove our main result that is the next theorem 10: it shows that the α-cuts of any fuzzy interval can be obtained by inverting the λ-ACFs of u and $-u$ for any value of $\lambda \in]0,1[$. Its proof will immediately show that the λ-ACFs corresponding to the values $\lambda = 0$ or $\lambda = 1$ are not able to capture completely the α-cuts $[u^-_\alpha, u^+_\alpha]$, as in fact $F^{(0)}_u$ and $F^{(1)}_{-u}$ loose information on u^R, while $F^{(1)}_u$ and $F^{(0)}_{-u}$ loose information on u^L.

Remark 8 *As we have previously shown, the two functions $F^{(\lambda)}_u$ and $F^{(1-\lambda)}_{-u}$ do not have, in general, the properties of a cdf (indeed, $F^{(\lambda)}_u$ is càdlàg on $[a,b]$ only if u^R is continuous and $F^{(1-\lambda)}_{-u}$ is càdlàg on $[-b,-a]$ only if u^L is continuous). But it is straightforward to verify that, for any value of $\lambda \in]0,1[$, the function $F^{(\lambda)}_u$ is càdlàg on $]-\infty,d[$ and the function $F^{(1-\lambda)}_{-u}$ is càdlàg on $]-\infty,-c[$ implying that, at least partially in their domains, they have the properties of a cdf.*

For a given nondecreasing function $F : [a,b] \longrightarrow [0,1]$, the generalized inverse (also called the *quantile function* of F in probability theory, see, e.g. [20], when F is càdlàg) is defined to be the function $F^{-1} : [0,1] \longrightarrow [a,b]$ such that

$$F^{-1}(\alpha) = \inf\{x | F(x) \geq \alpha\} \text{ for all } \alpha \in]0,1] \text{ and } F^{-1}(0) = a \tag{15}$$

Remark 9 *An equivalent definition of (15) for function F, called in [26] the pseudo-inverse $F^{(-)} : [0,1] \longrightarrow \mathbb{R}$ of F, is*

$$F^{(-)}(\alpha) = \begin{cases} a & \text{if } \{x|F(x) < \alpha\} \text{ is empty} \\ \sup\{x|F(x) < \alpha\} & \text{otherwise.} \end{cases} \tag{16}$$

Several properties of the pseudo-inverse of a nondecreasing function are analyzed in [26]. The equivalence between the two definitions in (15) and (16) can be deduced from Theorem 1 in [21], observing that in its proof F is only required to be nondecreasing.

Clearly, F^{-1} is not the ordinary inverse, unless F itself in strictly increasing from 0 to 1. The following well-known properties of F^{-1} (see [20]) have a role in a possibly statistical interpretation of the next Theorem 10:

1) F^{-1} is nondecreasing, left continuous and has right limits $\lim_{h\downarrow 0} F^{-1}(p+h) = \inf\{x|F(x) > p\}$;

2) $F^{-1}(F(x)) \leq x$ for all $x \in [a,b]$ with $0 < F(x) < 1$;

3) $F(F^{-1}(p)) \geq p$ for all $p \in]0,1[$ and for real x it is $F^{-1}(p) \leq x$ if and only if $p \leq F(x)$.

2.1 Main property of ACF representation

The main theorem below shows that the (partial) càdlàg property 1) is anyhow sufficient to determine all the relevant α-cuts $[u_\alpha^-, u_\alpha^+]$ of u, i.e., for $\alpha \in]0,1]$. Recall that the fuzzy interval $-u$ has α-cuts given by $[-u_\alpha^+, -u_\alpha^-]$, so that, in particular, $u_\alpha^+ = -(-u)_\alpha^-$.

Theorem 10 *Let $u \in \mathbb{R}_\mathcal{F}$ and let $F_u^{(\lambda)}$, $F_{-u}^{(1-\lambda)}$ be the λ-ACF of u and the $(1-\lambda)$-ACF of $-u$, respectively, for any given value $\lambda \in]0,1[$. For all $\alpha \in]0,1]$, the α-cut $[u_\alpha^-, u_\alpha^+]$ of u is given by*

$$\begin{aligned} u_\alpha^- &= \inf\left\{x \in [a,c] | F_u^{(\lambda)}(x) \geq (1-\lambda)\alpha\right\} && (17)\\ &= \left(F_u^{(\lambda)}|_*\right)^{-1}((1-\lambda)\alpha) \\ u_\alpha^+ &= -(-u)_\alpha^- = -\inf\left\{x \in [-b,-d] | F_{-u}^{(1-\lambda)}(x) \geq \lambda\alpha\right\} && (18)\\ &= -\left(F_{-u}^{(1-\lambda)}|_*\right)^{-1}(\lambda\alpha) \end{aligned}$$

where $\left(F_u^{(\lambda)}|_*\right)^{-1}$ and $\left(F_{-u}^{(1-\lambda)}|_*\right)^{-1}$ are the generalized inverses of the restrictions of $F_u^{(\lambda)}$ and $F_{-u}^{(1-\lambda)}$ to the subintervals $[a,c]$ and $[-b,-d]$, respectively (or more generally to $]-\infty,c]$ and $]-\infty,-d]$).
In the particular case of $\lambda = \frac{1}{2}$, we obtain, denoting $F_u = F_u^{(\frac{1}{2})}$ and $F_{-u} = F_{-u}^{(\frac{1}{2})}$,

$$\begin{aligned} u_\alpha^- &= \inf\left\{x | F_u(x) \geq \frac{\alpha}{2}\right\} = (F_u)^{-1}\left(\frac{\alpha}{2}\right) && (19)\\ u_\alpha^+ &= -\inf\left\{x | F_{-u}(x) \geq \frac{\alpha}{2}\right\} = -(F_{-u})^{-1}\left(\frac{\alpha}{2}\right). \end{aligned}$$

Proof. Let $\alpha \in]0,1]$ be fixed. Observe first that $F_u^{(\lambda)}(x) = 1-\lambda$ for all $x \in [c,d]$ so that $\inf\{x|F_u^{(\lambda)}(x) \geq (1-\lambda)\alpha\} \leq c$ and $\inf\{x|F_{-u}^{(1-\lambda)}(x) \geq \lambda\alpha\} \leq -d$. Due to (17), we can consider only $x \leq c$ and then the inequality $F_u^{(\lambda)}(x) \geq (1-\lambda)\alpha$ is equivalent to $(1-\lambda)u^L(x) \geq (1-\lambda)\alpha$, i.e., $u^L(x) \geq \alpha$ if $x \in]a,c]$; it follows that $\inf\{x|F_u^{(\lambda)}(x) \geq (1-\lambda)\alpha\} = \inf\{x|u^L(x) \geq \alpha\} = u_\alpha^-$. Analogously, due to (18), we can consider only $x \leq -d$, so that the inequality $F_{-u}^{(1-\lambda)}(x) \geq \lambda\alpha$, using Lemma 6, is equivalent to $\lambda u^R(-x) \geq \lambda\alpha$, i.e., $u^R(-x) \geq \alpha$; it follows that $-\inf\left\{x|F_{-u}^{(1-\lambda)}(x) \geq \lambda\alpha\right\} = \sup\{-x|F_{-u}^{(1-\lambda)}(x) \geq \lambda\alpha\} = \sup\{y|u^R(y) \geq \alpha\} = u_\alpha^+$. ■

Remark 11 *Considering $\lambda = \frac{1}{2}$, we have $F_{-u}(x) = 1 - F_u(-x)$ for all x, so that*

$$\begin{aligned} u_\alpha^+ &= -\inf\{x|1 - F_u(-x) \geq \frac{\alpha}{2}\} \\ &= -\inf\{-t|F_u(t) \leq 1 - \frac{\alpha}{2}\} \\ &= \sup\{x|F_u(x) \leq 1 - \frac{\alpha}{2}\}. \end{aligned}$$

A consequence is that, if $F_u(x)$ is continuous and strictly increasing, u_α^- is such that $F_u(u_\alpha^-) = \frac{\alpha}{2}$ and u_α^+ is such that $F_u(u_\alpha^+) = 1 - \frac{\alpha}{2}$; furthermore, if u has $\{c\}$ as the core and considering $\alpha = 1$, we obtain $c = \inf\{x|F_u(x) \geq \frac{1}{2}\} = \sup\{x|F_u(x) \leq \frac{1}{2}\}$ i.e. $c = \{x|F_u(x) = \frac{1}{2}\}$. The core value c (assumed to be unique) has the same property as the median of $F_u(x)$, when we consider F_u itself as a statistical cdf. In addition, with the same assumptions on F_u, it is immediate that

$$u_\alpha^- = F_u^{-1}(\frac{\alpha}{2}) \text{ and } u_\alpha^+ = F_u^{-1}(1 - \frac{\alpha}{2}). \tag{20}$$

Given any fixed value $\lambda \in]0,1[$, consider a nondecreasing function $F : \mathbb{R} \longrightarrow [0,1]$ satisfying the properties:

1) $a_F = \sup\{x|F(x) = 0\} \in \mathbb{R}$, $b_F = \inf\{x|F(x) = 1\} \in \mathbb{R}$ (clearly $a_F \leq b_F$);

2) $c_F = \inf\{x|F(x) \geq 1-\lambda\} \in \mathbb{R}$, $d_F = \sup\{x|F(x) \leq 1-\lambda\} \in \mathbb{R}$ (clearly $c_F \leq d_F$);

3) $a_F \leq c_F \leq d_F \leq b_F$ and F is right-continuous on $[a_F, c_F[$, left-continuous on $]d_F, b_F]$ and $F(x) = 1-\lambda$ for all $x \in [c_F, d_F]$.

In properties 1 and 2 we practically assume that the sets $\{x|F(x) = 0\}$ and $\{x|F(x) = 1\}$ are not empty implying that F represents a fuzzy number with a compact support and compact nonempty core.

Then there exists a unique fuzzy interval $u_F \in \mathbb{R}_\mathcal{F}$ with λ-ACF, for $\lambda \in]0,1[$ given by F. Indeed, the membership function of u_F is given by (compare with

Definition 1.)

$$u_F(x) = \begin{cases} 0 & \text{if} \quad x < a_F \\ \frac{1}{1-\lambda}F(x) & \text{if} \quad a_F \le x < c_F \\ 1 & \text{if} \quad c_F \le x \le d_F \\ \frac{1}{\lambda}\left(1 - F(x)\right) & \text{if} \quad d_F < x \le b_F \\ 0 & \text{if} \quad x > b_F \end{cases} \tag{21}$$

and, from the assumptions 1), 2) and 3) on F, u_F is a fuzzy interval (the proof is immediate by directly verifying that $u_F \in \mathbb{R}_{\mathcal{F}}$).

We denote by $\mathbb{F}_\lambda(\mathbb{R})$ the family of all functions $F : \mathbb{R} \longrightarrow [0,1]$ satisfying properties 1)-2)-3).

We soon deduce that, for any fixed $\lambda \in]0,1[$ there exists a bijection between the set of fuzzy intervals, $\mathbb{R}_{\mathcal{F}}$, and the family of nondecreasing functions $\mathbb{F}_\lambda(\mathbb{R})$:

$$\phi_\lambda \quad : \quad \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{F}_\lambda(\mathbb{R}),\ \phi_\lambda(u) = F_u^{(\lambda)} \tag{22}$$

$$\phi_\lambda^{-1} \quad : \quad \mathbb{F}_\lambda(\mathbb{R}) \longrightarrow \mathbb{R}_{\mathcal{F}},\ \phi_\lambda^{-1}(F) = u_F. \tag{23}$$

By construction, it is obvious that $\phi_\lambda^{-1}(\phi_\lambda(u)) = u$ for all $u \in \mathbb{R}_{\mathcal{F}}$, and that $\phi_\lambda(\phi_\lambda^{-1}(F)) = F$ for all $F \in \mathbb{F}_\lambda(\mathbb{R})$. We summarize the above result by the following proposition:

Proposition 12 *For any $u \in \mathbb{R}_{\mathcal{F}}$, its λ-AC function given by definition 1 satisfies properties 1),2),3) and, vice-versa, for any F satisfying 1),2),3) there exists a unique element $u_F \in \mathbb{R}_{\mathcal{F}}$, given by (21), having F as its λ-AC function.*

In the particular case of $\lambda = \frac{1}{2}$, the family $\mathbb{F}_\lambda(\mathbb{R})$ will be simply denoted by $\mathbb{F}(\mathbb{R})$ and the bijection ϕ_λ will be denoted by ϕ; the $\frac{1}{2}-$AC function of $u \in \mathbb{R}_{\mathcal{F}}$ is a nondecreasing function $F_u : \mathbb{R} \longrightarrow [0,1]$ such that $F_u(x) = \frac{1}{2}u^L(x)$ on $[a,c[$, $F_u(x) = 1 - \frac{1}{2}u^R(x)$ on $]d,b]$ and $F_u(x) = \frac{1}{2}$ on the core $[c,d]$ of u. On the other hand, if $F \in \mathbb{F}(\mathbb{R})$ is given, the membership function of the corresponding fuzzy number $u_F \in \mathbb{R}_{\mathcal{F}}$ has left and right branches given by $u^L(x) = 2F(x)$ and $u^R(x) = 2 - 2F(x)$

If $u \in \mathbb{R}_{\mathcal{F}}$ is continuous, then $F_u \in \mathbb{F}(\mathbb{R})$ is also continuous; vice-versa, if $F \in \mathbb{F}(\mathbb{R})$ is continuous, then also $u_F \in \mathbb{R}_{\mathcal{F}}$ is continuous. So, the bijection $\phi : \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{F}(\mathbb{R})$ transforms continuous fuzzy intervals into continuous $\frac{1}{2}$-AC functions and the bijection $\phi^{-1} : \mathbb{F}(\mathbb{R}) \longrightarrow \mathbb{R}_{\mathcal{F}}$ transforms continuous $F \in \mathbb{F}(\mathbb{R})$ into continuous $u_F \in \mathbb{R}_{\mathcal{F}}$.

The assumption of a proper fuzzy interval is not really restrictive in our construction and we can consider the λ-ACF representation for three special cases, according to the possible equalities for the values a, c, d, b and to the absence of proper definitions for some components in equation (1):

(i) $a = c \le d < b$ (right fuzzy intervals),
(ii) $a = c \le d = b$ (crisp intervals), and
(iii) $a < c \le d = b$ (left fuzzy intervals).

In case (i), u is a right fuzzy interval with membership function

$$u(x) = \begin{cases} 0 & \text{if} \quad x < a = c \\ 1 & \text{if} \quad c \le x \le d \\ u^R(x) & \text{if} \quad d < x \le b \\ 0 & \text{if} \quad x > b \end{cases} \tag{24}$$

so that

$$u^L_{ext}(x) = \begin{cases} 0 & \text{if} \quad x < a \\ 1 & \text{if} \quad x \ge c \end{cases}, \; u^R_{ext}(x) = \begin{cases} 1 & \text{if} \quad x \le d \\ u^R(x) & \text{if} \quad d < x \le b \\ 0 & \text{if} \quad x > b \end{cases} \tag{25}$$

and we have

$$F_u^{(\lambda)}(x) = \begin{cases} 0 & \text{if} \quad x < a = c \\ 1-\lambda & \text{if} \quad c \le x \le d \\ 1-\lambda u^R(x) & \text{if} \quad d < x \le b \\ 1 & \text{if} \quad x > b. \end{cases} \tag{26}$$

In case (ii), u is a crisp interval $[c, d]$ (improper fuzzy interval), or a crisp number if also $c = d$, with membership function

$$u(x) = \begin{cases} 0 & \text{if} \quad x < a = c \\ 1 & \text{if} \quad c \le x \le d \\ 0 & \text{if} \quad x > d = b \end{cases} \tag{27}$$

so that

$$u^L_{ext}(x) = \begin{cases} 0 & \text{if} \quad x < a = c \\ 1 & \text{if} \quad x \ge c \end{cases}, \; u^R_{ext}(x) = \begin{cases} 1 & \text{if} \quad x \le d \\ 0 & \text{if} \quad x > d = b \end{cases} \tag{28}$$

and we obtain

$$F_u^{(\lambda)}(x) = \begin{cases} 0 & \text{if} \quad x < a = c \\ 1-\lambda & \text{if} \quad c \le x \le d \\ 1 & \text{if} \quad x > d = b. \end{cases} \tag{29}$$

In case (iii), u is a left fuzzy interval with membership function

$$u(x) = \begin{cases} 0 & \text{if} \quad x < a \\ u^L(x) & \text{if} \quad a \le x < c \\ 1 & \text{if} \quad c \le x \le d \\ 0 & \text{if} \quad x > d = b. \end{cases} \tag{30}$$

so that

$$u^L_{ext}(x) = \begin{cases} 0 & \text{if} \quad x < a \\ u^L(x) & \text{if} \quad a \le x < c \\ 1 & \text{if} \quad c \le x \le d \end{cases}, \; u^R_{ext}(x) = \begin{cases} 1 & \text{if} \quad x \le d = b \\ 0 & \text{if} \quad x > b \end{cases} \tag{31}$$

and we have

$$F_u^{(\lambda)}(x) = \begin{cases} 0 & \text{if} \quad x < a \\ (1-\lambda)u^L(x) & \text{if} \quad a \le x < c \\ 1-\lambda & \text{if} \quad c \le x \le d \\ 1 & \text{if} \quad x > d = b. \end{cases} \tag{32}$$

2.2 Arithmetic operations with ACF

It is interesting to express fuzzy arithmetic operations in terms of $\frac{1}{2}$-AC function. If $\circledcirc$ is any binary operation defined on the space $\mathbb{R}_{\mathcal{F}}$, there exists a corresponding operation $\circledcirc'$ on the space $\mathbb{F}(\mathbb{R})$, such that, in terms of bijections ϕ (22) and ϕ^{-1} (23) it holds:

$$F_u \circledcirc' F_v = \phi(u \circledcirc v), \text{ and } u \circledcirc v = \phi^{-1}(F_u \circledcirc' F_v). \tag{33}$$

Among the most popular operations with fuzzy numbers there are the scalar multiplication (scalar different from zero) and the addition of fuzzy intervals. Below we show how the scalar multiplication and binary addition between fuzzy sets can be transformed to the respective operation between AC functions.

As we have seen in the previous section, the relationships between the membership function $u(x)$ and the corresponding AC function $F_u(x)$ are the following (we consider, for simplicity, the case with $a < c < d < b$):

$$u(x) = \begin{cases} 0 & \text{if} \quad x < a \\ u^L(x) & \text{if} \quad a \le x < c \\ 1 & \text{if} \quad c \le x \le d \\ u^R(x) & \text{if} \quad d < x \le b \\ 0 & \text{if} \quad x > b. \end{cases} \tag{34}$$

$$F_u(x) = \begin{cases} 0 & \text{if} \quad x < a \\ \frac{1}{2}u^L(x) & \text{if} \quad a \le x < c \\ \frac{1}{2} & \text{if} \quad c \le x \le d \\ 1 - \frac{1}{2}u^R(x) & \text{if} \quad d < x \le b \\ 1 & \text{if} \quad x > b. \end{cases} \tag{35}$$

The scalar multiplication ku of u is such that, for $k \neq 0$, $(ku)(x) = u\left(\frac{x}{k}\right)$ and for its AC function we have:

Proposition 13 *If u is given by (34) and $k \neq 0$, then the AC function of ku is given by*

$$F_{ku}(x) = \begin{cases} F_u(\frac{x}{k}) & \text{if} \quad k > 0 \\ 1 - F_u(\frac{x}{k}) & \text{if} \quad k < 0 \end{cases} \tag{36}$$

Proof. Consider first the case $k > 0$. We have $(ku)(x) = u\left(\frac{x}{k}\right)$, given by:

$$(ku)(x) = \begin{cases} 0 & \text{if} \quad x < ka \\ u^L\left(\frac{x}{k}\right) & \text{if} \quad ka \le x < kc \\ 1 & \text{if} \quad kc \le x \le kd \\ u^R\left(\frac{x}{k}\right) & \text{if} \quad kd < x \le kb \\ 0 & \text{if} \quad x > kb. \end{cases} \tag{37}$$

from (35) it follows that

$$F_{ku}(x) = \begin{cases} 0 & \text{if} \quad x < ka \\ \frac{1}{2}u^L\left(\frac{x}{k}\right) & \text{if} \quad ka \le x < kc \\ \frac{1}{2} & \text{if} \quad kc \le x \le kd \\ 1 - \frac{1}{2}u^R\left(\frac{x}{k}\right) & \text{if} \quad kd < x \le kb \\ 1 & \text{if} \quad x > kb. \end{cases} \tag{38}$$

i.e. $F_{ku}(x) = F_u(\frac{x}{k})$. To prove the case $k < 0$ (i.e. $(-k) > 0$), from Lemma 6,

$$F_{ku}(x) = 1 - F_{(-k)u}(-x) = 1 - F_u\left(\frac{-x}{-k}\right) = 1 - F_u\left(\frac{x}{k}\right).$$

■

Remark 14 *If $k = 0$ then $ku = \{0\}$ is the crisp $0 \in \mathbb{R}$ and its membership function is:*

$$\{0\}(x) = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases} \tag{39}$$

so that

$$F_{\{0\}}(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}. \tag{40}$$

For the addition of two different proper fuzzy intervals u, v with supports $[a, b]$ and $[a', b']$, we require an additional property of the generalized inverse that, in this case, to avoid confusing notations, it will be denoted by $\varphi^u = F_u^{-1}$ $\varphi^v = F_v^{-1}$. Their addition $\varphi = \varphi^u + \varphi^v$ has itself a generalized inverse φ^{-1} defined on $\mathbb{R}$ with values in $[0, 1]$ and such that $\varphi^{-1}(x) = 0$ if $x < a + a'$, $\varphi^{-1}(x) = 1$ if $x > b + b'$ and

$$\varphi^{-1}(x) = \inf\{\alpha \mid \varphi(\alpha) \geq x\} \tag{41}$$

if $x \in [a + a', b + b']$.

Proposition 15 *The function φ^{-1} defined in (41) is the AC function of $u + v$ and, according to (33), we have:*

$$(F_u \oplus' F_v)(x) = \varphi^{-1}(x) \quad \forall x \in \mathbb{R}$$

$$i.e. \quad (F_u \oplus' F_v) = \left(F_u^{-1} + F_v^{-1}\right)^{-1}$$

Proof. Considering that

$$\varphi_-(\alpha) = \lim_{h \nearrow 0} \varphi(\alpha + h) \text{ and } \varphi_+(\alpha) = \lim_{h \searrow 0} \varphi(\alpha + h)$$

we have

$$\varphi_-(\alpha) \leq \varphi_+(\alpha)$$

with strict inequality iff φ is discontinuous at α and

$$\varphi^{-1}(x) = \alpha \Leftrightarrow x \in \left[\varphi_-(\alpha), \varphi_+(\alpha)\right] \text{ for } x \in [a + a', b + b'].$$

From theorem 10, we have that $F_u^{-1}\left(\frac{\alpha}{2}\right) = u_\alpha^-$ and $F_v^{-1}\left(\frac{\alpha}{2}\right) = v_\alpha^-$. On the other hand, we have $-u_\alpha^+ = F_{-u}^{-1}\left(\frac{\alpha}{2}\right)$ and $-v_\alpha^+ = F_{-v}^{-1}\left(\frac{\alpha}{2}\right)$; but, using Lemma 6, we have:

$$\frac{\alpha}{2} = F_{-u}\left(-u_\alpha^+\right) = 1 - F_u\left(u_\alpha^+\right)$$

and

$$\frac{\alpha}{2} = F_{-v}\left(-v_\alpha^+\right) = 1 - F_v\left(v_\alpha^+\right)$$

so that

$$F_u\left(u_\alpha^+\right) = 1 - \frac{\alpha}{2} \text{ and } F_v\left(v_\alpha^+\right) = 1 - \frac{\alpha}{2}.$$

In terms of the inverses

$$u_\alpha^+ = F_u^{-1}\left(1 - \frac{\alpha}{2}\right) \text{ and } v_\alpha^+ = F_v^{-1}\left(1 - \frac{\alpha}{2}\right).$$

We conclude that

$$F_{u+v}^{-1}\left(\frac{\alpha}{2}\right) = u_\alpha^- + v_\alpha^- = F_u^{-1}\left(\frac{\alpha}{2}\right) + F_v^{-1}\left(\frac{\alpha}{2}\right) = (F_u \oplus' F_v)\left(\frac{\alpha}{2}\right)$$

$$F_{u+v}^{-1}\left(1 - \frac{\alpha}{2}\right) = u_\alpha^+ + v_\alpha^+ = F_u^{-1}\left(1 - \frac{\alpha}{2}\right) + F_v^{-1}\left(1 - \frac{\alpha}{2}\right) = (F_u \oplus' F_v)\left(1 - \frac{\alpha}{2}\right)$$

and

$$F_{u+v}^{-1} = (F_u \oplus' F_v)^{-1} = \varphi^{-1}.$$

■

Example 16 *For linear shaped trapezoidal fuzzy intervals such as $u = \langle a, c, d, b\rangle$ and $v = \langle a', c', d', b'\rangle$, we know that*

$$u + v = \langle a + a', c + c', d + d', b + b'\rangle . \tag{42}$$

Corresponding to the left and right branches of the membership functions, the ACF of u is $F_u(x) = \frac{1}{2}\frac{x-a}{c-a}$ on $[a, c[$, $F_u(x) = 1 - \frac{1}{2}\frac{b-x}{b-d}$ on $]d, b]$ and $F_u(x) = \frac{1}{2}$ on $[c, d]$ (similarly for v); then

$$F_u^{-1}(\alpha) = \begin{cases} a + 2\alpha(c - a) & \text{if } 0 \le \alpha \le \frac{1}{2} \\ d - 2\left(\frac{1}{2} - \alpha\right)(b - d) & \text{if } \frac{1}{2} < \alpha \le 1 \end{cases} \tag{43}$$

(similarly for $F_v^{-1}(\alpha)$). For the addition we have

$$\begin{aligned} F_{u+v}^{-1}(\alpha) &= F_u^{-1}(\alpha) + F_v^{-1}(\alpha) = \\ &= \begin{cases} a + a' + 2\alpha(c + c' - a - a') & \text{if } 0 \le \alpha \le \frac{1}{2} \\ d + d' - 2\left(\frac{1}{2} - \alpha\right)(b + b' - d - d') & \text{if } \frac{1}{2} < \alpha \le 1 \end{cases} \end{aligned}$$

and it holds $F_{u+v} = \left(F_u^{-1} + F_v^{-1}\right)^{-1}$.

3 AC function as a quantile function

In probability and statistical theory the following results are standard.

Consider a given probability space $(\Omega, \mathcal{A}, P)$. It is well known that (see [20]): if U is a real random variable with uniform distribution on $[0, 1]$, then $F : \mathbb{R} \to [0, 1]$ is a cdf with generalized inverse F^{-1} and if we consider the *quantile transformation* $X = F^{-1}(U)$, then the random variable X has exactly F as its cdf.

Let $X_1, X_2, ..., X_N$ be N independent and identically distributed real random variables on $(\Omega, \mathcal{A}, P)$ with common cdf F such that:

$$F_N(x) = \frac{1}{N}\sum_{i=1}^{N} I(X_i \le x), \, x \in \mathbb{R}, \tag{44}$$

where

$$I(X_i \le x) = \begin{cases} 1 & \text{if} \quad X_i \le x \\ 0 & \text{if} \quad x < X_i \end{cases},$$

is the corresponding so called *empirical distribution function*, i.e., the fraction (frequency) of observed values that are smaller or equal to x.

Theorem 17 *(Glivenko-Cantelli, Theorem 2.4.7 in [20]) Assuming that X_1, X_2, ..., X_N are independent and identically distributed with cdf $F(x)$, the following property holds*

$$\sup_{x\in\mathbb{R}} |F_N(x) - F(x)| \to 0 \ \textit{a.s. on} \ \Omega$$

where the limit is obtained as N tends to infinity.

For a real random variable X with cdf F_X, a *quantile* of order $p \in]0,1[$ is a real value x where F_X crosses or jumps over p.

Definition 18 *([42]) A quantile of order $p \in]0,1[$ for a cdf F_X (or for the associated random variable X) is a real value κ_p such that*

$$\lim_{x\uparrow\kappa_p} F_X(x) \le p \quad \textit{and} \quad F_X(\kappa_p) \ge p.$$

Consider a simple sample $x_1, x_2, ..., x_N$ from a real random variable X; for a value of $p \in]0,1]$, the (empirical) p-quantile $\widehat{\kappa}_p(N)$ is obtained by minimizing, with respect to k, (see [10], [29], [30], [43]) the following (empirical) function

$$S_{p,N}(k) = (1-p)\sum_{\substack{i=1 \\ x_i<k}}^{N}(k - x_i) + p\sum_{\substack{i=1 \\ x_i>k}}^{N}(x_i - k); \tag{45}$$

furthermore,

$$\widehat{\kappa}_p(N) = \arg\min_k S_{p,N}(k)$$

is an unbiased estimate of κ_p.

We have now all the elements to show that any AC function can be interpreted in terms of quantile functions and this property will be useful to estimate the membership function of a fuzzy number or interval $u \in \mathbb{R}_{\mathcal{F}}$ by an (empirical) estimate of its AC function in terms of finite samples of independent values in its support.

For simplicity, if not declared explicitly, in the rest of the paper we will consider the λ-AC functions only for $\lambda = \frac{1}{2}$ and the $\frac{1}{2}$-ACF of fuzzy intervals u and $-u$ will be denoted by $F_u(x)$ and $F_{-u}(x)$, respectively.

We will first consider the case of a fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ with a continuous membership function and a single-valued core. In this case, only the AC function F_u is needed to obtain the α-cuts of u and we can apply the Glivenko-Cantelli theorem directly to F_u. The general case will be considered in subsection 3.2.

3.1 The case of continuous u

According to Theorem 10 it is easy to deduce the following proposition.

Proposition 19 *Let $u \in \mathbb{R}_{\mathcal{F}}$ have continuous membership function (1); let $F_u(x)$, $x \in \mathbb{R}$ be its $\frac{1}{2}$-ACF. Then for all $\alpha \in]0,1]$, the α-cuts $[u_\alpha^-, u_\alpha^+]$ of u are such that u_α^- is the $\frac{\alpha}{2}$-quantile of $F_u(x)$ and u_α^+ is the $\frac{\alpha}{2}$-quantile of $F_{-u}(x)$.*

Proof. We have

$$F_u(x) = \frac{1}{2} u_{ext}^L(x) + \frac{1}{2}\left(1 - u_{ext}^R(x)\right)$$

and, from equality $F_{-u}(x) = 1 - F_u(-x)$,

$$F_{-u}(x) = \frac{1}{2} u_{ext}^R(-x) + \frac{1}{2}\left(1 - u_{ext}^L(-x)\right);$$

From the continuity of $u_{ext}^L(x)$ and $u_{ext}^R(x)$ it follows that both F_u and F_{-u} are continuous and their inverses are quantile functions. ■

Let us consider the case where the membership function is given at a finite number of points, i.e. suppose that the fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ is "measured" at N (independent) observations $(t_i, u(t_i))$; this is equivalent to consider a set of independent variables $X_1, X_2, ..., X_N$ identically distributed on the support $[a, b]$ and to extract a simple sample of N distinct values t_i from each X_i, $i = 1, 2, ..., N$.

Consider the decomposition $\mathbb{P}_N = \{x_1 < x_2 < ... < x_N\}$ of the support $[a, b]$, obtained by ordering the t_i such that $t_{(1)} < t_{(2)} ... < t_{(N)}$ and defining $x_i = t_{(i)}$ for $i = 1, 2..., N$. We define the corresponding *empirical AC function* as:

$$\widehat{F}_{\mathbb{P}_N}(x) = \frac{1}{N} \sum_{i=1}^{N} \widehat{I}(x \geq x_i) \tag{46}$$

where

$$\widehat{I}(x \geq x_i) = \begin{cases} 1 & \text{if} \quad x \geq x_i \\ 0 & \text{if} \quad x < x_i \end{cases} . \tag{47}$$

For $\alpha \in]0,1]$, the α-cuts of u can be estimated by computing the empirical $\frac{\alpha}{2}$-quantile of the sample data $\{x_i | i : 1, ..., N\}$ and the empirical $\frac{\alpha}{2}$-quantile of the data $\{-x_i | i : 1, ..., N\}$. To this issue, we have to minimize the two empirical functions, as in eq. (45),

$$S_\alpha^-(m) = (1 - \frac{\alpha}{2}) \sum_{\substack{i=1 \\ x_i < m}}^{N} (m - x_i) + \frac{\alpha}{2} \sum_{\substack{i=1 \\ x_i > m}}^{N} (x_i - m) \tag{48}$$

and

$$S_\alpha^+(m) = (1-\frac{\alpha}{2}) \sum_{\substack{i=1\\ -x_i<m}}^{N} (m+x_i) + \frac{\alpha}{2} \sum_{\substack{i=1\\ -x_i>m}}^{N} (-x_i - m). \tag{49}$$

The obtained values

$$m_\alpha^-(N) = \arg\min_m S_\alpha^-(m) \tag{50}$$

$$m_\alpha^+(N) = \arg\min_m S_\alpha^+(m) \tag{51}$$

give an estimate $[m_\alpha^-(N), m_\alpha^+(N)]$ of the α-cut $[u_\alpha^-, u_\alpha^+]$ of u and are obtained without computing directly the (empirical) AC function from the data.

For a given empirical AC function $\widehat{F}_{\mathbb{P}_N}$, the values $m_\alpha^-(N)$ and $m_\alpha^+(N)$ are called the *plug-in non parametric estimators* of u_α^- and u_α^+ respectively (as in [42]).

The Glivenko-Cantelli theorem can be applied to analyze the convergence of interval $[m_\alpha^-(N), m_\alpha^+(N)]$ to the $\alpha-$cut $[u_\alpha^-, u_\alpha^+]$.

Remark 20 *It is interesting to observe that the empirical function $S_\alpha^+(m)$ can also be written as*

$$S_\alpha^+(m) = \frac{\alpha}{2} \sum_{\substack{i=1\\ x_i<m}}^{N} (m-x_i) + (1-\frac{\alpha}{2}) \sum_{\substack{i=1\\ x_i>m}}^{N} (x_i - m) \tag{52}$$

and, comparing with eq. (45), the estimated value $m_\alpha^+(N)$ is exactly the $(1-\frac{\alpha}{2})$-quantile of the sample data $\{x_i | i : 1, ..., N\}$; so, the extreme values u_α^- and u_α^+ of each α-cut of u can be estimated, statistically, by an $\frac{\alpha}{2}$-quantile and an $(1-\frac{\alpha}{2})$-quantile, respectively.

In the case of a continuous ACF F_u, we have that the $\frac{\alpha}{2}$-quantile and the $\left(1-\frac{\alpha}{2}\right)$-quantile of F_u give exactly the α-cuts of u; as a consequence, the Glivenko-Cantelli theorem ensures that:

Proposition 21 *$\widehat{F}_{\mathbb{P}_N}$ converges to F almost surely uniformly on the support of u and this implies that the empirical intervals $[m_\alpha^-(N), m_\alpha^+(N)]$ will converge to $[u_\alpha^-, u_\alpha^+]$ as $N \longrightarrow \infty$ (assuming that $\{t_1, t_2, ..., t_N\}$ is a simple sample).*

In practical applications the assumption of continuity of F_u is not restrictive and it is a standard approach in statistics to estimate the quantiles and the cumulative distribution function.

Remark 22 *In several statistical software procedures, the quantiles of the empirical distribution of observations t_i, $i = 1, ..., N$, are frequently computed by sorting the data in ascending order, taking the sorted values $x_i = t_{(i)}$ as the quantiles corresponding to probability $p_i = \frac{2i-1}{2N}$, $i = 1, , ..., N$ and using linear interpolation for quantiles corresponding to intermediate probabilities. A similar algorithm is not exact; this is why we adopt (50) and (51). An example is the* *following with* $N = 8$, $x_1 = 2$, $x_2 = 4$, $x_3 = 5$, $x_4 = 6$, $x_5 = 8$, $x_6 = 9$, $x_7 = 11$, $x_8 = 13$*; the empirical fuzzy interval has (piecewise constant) membership function given by*

$$u(x) = \begin{cases} 0 & \text{if} \quad x < 2 \text{ or } x > 13 \\ \frac{1}{4} & \text{if} \quad x \in [2,4[\text{ or } x \in]11,13] \\ \frac{1}{2} & \text{if} \quad x \in [4,5[\text{ or } x \in]9,11] \\ \frac{3}{4} & \text{if} \quad x \in [5,6[\text{ or } x \in]8,9] \\ 1 & \text{if} \quad x \in [6,8] \text{ (the core)} \end{cases} .$$

The core is $[6, 8]$ *and the support is* $[2, 13]$ *so that* $a = 2, b = 13, c = 6, d = 8$.

3.2 The case of a general fuzzy interval u

The two functions defined in the next proposition satisfy the properties of a cdf.

Proposition 23 *Let* $\lambda \in]0, 1[$ *be fixed and* $F_u^{(\lambda)}$ *be the* λ*-ACF of* $u \in \mathbb{R}_\mathcal{F}$ *with membership function (1). Then, the two functions* $\Phi_u^{(\lambda)}, \Phi_{-u}^{(1-\lambda)} : \mathbb{R} \longrightarrow [0, 1]$ *defined by*

$$\Phi_u^{(\lambda)}(x) = \begin{cases} \frac{1}{1-\lambda} F_u^{(\lambda)}(x) & \text{if} \quad x < c \\ 1 & \text{if} \quad x \geq c \end{cases} \tag{53}$$

and

$$\Phi_{-u}^{(1-\lambda)}(x) = \begin{cases} \frac{1}{\lambda} F_{-u}^{(1-\lambda)}(x) & \text{if} \quad x < -d \\ 1 & \text{if} \quad x \geq -d \end{cases} \tag{54}$$

are càdlàg on $\mathbb{R}$ *with* $\Phi_u^{(\lambda)}(x) = 0$ *if* $x < a$ *and* $\Phi_{-u}^{(1-\lambda)}(x) = 0$ *if* $x < -b$.

Proof. For function $\Phi_u^{(\lambda)}$ the proof is immediate because $F_u^{(\lambda)}$ is càdlàg on $]-\infty, d[$; for function $\Phi_{-u}^{(1-\lambda)}$ we have $F_{-u}^{(1-\lambda)}(x) = \lambda F_u^L(x) + (1-\lambda) F_{-u}^R(x)$ and, from equations (13)-(14), for $x < -d$, it is $F_{-u}^{(1-\lambda)}(x) = \lambda F_{-u}^L(x)$ with F_{-u}^L right continuous and càdlàg. ■

For the value $\lambda = \frac{1}{2}$ we will denote the two functions (53) and (54) by $\Phi_u(x)$ and $\Phi_{-u}(x)$, respectively. We have

$$\Phi_u(x) = \begin{cases} 2F_u(x) & \text{if} \quad x < c \\ 1 & \text{if} \quad x \geq c \end{cases} \tag{55}$$

and

$$\Phi_{-u}(x) = \begin{cases} 2F_{-u}(x)) & \text{if} \quad x < -d \\ 1 & \text{if} \quad x \geq -d \end{cases} ; \tag{56}$$

the two functions above are both càdlàg on $\mathbb{R}$ (a similar result has been obtained in [17]).

Consider now the simple samples $\{t_i^L,\ i = 1, 2, ..., N_L\}$ from $\Phi_u(x)$ on its support $[a, c]$ and $\{t_j^R,\ j = 1, 2, ..., N_R\}$ from $\Phi_{-u}(x)$ on its support $[-b, -d]$; let $x_i^L = t_{(i)}^L$ and $x_j^R = t_{(j)}^R$ be the corresponding ascending ordered values and construct the decompositions (eventually by extending the samples to obtain partitions) $\mathbb{P}^L = \{x_1^L < x_2^L < ... < x_{N_L}^L\}$ and $\mathbb{P}^R = \{x_1^R < x_2^R < ... < x_{N_R}^R\}$.

Again, we are assuming that the simple samples are obtained from N_L independent random variables with common cdf Φ_u and N_R independent random variables with cdf Φ_{-u} on the same probability space $(\Omega, \mathcal{A}, P)$.

The empirical AC functions $\widehat{\Phi}_{\mathbb{P}^L}(x)$ and $\widehat{\Phi}_{\mathbb{P}^R}(x)$ are obtained according to eq. (46). Together with the Proposition 23, the Glivenko-Cantelli theorem implies the following property.

Proposition 24 *$\widehat{\Phi}_{\mathbb{P}^L}$ converges to Φ_u almost surely and uniformly and $\widehat{\Phi}_{\mathbb{P}^R}$ converges to Φ_{-u} almost surely and uniformly as N_L and N_R tend to infinity*

$$\begin{aligned}\sup_{x\in\mathbb{R}}\left|\widehat{\Phi}_{\mathbb{P}^L}(x)-\Phi_u(x)\right| &\rightarrow 0 \text{ a.s. on } \Omega\\ \sup_{x\in\mathbb{R}}\left|\widehat{\Phi}_{\mathbb{P}^R}(x)-\Phi_{-u}(x)\right| &\rightarrow 0 \text{ a.s. on } \Omega.\end{aligned}$$

It is immediate to obtain a Glivenko-Cantelli result for the AC function F_u of a general $u \in \mathbb{R}_{\mathcal{F}}$.

Proposition 25 *Let $u \in \mathbb{R}_{\mathcal{F}}$ have AC function F_u and let Φ_u, Φ_{-u} be as in (55)-(56); define the empirical AC function of F_u as*

$$\widehat{F}_{\mathbb{P}_N}(x) = \frac{\widehat{\Phi}_{\mathbb{P}^L}(x) + 1 - \widehat{\Phi}_{\mathbb{P}^R}(-x)}{2}$$

where $\mathbb{P}_N = \mathbb{P}^L \cup \mathbb{P}^R$ is the union of decompositions $\mathbb{P}^L$, $\mathbb{P}^R$ defined above and $N = N_L + N_R$. Then, for $N_L \to \infty$ and $N_R \to \infty$,

$$\sup_{x\in\mathbb{R}}\left|\widehat{F}_{\mathbb{P}_N}(x)-F_u(x)\right| \rightarrow 0 \text{ a.s. on } \Omega.$$

Proof. From the definition of Φ_u and Φ_{-u} we have that, for all $x \in \mathbb{R}$,

$$\Phi_u(x) = u^L_{ext}(x) \text{ and } \Phi_{-u}(x) = \begin{cases} u^R_{ext}(-x) & \text{if } x < -d \\ 1 & \text{if } x \geq -d \end{cases}$$

so that

$$\begin{aligned}\Phi_u(x) &= \begin{cases} 0 & \text{if } x < a \\ u^L(x) & \text{if } a \leq x < c \\ 1 & \text{if } x \geq c \end{cases} \text{ and}\\ 1-\Phi_{-u}(-x) &= \begin{cases} 0 & \text{if } x \leq d \\ 1-u^R(x) & \text{if } d < x \leq b \\ 1 & \text{if } x > b \end{cases} ;\end{aligned}$$

then

$$\Phi_u(x) + 1 - \Phi_{-u}(-x) = \begin{cases} 0 & \text{if } x < a \\ u^L(x) & \text{if } a \leq x < c \\ 1 & \text{if } c \leq x \geq d \\ 2-u^R(x) & \text{if } d < x \leq b \\ 2 & \text{if } x > b \end{cases}, \text{ i.e.,} \tag{57}$$

$$\Phi_u(x) + 1 - \Phi_{-u}(-x) = 2F_u(x). \tag{58}$$

It follows that, for all $x \in \mathbb{R}$,

$$\begin{aligned}\left|\widehat{F}_{\mathbb{P}_N}(x) - F_u(x)\right| &= \left|\frac{\widehat{\Phi}_{\mathbb{P}^L}(x) + 1 - \widehat{\Phi}_{\mathbb{P}^R}(-x)}{2} - \frac{\Phi_u(x) + 1 - \Phi_{-u}(-x)}{2}\right| \\ &\leq \frac{1}{2}\left(\left|\widehat{\Phi}_{\mathbb{P}^L}(x) - \Phi_u(x)\right| + \left|\Phi_{-u}(-x) - \widehat{\Phi}_{\mathbb{P}^R}(-x)\right|\right).\end{aligned}$$

Then

$$\sup_{x\in\mathbb{R}}\left|\widehat{F}_{\mathbb{P}_N}(x) - F_u(x)\right| \leq \frac{1}{2}\sup_{x\in\mathbb{R}}\left|\widehat{\Phi}_{\mathbb{P}^L}(x) - \Phi_u(x)\right| + \frac{1}{2}\sup_{x\in\mathbb{R}}\left|\Phi_{-u}(-x) - \widehat{\Phi}_{\mathbb{P}^R}(-x)\right|$$

and the proof follows from Proposition 24, i.e. by the simultaneous application of the Glivenko-Cantelli theorem to Φ_u and Φ_{-u}. ∎

The final step is now to apply the empirical quantile procedure to obtain the estimates $m_\alpha^-(N_L)$ and $m_\alpha^+(N_R)$ of u_α^- and u_α^+, respectively. From eq. (57) the $\frac{\alpha}{2}$-quantile of $F_u(x)$ and $F_{-u}(x)$ will correspond to the α-quantile of $\Phi_u(x)$ and $\Phi_{-u}(x)$, respectively; we define the two (empirical) objective functions

$$S_\alpha^L(m) = (1-\alpha)\sum_{\substack{i=1\\x_i^L<m}}^{N_L}(m - x_i^L) + \alpha\sum_{\substack{i=1\\x_i^L>m}}^{N_L}(x_i^L - m) \tag{59}$$

and

$$S_\alpha^R(m) = \alpha\sum_{\substack{i=1\\x_i^R<m}}^{N_R}(m - x_i^R) + (1-\alpha)\sum_{\substack{i=1\\x_i^R>m}}^{N_R}(x_i^R - m). \tag{60}$$

The obtained values

$$m_\alpha^-(N_L) = \arg\min_m S_\alpha^L(m) \tag{61}$$

$$m_\alpha^+(N_R) = \arg\min_m S_\alpha^R(m) \tag{62}$$

give an estimate $[m_\alpha^-(N_L), m_\alpha^+(N_R)]$ of the α-cut $[u_\alpha^-, u_\alpha^+]$ of u and, also in this case, are obtained without computing directly the (empirical) AC function from the data.

3.3 Computational experiments

In order to evaluate the applicability of equations (50) and (51) to approximate the α-cuts of a fuzzy number we show a series of five experiments by generating 100 random samples from the ACF of a continuous fuzzy interval for different sampling techniques.

We consider the fuzzy number $u \in \mathbb{R}_\mathcal{F}$ having α-cuts

$$[u_\alpha^-, u_\alpha^+] = [10\alpha^{0.5}, 12 - 2\alpha^{1.5}],\ \alpha \in [0,1].$$

The core of u is $c = 10$ and the support is $[0, 12]$. Observations from u can be generated by sampling u_α^- and u_α^+ at n values α_i, $i = 1, ..., n$, uniformly between

0 and 1, for different values of n (so than a total of $N = 2n$ data are obtained). Furthermore, to verify robustness, we apply (50),(51) to randomly generated fuzzy numbers $u^{(k)}$, $k = 1, ..., K$ by perturbing the core and/or the support of u, for a given number K of replications.

In each experiment, for a fixed $n \in \{11, 21, 51, 101\}$, $K = 100$ random samples $[(u^{(k)})_i^-, (u^{(k)})_i^+]$ are generated ($i = 1, ..., n$, $k = 1, ..., K$) and (50-51) are applied for each k to obtain $L = 41$ estimated *level*-cuts $[\widehat{u}_{\beta_j}^{(k)-}, \widehat{u}_{\beta_j}^{(k)+}]$ of u, with $\beta_j = \frac{j-1}{L-1}$ ($j = 1, ..., L$); finally, the averages $(\overline{u})_{\beta_j}^- = \frac{1}{K}\sum_{k=1}^{K} \widehat{u}_{\beta_j}^{(k)-}$ and $(\overline{u})_{\beta_j}^+ = \frac{1}{K}\sum_{k=1}^{K} \widehat{u}_{\beta_j}^{(k)+}$ are compared with the exact α-cuts of u, $u_{\beta_j}^- = 10\beta_j^{0.5}$, $u_{\beta_j}^+ = 12 - 2\beta_j^{1.5}$. The percentage average absolute error (being $u_{\beta_j}^- \geq 0$, the denominator is set to $1 + u_{\beta_j}^-$ to avoid possible division by zero)

$$AERR = \frac{100}{2L}\sum_{j=1}^{L}\left(\left|\frac{(\overline{u})_{\beta_j}^- - u_{\beta_j}^-}{1 + u_{\beta_j}^-}\right| + \left|\frac{(\overline{u})_{\beta_j}^+ - u_{\beta_j}^+}{1 + u_{\beta_j}^+}\right|\right)$$

is computed for each experiment and different values of n.

In all figures describing the results, the top subplot shows the $K = 100$ replications of the sampled observation points $\left((u^{(k)})_i^-, \alpha_i^{(k)}\right)$ and $\left((u^{(k)})_i^+, \alpha_i^{(k)}\right)$, for $i = 1, ..., n$ and for the selected n (i.e. the membership functions of each $u^{(k)}$). The bottom-left subplot reproduces the analogous points $\left((\widehat{u}^{(k)})_j^-, \beta_j^{(k)}\right)$ and $\left((\widehat{u}^{(k)})_j^+, \beta_j^{(k)}\right)$, i.e. the K membership functions of the fuzzy numbers obtained by (50-51). Finally, the bottom-right subplot reproduces the estimated membership functions (black crosses) and the original membership corresponding to the β_j-cuts, $j = 1, ..., L$ (blue circles).

In the first experiment, the samples $[(u^{(k)})_i^-, (u^{(k)})_i^+]$ are obtained from u simply for different selections of the "observed" levels $\alpha_i^{(k)}$, $i = 1, ..., n$, randomly generated from a uniform distribution between 0 and 1, and replicated K times. In figure 2, the case with $n = 21$ is pictured.

In table 1, the AERR is given for the different values of n; we see that it decreases quickly as the number of the observed levels increases from $n = 11$ to $n = 101$.

Table 1: AERR for different values of n in first experiment.

n	11	21	51	101
$AERR$	4.33%	2.26%	0.90%	0.36%

The aim of the next four experiments is to verify the robustness of the introduced estimation of u through its α-cuts, with respect to different ways to perturb data:

1. the core is not the constant $c = 10$, but $c^{(k)} = c + \xi_{c,k}$ where each $\xi_{c,k}$ is a normal variable with distribution $N(0, \sigma_c)$ (mean 0 and variance σ_c^2);

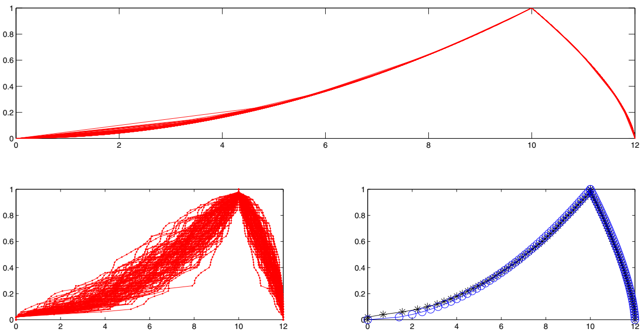

Figure 2: First experiment, = 21.

2. the left value of the support is not the constant $a = 0$, but $a^{(k)} = \xi_{a,k}$, where each $\xi_{a,k}$ is a normal variable randomly generate from a distribution $N(0, \sigma_a)$ (mean 0 and variance σ_a^2);

3. the right value of the support is not the constant $b = 12$, but $b^{(k)} = 12 + \xi_{b,k}$, where each $\xi_{b,k}$ is a normal variable with distribution $N(0, \sigma_b)$ (mean 0 and variance σ_b^2).

As for the first experiment, we sample u_α^- and u_α^+ at n points α_i generated from uniform distribution on [0, 1] as follows: $\alpha_1 = 0$, $\alpha_i = rand()$ and $\alpha_n = 1$ ($N = 2n$ and $rand()$ is a uniform pseudo-random number generator); the data are then computed with the same shape as for u but with the modified core $c^{(k)}$ and support $[a^{(k)}, b^{(k)}]$ (provided that $a^{(k)} < c^{(k)} < b^{(k)}$)

$$\begin{aligned}(u^{(k)})_i^- &= a^{(k)} + (c^{(k)} - a^{(k)})\alpha_i^{0.5} \\ (u^{(k)})_i^+ &= b^{(k)} - (b^{(k)} - a^{(k)})\alpha_i^{1.5}\end{aligned}$$

The second experiment uses $\sigma_c = 1.0$, $\sigma_a = 0.0$ and $\sigma_b = 0.0$, i.e. only the core is perturbed. Consider that $\sigma_c = 1.0$ produces a relatively big perturbation with respect to $c = 10.0$. This appears in figure 3, where the case with $n = 21$ is pictured.

In table 2, the AERR for the second experiment is given for the different values of n; also for this experiment the AERR rapidly decreases for $n = 11$ to $n = 101$.

Table 2: AERR for different values of n in second experiment.

n	11	21	51	101
$AERR$	4.38%	2.29%	0.90%	0.35%

The third experiment uses $\sigma_c = 1.0$, $\sigma_a = 0.5$ and $\sigma_b = 1.0$, i.e. the core and the support are both changed with relatively big perturbations. This appears in figure 4, where the case with $n = 21$ is pictured, and in table 3, for different values of n.

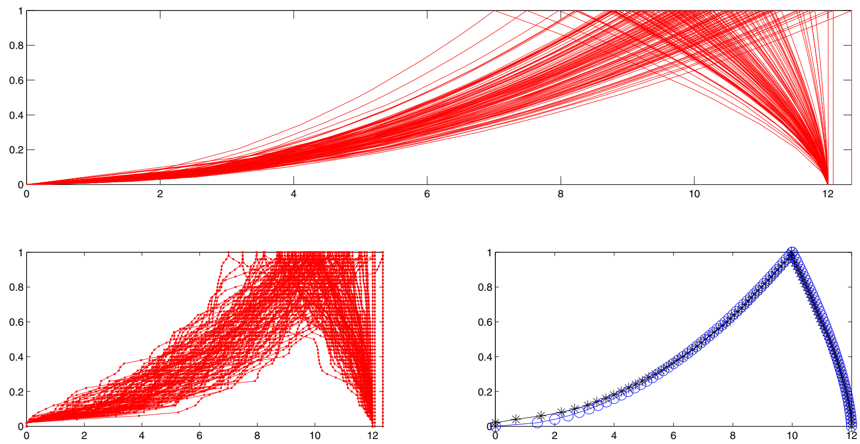

Figure 3: Second experiment, with $\sigma_c = 1.0$, $\sigma_a = 0.0$, $\sigma_b = 0.0$ and $n = 21$.

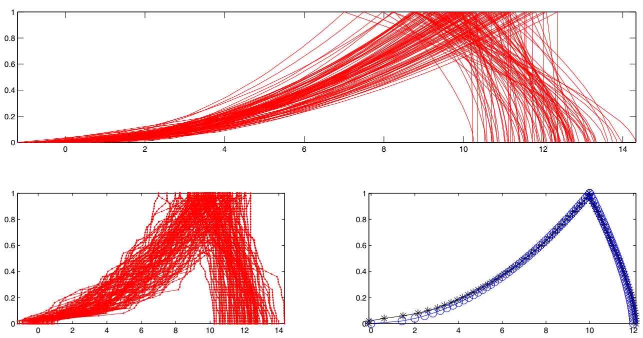

Figure 4: Third experiment, with $\sigma_c = 1.0$, $\sigma_a = 0.5$, $\sigma_b = 1.0$ and $n = 21$.

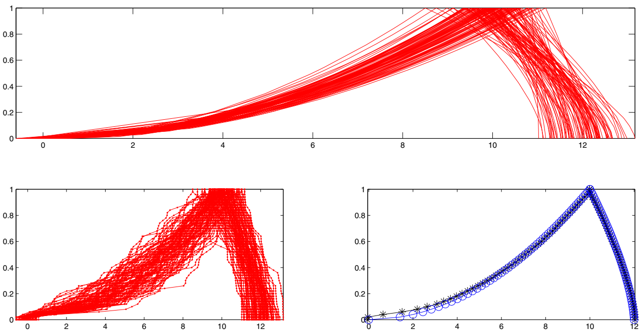

Figure 5: Fourth experiment, with $\sigma_c = 0.5$, $\sigma_a = 0.25$, $\sigma_b = 0.5$ and $n = 21$.

Table 3: AERR for different values of n in third experiment.

n	11	21	51	101
$AERR$	4.81%	2.71%	1.26%	0.81%

Finally, the last two experiments are obtained by progressively reducing the perturbations of the core and the support; in the fourth case, we chose $\sigma_c = 0.5$, $\sigma_a = 0.25$ and $\sigma_b = 0.5$, in the fifth case we chose $\sigma_c = 0.2$, $\sigma_a = 0.1$ and $\sigma_b = 0.2$. Tables 4 and 5 give the corresponding $AERR$ for different n.

Table 4: AERR for different values of n in fourth experiment.

n	11	21	51	101
$AERR$	4.54%	2.42%	1.03%	0.46%

Table 5: AERR for different values of n in fifth experiment.

n	11	21	51	101
$AERR$	4.41%	2.32%	0.94%	0.37%

Resuming the results of the five experiments it follows that the AERR has the same order of magnitude independently from the perturbations we apply and this may be viewed as a good robustness property.

4 Parametric representation of ACFs

As in section 2, we consider fuzzy intervals $u \in \mathbb{R}_\mathcal{F}$ with membership function $u : \mathbb{R} \longrightarrow [0, 1]$, with compact support $[a, b]$ (where $a = \inf\{x|u(x) > 0\}$ and

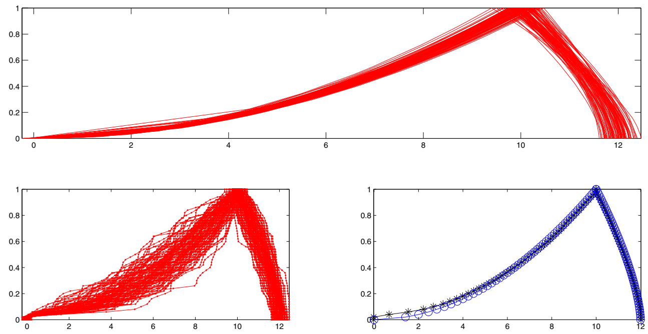

Figure 6: Fifth experiment, with $\sigma_c = 0.2$, $\sigma_a = 0.1$, $\sigma_b = 0.2$ and $n = 21$.

$b = \sup\{x|u(x) > 0\}$) and nonempty compact core $[c, d]$, $c \leq d$ (where $c = \inf\{x|u(x) = 1\}$ and $d = \sup\{x|u(x) = 1\}$).

From the same section it comes to light that any $u \in \mathbb{R}_{\mathcal{F}}$ has tree parametric representations:

▶ LR-parametric u_{LR}, with decompositions of the support for left and right sides;

▶ LU-parametric u_{LU}, with decompositions of $[0, 1]$ for the lower and upper branches;

▶ ACF-parametric u_{ACF}, i.e.,

$$u_{ACF} = \{(x_i^L, F_i^L), (x_j^R, F_j^R)|i = 0, 1, ..., N_L, j = 0, 1, ..., N_R\}$$

with decompositions of left and right subintervals for *quasi-càdlàg* function F_u. Using appropriate transformations of the parameters we can obtain each one from the other; for example from LR we can obtain ACF as in (21).

Remark 26 *The ACF-representation can be related to the horizontal membership functions introduced in [35] and to the RDM arithmetic in [34]. Essentially, the so called horizontal membership function of a fuzzy interval* $u \in \mathbb{R}_{\mathcal{F}}$ *with* α*-cuts* $[u_\alpha^-, u_\alpha^+]$ *is represented in terms of the function*

$$H_u(\alpha, t_u) = (1 - t_u)u_\alpha^- + t_u u_\alpha^+ \text{ for all } t_u \in [0, 1],\ \alpha \in]0, 1].$$

We then have, in general, for $\alpha \in]0, 1]$,

$$H_u(\alpha, t_u) = (1 - t_u)(F_u|_*)^{-1}(\frac{\alpha}{2}) - t_u(F_{-u}|_*)^{-1}(\frac{\alpha}{2}),\ t_u \in [0, 1]$$

and, for a continuous strictly increasing AC function F_u *with ordinary inverse* F_u^{-1},

$$H_u(\alpha, t_u) = (1 - t_u)F_u^{-1}(\frac{\alpha}{2}) + t_u F_u^{-1}(1 - \frac{\alpha}{2}), t_u \in [0, 1].$$

In the rest of this paper, we assume that F is continuous and we denote by $\mathbb{F}_c(\mathbb{R})$ the subfamily of continuous functions of $\mathbb{F}(\mathbb{R})$.

A general approximation for functions $F \in \mathbb{F}_c(\mathbb{R})$ can be obtained by adopting parametric monotonic functions of the same type as suggested in [39] and [40], e.g., the (2,2)-rational function $p : [0,1] \longrightarrow [0,1]$ defined, for fixed but arbitrary $\beta_0, \beta_1 \geq 0$, by

$$p(t;\beta_0,\beta_1) = \frac{t^2 + \beta_0 t(1-t)}{1 + (\beta_0 + \beta_1 - 2)t(1-t)},\ t \in [0,1] \tag{63}$$

The basic properties of p are that, for all $\beta_0, \beta_1 \geq 0$, $p(0;\beta_0,\beta_1) = 0$, $p(1;\beta_0,\beta_1) = 1$, its derivative is nonnegative (considering right derivative at $t = 0$ and left derivative at $t = 1$) and $p'(0;\beta_0,\beta_1) = \beta_0$, $p'(1;\beta_0,\beta_1) = \beta_1$. By changing the values of $\beta_0, \beta_1 \geq 0$, functions (63) generate an infinite number of monotonic increasing functions.

The "shape" functions $p(t;\beta_0,\beta_1)$ can be adopted to represent functions $F \in \mathbb{F}_c(\mathbb{R})$ "piecewise" on two decompositions of the intervals $[a_F, c_F]$ and $[d_F, b_F]$ into N_L subintervals $a_F = x_0^L < x_1^L < ... < x_{N_L}^L = c_F$ and N_R subintervals $d_F = x_0^R < x_1^R < ... < x_{N_R}^R = b_F$; at the extreme points of each subinterval $I_i^L = \left[x_{i-1}^L, x_i^L\right]$ and $I_j^R = \left[x_{j-1}^R, x_j^R\right]$ the values of F are fixed to the nondecreasing values F_i^L, $i = 0,1,...,N_L$ and F_j^R, $j = 0,1,...,N_R$ with $F_0^L = 0 < F_i^L < F_{N_L}^L = \frac{1}{2}$ (with $F_{i-1}^L \leq F_i^L$ for $i = 2,...,N_L - 1$) and $F_0^R = \frac{1}{2} < F_j^R < F_{N_R}^R$ (with $F_{j-1}^R \leq F_j^R$ for $j = 2,...,N_R - 1$).

Finally, $F \in \mathbb{F}_c(\mathbb{R})$, is constructed by choosing $N_L + N_R$ pairs of nonnegative parameters $(\beta_{0,i}^L, \beta_{1,i}^L)$, $(\beta_{0,j}^R, \beta_{1,j}^R)$ for all $i = 1,...,N_L$ and $j = 1,...,N_R$ (the slopes of F at the extremes of each subinterval I_i^L and I_j^R) and by setting

$$F(x) = \begin{cases} 0 & \text{if} \quad x < a_F \\ F_{i-1}^L + (F_i^L - F_{i-1}^L)p(\frac{x - x_{i-1}^L}{x_i^L - x_{i-1}^L}; \beta_{0,i}^L, \beta_{1,i}^L) & \text{if} \quad x_{i-1}^L \leq x < x_i^L \\ \frac{1}{2} & \text{if} \quad c_F \leq x \leq d_F \\ F_{j-1}^R + (F_j^R - F_{j-1}^R)p(\frac{x - x_{j-1}^R}{x_j^R - x_{j-1}^R}; \beta_{0,j}^R, \beta_{1,j}^R) & \text{if} \quad x_{j-1}^R < x \leq x_j^R \\ 1 & \text{if} \quad x > b_F \end{cases} \tag{64}$$

It is easy to check that the construction leads to functions $F \in \mathbb{F}_c(\mathbb{R})$.

For the case when $N_L = N_R = 1$, we have a simple construction where a single standardized function $p\left(t;\beta_0^L,\beta_1^L\right)$ is used to represent F on $[a_F, c_F]$ and another $p\left(t;\beta_0^R,\beta_1^R\right)$ to represent F on $[d_F, b_F]$

$$F(x) = \begin{cases} 0 & \text{if} \quad x < a_F \\ p(\frac{x - a_F}{c_F - a_F}; \beta_0^L, \beta_1^L) & \text{if} \quad a_F \leq x < c_F \\ \frac{1}{2} & \text{if} \quad c_F \leq x \leq d_F \\ \frac{1}{2}\left(1 + p(\frac{x - d_F}{b_F - d_F}; \beta_0^R, \beta_1^R)\right) & \text{if} \quad d_F < x \leq b_F \\ 1 & \text{if} \quad x > b_F \end{cases}. \tag{65}$$

This simple construction requires 8 parameters, i.e. the four values $a_F \leq c_F \leq d_F \leq b_F$ for the support and the core of the corresponding u_F and the four nonnegative parameters (β_0^L, β_1^L) and (β_0^R, β_1^R) used to fix the slopes of F at the points a_F, c_F and d_F, b_F, respectively.

Remark 27 *The described method allows a way to generate fuzzy random intervals, as introduced and analyzed by several authors to model imprecisely valued (fuzzy valued) random variables ([32], [37], [24], [25], [22]). In this setting, given a probability space $(\Omega, \mathcal{A}, P)$, a real fuzzy valued interval can be parametrized using (65) by defining a random support $[a_F(\omega), b_F(\omega)]_{\omega\in\Omega}$, and a random core $[c_F(\omega), d_F(\omega)]$ such that $a_F(\omega) \leq c_F(\omega) \leq d_F(\omega) \leq b_F(\omega)$ for all $\omega \in \Omega$, and random standardized functions $p_\omega\left(t; \beta_0^L, \beta_1^L\right), q_\omega(t; \beta_0^R, \beta_1^R)$ for $\omega \in \Omega$. The immediate way is to generate the support $[a_F, b_F]$, the core $[c_F, d_F]$ and the four parameters β_0^L, β_1^L, $\beta_0^R, \beta_1^R \geq 0$ and use (65) to obtain the corresponding random AC function $F_\omega(x)$. More generally, a random AC function can be parametrized by $2+N_L+N_R$ random variables $x_i^L(\omega)$, $i = 0, 1, ..., N_L$ and $x_j^R(\omega)$, $j = 0, 1, ..., N_R$ together with $2N_L$ random parameters $\beta_{0,i}^L(\omega), \beta_{1,i}^L(\omega)$, $i = 1, ..., N_L$ and $2N_R$ random parameters $\beta_{0,j}^R(\omega), \beta_{1,j}^R(\omega)$, $j = 1, , .N_R$; the corresponding AC function F_ω is then obtained by equation (64).*

Examples of functions F and corresponding u_F for different pairs (β_0^L, β_1^L) and (β_0^R, β_1^R) are pictured in figure 7, where $a_F = 1$, $b_F = 5$, $c_F = 2+0.5\,\text{rand}()$, $d_F = 3+0.5\,\text{rand}()$ and the various parameters βs are generated between 0 and 2 by $\beta = 2\,\text{rand}()$.

In applications where we are interested in generating fuzzy numbers u_F with a single-valued core, without specifying its value *a priori*, we can model the ACF by fixing the support $[a_F, b_F]$, $a_F < b_F$, and the two end-slope parameters $\beta_0^L = \beta_a \geq 0$, $\beta_1^R = \beta_b \geq 0$ so that

$$F(x) = \begin{cases} 0 & \text{if} \quad x < a_F \\ p(\frac{x-a_F}{b_F-a_F}; \beta_a, \beta_b) & \text{if} \quad a_F \leq x \leq b_F \\ 1 & \text{if} \quad x > b_F \end{cases} ; \tag{66}$$

the core of u_F can be computed simply by solving for the unique value of $x \in [a_F, b_F]$ that solves the equation $F(x) = \frac{1}{2}$, i.e. by solving the equation

$$q(t) = t^2 + \beta_a t(1-t) - \frac{1}{2} - \frac{1}{2}(\beta_a + \beta_b - 2)t(1-t) = 0$$

for the unique root $t_F \in]0, 1[$ (observe that $q(0) = -\frac{1}{2}$ and $q(1) = \frac{1}{2}$ and $q(t)$ is quadratic), then the core of u_F is obtained as $c_F = a_F + t_F(b_F - a_F)$.

Some examples for different values of β_a, β_b are given in figure 8, where $a_F = 1$, $b_F = 5$, $c_F = d_F$ (the core is a singleton) and β_a, β_b are generated between 0 and 5.

A final interesting case with $\beta_a = \beta_b = \beta$ is in figure 9. Again, $a_F = 1$, $b_F = 5$, and β is generated randomly between 0 and 2.

It is immediate to see that in the last case (if $\beta_a = \beta_b$ in (66)) the associated fuzzy number is symmetric with respect to the core $c_F = \frac{a_F+b_F}{2}$.

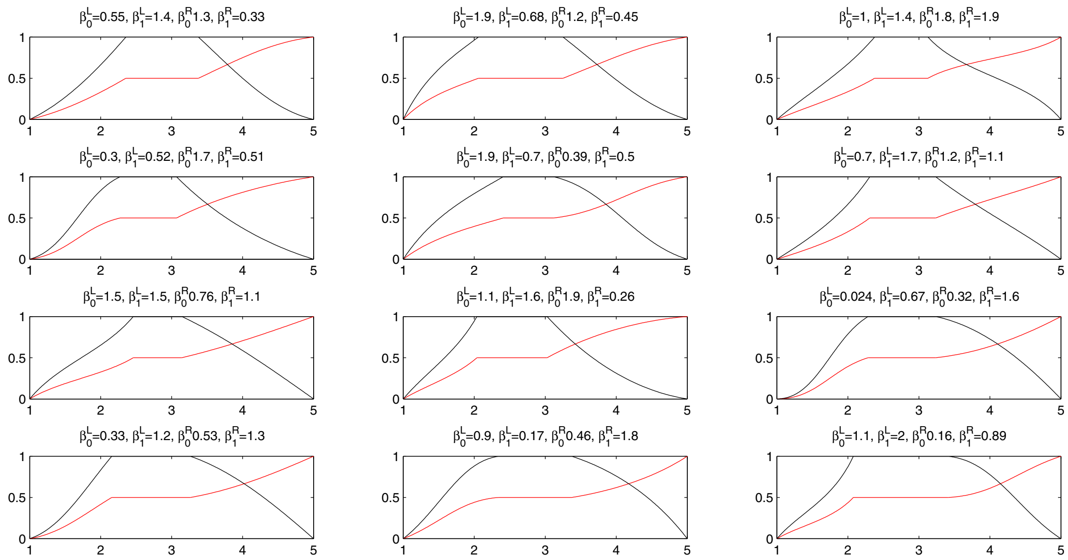

Figure 7: Randomly generated $F \in \mathbb{F}_c(\mathbb{R})$ and corresponding fuzzy intervals u_F

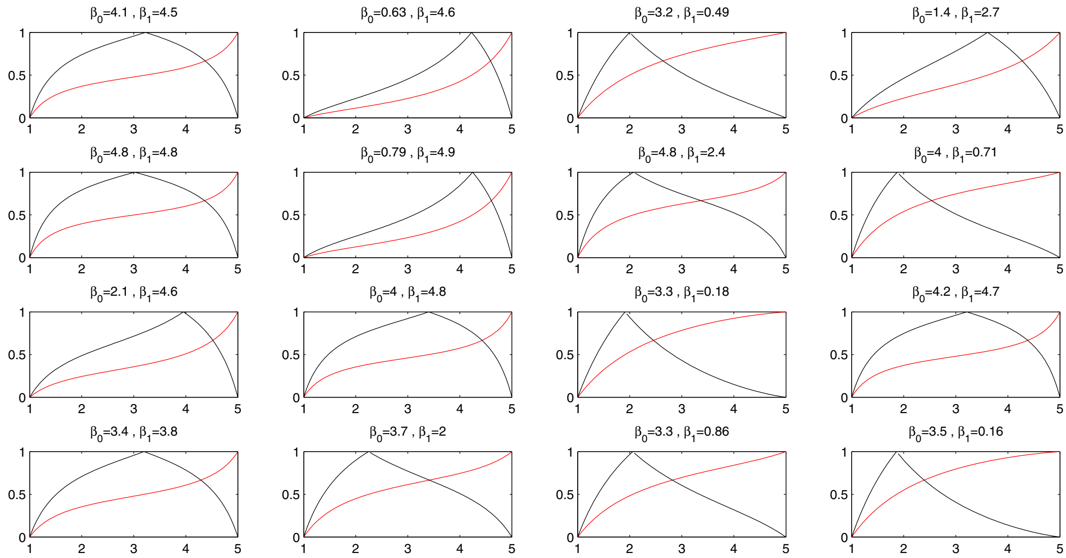

Figure 8: Randomly generated $F \in \mathbb{F}_c(\mathbb{R})$ and corresponding fuzzy numbers u_F

5 Conclusions and further research

We introduce the ACF-representation of fuzzy intervals, its parametric form and its properties; the ACF is based on some concepts shared with possibility theory.

We prove that the ACF can be uniquely defined for any fuzzy interval and we establish a relationship between ACF and quantile functions with a possibly statistical interpretation.

Further research involves some computational, empirical and theoretical aspects about several topics:

- arithmetic operations: using an approach similar to probabilistic arithmetic, which is based on convolutions with density functions (as in [31], [23]) we will try to express fuzzy arithmetic in terms of AC functions, as we have done for scalar multiplication and addition in subsection 2.2. This approach has been extensively addressed by developing a very efficient software like the packages "**distr**" and "**distrEx**" in R language ([28]);
- membership estimation through observations (see for example [17] and [9]); as we have seen, the empirical AC function $\widehat{F}_{\mathbb{P}_N}(x)$ is an unbiased

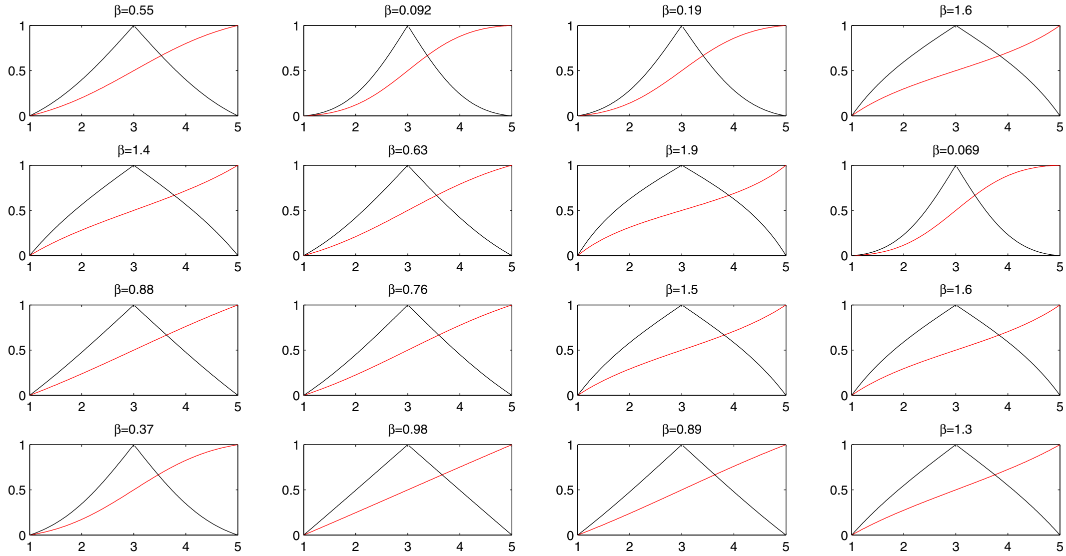

Figure 9: Randomly generated $F \in \mathbb{F}_c(\mathbb{R})$ and corresponding core-symmetric u_F

estimator of the AC function $F_u(x)$ of a fuzzy interval $u \in \mathbb{R}_{\mathcal{F}}$, for all fixed $x \in \mathbb{R}$. In the case of a simple sample, the average and variance of $\widehat{F}_{\mathbb{P}_N}(x)$ (for any fixed x) can be easily estimated by standard statistical procedure as indeed $\mathbb{E}[\widehat{F}_{\mathbb{P}_N}(x)] = F_u(x)$ and $Var[\widehat{F}_{\mathbb{P}_N}(x)] = \frac{1}{N} F_u(x)(1 - F_u(x))$ (see [42], Chapter 3). This is a possible starting point to compute confidence intervals for the estimated α-cuts $[m_\alpha^-(N), m_\alpha^+(N)]$, e.g. by the well known bootstrap method.

- generation of random fuzzy intervals and possible metrics on ACFs that focus on useful topological structures (see for example [38] and [41]);
- ACF approximation through F-transform: the ACF-representation based on monotonic functions eases the search of approximation methods and algorithms (as in [5] and [6]);
- relationship between probability distributions and membership functions ([36], [17]).

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