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HÖRMANDER VECTOR FIELDS EQUIPPED WITH DILATIONS: LIFTING, LIE-GROUP CONSTRUCTION, APPLICATIONS

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Let $X = \{X_1, \dots, X_m\}$ be a set of Hörmander vector fields in $\mathbb{R}^n$, where any X_j is homogeneous of degree 1 with respect to a family of non-isotropic dilations in $\mathbb{R}^n$. If N is the dimension of $\mathrm{Lie}\{X\}$, we can either lift X to a system of generators of a higher dimensional Carnot group on $\mathbb{R}^N$ (if $N > n$), or we can equip $\mathbb{R}^n$ with a Carnot group structure with Lie algebra equal to $\mathrm{Lie}\{X\}$ (if $N = n$). We shall deduce these facts via a local-to-global procedure (available in the homogeneous setting), starting from general results on the lifting of finite-dimensional Lie algebras of vector fields. The use of the Baker-Campbell-Hausdorff Theorem is crucial. Due to homogeneity, the lifting procedure is simpler than Rothschild-Stein's lifting technique. We finally provide applications to the study of the fundamental solution Γ for the Hörmander sum of squares $\sum_{j=1}^{m} X_j^2$, including global pointwise estimates of Γ and of its X-derivatives in terms of the Carnot-Carathéodory distance induced by X.

We review some recent results obtained with Stefano Biagi [1–3], and with Stefano Biagi and Marco Bramanti [5].

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1. Hörmander vector fields equipped with dilations δ_λ

In this paper we assume that $X = \{X_1, \dots, X_m\}$ (with $m \geq 2$) is a set of smooth vector fields on space $\mathbb{R}^n$ (with $n \geq 2$) fulfilling suitable conditions. By smooth 'vector field' $Y = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j}$ we mean both a linear differential operator acting on the set of the smooth functions as

$$C^\infty(\mathbb{R}^n) \ni f \mapsto Yf(x) = \sum_{j=1}^n a_j(x) \frac{\partial f}{\partial x_j}(x), \quad x \in \mathbb{R}^n, \tag{1}$$

or, occasionally, we mean a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^n$

$$\mathbb{R}^n \ni x \mapsto Y(x) = (a_1(x), \dots, a_n(x)), \quad x \in \mathbb{R}^n.$$

If $\mathcal{X}(\mathbb{R}^n)$ is the Lie algebra of all the smooth vector fields on $\mathbb{R}^n$, we denote by $\mathrm{Lie}\{X\}$ the smallest Lie subalgebra of $\mathcal{X}(\mathbb{R}^n)$ containing X. We set once and for all the notation

$$N := \dim(\mathrm{Lie}\{X\}), \tag{2}$$

where N can be ∞. In due course, we shall fix on X the following two assumptions (H.1) and (H.2) (whilst in Sections 2 and 3 we shall considerably weaken these assumptions):

(H.1) There exists a family of (non-isotropic) "dilations" $\{\delta_\lambda\}_{\lambda>0}$ of the form

$$\delta_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad \delta_\lambda(x) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_n} x_n), \tag{3}$$

where $1 = \sigma_1 \leq \cdots \leq \sigma_n$, such that $X_1, \dots, X_m$ are δ_λ-homogeneous of degree 1, i.e.,

$$X_j(f \circ \delta_\lambda) = \lambda\, (X_j f) \circ \delta_\lambda, \quad \text{for } \lambda > 0,\, f \in C^\infty(\mathbb{R}^n),\, j = 1, \dots, m.$$

In what follows, we denote by

$$q := \textstyle\sum_{j=1}^n \sigma_j \tag{4}$$

the so-called homogeneous dimension of $(\mathbb{R}^n, \delta_\lambda)$.

(H.2) $X_1, \dots, X_m$ are linearly independent[1] and satisfy Hörmander's rank condition at 0, i.e.,

$$\dim\big\{Y(0) : Y \in \mathrm{Lie}\{X\}\big\} = n.$$

[1]The linear independence of the X_i's is meant with respect to the vector space of the smooth vector fields on $\mathbb{R}^n$; this must not be confused with the linear independence of the vectors $X_1(x), \dots, X_m(x)$ in $\mathbb{R}^n$ (when $x \in \mathbb{R}^n$): the latter is sufficient but not necessary to the former linear independence. Thus, $X_1 = \partial_{x_1}$ and $X_2 = x_1\, \partial_{x_2}$ are linearly independent vector fields, even if $X_1(0, x_2) \equiv (1, 0)$ and $X_2(0, x_2) \equiv (0, 0)$ are dependent vectors of $\mathbb{R}^2$.

Note that, due to assumption (H.2), one necessarily has

$$N \geq n.$$

Strict inequality may hold, as in some of the following examples.

Example 1.1. In the following list we provide examples of sets of vector fields X on $\mathbb{R}^n$ fulfilling assumptions (H.1) and (H.2) w.r.t. the family of dilations δ_λ; N is also exhibited:

$$\begin{aligned}
n=3,\quad N=3,\qquad & X=\{\partial_{x_1},\ \partial_{x_2}+x_1\,\partial_{x_3}\},\\
& \delta_\lambda(x)=(\lambda x_1,\lambda x_2,\lambda^2 x_3);\\
n=2,\quad N=3,\qquad & X=\{\partial_{x_1},\ x_1\,\partial_{x_2}\},\\
& \delta_\lambda(x)=(\lambda x_1,\lambda^2 x_2);\\
n=2,\quad N=5,\qquad & X=\{\partial_{x_1},\ x_1^3\,\partial_{x_2}\},\\
& \delta_\lambda(x)=(\lambda x_1,\lambda^4 x_2);\\
n=3,\quad N=4,\qquad & X=\{\partial_{x_1},\ x_1\partial_{x_2}+x_2\partial_{x_3}\},\\
& \delta_\lambda(x)=(\lambda x_1,\lambda^2 x_2,\lambda^3 x_3).\\
n=4,\quad N=5,\qquad & X=\{\partial_{x_1},\ x_1\,\partial_{x_2}+x_1^2\,\partial_{x_3}+x_1^3\,\partial_{x_4}\},\\
& \delta_\lambda(x)=(\lambda x_1,\lambda^2 x_2,\lambda^3 x_3,\lambda^4 x_4).
\end{aligned}$$

It is not by chance that the above vector fields have polynomial coefficients; indeed, it is easy to prove that, under assumption (H.1), if we write X_j ($j=1,\dots,m$) in its coordinate form

$$X_j=\sum_{k=1}^{n} a_{j,k}(x)\,\frac{\partial}{\partial x_k},$$

then $a_{j,k}(x)$ is a polynomial function,[2] δ_λ-homogeneous of degree σ_k-1. Incidentally,

$$a_{j,k}(x) \text{ depends on those } x_i\text{'s such that } \sigma_i \leq \sigma_k-1. \tag{5}$$

Thus, $a_{j,k}(x)$ is independent of x_k, so that $\operatorname{div}(X_j(x))\equiv 0$ for any $j=1,\dots,m$. In particular, the formal adjoint of X_j (with respect to Lebesgue measure on $\mathbb{R}^n$)

[2]This is a consequence of the following facts:

- a smooth function a is δ_λ-homogeneous of degree r if and only if a is a polynomial $a(x)=\sum_\alpha c_\alpha x^\alpha$, where the sum is extended over the multi-indices α such that $\sum_{j=1}^n \alpha_j\sigma_j=r$ (so that $r\geq 0$ if $a\not\equiv 0$);
- a smooth vector field $X=\sum_{j=1}^n a_j(x)\,\partial_{x_j}$ is δ_λ-homogeneous of degree β if and only if a_j is δ_λ-homogeneous of degree $\sigma_j-\beta$; hence $a_j\equiv 0$ whenever $\sigma_j<\beta$ (so that $X\equiv 0$ if $\beta>\sigma_n$).

is $-X_j$, and the following operator, a sum of squares of vector fields,

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 \tag{6}$$

is a second order divergence form PDO, formally self-adjoint on test functions.

Remark 1.2. Property (5) has another important consequence: any vector field $X_1, \dots, X_m$ (and analogously any vector field in $\mathrm{Lie}\{X\}$) is complete, i.e., its integral curves are all defined on the whole of $\mathbb{R}$ (see e.g., [4, Example 1.20]).

For brevity, in the sequel we use the notation

$$\mathfrak{a} := \mathrm{Lie}\{X\}.$$

Remark 1.3. Thanks to (H.1) and (H.2), the Lie algebra $\mathfrak{a}$ enjoys the following properties:[3]

- $\mathfrak{a}$ **is graded** If we set

  $$\mathfrak{a}_1 := \mathrm{span}\big\{X_1, \dots, X_m\big\}, \qquad \mathfrak{a}_k := [\mathfrak{a}_1, \mathfrak{a}_{k-1}] \quad (\text{for } k \geq 2),$$

  then $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2 + \cdots$ (in the sense of the sum of vector spaces); moreover, since any commutator of length k of $X_1, \dots, X_m$ is δ_λ-homogeneous of degree k, any element of $\mathfrak{a}_k$ is δ_λ-homogeneous of degree k.

- $\mathfrak{a}$ **is nilpotent** Indeed, $\mathfrak{a}_k = \{0\}$ whenever $k > \sigma_n$ so that $\mathfrak{a}$ is nilpotent, and its step (r, say) satisfies $r \leq \sigma_n$; furthermore, if we group the exponents σ_i's of δ_λ as follows

  $$\underbrace{\sigma_1, \dots, \sigma_{n_1}}_{=\sigma_1^*}, \quad \underbrace{\sigma_{n_1+1}, \dots, \sigma_{n_1+n_2}}_{=\sigma_2^*}, \quad \underbrace{\sigma_{n_1+n_2+1}, \dots, \sigma_{n_1+n_2+n_3}}_{=\sigma_3^*}, \quad \dots,$$

  with $1 = \sigma_1^* \lneqq \sigma_2^* \lneqq \sigma_3^* \lneqq \cdots$ and $n = n_1 + n_2 + n_3 + \cdots$, then the typical element of $\mathfrak{a}_k$ is

  $$\sum_{i=1}^{n_1} a_i^{(1)}(x) \frac{\partial}{\partial x_i} + \sum_{i=n_1+1}^{n_1+n_2} a_i^{(2)}(x) \frac{\partial}{\partial x_i} + \sum_{i=n_1+n_2+1}^{n_1+n_2+n_3} a_i^{(3)}(x) \frac{\partial}{\partial x_i} + \cdots,$$

  with $a_i^{(1)}$ δ_λ-homogeneous of degree $\sigma_1^* - k$, $a_i^{(2)}$ δ_λ-homogeneous of degree $\sigma_2^* - k$, etc. Thus $a_i^{(1)}, a_i^{(2)}, \dots$ are identically zero if $\sigma_1^*, \sigma_2^*, \dots < k$

[3] To prove these facts, one repeatedly applies the remarks in footnote 2.

(resp.), and, $a_i^{(1)}(0), a_i^{(2)}(0), \dots$ are 0 if $\sigma_1^*, \sigma_2^*, \dots > k$ (resp.). Thus, assumption (H.2) implies that the σ_i^*'s are all positive integers (that is, all the exponents σ_i's of δ_λ are positive integers), and the largest of the σ_i^*'s (i.e., σ_n), satisfies $\mathfrak{a}_{\sigma_n} \neq \{0\}$. This gives $r \geq \sigma_n$, so that $r = \sigma_n$, that is, the last exponent in the dilations δ_λ is exactly the step of nilpotence of $\mathfrak{a}$.

- $\mathfrak{a}$ **is finite-dimensional** This follows from the fact that $\mathfrak{a}$ is a nilpotent Lie algebra that is Lie-generated by a finite set. Thus N in (2) is finite.

- $\mathfrak{a}$ **is stratified** This follows by gathering together the previous facts on $\mathfrak{a}$, observing that, if $Y_1, \dots, Y_k \in \mathcal{X}(\mathbb{R}^n)$ are δ_λ-homogeneous of pairwise distinct degrees (and if any Y_j is not identically 0), then $Y_1, \dots, Y_k$ are linearly independent: summing up, this gives

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r, \quad \text{with} \quad \begin{cases} \mathfrak{a}_1 := \text{span}\{X_1, \dots, X_m\}, \\ \mathfrak{a}_k := [\mathfrak{a}_1, \mathfrak{a}_{k-1}] \quad \text{for } 2 \leq k \leq r; \\ \mathfrak{a}_r \neq \{0\}, \ [\mathfrak{a}_1, \mathfrak{a}_r] = \{0\}. \end{cases} \tag{7}$$

Thus $\mathfrak{a}_k$ is exactly the set of vector fields in $\text{Lie}\{X\}$ that are δ_λ-homogeneous of degree k.

- $\mathfrak{a}$ **is endowed with dilations** By (7), we can define a family $\{\Delta_\lambda\}_{\lambda>0}$ of dilations on $\mathfrak{a}$ in the following way:

$$\Delta_\lambda(Y) = \sum_{k=1}^r \lambda^k Y_k, \quad \text{where } Y = \sum_{k=1}^r Y_k \text{ and } Y_k \in \mathfrak{a}_k \text{ for any } k = 1, \dots, r. \tag{8}$$

It is not difficult to prove that $\{\Delta_\lambda\}_{\lambda>0}$ is a family Lie algebra morphisms of $\mathfrak{a}$:

$$\Delta_\lambda[X,Y] = [\Delta_\lambda X, \Delta_\lambda Y], \quad \forall\, X, Y \in \mathfrak{a},\ \lambda > 0. \tag{9}$$

Remark 1.4. It is rather unusual to handle with a set of vector fields that fulfil Hörmander's rank condition at one point only; as a matter of fact, (H.1) and (H.2) together imply that the rank condition is fulfilled *at every* $x \in \mathbb{R}^n$, as we now describe. Indeed, by (H.2) we can find a family $Y_1, \dots, Y_n \in \text{Lie}\{X\}$ such that $Y_1(0), \dots, Y_n(0)$ is a basis of $\mathbb{R}^n$. Thus, the matrix-valued function

$$x \mapsto \mathbf{M}(x) := \big(Y_1(x)^T \cdots Y_n(x)^T\big)$$

is non-singular at $x = 0$; therefore, there exists a neighborhood Ω of 0 such that $\det(\mathbf{M}(x)) \neq 0$ for every $x \in \Omega$. Furthermore, since the left-nested brackets of length k, say

$$[\cdots[[X_{i_1}, X_{i_2}], X_{i_3}], \cdots X_{i_k}] \qquad (\text{with } i_1, \dots, i_k \in \{1, \dots, m\}),$$

span Lie$\{X\}$ as k ranges in $\mathbb{N}$, we can suppose that any Y_j is left-nested, and we denote its length by $\ell(j)$. It is simple to check that, under assumption (H.1), Y_j is δ_λ-homogeneous of degree $\ell(j)$, i.e.,

$$Y_j(f\circ\delta_\lambda)=\lambda^{\ell(j)}\,(Y_jf)\circ\delta_\lambda,\quad \text{for every } \lambda>0 \text{ and } f\in C^\infty(\mathbb{R}^n).$$

This is in turn equivalent to

$$Y_j(\delta_\lambda(x))=\lambda^{-\ell(j)}\delta_\lambda(Y_j(x)),\quad \forall\,\lambda>0,\, x\in\mathbb{R}^n. \tag{10}$$

Fixing $x\in\mathbb{R}^n$ and taking a small $\lambda>0$ such that $\delta_\lambda(x)\in\Omega$, we have

$$\begin{aligned}
0\neq \det\big(\mathbf{M}(\delta_\lambda(x))\big) &\overset{(10)}{=} \det\Big(\lambda^{-\ell(1)}\,\delta_\lambda\big(Y_1(x)\big)^T\cdots\lambda^{-\ell(n)}\,\delta_\lambda\big(Y_n(x)\big)\Big)^T\\
&= \lambda^{-\ell(1)-\cdots-\ell(n)}\det\Big(\delta_\lambda\big(Y_1(x)\big)^T\cdots\delta_\lambda\big(Y_n(x)\big)^T\Big)\\
&\overset{(4)}{=} \lambda^{q-\ell(1)-\cdots-\ell(n)}\det\big(Y_1(x)^T\cdots Y_n(x)^T\big).
\end{aligned}$$

This implies that the vectors $Y_1(x),\dots,Y_n(x)$ form a basis of $\mathbb{R}^n$, i.e, $X_1,\dots,X_m$ satisfy Hörmander's rank condition at any $x\in\mathbb{R}^n$. As a consequence, $\mathcal{L}$ in (6) is C^∞-hypoelliptic on every open subset of $\mathbb{R}^n$, due to Hörmander's Hypoellipticity Theorem, [11].

In its simplicity, the argument in Remark 1.4 shows how δ_λ-homogeneity can serve as a "local-to-global" tool; we shall invoke similar arguments so frequently that we state this globalizing property of δ_λ-homogeneity as an independent remark:

Remark 1.5 (**Propagation of inequalities via homogeneity**). Let $A\subseteq\mathbb{R}^n$ be a set which is closed under $\{\delta_\lambda\}_\lambda$, that is,

$$\delta_\lambda(x)\in A \text{ for every } x\in A \text{ and every } \lambda>0. \tag{11}$$

Suppose that $F,G:A\to\mathbb{R}$ are two δ_λ-homogeneous functions of the same degree, say α, i.e.,

$$F(\delta_\lambda(x))=\lambda^\alpha F(x),\quad G(\delta_\lambda(x))=\lambda^\alpha G(x),\quad \text{for } x\in A \text{ and } \lambda>0.$$

Finally, suppose that there exists a neighborhood Ω of $0\in\mathbb{R}^n$ such that $\Omega\cap A\neq\emptyset$ and $F\le G$ on $\Omega\cap A$; then $F\le G$ on A. Indeed, let $x\in A$ be arbitrary; then there exists a small $\lambda>0$ such that $\delta_\lambda(x)\in\Omega\cap A$ (this follows from (11) and since $\delta_\lambda(x)\to 0\in\mathbb{R}^n$ as $\lambda\to 0^+$). As $F\le G$ on $\Omega\cap A$ we infer that $F(\delta_\lambda(x))\le G(\delta_\lambda(x))$; due to the δ_λ-homogeneity of F and G, this is equivalent to $\lambda^\alpha F(x)\le\lambda^\alpha G(x)$. Canceling out $\lambda^\alpha>0$, this gives $F(x)\le G(x)$.

A completely analogous result holds true if we replace "$F \leq G$" with any of

$$\text{"}F \geq G\text{"}, \quad \text{"}F = G\text{"}, \quad \text{"}F \neq G\text{"}, \quad \text{"}F < G\text{"}, \quad \text{"}F > G.\text{"}$$

Thus, we recognize that the argument in Remark 1.4 is a particular case of Remark 1.5 relative to the maps $F(x) = \det(\mathbf{M}(x))$ and $G(x) \equiv 0$: they are both δ_λ-homogeneous of degree $q - \sum_{j=1}^n \ell(j)$, so that the information $F \neq G$ on Ω "propagates" to the whole of $A = \mathbb{R}^n$.

Supposing the reader is familiar with the following topics, other examples of meaningful inequalities with two sides that are homogeneous of the same degree w.r.t. some dilations are listed below (d_X denotes the Carnot-Carathodory distance in $\mathbb{R}^n$ associated with $X = \{X_1, \dots, X_m\}$ and $B_X(x,r)$ denotes the d_X-ball of center $x \in \mathbb{R}^n$ and radius $r > 0$):

(i) the *doubling inequality*

$$\operatorname{meas}(B_X(x, 2r)) \leq C \operatorname{meas}(B_X(x,r))$$

for the Lebesgue measure of the balls of the Carnot-Carathodory distance d_X: both sides are homogeneous of degree λ^q w.r.t. the dilations

$$\mathbb{R}^n \times \mathbb{R} \ni (x,r) \mapsto (\delta_\lambda(x), \lambda r);$$

the same can be said of the *reverse doubling inequality*: there exists $\theta \in (0,1)$ such that

$$\operatorname{meas}(B_X(x,r)) \leq \theta \operatorname{meas}(B_X(x,2r));$$

(ii) when $n > 2$, the *lower/upper estimates for the fundamental solution* Γ of $\mathcal{L} = \sum_{j=1}^m X_j^2$

$$C^{-1} \frac{d_X(x,y)^2}{\operatorname{meas}\Big(B_X(x, d_X(x,y))\Big)} \leq \Gamma(x;y) \leq C \frac{d_X(x,y)^2}{\operatorname{meas}\Big(B_X(x, d_X(x,y))\Big)};$$

all members are homogeneous of degree $2 - q$ w.r.t. the dilations

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x,y) \mapsto (\delta_\lambda(x), \delta_\lambda(y));$$

(iii) when $q > 2$, the *upper estimates for the X-derivatives of* Γ

$$\Big| X_{i_1} \cdots X_{i_k} \Gamma(x;y) \Big| \leq C \frac{d_X(x,y)^{2-k}}{\operatorname{meas}\Big(B_X(x, d_X(x,y))\Big)},$$

where the X_i's can act on x and/or on y; both sides are homogeneous of degree $2 - q - k$ w.r.t. the dilations in (ii);

(iv) the *Nagel-Stein-Wainger estimates for the volume of* $B_X(x,r)$ (see [13]):

$$C^{-1}\sum_{k=n}^{q} f_k(x)\,r^k \le \mathrm{meas}\big(B_X(x,r)\big) \le C\sum_{k=n}^{q} f_k(x)\,r^k;$$

indeed, one can prove that f_k is δ_λ-homogeneous of degree $q-k$, so that all members of the above two inequalities are homogeneous of degree q w.r.t. the dilations in (i);

(v) the *Poincar inequality* associated with X:

$$⨍_{B_X(x,r)} \Big|u(y)-u_{B_X(x,r)}\Big|\,\mathrm{d}y \le C_P\, r ⨍_{B_X(x,2r)} \sqrt{\sum_{j=1}^{m} |X_j u(y)|^2}\,\mathrm{d}y,$$

valid for every u which is C^1 in a neighborhood of $B_X(x,2r)$, and where we have set

$$u_B := ⨍_B u := \frac{1}{|B|}\int_B u(y)\,\mathrm{d}y \qquad \text{for any } d_X\text{-ball } B;$$

indeed, one considers the dilations in (i) on (x,r) and one replaces u with $v=u\circ\delta_{1/\lambda}$.

Thus, due to Remark 1.5, once one knows that one of the above inequalities is true in the small scale, then it is globally valid. For instance, due to profound results contained in the seminal papers [10, 12, 13], the inequalities in (i), (iv), (v) are valid for x in a neighborhood of the origin and for small r's, so that Remark 1.5 implies that they are globally valid for all $x\in\mathbb{R}^n$ and all $r>0$ in our homogeneous setting. We shall investigate the global validity of (ii) and (iii) in Section 5.

Not always are we so lucky to handle with inequalities with two members with the same homogeneity: an example of a meaningful inequality that does *not* rescale suitably is the X*-Sobolev inequality* for $\mathcal{L}$ (here L^p norms are meant on $\mathbb{R}^n$)

$$\|u\|_{W^{2,p}_X} \le c\Big(\|\mathcal{L}u\|_{L^p}+\|u\|_{L^p}\Big), \tag{12}$$

with the Sobolev norm $\|u\|_{W^{2,p}_X} = \|u\|_{L^p}+\sum_{j=1}^m \|X_j u\|_{L^p}+\sum_{i,j=1}^m \|X_iX_j u\|_{L^p}$. Indeed[4]

$$\|u\circ\delta_\lambda\|_{L^p} = \lambda^{-q/p}\|u\|_{L^p}, \qquad \text{whereas} \quad \|\mathcal{L}(u\circ\delta_\lambda)\|_{L^p} = \lambda^{2-q/p}\|\mathcal{L}u\|_{L^p}.$$

[4] One clearly performs the change of variable $\delta_\lambda(x)=y$ in the integral defining $\|u\circ\delta_\lambda\|_{L^p}$, so that $\mathrm{d}x=\lambda^{-q}\,\mathrm{d}y$. In rescaling $\|\mathcal{L}(u\circ\delta_\lambda)\|_{L^p}$, one also exploits (H.1), which gives $\mathcal{L}(u\circ\delta_\lambda) = \lambda^2\,(\mathcal{L}u)\circ\delta_\lambda$.

Similarly, the $W_X^{2,p}$-norm rescales not so satisfactorily:

$$\|u\circ\delta_\lambda\|_{W_X^{2,p}} = \lambda^{-q/p}\|u\|_{L^p} + \lambda^{1-q/p}\sum_{j=1}^{m}\|X_j u\|_{L^p} + \lambda^{2-q/p}\sum_{i,j=1}^{m}\|X_iX_j u\|_{L^p}.$$

The fact that the inequality (12) does not behave encouragingly under δ_λ-rescaling does not mean that it does not hold true: indeed, we recently proved in [6] that (12) is valid (more generally, we proved this when $W_X^{k,p}$ norms are involved, for any $k \geq 0$ and any $p \in (1,\infty)$).

Remark 1.6 (Propagation of injectivity/surjectivity via homogeneity). Suppose that $F : \mathbb{R}^n \to \mathbb{R}^m$ is a map with the following property: every component function F_i of F (for $i = 1,\dots,m$) is δ_λ-homogeneous of some *positive* degree, say α_i. This is equivalent to saying that

$$F(\delta_\lambda(x)) = \Delta_\lambda(F(x)), \quad \forall\, x \in \mathbb{R}^n,\ \lambda > 0, \tag{13}$$

where we have set

$$\Delta_\lambda : \mathbb{R}^m \to \mathbb{R}^m, \quad \Delta_\lambda(y_1,\dots,y_m) = (\lambda^{\alpha_1}y_1,\dots,\lambda^{\alpha_m}y_m).$$

Then the following facts hold:

1. *If there exists an open neighborhood* Ω *of* $0 \in \mathbb{R}^n$ *such that* $F|_\Omega$ *is injective, then* F *is globally injective.* Indeed, if $x,y \in \mathbb{R}^n$ are such that $F(x) = F(y)$, then take some small $\lambda > 0$ such that $\delta_\lambda(x), \delta_\lambda(y) \in \Omega$; then we have

$$F(\delta_\lambda(x)) \overset{(13)}{=} \Delta_\lambda(F(x)) = \Delta_\lambda(F(y)) \overset{(13)}{=} F(\delta_\lambda(y)).$$

Thus (as $F|_\Omega$ is injective) $\delta_\lambda(x) = \delta_\lambda(y)$, which implies that $x = y$, since δ_λ is injective.

2. *If there exist open neighborhoods* U *and* V *of* $0 \in \mathbb{R}^n$ *and of* $0 \in \mathbb{R}^m$ (*respectively*) *such that* $V \subseteq F(U)$, *then* F *is globally surjective*. Indeed, given any $y \in \mathbb{R}^m$, take some small $\lambda > 0$ such that $\Delta_\lambda(y) \in V$ (here we have made use of the positivity of the α_i's). Since $V \subseteq F(U)$, there exists $u_\lambda \in U$ such that $\Delta_\lambda(y) = F(u_\lambda)$. Next we set $x := \delta_{1/\lambda}(u_\lambda)$ and we notice that

$$F(x) = F(\delta_{1/\lambda}(u_\lambda)) \overset{(13)}{=} \Delta_{1/\lambda}F(u_\lambda) = \Delta_{1/\lambda}\Delta_\lambda(y) = y.$$

Thus F is surjective.

Homogeneity of vector fields is not only a technical tricky tool (as it may seem from a rapid glance to Remarks 1.5 and 1.6), but much more can be done in its presence, as glaringly appears from the following theorem (one of the main results of this review), a combination of two theorems results proved in [1, 2].

Theorem 1.7. *Assume that* $X = \{X_1, \dots, X_m\}$ *satisfies assumptions* (H.1) *and* (H.2)*, of which we inherit the notation. As usual,* $N = \dim(\mathrm{Lie}\{X\})$*. The following facts hold:*

(1). *Suppose that* $N = n$*. Then there exists a homogeneous Carnot group* $\mathbb{G} = (\mathbb{R}^n, \star, \delta_\lambda)$ *(with the same dilations as in* (3)*) such that* $\mathrm{Lie}(\mathbb{G})$ *coincides with* $\mathrm{Lie}\{X\}$*. Thus the vector fields* $X_1, \dots, X_m$ *are left-invariant on* $\mathbb{G}$*, and the operator* $\mathcal{L}$ *in* (6) *is a sub-Laplacian on* $\mathbb{G}$*.*

(2). *Suppone that* $N > n$ *and, setting* $p := N - n$*, denote the variables of* $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p$ *by* (x, ξ)*. There exist a homogeneous Carnot group* $\mathbb{G} = (\mathbb{R}^N, \star, D_\lambda)$ *of homogeneous dimension* $Q > q$ *and a system* $\{\widetilde{X}_1, \dots, \widetilde{X}_m\}$ *of Lie-generators of* $\mathrm{Lie}(\mathbb{G})$ *such that* $\widetilde{X}_i$ *is a lifting of* X_i *for every* $i = 1, \dots, m$*; by this we mean that*

$$\widetilde{X}_i(x, \xi) = X_i(x) + R_i(x, \xi), \tag{14}$$

where $R_i(x, \xi)$ *is a smooth vector field operating only in the variables* $\xi \in \mathbb{R}^p$*, with coefficients possibly depending on* (x, ξ)*. Moreover the dilations* $\{D_\lambda\}_{\lambda>0}$ *and the dilations* $\{\delta_\lambda\}_{\lambda>0}$ *are related as follows:*

$$D_\lambda(x, \xi) = (\delta_\lambda(x), \delta^*_\lambda(\xi)),$$

with $\delta^*_\lambda(\xi) = (\lambda^{\tau_1}\xi_1, \dots, \lambda^{\tau_p}\xi_p)$*, for suitable integers* $1 \le \tau_1 \le \dots \le \tau_p$*.*

We shall describe how to obtain Theorem 1.7 in the next sections, where we considerably relax our assumptions on the vector fields involved. We observe that, unlike Rothschild-Stein's local lifting technique, [14], the above lifting is globally valid. See also Folland's global lifting for homogeneous vector fields, [9].

2. A general result on the local lifting of vector-field algebras

In this section, we make the effort to handle with less restrictive assumptions than the δ_λ-homogeneous framework of Section 1. Thus, we only assume that

$$\mathfrak{g} \text{ is a Lie subalgebra of } \mathcal{X}(\mathbb{R}^n) \text{ of finite dimension, and any } X \in \mathfrak{g} \text{ is a complete vector field.}$$

The above assumptions have useful consequences.

Consequence (I). Since $\mathfrak{g}$ is a finite-dimensional Lie algebra, one can equip $\mathfrak{g}$ with a local operation by means of the celebrated Baker-Campbell-Hausdorff series (see e.g., [7])

$$a\diamond b=a+b+\frac{1}{2}[a,b]+\frac{1}{12}[a,[a,b]]-\frac{1}{12}[b,[a,b]]+\frac{1}{24}[a,[b,[b,a]]]+\cdots, \quad (15)$$

the series being convergent for any a,b in a small neighborhood[5], say $\mathfrak{U}$, of $0\in\mathfrak{g}$ (see [4, Chap. 5]). Moreover, $\diamond$ defines a local-Lie-group (see e.g, [4, Thm. 5.9]), that is, the following facts hold:

- $a\diamond 0=0\diamond a=a$ and $a\diamond(-a)=(-a)\diamond a=0$ for every $a\in\mathfrak{g}$;
- there exists a (smaller) neighborhood of 0, say $\mathfrak{V}\subseteq\mathfrak{U}$, such that $a\diamond b\in\mathfrak{U}$ whenever $a,b\in\mathfrak{V}$;
- $a\diamond(b\diamond c)=(a\diamond b)\diamond c$ for every $a,b,c\in\mathfrak{V}$, the local associativity of $\diamond$.

We use (not by chance!) the "left-translation" notation

$$\tau_a(b):=a\diamond b,\quad a,b\in\mathfrak{U}.$$

It can be proved, by using the magnificent properties of the Baker-Campbell-Hausdorff series, that one can define a Lie algebra $L_{\mathfrak{V}}(\mathfrak{g})$ of vector fields on $\mathfrak{V}$ (analogous to the "left invariant" vector fields associated with the local left translations τ_a) which is isomorphic to $\mathfrak{g}$ (see [4, §15.1]): this provides a proof of the local version of Lie's Third Theorem, in that we construct a local Lie group on the neighborhood $\mathfrak{V}$ whose "local Lie algebra" $L_{\mathfrak{V}}(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$.

More precisely, $L_{\mathfrak{V}}(\mathfrak{g})$ can be defined as follows (see [4, Thm. 15.3]): an element of $L_{\mathfrak{V}}(\mathfrak{g})$ is the restriction to $\mathfrak{V}$ of a vector field Z on $\mathfrak{U}$ satisfying the following identity

$$\mathrm{d}_b\tau_a(Z_b)=Z_{\tau_a(b)},\quad \text{for every } a,b\in\mathfrak{V}. \quad (16)$$

Such a Z can always be constructed: namely, for any tangent vector $\mathbf{v}\in T_0\mathfrak{g}$, the vector field

$$Z_a:=\mathrm{d}_0\tau_a(\mathbf{v})\quad (a\in\mathfrak{U}) \quad (17)$$

is smooth on $\mathfrak{U}$ and (thanks to the local associativity of $\diamond$) it satisfies (16); moreover the map

$$\Lambda: L_{\mathfrak{V}}(\mathfrak{g})\to T_0\mathfrak{g},\quad \Lambda(Z):=Z_0 \quad (18)$$

[5]We tacitly equip $\mathfrak{g}$ with a metric structure resulting from its being a real finite-dimensional vector space.

is an isomorphism of vector spaces; in particular $\dim(L_{\mathfrak{V}}(\mathfrak{g})) = \dim(\mathfrak{g})$.

Consequence (II). In (I) above we have not used the fact that $\mathfrak{g}$ is made of vector fields: we shall do it now. Since any element $X \in \mathfrak{g}$ is a complete vector field in $\mathbb{R}^n$, then, for every $x \in \mathbb{R}^n$, the integral curve[6] $t \mapsto \Psi^X_t(x)$ of X starting (at null time) from x is defined for every time $t \in \mathbb{R}$. Thus, time $t = 1$ is always allowed, and we use the notation

$$\exp(X)(x) := \Psi^X_1(x). \tag{19}$$

For example, if $x = 0$, we shall soon make crucial use of the map

$$\mathrm{Exp} : \mathfrak{g} \to \mathbb{R}^n, \quad \mathrm{Exp}(X) := \exp(X)(0). \tag{20}$$

As in the old days of Sophus Lie's theory of continuous transformations, the family

$$\{\exp(X)\}_{X\in\mathfrak{g}}$$

is a subset of the smooth diffeomorphisms of $\mathbb{R}^n$.

Indeed, notice that $\exp(X)^{-1} = \exp(-X)$ and $\exp(0) = \mathrm{id}_{\mathbb{R}^n}$. Unfortunately, this family is not always closed under composition, but this is true "in the small", as we now describe:

Link between (I) and (II). A very remarkable fact links the operation diamond in (I) and the exp-like maps in (II): there exists a neighborhood of 0 in $\mathfrak{g}$, say $\mathfrak{W} \subseteq \mathfrak{V}$, such that

$$\exp(Y)(\exp(X)(x)) = \exp(X \diamond Y)(x), \quad \text{for every } X,Y \in \mathfrak{W} \text{ and every } x \in \mathbb{R}^n. \tag{21}$$

This can be referred to as the Baker-Campbell-Hausdorff Theorem for ODE's (see [4, Sec. 13.3]).

Our next step is to show that any vector field X in $\mathfrak{g}$ admits a "local lifting" $\widetilde{X}$ (via the map Exp in (20)), where $\widetilde{X}$ is a suitable vector field defined on the open neighborhood $\mathfrak{V}$ of $0 \in \mathfrak{g}$ introduced above. Indeed, any $X \in \mathfrak{g}$ defines an element $\mathbf{x}$ of $T_0\mathfrak{g}$ (the tangent space of $\mathfrak{g}$ at 0) as follows:

$$\mathbf{x}f = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(tX), \qquad \forall\, f \in C^\infty(\mathfrak{g}). \tag{22}$$

The map $X \mapsto \mathbf{x}$ of $\mathfrak{g}$ into $T_0\mathfrak{g}$ is an isomorphism of vector spaces.

[6]By this we mean that $\gamma(t) = \Psi^X_t(x)$ is the solution of the Cauchy problem $\dot\gamma(t) = X(\gamma(t))$, $\gamma(0) = x$.

We define $\widetilde{X} \in L_{\mathfrak{V}}(\mathfrak{g})$ as the unique vector field on $\mathfrak{V}$ corresponding to $\mathbf{x}$ via the linear isomorphism Λ in (18): by unraveling the definitions (see (17)), this means that $\widetilde{X}$ is the restriction to $\mathfrak{V}$ of the vector field on $\mathfrak{U}$ defined by

$$\widetilde{X}_Z = \mathrm{d}_0 \tau_Z(\mathbf{x}), \quad \text{for every } Z \in \mathfrak{U}. \tag{23}$$

With this definition at hand, we claim that X *and* $\widetilde{X}$ *are* Exp*-related on* $\mathfrak{W}$, i.e.,

$$\mathrm{d}_w \mathrm{Exp}(\widetilde{X}_w) = X_{\mathrm{Exp}(w)}, \quad \text{for every } w \in \mathfrak{W}. \tag{24}$$

We prove (24) by showing that both members of this identity act in the same way on $f \in C^\infty(\mathbb{R}^n)$; this is a consequence of the following computation

$$\begin{aligned}
\mathrm{d}_w \mathrm{Exp}(\widetilde{X}_w) f &= \widetilde{X}_w(f \circ \mathrm{Exp}) \overset{(23)}{=} \mathrm{d}_0 \tau_w(\mathbf{x})(f \circ \mathrm{Exp}) = \mathbf{x}(f \circ \mathrm{Exp} \circ \tau_w) \\
&\overset{(22)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f \circ \mathrm{Exp} \circ \tau_w)(tX) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f\big(\mathrm{Exp}(w \diamond (tX))\big) \\
&\overset{(21)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f\Big(\exp(tX)(\mathrm{Exp}(w))\Big) \overset{(19)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f\Big(\Psi_1^{tX}(\mathrm{Exp}(w))\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f\Big(\Psi_t^X(\mathrm{Exp}(w))\Big) = X_{\mathrm{Exp}(w)} f.
\end{aligned}$$

The last identity derives from the fact that $t \mapsto \Psi_t^X(\mathrm{Exp}(w))$ is the integral curve of X starting at the point $\mathrm{Exp}(w)$. We remark the essential use of the Baker-Campbell-Hausdorff identity (21).

Whilst, in general, (24) is referred to as the Exp-relatedness of X and $\widetilde{X}$, one can call this identity a (local) 'lifting' of X to $\widetilde{X}$ if Exp is (locally) surjective near $0 \in \mathfrak{g}$. In turn, it is not difficult to recognize (see e.g., [4, Thm. 13.4]) that the differential of Exp at 0 is the map

$$\mathrm{d}_0 \mathrm{Exp} : T_0 \mathfrak{g} \to T_0 \mathbb{R}^n, \qquad T_0 \mathfrak{g} \simeq \mathfrak{g} \ni Y \mapsto Y(0) \in \mathbb{R}^n \simeq T_0 \mathbb{R}^n. \tag{25}$$

Thus the image set of Exp on $\mathfrak{W}$ (the latter being a neighborhood of $0 \in \mathfrak{g}$) contains an open ball centered at the origin in $\mathbb{R}^n$ if and only if

$$\dim \big\{ Y(0) \in \mathbb{R}^n \,|\, Y \in \mathfrak{g} \big\} = n,$$

which is Hörmander's rank condition at 0 for the algebra of vector fields $\mathfrak{g}$.

Summing up, we have proved the following result on the local lifting of finite-dimensional Lie algebras of complete vector fields:

Theorem 2.1. *Let* $\mathfrak{g}$ *be a Lie subalgebra of* $\mathcal{X}(\mathbb{R}^n)$ *of finite dimension, and suppose that every* $X \in \mathfrak{g}$ *is a complete vector field. Let* $\mathrm{Exp} : \mathfrak{g} \to \mathbb{R}^n$ *be the* *map introduced in* (20)*, obtained by letting the vector fields of* $\mathfrak{g}$ *flow up to time* $t = 1$ *starting from* $0 \in \mathbb{R}^n$.

Then, there exists an open neighborhood $\mathfrak{W}$ *of* $0 \in \mathfrak{g}$ *with the following properties: for every* $X \in \mathfrak{g}$ *there exists a smooth vector field* $\widetilde{X}$ *defined on* $\mathfrak{W}$ *such that*

$$\mathrm{d}_w \mathrm{Exp}(\widetilde{X}_w) = X_{\mathrm{Exp}(w)}, \quad \textit{for every } w \in \mathfrak{W}. \tag{26}$$

If the algebra of vector fields $\mathfrak{g}$ *satisfies Hörmander's rank condition at* 0*, then* $\mathrm{Exp}(\mathfrak{W})$ *is a neighborhood of* $0 \in \mathbb{R}^n$.

The vector field $\widetilde{X}$ *can be constructed as follows: if* $\tau_v(w) := v \diamond w$ *is the local left translation defined by the Baker-Campbell-Hausdorff series* $\diamond$ *in* (15)*, then to any* $X \in \mathfrak{g}$ *we can associate the smooth vector field* $\widetilde{X}$ *on* $\mathfrak{W}$ *defined as follows*

$$\widetilde{X}_w f = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f \circ \tau_w)(tX), \tag{27}$$

for any $w \in \mathfrak{W}$ *and any* $f \in C^\infty(\mathfrak{g})$*. Thus,* $\widetilde{X}$ *enjoys the left-invariance property*

$$\mathrm{d}_w \tau_v(\widetilde{X}_w) = \widetilde{X}_{v \diamond w}, \quad \textit{for every } v, w \in \mathfrak{W}. \tag{28}$$

Moreover, the map $X \mapsto \widetilde{X}$ *of* $\mathfrak{g}$ *onto its image set is an isomorphism of Lie algebras.*

3. A general result on local Lie groups for vector-field algebras

In this section, together with the same assumptions on $\mathfrak{g}$ made in Section 2 (i.e., $\mathfrak{g}$ is a finite-dimensional Lie subalgebra of $\mathcal{X}(\mathbb{R}^n)$ made of complete vector fields), we also assume that

the dimension of $\mathfrak{g}$ *is* n,
and $\mathfrak{g}$ *satisfies Hörmander's rank condition at any* $x \in \mathbb{R}^n$.

Under all these assumptions, due to (25) (and the Inverse Function Theorem) we can infer that Exp is a diffeomorphism of a neighborhood of $0 \in \mathfrak{g}$ onto a neighborhood of $0 \in \mathbb{R}^n$. Resuming the notation of Section 2, we can assume from the very start that the set $\mathfrak{U}$ where the Baker-Campbell-Hausdorff series converges is contained in the open neighborhood of $0 \in \mathfrak{g}$ on which Exp is a diffeomorphism. Thus, we can transfer the local Lie group $(\mathfrak{U}, \diamond)$ on a neighborhood Ω of $0 \in \mathbb{R}^n$. This amounts to introduce the local operation $\star$ defined by

$$x \star y := \mathrm{Exp}(\mathrm{Log}(x) \diamond \mathrm{Log}(y)), \quad \text{for } x, y \in \Omega, \tag{29}$$

where Log denotes the inverse function of $\mathrm{Exp}|_{\mathfrak{U}}$. By replacing Ω with $\mathrm{Exp}(\mathfrak{W})$, and taking into account the notable formula (21), one recognizes that

$$x \star y = \exp(\mathrm{Log}(y))(x), \quad \text{for } x, y \in \Omega.$$

This immediately provides a prolongation of $\star$ to $\Omega \times \mathbb{R}^n$. Now, a result proved in [4, Chap. 17] (see also [3] and [1]; in the latter paper, real-analytic vector fields are involved) shows that

$\star$ can be smoothly prolonged to the whole of $\mathbb{R}^n \times \mathbb{R}^n$,

and the prolongation can be chosen as to define on $\mathbb{R}^n$ a Lie group, say $\mathbb{G}$. In [3] we obtained this result by considering a suitable ODE solved by the curve

$$t \mapsto x \star (ty),$$

and by showing that this ODE admits a global solution defined throughout $\mathbb{R}$. Now, a natural question arises: what is the relationship between $\mathrm{Lie}(\mathbb{G})$ and $\mathfrak{g}$? Clearly, from the arguments in Section 2, it appears that $\mathrm{Lie}(\mathbb{G})$ and $\mathfrak{g}$ are isomorphic Lie algebras, which is however only a partially satisfactory fact. Furthermore, it is not difficult to recognize from the very definition of $\star$, and thanks to Theorem 2.1, that any $X \in \mathfrak{g}$ is locally left invariant for the $\star$ operation: indeed, for any $x, y \in \Omega$ (denoting by $\tau^\star$ and $\tau^\diamond$ the left translations associated with $\star$ and $\diamond$ respectively)

$$\begin{aligned}
\mathrm{d}_y \tau^\star_x(X_y) &\overset{(29)}{=} \mathrm{d}_y(\mathrm{Exp} \circ \tau^\diamond_{\mathrm{Log}(x)} \circ \mathrm{Log})(X_y) \\
&= \mathrm{d}_{\mathrm{Log}(x)\diamond\mathrm{Log}(y)}\mathrm{Exp} \circ \mathrm{d}_{\mathrm{Log}(y)}\, \tau^\diamond_{\mathrm{Log}(x)} \circ \mathrm{d}_y \mathrm{Log}(X_y) \\
&\overset{(26)}{=} \mathrm{d}_{\mathrm{Log}(x)\diamond\mathrm{Log}(y)}\mathrm{Exp} \circ \mathrm{d}_{\mathrm{Log}(y)}\, \tau^\diamond_{\mathrm{Log}(x)}(\widetilde{X}_{\mathrm{Log}(y)}) \\
&\overset{(28)}{=} \mathrm{d}_{\mathrm{Log}(x)\diamond\mathrm{Log}(y)}\mathrm{Exp}(\widetilde{X}_{\mathrm{Log}(x)\diamond\mathrm{Log}(y)}) \\
&\overset{(26)}{=} X_{\mathrm{Exp}(\mathrm{Log}(x)\diamond\mathrm{Log}(y))} \overset{(29)}{=} X_{x\star y}.
\end{aligned}$$

Actually, the identity $\mathrm{d}_y \tau^\star_x(X_y) = X_{x\star y}$ remains valid if $\star$ is replaced by its mentioned prolongation, so that we can prove that[7]

$$\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}.$$

Summing up, we have the following result:

Theorem 3.1. *Suppose that $\mathfrak{g}$ is a Lie algebra of smooth vector fields on $\mathbb{R}^n$ such that:*

[7] Here we are thinking of $\mathrm{Lie}(\mathbb{G})$ as the Lie algebra of the left invariant vector fields on $\mathbb{G}$, where vector fields are always meant as linear first order PDOs, as in (1).

1. *every* $X \in \mathfrak{g}$ *is a complete vector field;*

2. $\mathfrak{g}$ *satisfies Hörmander's rank condition at any* $x \in \mathbb{R}^n$*;*

3. $\dim(\mathfrak{g}) = n$.

Then there exists a Lie group $\mathbb{G} = (\mathbb{R}^n, \star)$ *such that* $\mathrm{Lie}(\mathbb{G}) = \mathfrak{g}$. *The operation* $\star$ *is a prolongation of the local operation of Baker-Campbell-Hausdorff type*

$$x \star y = \mathrm{Exp}(\mathrm{Log}(x) \diamond \mathrm{Log}(y)), \quad \textit{for } x, y \in \mathrm{Exp}(\mathfrak{W}),$$

where $\diamond$, Exp *and* $\mathfrak{W}$ *are as in Theorem 2.1.*

In the presence of homogeneity, the results in Theorems 2.1 and 3.1 produce a *global* lifting Carnot group, as we show in the next section.

4. Back to homogeneity: the proof of Theorem 1.7

Let us return to a Lie algebra of vector fields $\mathfrak{a} = \mathrm{Lie}\{X\}$, where we set as in the previous sections $X = \{X_1, \ldots, X_m\}$, and X satisfies axioms (H.1) and (H.2) in Section 1. Then the following facts hold true:

(i) $\mathfrak{a}$ has finite dimension, say N as usual (see Remark 1.3);

(ii) every $X \in \mathfrak{a}$ is a complete vector field (see Remark 1.2);

(iii) $\mathfrak{a}$ satisfies Hörmander's rank condition at any point of $\mathbb{R}^n$ (see Rem. 1.4);

(iv) since $\mathfrak{a}$ is nilpotent, the Baker-Campbell-Hausdorff series $X \diamond Y$ is convergent for every $X, Y \in \mathfrak{a}$ (actually, it is a finite sum).

Now, if Δ_λ are the dilations on $\mathfrak{a}$ introduced in (8), it is not difficult to prove that[8]

$$\delta_\lambda(\exp(X)(x)) = \exp(\Delta_\lambda(X))(\delta_\lambda(x)), \qquad \forall\, X \in \mathfrak{a},\ x \in \mathbb{R}^n,\ \lambda > 0. \tag{30}$$

Moreover, Δ_λ is a Lie-group morphism of $(\mathfrak{a}, \diamond)$ (see (9)):

$$\Delta_\lambda(X \diamond Y) = \Delta_\lambda(X) \diamond \Delta_\lambda(Y), \qquad \forall\, X, Y \in \mathfrak{a},\ \lambda > 0. \tag{31}$$

Property (iii) ensures that, defining $\mathrm{Exp} : \mathfrak{a} \to \mathbb{R}^n$ by $\mathrm{Exp}(X) = \exp(X)(0)$, then

$$\text{the image under Exp of any neighb. of } 0 \in \mathfrak{a} \text{ is a neighb. of } 0 \in \mathbb{R}^n. \tag{32}$$

We claim that (30)-*to*-(32) *allow us to globalize the local results obtained in Sections 2 and 3*, via suitable applications of the homogeneity arguments in Remarks 1.5 and 1.6. Indeed we have the following list of facts:

[8] Indeed, starting from (10) one can easily show that $\delta_\lambda(X(x)) = (\Delta_\lambda X)(\delta_\lambda(x))$, for every $X \in \mathfrak{a}$, every $x \in \mathbb{R}^n$ and $\lambda > 0$. In its turn, this gives $\Psi_t^{\Delta_\lambda X}(\delta_\lambda(x)) = \delta_\lambda(\Psi_t^X(x))$.

- The i-th component functions of both sides of (21) are homogeneous of degree σ_i w.r.t.

$$\mathfrak{a} \times \mathfrak{a} \times \mathbb{R}^n \ni (X, Y, x) \mapsto (\Delta_\lambda(X), \Delta_\lambda(Y), \delta_\lambda(x)).$$

  Thus, the local identity (21) is globally true:

$$\exp(Y)(\exp(X)(x)) = \exp(X \diamond Y)(x), \quad \text{for } X, Y \in \mathfrak{a} \text{ and } x \in \mathbb{R}^n.$$

- $\mathrm{Exp} : \mathfrak{a} \to \mathbb{R}^n$ is surjective: we argue as in Remark 1.6 starting from (31), which also gives

$$\delta_\lambda \circ \mathrm{Exp} = \mathrm{Exp} \circ \Delta_\lambda \quad \text{on } \mathfrak{a}. \tag{33}$$

- Properties (i) and (ii) imply that Theorem 2.1 is valid for $\mathfrak{a}$; the vector field $\widetilde{X}$ which locally lifts $X \in \mathfrak{a}$ is defined via (27) as a vector field defined on the whole of $\mathfrak{a}$, since τ_w is defined by the Baker-Campbell-Hausdorff (global) operation $\diamond$. We claim that (26) holds globally:

$$\mathrm{d}_w \mathrm{Exp}(\widetilde{X}_w) = X_{\mathrm{Exp}(w)}, \quad \text{for every } w \in \mathfrak{a} \text{ and every } X \in \mathfrak{a}.$$

  This can be proved via a homogeneity argument w.r.t. the dilations

$$\mathfrak{a} \times \mathfrak{a} \ni (X, w) \mapsto (\Delta_\lambda(X), \Delta_\lambda(w)).$$

- The operation $\diamond$ endows $\mathfrak{a}$ of a Lie group structure: once again one can prove the associativity of $\diamond$ by globalizing the local associativity via a homogeneity argument, based on (31). Moreover, the Lie algebra of this group is isomorphic to $\mathfrak{a}$, hence it is stratified (Rem. 1.3).

- The vector fields $\widetilde{X}_1, \ldots, \widetilde{X}_m$ are Δ_λ-homogeneous of degree 1 (see (27)), and are Lie-generators of the Lie algebra of $(\mathfrak{a}, \diamond)$.

From what we have proved so far, it follows that

$$(\mathfrak{a}, \diamond, \Delta_\lambda) \textit{ is a homogeneous Carnot group.}$$

We can prove part (2) of Theorem 1.7 starting from this fact. The special form (14) under which the lifting $X_i \mapsto \widetilde{X}_i$ can be put is however subtler: this needs a suitable change of variable on $\mathfrak{a} \equiv \mathbb{R}^N$, for which the reader is directly referred to [2].

For what concerns part (1) of Theorem 1.7, if $N = n$ then we are entitled to apply Theorem 3.1, together with all the above facts (which hold true whatever the N is). The local invertibility of Exp, together with (33), proves that Exp is actually globally invertible: this is a consequence of Remark 1.6-(1). Thus we can globally transfer the dilations Δ_λ to $\mathbb{R}^n$ via Exp; these dilations on $\mathbb{R}^n$ coincide with δ_λ, owing to (33). This shows that $\mathbb{G} = (\mathbb{R}^n, \star, \delta_\lambda)$ is a homogeneous Carnot group. Since the group $(\mathbb{R}^n, \star)$ is obtained from $(\mathfrak{a}, \diamond)$ via Exp, the Lie algebra Lie$(\mathbb{G})$ is obtained from Lie$(\mathfrak{a})$ via dExp. The identity (26) says that the vector field of Lie$(\mathbb{G})$ corresponding to $\widetilde{X} \in \text{Lie}(\mathfrak{a})$ is exactly X; this proves that Lie$(\mathbb{G}) = \mathfrak{g}$.

5. Applications to the study of the fundamental solution of $\mathcal{L}$

In this section we apply Theorem 1.7 in order to get precious information on the existence and on the estimates of a global fundamental solution Γ for $\mathcal{L} = \sum_{i=1}^m X_i^2$; in the sequel we suppose that $X = \{X_1, \dots, X_m\}$ satisfy axioms (H.1) and (H.2) in Section 1. If $N = n$, Theorem 1.7-(1) says that $\mathcal{L}$ is a sub-Laplacian on a Carnot group, and all that is worthy of note about Γ is contained in the paper [8] by Folland. Thus we suppose that

$$N > n \quad \text{and} \quad q > 2.$$

Indeed the latter assumption is not restrictive since the case $q = 2$ only happens when $\mathcal{L}$ is a strictly-elliptic constant-coefficient operator in $\mathbb{R}^2$ (which is also left invariant on $(\mathbb{R}^2, +)$), another well-known setting, where everything is known about the associated Γ.

Thus we are entitled to apply Theorem 1.7-(2), which grants the existence of a lifting Carnot group $\mathbb{G} = (\mathbb{R}^N, \star, D_\lambda)$ on $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p$, and of a sub-Laplacian $\mathcal{L}_\mathbb{G} = \sum_{i=1}^m \widetilde{X}_i^2$ which lifts $\mathcal{L}$ (in the sense of (14)). Thanks to the aforementioned paper [8], we know of the existence of a unique fundamental solution $\Gamma_\mathbb{G}$ for $\mathcal{L}_\mathbb{G}$ with pole at the origin and δ_λ-homogeneous of degree $2 - Q < 0$. By a "saturation" argument over the lifting variables of $\mathbb{R}^p$, in [2] it is proved that $\mathcal{L}$ has a unique global fundamental solution Γ vanishing at infinity, which admits the following integral representation

$$\Gamma(x;y) = \int_{\mathbb{R}^p} \Gamma_\mathbb{G}\Big((x,0)^{-1} \star (y,\eta)\Big)\, \mathrm{d}\eta \qquad (\text{for } x \neq y \text{ in } \mathbb{R}^n). \tag{34}$$

By saying that Γ is a global fundamental solution of $\mathcal{L}$ we mean that the map $y \mapsto \Gamma(x;y)$ is locally integrable on $\mathbb{R}^n$ and that

$$\int_{\mathbb{R}^n} \Gamma(x;y)\, \mathcal{L}\varphi(y)\, \mathrm{d}y = -\varphi(x) \qquad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^n) \text{ and every } x \in \mathbb{R}^n.$$

Γ enjoys further properties: it is smooth out of the diagonal; it is symmetric in x, y; it is strictly positive; it is locally integrable on $\mathbb{R}^n \times \mathbb{R}^n$; it vanishes when x

or y go to infinity; it is jointly homogeneous of degree $2-q<0$, i.e.,

$$\Gamma\big(\delta_\lambda(x);\delta_\lambda(y)\big)=\lambda^{2-q}\,\Gamma(x,y),\qquad x\neq y,\ \lambda>0.$$

In the sequel we denote by d_X the Carnot-Carathodory distance on $\mathbb{R}^n$ associated with X, and by $d_{\widetilde{X}}$ the Carnot-Carathodory distance induced on $\mathbb{R}^N$ by the lifted vector fields $\widetilde{X}_1,\dots,\widetilde{X}_m$. Accordingly, the balls in the cited metrics are denoted by $B_X(x,r)$ and $B_{\widetilde{X}}((x,\xi),r)$.

By means of the local-to-global Remark 1.5, one can prove the following result, starting from profound (local) results concerning the geometry of Hörmander vector fields, contained in the seminal papers [13] by Nagel, Stein, Wainger, and [15] by Sánchez-Calle:

Theorem 5.1. *With the above notation and assumptions, the following global results hold.*

(A). *Let q be as in* (4). *For any $k\in\{n,\dots,q\}$ there exists a function $f_k:\mathbb{R}^n\to\mathbb{R}$ which is continuous, nonnegative and δ_λ-homogeneous of degree $q-k$, and there exist structural constants $\gamma_1,\gamma_2>0$ such that*

$$\gamma_1\sum_{k=n}^{q} f_k(x)\,r^k\le \big|B_X(x,r)\big|\le \gamma_2\sum_{k=n}^{q} f_k(x)\,r^k, \tag{35}$$

for every $x\in\mathbb{R}^n$ and every $r>0$. Moreover, $f_q(x)$ is constant in x, and strictly positive.

(B). *There exist constants $\kappa\in(0,1)$ and $c_1,c_2>0$ such that, for every $x\in\mathbb{R}^n$, $\xi\in\mathbb{R}^p$ and $r>0$ one has the following estimates:*

$$\Big|\{\eta\in\mathbb{R}^p:(y,\eta)\in B_{\widetilde{X}}((x,\xi),r)\}\Big|\le c_1\frac{\big|B_{\widetilde{X}}((x,\xi),r)\big|}{\big|B_X(x,r)\big|},\quad \textit{for all } y\in\mathbb{R}^n, \tag{36}$$

$$\Big|\{\eta\in\mathbb{R}^p:(y,\eta)\in B_{\widetilde{X}}((x,\xi),r)\}\Big|\ge c_2\frac{\big|B_{\widetilde{X}}((x,\xi),r)\big|}{\big|B_X(x,r)\big|},\quad \textit{for all } y\in B_X(x,\kappa r). \tag{37}$$

It is nice to observe that all terms in (35) have the same homogeneity (of degree q) w.r.t. the dilations $(\delta_\lambda x,\lambda\,r)$, and that all terms in (36) and (37) have the same homogeneity (of degree $Q-q$) w.r.t. the dilations $(\delta_\lambda x,\delta^*_\lambda\xi,\lambda\,r)$: thus one obtains the global inequalities (35)-to-(36) starting from the local results in [13]-[15] by means of Remark 1.5.

By using the representation (34), and by a crucial use of Theorem 5.1, in [5] it is proved the following result:[9]

[9]By 'structural constant' we mean a constant only depending on the objects introduced in axioms (H.1)-(H.2).

Theorem 5.2. *Under the above assumptions on X, N, q, the following results hold true.*

(I). Representation of the X-derivatives of Γ. *For any $s, t \geq 1$, and any choice of $i_1, \dots, i_s$ and $j_1, \dots, j_t$ in $\{1, \dots, m\}$, we have the following representation formulas (holding true for $x \neq y$):*

$$X^y_{i_1} \cdots X^y_{i_s}\big(\Gamma(x;\cdot)\big)(y) = \int_{\mathbb{R}^p} \Big(\widetilde{X}_{i_1} \cdots \widetilde{X}_{i_s} \Gamma_{\mathbb{G}}\Big)\Big((x,0)^{-1} \star (y,\eta)\Big)\, \mathrm{d}\eta\,;$$

$$X^x_{j_1} \cdots X^x_{j_t}\big(\Gamma(\cdot;y)\big)(x) = \int_{\mathbb{R}^p} \Big(\widetilde{X}_{j_1} \cdots \widetilde{X}_{j_t} \Gamma_{\mathbb{G}}\Big)\Big((y,0)^{-1} \star (x,\eta)\Big)\, \mathrm{d}\eta\,;$$

$$\begin{aligned} &X^x_{j_1} \cdots X^x_{j_t} X^y_{i_1} \cdots X^y_{i_s} \Gamma(x;y) \\ &\qquad = \int_{\mathbb{R}^p} \Big(\widetilde{X}_{j_1} \cdots \widetilde{X}_{j_t} \Big(\big(\widetilde{X}_{i_1} \cdots \widetilde{X}_{i_s} \Gamma_{\mathbb{G}}\big) \circ \iota\Big)\Big)\Big((y,0)^{-1} \star (x,\eta)\Big)\, \mathrm{d}\eta\,. \end{aligned}$$

Here ι denotes the inversion map of the Lie group $\mathbb{G}$. (Superscripts on the vector fields denote the variables w.r.t. which differentiation is performed.)

(II). Estimates for the X-derivatives of Γ. *For any integer $r \geq 1$ there exists a constant $C_r > 0$ (only depending on r, otherwise structural) such that*

$$\Big| Z_1 \cdots Z_r \Gamma(x;y) \Big| \leq C_r \, \frac{d_X(x,y)^{2-r}}{\big|B_X(x, d_X(x,y))\big|},$$

for any $x, y \in \mathbb{R}^n$ (with $x \neq y$) and any $Z_1, \dots, Z_r \in \big\{X^x_1, \dots, X^x_m, X^y_1, \dots, X^y_m\big\}$. In particular, for every fixed $x \in \mathbb{R}^n$ we have

$$\lim_{|y| \to \infty} Z_1 \cdots Z_r \Gamma(x;y) = 0.$$

(III). Estimates of Γ when $n > 2$. *Suppose that $n > 2$. Then one has*

$$C^{-1} \frac{d_X(x,y)^2}{\big|B_X\big(x, d_X(x,y)\big)\big|} \leq \Gamma(x;y) \leq C \, \frac{d_X(x,y)^2}{\big|B_X(x, d_X(x,y))\big|},$$

for any $x, y \in \mathbb{R}^n$ (with $x \neq y$). Here $C \geq 1$ is a structural constant.

(IV). Estimates of Γ when $n = 2$. *Suppose that $n = 2$. For every compact set $K \subseteq \mathbb{R}^2$ there exist structural constants $c_1, c_2 > 0$ and real numbers $R_1, R_2 > 0$ (all depending on K) such that*

$$c_1 \log\Big(\frac{R_1}{d_X(x,y)}\Big) \leq \Gamma(x;y) \leq c_2 \, \frac{d_X(x,y)^2}{\big|B_X(x, d_X(x,y))\big|} \cdot \log\Big(\frac{R_2}{d_X(x,y)}\Big),$$

uniformly for $x \neq y$ in K.

(V). On-diagonal estimates of Γ when $n = 2$. *For every fixed pole $x \in \mathbb{R}^2$, there exist positive constants $\gamma_1(x), \gamma_2(x)$ and $0 < \varepsilon(x) < 1$ such that*

$$\gamma_1(x)\, F(x,y) \leq \Gamma(x;y) \leq \gamma_2(x)\, F(x,y),$$

for any y such that $0 < d_X(x,y) < \varepsilon(x)$, where ($f_2$ being as in Theorem 5.1)

$$F(x,y) = \begin{cases} \log\left(\dfrac{1}{d_X(x,y)}\right) & \text{if } f_2(x) > 0, \\ \dfrac{d_X(x,y)^2}{\big|B_X(x, d_X(x,y))\big|} & \text{if } f_2(x) = 0. \end{cases}$$

(VI). Blowing-up property of Γ at the pole. *For any $n \geq 2$, $\Gamma(x;\cdot)$ has a pole at $x \in \mathbb{R}^n$:*

$$\lim_{y \to x} \Gamma(x;y) = \infty.$$

Let us now say a few words about the proof of Theorem 5.2.

- The representation formulas in (I) follow from (34) and a passage-under-the-integral argument that, for the case of mixed derivatives, is particularly delicate.
- For simple homogeneity reasons on the group $\mathbb{G}$, $\Gamma_{\mathbb{G}}$ and its derivatives satisfy global growth estimates, which, combined with the representations in (I), give

$$\Big|Z_1 \cdots Z_r \Gamma(x;y)\Big| \leq c_r \int_{\mathbb{R}^p} d_{\widetilde{X}}^{2-Q-r}\Big((x,0)^{-1} \star (y,\eta)\Big)\, \mathrm{d}\eta, \quad \text{for } x \neq y.$$

- Again via the homogeneity arguments in Remark 1.5, it is sufficient to provide estimates of Γ and its derivatives when x, y are confined to a compact set.

Deferring all the details to [5], we give a rough idea of the proof of (II). As said, we can take x, y in some compact set, say K. Then one has

$$\begin{aligned} \Big|Z_1 \cdots Z_r \Gamma(x;y)\Big| &\leq c_r \int_{\mathbb{R}^p} d_{\widetilde{X}}^{2-Q-r}\Big((x,0)^{-1} \star (y,\eta)\Big)\, \mathrm{d}\eta \\ &= c_r \int_{|\eta| \geq 1} d_{\widetilde{X}}^{2-Q-r}\Big((x,0)^{-1} \star (y,\eta)\Big)\, \mathrm{d}\eta \\ &\quad + c_r \int_{|\eta| < 1} d_{\widetilde{X}}^{2-Q-r}\Big((x,0)^{-1} \star (y,\eta)\Big)\, \mathrm{d}\eta. \end{aligned}$$

Then one turns to prove that both summands in the above far right-hand term are bounded by
$$C_r \frac{d_X(x,y)^{2-r}}{\left|B_X(x,d_X(x,y))\right|}.$$
Indeed, as for the first summand, one can easily prove that it is finite (notice that, as $|\eta|\geq 1$, $(x,0)^{-1}\star(y,\eta)$ is far from the singularity of $d_{\widetilde{X}}^{2-Q-r}$), and that (see Theorem 5.1-(A))
$$\inf_{\substack{x,y\in K\\ x\neq y}} \frac{d_X(x,y)^{2-r}}{\left|B_X(x,d_X(x,y))\right|} \geq \frac{1}{\gamma_2} \inf_{\substack{x,y\in K\\ x\neq y}} \left(\sum_{k=n}^{q} f_k(x)\, d_X(x,y)^{k+r-2}\right)^{-1} =: M(K,r)>0.$$
Thus the really difficult task is to estimate the second integral summand: to this regard, one can prove the following estimates
$$\begin{aligned}\int_{|\eta|<1} d_{\widetilde{X}}^{2-Q-r}\Big((x,0)^{-1}\star(y,\eta)\Big)\,\mathrm{d}\eta &\leq C(K,r)\int_{d_X(x,y)}^{R_0(K)} \frac{\rho^{1-r}}{\left|B_X(x,\rho)\right|}\,\mathrm{d}\rho\\ &\leq C'(K,r)\,\frac{d_X(x,y)^{2-r}}{\left|B_X(x,d_X(x,y))\right|};\end{aligned}$$
the second inequality easily follows from the doubling inequality (a corollary of Theorem 5.1-(A)), while the first inequality follows from a delicate argument based on (A) and (B) in Theorem 5.1. It is out of the scope of this review to enter the details of the proof of the latter inequality, but we think it is worthy of note to know that it can be proved by means of the sole information on the geometry of Hörmander vector fields contained in the mentioned Theorem 5.1.

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