

A GIT CONSTRUCTION OF DEGENERATIONS OF HILBERT SCHEMES OF POINTS

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Received: July 10, 2017

Revised: October 18, 2017

Communicated by Thomas Peternell

ABSTRACT. We present a Geometric Invariant Theory (GIT) construction which allows us to construct good projective degenerations of Hilbert schemes of points for simple degenerations. A comparison with the construction of Li and Wu shows that our GIT stack and the stack they construct are isomorphic, as are the associated coarse moduli schemes. Our construction is sufficiently explicit to obtain good control over the geometry of the singular fibres. We illustrate this by giving a concrete description of degenerations of degree n Hilbert schemes of a simple degeneration with two components.

2010 Mathematics Subject Classification: Primary 14D06; Secondary 14C05, 14L24, 14D23

Keywords and Phrases: Geometric Invariant Theory, degeneration, Hilbert scheme

Constructing and understanding degenerations of moduli spaces is a crucial problem in algebraic geometry, as well as a vitally important technique, going back to the classical German and Italian schools where it was used for solving enumerative problems. New techniques for studying degenerations were introduced by Li and Li–Wu respectively. Their approach is based on the technique of *expanded degenerations*, which first appeared in [Li01]. This method is very general and can be used to study degenerations of various types of moduli problems, including Hilbert schemes and moduli spaces of sheaves. In [LW11] Li and Wu used degenerations of Quot-schemes and coherent systems to obtain degeneration formulae for Donaldson–Thomas invariants and Pandharipande–Thomas stable pairs. The reader can find a good introduction to these techniques in Li’s article [Li13].

The motivation for our work was a concrete geometric question: we wanted to understand degenerations of irreducible holomorphic symplectic manifolds. Clearly, a starting point for this is to study degenerations of $K3$ surfaces and their Hilbert schemes. Our guiding example were type II degenerations of $K3$ surfaces, but we were soon led to investigate the degeneration of *Hilbert schemes of points* for simple degenerations $X \rightarrow C$ where we make no a priori restriction on the type of the fibre nor its dimension. A simple degeneration means in particular that the total space is smooth and that the central fibre X_0 over the point $0 \in C$ of the 1-dimensional base C has normal crossing along smooth varieties. The aim of this paper is to develop the technique for the *construction* of degenerations of Hilbert schemes which give us not only abstract existence results, but also allow us to control the geometry of the degenerate fibres. In the forthcoming paper [GHHZ] we will then investigate the *properties* of these degenerations.

At this point we would like to explain the common ground, but also the differences of our approach to that of Li and Wu. First of all we only consider Hilbert schemes of points, whereas Li and Wu consider more generally Hilbert schemes of ideal sheaves with arbitrary Hilbert polynomial, and even Quot schemes. We have not investigated in how far our techniques can be extended to non-constant Hilbert polynomials. This might indeed be a question well worth pursuing, but one which would go far beyond the scope of this paper. The common ground with the approach of Li and Wu is that we also use Li's method of expanded degenerations $X[n] \rightarrow C[n]$. In the case of constant Hilbert polynomial the relevance of this construction is the following: ideally, one wants to construct a family whose special fibre over 0 parametrizes length n subschemes of the degenerate fibre X_0 . Clearly, the difficult question is how to describe subschemes whose support meets the singular locus of X_0 . The main idea of the construction of expanded degenerations $X[n] \rightarrow C[n]$ is that, whenever a subscheme approaches a singularity in X_0 , a new ruled component is inserted into X_0 and thus it will be sufficient to work with subschemes supported on the smooth loci of the fibres of $X[n] \rightarrow C[n]$. The price one pays for this is that the dimension of the base $C[n]$ is increased at each step of increasing n , and finally one has to take equivalence classes of subschemes supported on the fibres of $X[n] \rightarrow C[n]$. Indeed, the construction of expanded degenerations also includes the action of an n -dimensional torus $G[n]$ which acts on $X[n] \rightarrow C[n]$ such that $C[n]/G[n] = C$.

The way Li and Wu then proceed is by constructing the *stack* $\mathfrak{X}/\mathfrak{C}$ of *expanded degenerations* associated to $X \rightarrow C$, which is done by introducing a suitable notion of equivalence on expanded degenerations. For fixed Hilbert polynomial P they then introduce the notion of *stable* ideal sheaves with Hilbert polynomial P , and use this to define a stack $\mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^P$ over C parametrizing such stable ideal sheaves. In the case of constant Hilbert polynomial $P = n$ this leads to subschemes of length n supported on the smooth locus of a fibre of an expanded degeneration, and having finite automorphism group. We call the stack $\mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^n$ the *Li–Wu stack*. For details see [LW11] and, for a survey, also [Li13]. In con-

trast to this approach our method does not use the Li–Wu stack, but is based on a Geometric Invariant Theory approach (GIT, [MFK94]), which we will now outline.

ACKNOWLEDGEMENTS

The first author would like to thank the Research Council of Norway (NFR) for partial support under grant 230986. The third author is grateful to Deutsche Forschungsgemeinschaft (DFG) for partial support under grant Hu 337/7-1.

0.1 THE MAIN RESULTS

The main technical achievement of the paper is to construct a suitable set-up which allows us to apply GIT methods. To perform this we must make one assumption on the dual graph $\Gamma(X_0)$ associated to the singular fibre X_0 , namely that it is *bipartite*, or equivalently it has no cycles of odd length. This is not a crucial restriction as we can always perform a quadratic base change to get into this situation. We first construct a relatively ample line bundle \mathcal{L} on $X[n] \rightarrow C[n]$. The bipartite assumption allows us to construct a particular $G[n]$ -linearization on \mathcal{L} which will then turn out to be well adapted for our applications to Hilbert schemes. The definition of the correct $G[n]$ -linearization is the most important technical tool of this paper. Using \mathcal{L} we can construct an ample line bundle \mathcal{M}_ℓ on the relative Hilbert scheme $\mathbf{H}^n := \text{Hilb}^n(X[n]/C[n])$, which comes equipped with a natural $G[n]$ -linearization. (The integer $\ell \gg 0$ only plays an auxiliary role.) This construction is sufficiently explicit to allow us to analyse GIT stability, using a relative version of the Hilbert–Mumford numerical criterion (see [GHH15, Cor. 1.1]). In particular, we are able to prove that (semi-)stability of a point $[Z] \subset \mathbf{H}^n$ only depends on the degree n cycle associated to Z (and not on its scheme structure).

After having fixed the $G[n]$ -linearized sheaf \mathcal{L} , our construction depends a priori on several choices. One choice is the orientation of the dual graph $\Gamma(X_0)$. As we work with a bipartite graph, it admits exactly two bipartite orientations and we will show that these lead to isomorphic GIT quotients. We moreover need to select a suitable ℓ in the construction of \mathcal{M}_ℓ . Our characterization of stable n cycles will a posteriori show that the final result is independent also of this choice.

This characterization is indeed crucial and in order to formulate this theorem, we first need some notation. Let $[Z] \in \mathbf{H}^n$ be represented by a subscheme $Z \subset X[n]_q$ for some point $q \in C[n]$. Using a local étale coordinate t on C we obtain coordinates $t_i, i \in \{1, \dots, n+1\}$ on $C[n]$ and we define $\{a_1, \dots, a_r\}$ to be the subset indexing coordinates with $t_i(q) = 0$. Setting $a_0 = 1$ and $a_{r+1} = n+1$ we obtain a vector $\mathbf{a} = (a_0, \dots, a_{r+1}) \in \mathbb{Z}^{r+2}$, which, in turn, determines a vector $\mathbf{v}_\mathbf{a} \in \mathbb{Z}^{r+1}$ whose i -th component is $a_i - a_{i-1}$.

We say that Z has *smooth support* if Z is supported in the smooth part of the fibre $X[n]_q$. Then each point P_i in the support of Z is contained in a unique

component of $X[n]_q$ with some multiplicity n_i . This allows us to define the numerical support $\mathbf{v}(Z) \in \mathbb{Z}^{r+1}$, see Definition 2.6. Our characterization then reads as follows:

THEOREM 0.1. *Let $\ell \gg 2n^2$. The (semi-)stable locus in \mathbf{H}^n with respect to \mathcal{M}_ℓ can be described as follows:*

1. *If $[Z] \in \mathbf{H}^n$ has smooth support, then $[Z] \in \mathbf{H}^n(\mathcal{M}_\ell)^{ss}$ if and only if*

$$\mathbf{v}(Z) = \mathbf{v}_a.$$

In this case, it also holds that $[Z] \in \mathbf{H}^n(\mathcal{M}_\ell)^s$.

2. *If $[Z] \in \mathbf{H}^n$ does not have smooth support, then $[Z] \notin \mathbf{H}^n(\mathcal{M}_\ell)^{ss}$.*

We denote the locus of stable points by $\mathbf{H}_{\text{GIT}}^n := \mathbf{H}^n(\mathcal{M}_\ell)^s$ (it does not depend on ℓ). It is interesting to note that our GIT approach independently also leads to the property that stable cycles have smooth support, a condition also appearing in Li–Wu stability. In fact, GIT stable cycles are always Li–Wu stable, but the converse does not hold in general. In other words we obtain an inclusion $\mathbf{H}_{\text{GIT}}^n \subset \mathbf{H}_{\text{LW}}^n$ of GIT stable cycles in Li–Wu stable cycles, which, in general, is strict, see Lemma 3.7 and the comment following it.

We can now form the *GIT-quotient*

$$I_{X/C}^n = \mathbf{H}_{\text{GIT}}^n/G[n].$$

This is the main new object which we construct in this paper. The advantage of our method is that we can control the GIT stable points very explicitly and this allows us to analyse the geometry of the fibres of the degenerate Hilbert schemes in great detail. Moreover, we can also use the results of [GHH15], where it was shown, in particular, that $I_{X/C}^n$ is projective over C .

We can also form the *stack quotient*

$$\mathcal{I}_{X/C}^n = [\mathbf{H}_{\text{GIT}}^n/G[n]].$$

Our main result about this stack is

THEOREM 0.2. *The GIT quotient $I_{X/C}^n$ is projective over C . The stack $\mathcal{I}_{X/C}^n$ is a Deligne–Mumford stack, proper and of finite type over C , having $I_{X/C}^n$ as coarse moduli space.*

We also investigate how the GIT stack quotient and the Li–Wu stack compare. For this we construct a natural morphism $f: \mathcal{I}_{X/C}^n \rightarrow \mathcal{I}_{\mathbf{x}/\mathfrak{e}}^n$ between the two stacks and show

THEOREM 0.3. *The morphism $f: \mathcal{I}_{X/C}^n \rightarrow \mathcal{I}_{\mathbf{x}/\mathfrak{e}}^n$ is an isomorphism of Deligne–Mumford stacks.*

In this way our approach gives an alternative proof of the properness over the base curve C of the Li–Wu stack $\mathcal{I}_{x/C}^n$ for Hilbert schemes of points, see [Li13, Thm. 3.54]. It thus turns out that our GIT approach and the Li–Wu construction of degenerations of Hilbert schemes of points are in fact equivalent. The main advantage which we have thus gained is, in addition to constructing a relatively projective coarse moduli space for the Li–Wu stack, that we have the tools to explicitly describe the degenerate Hilbert schemes. We will illustrate this with the example of degree n Hilbert schemes on two components, which we treat in detail in Section 4.

Of course, one of the main objectives of this research is to construct good degenerations of, in particular, Hilbert schemes of $K3$ surfaces, such as in the work of Nagai [Nag08] who used an ad hoc approach which works very well in degree 2. At this point it is also worth noting that the simple approach of taking the relative Hilbert scheme will not work as this is hard to control and badly behaved. We will study the properties of our degenerations in detail in [GHHZ], but we would like to mention at least the main results in support of our approach. First of all, starting with a strict simple degeneration $X \rightarrow C$ of surfaces, the GIT stack $\mathcal{I}_{X/C}^n$ is smooth and semi-stable as a DM -stack over C . The scheme $I_{X/C}^n$ has finite quotient singularities – see also Section 4 for the degree 2 case – and $(I_{X/C}^n, (I_{X/C}^n)_0)$ is simple normal crossing up to finite group actions. In particular, this allows us to attach a dual complex to the central fibre and due to our good control of the degenerations we can describe this complex explicitly. If $X \rightarrow C$ is a type II degeneration of $K3$ surfaces, then the stack $\mathcal{I}_{X/C}^n$ carries a nowhere degenerate relative logarithmic 2-form. If $X \rightarrow C$ is any strict simple degeneration of surfaces, then $(I_{X/C}^n, (I_{X/C}^n)_0)$ is a dlt (divisorial log terminal) pair. Moreover, if we start with a type II Kulikov model of $K3$ surfaces $I_{X/C}^n \rightarrow C$ is a minimal dlt model. In this case the dual complex can be identified with the Kontsevich–Soibelman skeleton.

Lastly, let us remark that for a simple degeneration $f: X \rightarrow C$, it is also natural to consider configurations of n points in the fibres of f (rather than length n subschemes, as in this article). This has been studied thoroughly by Abramovich and Fantechi in [AF14]. In particular, they exhibit a moduli space, which is projective over C , parametrizing *stable* configurations.

0.2 ORGANIZATION OF THE PAPER

The paper is organized as follows. Section 1 introduces most of the main concepts and technical tools. In particular, we will review the notions of a simple degeneration $X \rightarrow C$ and of expanded degenerations $X[n] \rightarrow C[n]$, as well as the action of the rank n torus $G[n]$ on $X[n] \rightarrow C[n]$. The construction of $X[n] \rightarrow C[n]$ depends on the choice of an orientation of the dual graph $\Gamma(X_0)$ of the central fibre X_0 . In Proposition 1.7 we shall give a concrete description and local equations for $X[n] \rightarrow C[n]$, see also [Wu07, §4.2]. We then enter into a discussion of the properties of $X[n] \rightarrow C[n]$. We will prove in Proposition 1.9 that the algebraic space $X[n]$ is a scheme if and only if the

degeneration $X \rightarrow C$ is *strict*, i.e. all components of X_0 are smooth. Our next aim is to understand when the morphism $X[n] \rightarrow C[n]$ is projective. It turns out that this is the case if and only if the directed graph $\Gamma(X_0)$ contains *no directed cycles*, see Proposition 1.10. Since we aim at a GIT approach we need a relatively ample line bundle \mathcal{L} on $X[n] \rightarrow C[n]$ together with a suitable $G[n]$ -linearization. This will be achieved in Section 1.4 and this construction is the technical core of our approach. At this point we must impose another condition on the degeneration $X \rightarrow C$, namely that the dual graph $\Gamma(X_0)$ can be equipped with a *bipartite orientation*, see Section 1.4. In Proposition 1.11 we shall prove that reversing the orientation of the graph $\Gamma(X_0)$ leads to an isomorphic quotient. Finally, we investigate the fibres of the morphism $X[n] \rightarrow C[n]$ in detail and enumerate their components in Proposition 1.12, an essential tool for all practical computations, in particular also for the GIT analysis.

In Section 2 we perform a careful analysis of GIT stability. Using the line bundle \mathcal{L} we construct the relatively ample line bundle \mathcal{M}_ℓ on $\mathbf{H}^n = \text{Hilb}^n(X[n]/C[n])$, which inherits a $G[n]$ -linearization. The main result of this section is Theorem 2.10 (Theorem 0.1) where we characterize the stable locus. For these calculations we will make extensive use of the local coordinates which we introduce in Section 1.1.5.

Section 3 is devoted to the comparison of our construction with the Li–Wu stack. For this we introduce the GIT quotient stack $\mathcal{I}_{X/C}^n$ and prove the properness Theorem 3.2 (Theorem 0.2). Finally we construct a map between the GIT quotient stack and the Li–Wu stack and prove their equivalence in Theorem 3.10 (Theorem 0.3).

In Section 4 we discuss one example in detail in order to illustrate how the machinery works. The example we have chosen is a simple degeneration $X \rightarrow C$ where X_0 has two components. We shall describe the geometry of the central fibre $(I_{X/C})_0$ in detail and, in case of degree 2 give a complete classification of the singularities of the total space. We also compute the dual complex for arbitrary degree n , which turns out to be the standard n -simplex.

0.3 NOTATION

We work over a field k which is algebraically closed of characteristic zero. By a *point* of a k -scheme of finite type, we will always mean a closed point, unless further specification is given. The projectivization $\mathbb{P}(\mathcal{E})$ of a coherent sheaf \mathcal{E} is Grothendieck’s contravariant one, parametrizing rank one quotients.

For an integer n we denote $[n] = \{1, \dots, n\}$.

1 EXPANDED DEGENERATIONS

Here we recall a construction, due to Li [Li01], which to a simple degeneration $X \rightarrow C$ over a curve (Definition 1.1), together with an orientation of the dual

graph Γ of the degenerate fibre X_0 , associates a family of *expanded degenerations* $X[n] \rightarrow C[n]$ over an $(n+1)$ -dimensional base $C[n]$, equipped with an action by an algebraic torus $G[n] \cong \mathbb{G}_m^n$.

The aim of the current section is to set the stage for applying GIT to the induced $G[n]$ -action on the relative Hilbert scheme $\text{Hilb}^n(X[n]/C[n])$. Thus, after recalling Li's construction in Section 1.1 we study under which circumstances $X[n] \rightarrow C[n]$ is a projective scheme in Section 1.2. In Section 1.3 we study its fibres. Finally, in Section 1.4 we restrict to the case of a bipartite oriented graph Γ and in this situation equip $X[n]$ with a particular linearization of the $G[n]$ -action and compute the associated Hilbert–Mumford invariants.

1.1 CONSTRUCTION OF THE FAMILY $X[n] \rightarrow C[n]$

In this section we summarize work from Li [Li01, Li13], Wu [Wu07] and Li–Wu [LW11] on expanded degenerations.

1.1.1 SETUP

Let C be a smooth curve with a distinguished point $0 \in C$. Following the terminology of Li–Wu [LW11, Def. 1.1] we define:

DEFINITION 1.1. A *simple degeneration* is a flat morphism $\pi: X \rightarrow C$ from a smooth algebraic space X to a k -smooth curve C with a distinguished point $0 \in C$, such that

- (i) π is smooth outside the central fibre $X_0 = \pi^{-1}(0)$ and
- (ii) the central fibre X_0 has normal crossing singularities and its singular locus $D \subset X_0$ is smooth.

We call a simple degeneration *strict* if all components of X_0 are smooth.

In étale local coordinates, a simple degeneration $X \rightarrow C$ is thus of the form $t = xy$. The main motivation for us is degenerations of K3 surfaces: Kulikov models of type II are simple degenerations, whereas Kulikov models of type III are not, because of triple intersections in the central fibre.

INPUT. The input data to Li's construction are

- a smooth base curve C with a distinguished point $0 \in C$ and an étale morphism $t: C \rightarrow \mathbb{A}^1$ with $t^{-1}(0) = \{0\}$,
- a strict simple degeneration $X \rightarrow C$ and
- an orientation of the dual graph Γ of the special fibre X_0 .

Let $G[n] \subset \text{SL}(n+1)$ be the diagonal maximal torus and let $C[n]$ be the fibre product $C \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ with respect to $t: C \rightarrow \mathbb{A}^1$ and the multiplication morphism $\mathbb{A}^{n+1} \rightarrow \mathbb{A}^1$. Then $G[n]$ acts naturally on \mathbb{A}^{n+1} such that $\mathbb{A}^{n+1} \rightarrow \mathbb{A}^1$ is invariant. Hence there are induced actions on $C[n]$ and on $X \times_C C[n]$.

OUTPUT. The output of Li’s construction is an explicit small $G[n]$ -equivariant resolution $X[n]$ of the fibre product $X \times_C C[n]$.

REMARK 1.2.

- The logic here is not axiomatic; rather we give the explicit construction of $X[n]$ first and study its properties afterwards.
- Li’s construction applies also to non-strict simple degenerations, but this requires an additional hypothesis and is more cumbersome to state. We briefly treat the non-strict case in Section 1.2.2.
- It suffices to treat $C = \mathbb{A}^1$ since $X \times_C C[n] \cong X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$. We occasionally take advantage of this, notably in Section 1.1.2, but otherwise keep the base curve C as this is notationally convenient.

REMARK 1.3. Li does not use the language of an orientation of the dual graph Γ . Instead, let $\nu: \widetilde{X}_0 \rightarrow X_0$ be the normalization morphism. For each component D of the singular locus of X_0 the inverse image $\nu^{-1}(D)$ is a disjoint union of two copies of D . Li fixes a labelling $\nu^{-1}(D) = D^+ \cup D^-$ for each such D , and in fact considers this labelling as part of the data defining a simple degeneration. This is equivalent to our orientation of Γ : viewing the nodes of Γ as the components of \widetilde{X}_0 , the corresponding orientation of the edge $[D]$ points from the (node corresponding to the) component containing D^- to that containing D^+ .

1.1.2 THE BASIC CASE

We first define $X[n]$ in the special case where $X_0 = Y_1 \cup Y_2$ has two smooth components with irreducible intersection $D = Y_1 \cap Y_2$. The dual graph Γ consists of two nodes $[Y_i]$ connected by one edge $[D]$. Fix one out of the two possible orientations and then relabel the two components Y_i if necessary so that $[D]$ points from $[Y_1]$ to $[Y_2]$:

$$[Y_1] \xrightarrow{[D]} [Y_2]$$

Recursively assume a small resolution $X[n - 1] \rightarrow X \times_{\mathbb{A}^1} \mathbb{A}^n$ has been constructed. View this first as a morphism over \mathbb{A}^n and then via the last coordinate $t_n: \mathbb{A}^n \rightarrow \mathbb{A}^1$ as a morphism over \mathbb{A}^1 . Now pull back along the multiplication morphism $m: \mathbb{A}^2 \rightarrow \mathbb{A}^1$ to obtain a partial resolution

$$X[n - 1] \times_{\mathbb{A}^1} \mathbb{A}^2 \rightarrow X \times_{\mathbb{A}^1} \mathbb{A}^{n+1} \tag{1}$$

as in the diagram

$$\begin{array}{ccccccc} X[n - 1] \times_{\mathbb{A}^1} \mathbb{A}^2 & \longrightarrow & X \times_{\mathbb{A}^1} \mathbb{A}^{n+1} & \longrightarrow & \mathbb{A}^{n+1} & \xrightarrow{(t_n, t_{n+1})} & \mathbb{A}^2 \\ & & \downarrow & & \downarrow (t_1, \dots, t_{n-1}, m) & & \downarrow m \\ X[n - 1] & \longrightarrow & X \times_{\mathbb{A}^1} \mathbb{A}^n & \longrightarrow & \mathbb{A}^n & \xrightarrow{t_n} & \mathbb{A}^1 \end{array}$$

where all squares are Cartesian.

PROPOSITION 1.4. *The following recursion defines a small $G[n]$ -equivariant resolution $X[n]$ of the fibre product $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ for all natural numbers n :*

- (i) $X[0] = X$
- (ii) $X[n]$ is the blow-up of $X[n - 1] \times_{\mathbb{A}^1} \mathbb{A}^2$ along the strict transform of

$$Y_1 \times V(t_{n+1}) \subset X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$$

under the partial resolution (1). Here the chosen orientation of the dual graph Γ is used to order the components Y_i of X_0 by the convention that the arrow $[D]$ has source $[Y_1]$ and target $[Y_2]$.

Proof. First consider $X[1]$. This is defined to be the blow-up of $X \times_{\mathbb{A}^1} \mathbb{A}^2$ along $Y_1 \times V(t_2)$. In étale local coordinates $X \rightarrow \mathbb{A}^1$ is $t = xy$ and the fibre product $X \times_{\mathbb{A}^1} \mathbb{A}^2$ is $t_1 t_2 = xy$. Blowing up either of the Weil divisors $x = t_2 = 0$ or $y = t_2 = 0$ yields a small resolution. Explicitly, let U be an étale local neighbourhood in X with coordinates x and y such that Y_1 is given by the equation $y = 0$. The blow-up along $y = t_2 = 0$ is then the locus

$$\begin{aligned} ux &= t_1 v \\ vy &= t_2 u \end{aligned}$$

in $U \times \mathbb{A}^2 \times \mathbb{P}^1$, where $(u : v)$ are homogeneous coordinates on \mathbb{P}^1 . Thus $X[1]$ is indeed a small resolution.

This basic construction is now repeated: view $X[n - 1]$ as a family over \mathbb{A}^1 via the last coordinate $t_n : \mathbb{A}^n \rightarrow \mathbb{A}^1$.

Claim: $X[n - 1] \rightarrow \mathbb{A}^1$ is a simple degeneration. Its central fibre has two components, the strict transforms $Y_i^{(n-1)}$ of $Y_i \times V(t_n) \subset X \times_{\mathbb{A}^1} \mathbb{A}^n$. Their intersection $D^{(n-1)}$ is the strict transform of $D \times V(t_n)$ and it is irreducible.

Granted this, the dual graph of $X[n - 1]|_{t_n=0}$ is identified with that of X_0 and inherits an orientation

$$[Y_1^{(n-1)}] \xrightarrow{[D^{(n-1)}]} [Y_2^{(n-1)}].$$

Now the strict transform of $Y_1 \times V(t_{n+1})$ in $X[n - 1] \times_{\mathbb{A}^1} \mathbb{A}^2$ is precisely

$$Y_1^{(n-1)} \times V(t_{n+1}) \subset X[n - 1] \times_{\mathbb{A}^1} \mathbb{A}^2$$

(writing (t_n, t_{n+1}) for the coordinates on the last factor \mathbb{A}^2). Thus the recipe says $X[n] = X[n - 1][1]$. In view of this it suffices to verify the claim for $X[1]$ viewed as a family over \mathbb{A}^1 via t_2 : in fact under the blow-up

$$X[1] \rightarrow X \times_{\mathbb{A}^1} \mathbb{A}^2$$

the strict transform of $Y_1 \times V(t_2)$ is an isomorphic copy of it, the strict transform of $Y_2 \times V(t_2)$ is its blow-up along $D \times \{(0, 0)\}$, and they are normal crossing divisors intersecting along the strict transform of $D \times V(t_2)$, which is an isomorphic copy of it. These statements are readily verified with the help of the local equations for $X[1]$ given above. \square

1.1.3 THE GENERAL CASE

Return to the situation of an arbitrary strict simple degeneration $X \rightarrow C$, where the central fibre $X_0 \subset X$ and its singular locus are allowed to have several components. Fix an orientation of the dual graph Γ . We phrase the definition of $X[n]$ in somewhat informal language and expand the precise meaning below.

DEFINITION 1.5. Let $X[n] \rightarrow X \times_C C[n]$ be the small resolution obtained by applying Proposition 1.4 locally around each component of the singular locus of X_0 .

Explicitly:

- For each component D of the singular locus of X_0 there are unique components Y_1 and Y_2 of X_0 such that D is a component of $Y_1 \cap Y_2$. Use the chosen orientation of Γ to distinguish the role of the two components Y_i , by labelling them so that the arrow $[D]$ in the oriented dual graph Γ points from $[Y_1]$ to $[Y_2]$.
- Define $U \subset X$ to be the Zariski open subset whose complement is the union of all components of X_0 except Y_1 and Y_2 together with all components of $Y_1 \cap Y_2$ except D . Thus U is a Zariski open neighbourhood around D such that U_0 has exactly two components $Y_i \cap U$ with intersection D .
- Apply Proposition 1.4 to obtain the small resolution $U[n] \rightarrow U \times_C C[n]$.
- This small resolution is an isomorphism away from $D \times_C C[n]$, and the collection of all U as D varies covers X . Thus the various $U[n]$ glue to yield the global small resolution $X[n] \rightarrow X \times_C C[n]$.

1.1.4 THE GROUP ACTION

We equip $X[n]$ with a $G[n]$ -action such that $X[n] \rightarrow X \times_C C[n]$ is equivariant. The target is here equipped with the action induced by the natural action of $G[n]$ on \mathbb{A}^{n+1} and (hence) on $C[n]$.

Recall that an element in $G[n]$ is an $(n+1)$ -tuple $(\sigma_1, \dots, \sigma_{n+1})$ of elements in \mathbb{G}_m such that $\prod_i \sigma_i = 1$. There is a Cartesian diagram

$$\begin{array}{ccc} G[n] & \xrightarrow{\text{pr}_{n,n+1}} & \mathbb{G}_m^2 \\ \downarrow m_{n,n+1} & & \downarrow m \\ G[n-1] & \xrightarrow{\text{pr}_n} & \mathbb{G}_m \end{array}$$

where pr_n and $\text{pr}_{n,n+1}$ are projections onto the last and the last two coordinates, m is multiplication and $m_{n,n+1}$ multiplies together the last two coordinates.

PROPOSITION 1.6. *There is a unique $G[n]$ -action on $X[n]$ such that*

$$X[n] \rightarrow X[n-1] \times_{\mathbb{A}^1} \mathbb{A}^2$$

is equivariant with respect to the natural action of $G[n] \cong G[n-1] \times_{\mathbb{G}_m} \mathbb{G}_m^2$ on the target.

Proof. In view of the local nature of Definition 1.5 it suffices to treat the basic situation in Proposition 1.4. Then $X[n] \rightarrow X[n-1] \times_{\mathbb{A}^1} \mathbb{A}^2$ is the blow-up along the strict transform of $Y_1 \times V(t_2)$, which is $G[n]$ -invariant. Hence the action lifts uniquely. \square

1.1.5 LOCAL EQUATIONS

It is useful to have explicit equations for $X[n]$ in the case of the local model

$$X = \text{Spec } k[x, y, z, \dots] \xrightarrow{t=xy} C = \text{Spec } k[t]. \tag{2}$$

Consider the product

$$(X \times \mathbb{A}^{n+1}) \times (\mathbb{P}^1)^n = \text{Spec } k[x, y, z, \dots, t_1, \dots, t_{n+1}] \times \prod_i \text{Proj } k[u_i, v_i]$$

and its subvariety $(X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}) \times (\mathbb{P}^1)^n$ defined by $xy = t_1 t_2 \cdots t_{n+1}$.

PROPOSITION 1.7 (Wu [Wu07, §4.2]). *Let $X \rightarrow C$ be the simple degeneration (2), with dual graph oriented as $[V(y)] \rightarrow [V(x)]$. Then*

(i) *$X[n]$ is the subvariety of $(X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}) \times (\mathbb{P}^1)^n$ defined by the equations*

$$\begin{aligned} u_1 x &= v_1 t_1 \\ u_i v_{i-1} &= v_i u_{i-1} t_i & (1 < i \leq n) \\ v_n y &= u_n t_{n+1}. \end{aligned}$$

(ii) *The $G[n]$ -action on $X[n] \subset (X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}) \times (\mathbb{P}^1)^n$ is the restriction of the action which is trivial on X , given by*

$$(t_1, \dots, t_{n+1}) \xrightarrow{\sigma} (\sigma_1 t_1, \dots, \sigma_{n+1} t_{n+1})$$

on \mathbb{A}^{n+1} , and given by

$$(u_i : v_i) \xrightarrow{\sigma} (\sigma_1 \sigma_2 \cdots \sigma_i u_i : v_i)$$

on the i 'th copy of \mathbb{P}^1 .

This is straight forward to verify from the construction in Proposition 1.4.

REMARK 1.8. There is an isomorphism:

$$\begin{aligned} \mathbb{G}_m^n &\cong G[n] \subset \mathbb{G}_m^{n+1} \\ (\tau_1, \tau_2, \dots, \tau_n) &\mapsto (\tau_1, \tau_1^{-1}\tau_2, \tau_2^{-1}\tau_3, \dots, \tau_{n-1}^{-1}\tau_n, \tau_n^{-1}) \end{aligned}$$

In τ -coordinates, the action on the \mathbb{P}^1 -factors above is conveniently written as $(u_i : v_i) \mapsto (\tau_i u_i : v_i)$. (Note that Li [Li01] writes $\bar{\sigma}$ for our σ , and σ for our τ .)

Let $(u_0 : v_0) = (1 : x)$ and $(u_{n+1} : v_{n+1}) = (y : 1)$, so that the equations in Proposition 1.7 can be written uniformly as

$$u_i v_{i-1} = v_i u_{i-1} t_i, \quad (1 \leq i \leq n+1). \quad (3)$$

These local equations immediately lead to the explicit affine open cover given by Li [Li01, Lemma 1.2]: we have $X[n] = \bigcup_{k=1}^{n+1} W_k$, where

$$W_k: \begin{cases} u_i \neq 0 & \text{for } i < k \\ v_i \neq 0 & \text{for } i \geq k. \end{cases}$$

In each chart W_k , most of the equations (3) result in elimination of either u_i/v_i or v_i/u_i , so that W_k has coordinates

$$t_1, \dots, t_{n+1}, \frac{v_{k-1}}{u_{k-1}}, \frac{u_k}{v_k}$$

(together with z, \dots) subject to the single relation $t_k = \frac{u_k v_{k-1}}{v_k u_{k-1}}$. Each W_k is clearly $G[n]$ -invariant, and the $G[n]$ -action is

$$\begin{aligned} (t_1, \dots, t_n) &\xrightarrow{\sigma} (\sigma_1 t_1, \dots, \sigma_n t_n) \\ \left(\frac{v_{k-1}}{u_{k-1}}, \frac{u_k}{v_k} \right) &\xrightarrow{\sigma} \left(\tau_{k-1}^{-1} \frac{v_{k-1}}{u_{k-1}}, \tau_k \frac{u_k}{v_k} \right) \end{aligned}$$

on points; here $\tau_i = \sigma_1 \cdots \sigma_i$ as in Remark 1.8.

1.2 PROJECTIVITY CRITERIA

1.2.1 PRELIMINARIES

In the arguments that follow, we shall frequently make use of the exactness of

$$\bigoplus_i \mathbb{Z}D_i \rightarrow \text{Pic}(Y) \rightarrow \text{Pic}(Y \setminus \cup_i D_i) \rightarrow 0$$

whenever $\{D_i\}$ is a finite set of effective prime divisors in a nonsingular variety Y (see e.g. [Har77, Prop. II.6.5]). Whenever $P \rightarrow Y$ is a principal \mathbb{G}_m -bundle, with associated line bundle L , we also have an exact sequence

$$\mathbb{Z} \xrightarrow{c_1(L)} \text{Pic}(Y) \rightarrow \text{Pic}(P) \rightarrow 0.$$

This follows by applying the first short exact sequence to the line bundle L and its 0-section H_0 , together with the fact that pullback defines an isomorphism $\text{Pic}(Y) \rightarrow \text{Pic}(L)$ which identifies $c_1(L)$ with H_0 .

1.2.2 CRITERION FOR $X[n]$ TO BE A SCHEME.

Let $X \rightarrow C$ be a non-strict simple degeneration. Thus the central fibre X_0 contains at least one singular (“self intersecting”) component Y . Let $D \subset Y$ be a component of the singular locus of such a component.

We can no longer apply Proposition 1.4 Zariski locally around D , but we still can do so in an étale local sense provided we can distinguish étale locally around D between the two branches of Y meeting there. This blends well with Li’s use of the normalization morphism $\nu: \tilde{X}_0 \rightarrow X_0$ explained in Remark 1.3: we define a non-strict simple degeneration as in Definition 1.1, but replace strictness by the following condition on each component D of the singular locus of X_0 :

Assume the preimage $\nu^{-1}(D)$ is a disjoint union of two copies of D .

Under this condition Definition 1.5 of $X[n] \rightarrow C[n]$ applies to non-strict simple degenerations when the word “locally” is interpreted in the étale topology.

We refrain from giving further details as our aim here is just to point out that in the non-strict situation $X[n]$ is an algebraic space but never a scheme.

PROPOSITION 1.9. *Let $X \rightarrow C$ be a simple degeneration where X is a scheme. Then, for all $n > 0$, the algebraic space $X[n]$ is a scheme if and only if $X \rightarrow C$ is strict, i.e. if and only if the graph $\Gamma(X_0)$ contains no loops.*

Proof. If $X \rightarrow C$ is strict, then $X[n]$ is a scheme by construction: it is recursively defined as a resolution $X[n] \rightarrow X[n-1] \times_{\mathbb{A}^1} \mathbb{A}^2$ given Zariski locally on the target by blowing up Weil divisors.

Conversely let $X \rightarrow C$ be non-strict. We reduce to $n = 1$: let $\mathbb{A}^2 \rightarrow \mathbb{A}^{n+1}$ be the map $(t_1, t_2) \mapsto (t_1, t_2, 1, \dots, 1)$ and let $C[1] \rightarrow C[n]$ be the induced map. Then there is a Cartesian diagram (see e.g. [Li13, 2.14 and 2.15])

$$\begin{array}{ccc} X[1] & \longrightarrow & X[n] \\ \downarrow & & \downarrow \\ C[1] & \longrightarrow & C[n] \end{array}$$

so that if $X[1]$ fails to be a scheme, then so does $X[n]$.

As X_0 is non-strict there exists a singular component $Y \subset X_0$. Fix a singular point $P \in Y$. The inverse image of $(P; 0, 0)$ by $X[1] \rightarrow X \times_{\mathbb{A}^1} \mathbb{A}^2$ is a \mathbb{P}^1 . If $X[1]$ is a scheme, hence a nonsingular variety, then there exists an effective divisor $H \subset X[1]$ intersecting this \mathbb{P}^1 in a positive number of points. In particular the corresponding line bundle $\mathcal{L} = \mathcal{O}_{X[1]}(H)$ has nontrivial restriction to \mathbb{P}^1 . Choose a Zariski open neighbourhood $U \subset X$ of P , such that U does not intersect any other component of X_0 besides Y . Then $U[1]$ is a Zariski open neighbourhood of $\mathbb{P}^1 \subset X[1]$. The inverse image by $U[1] \rightarrow \mathbb{A}^2$ of each of the coordinate axes $V(t_i) \subset \mathbb{A}^2$ is a principal prime divisor $D_i \subset U[1]$. Thus $\text{Pic}(U[1]) \cong \text{Pic}(U[1] \setminus (D_1 \cup D_2))$ by restriction. Also, the fibre $U_0 \subset U$ over $0 \in \mathbb{A}^1$ is a principal prime divisor, so $\text{Pic}(U) \cong \text{Pic}(U \setminus U_0)$ by restriction. But

$U[1] \setminus D_1 \cup D_2$ is a principal \mathbb{G}_m -bundle over $U \setminus U_0$, being a pullback of the multiplication map $\mathbb{A}^2 \setminus \{t_1 t_2 = 0\} \rightarrow \mathbb{A}^1 \setminus \{0\}$. Thus pullback $\pi^*: \text{Pic}(U) \rightarrow \text{Pic}(U[1])$ is surjective, and thus there is a line bundle \mathcal{M} on U such that $\pi^*(\mathcal{M}) \cong \mathcal{L}|_{U[1]}$. But the restriction of $\pi^*(\mathcal{M})$ to \mathbb{P}^1 is trivial. \square

1.2.3 CRITERION FOR $X[n] \rightarrow C[n]$ TO BE PROJECTIVE

PROPOSITION 1.10. *Let $X \rightarrow C$ be a projective strict simple degeneration with oriented dual graph $\Gamma(X_0)$. Then, for each $n > 0$, the morphism $X[n] \rightarrow C[n]$ is projective if and only if $\Gamma(X_0)$ contains no directed cycles.*

Proof that no cycles \implies projective. By induction we can assume that $X[n-1] \rightarrow C[n-1]$ is projective. Then $X[n-1] \times_{C[n-1]} C[n] \rightarrow C[n]$ is projective as well. It thus suffices to show that the resolution $X[n]$ of $X[n-1] \times_{C[n-1]} C[n] \cong X[n-1] \times_{\mathbb{A}^1} \mathbb{A}^2$ is globally given by blowing up Weil divisors and their strict transforms in a certain order. The required order will be dictated by the oriented graph $\Gamma(X_0)$.

As $\Gamma(X_0)$ has no directed cycles, the components Y of X_0 are partially ordered by the rule $Y \leq Y'$ if there is a directed path from the node $[Y]$ to $[Y']$ in $\Gamma(X_0)$. We first claim that the resolution

$$X[1] \rightarrow X \times_{\mathbb{A}^1} \mathbb{A}^2$$

is the blow-up of all the Weil divisors $Y \times V(t_2)$ and their strict transforms, in increasing order with respect to the partial order of the components Y just introduced. In fact, if $D \subset Y \cap Y'$ is a component corresponding to an arrow in the direction

$$[Y] \xrightarrow{[D]} [Y']$$

then the construction of $X[1]$ in Definition 1.5 and Proposition 1.4 instructs us to blow up along $Y \times V(t_2)$ in a neighbourhood of $D \times \{0, 0\}$. Thus globally blowing up along $Y \times V(t_2)$ has the required effect there. Moreover, having resolved the singularity at $D \times \{(0, 0)\}$, the strict transform of the Weil divisor $Y' \times V(t_2)$ is now Cartier over $D \times_{\mathbb{A}^1} \mathbb{A}^2$, so a later blow-up in the given partial order has no effect over those loci $D \times \{(0, 0)\}$ already resolved. Lastly, if Y and Y' are unrelated in the partial order they are disjoint, so the blow-up order is irrelevant. This proves the claim for $X[1]$.

$X[1]$ viewed as a family over \mathbb{A}^1 via second projection $\mathbb{A}^2 \rightarrow \mathbb{A}^1$ is again simple with dual graph $\Gamma(X[1]|_{t_2=0})$ canonically isomorphic to $\Gamma(X_0)$. More precisely, for Y and D running through the components of X_0 and the components of its singular locus respectively, let $Y^{(1)}$ and $D^{(1)}$ denote the strict transforms of $Y \times V(t_2)$ and of $D \times V(t_2)$ by the resolution $X[1] \rightarrow X \times_{\mathbb{A}^1} \mathbb{A}^2$. Then $Y^{(1)}$ are precisely the components of $X[1]|_{t_2=0}$ and $D^{(1)}$ are precisely the components of its singular locus. This follows by applying the claim in the proof of Proposition 1.4 to Zariski open neighbourhoods U around D . By induction the resolution

$$X[n] = X[n-1][1] \rightarrow X[n-1] \times_{\mathbb{A}^1} \mathbb{A}^2$$

is thus the composition of the blow-ups along the strict transforms of $Y^{(n-1)} \times V(t_2)$ in increasing order with respect to the partial order of the components Y . It is thus projective. \square

Proof that cycle \implies not projective. As in the proof of Proposition 1.9, we reduce to $n = 1$ by pullback along $C[1] \rightarrow C[n]$.

Let $X_0 = \bigcup Y_i$ and $D = \bigcup D_u$ be the decompositions of the special fibre and its singular locus into irreducible components. Fix a point P_u on each D_u . The fibre of $X[1] \rightarrow X \times_{\mathbb{A}^1} \mathbb{A}^2$ over each $(P_u; 0, 0)$ is a \mathbb{P}^1 , which we denote \mathbb{P}_u^1 . If $X[1] \rightarrow C[1]$ is projective, there exists a relatively ample line bundle \mathcal{L} on $X[1]$. Thus \mathcal{L} restricts to an ample line bundle on the fibres of $X[1] \rightarrow C[1]$; in particular it restricts to a line bundle of positive degree on each \mathbb{P}_u^1 .

For $m = 1, 2$, let $H_{im} \subset X[1]$ be the divisor obtained as strict transform of $Y_i \times_{\mathbb{A}^1} V(t_m) \subset X \times_{\mathbb{A}^1} \mathbb{A}^2$. Arguing as in the proof of Proposition 1.9, we arrive at the diagram

$$\begin{CD} \bigoplus \mathbb{Z}H_{im} @>>> \text{Pic}(X[1]) @>>> \text{Pic}(X[1] \setminus \cup H_{im}) @>>> 0 \\ @. @. @. @. \\ @. @. @. @. \\ @. @. @. @. \\ \bigoplus \mathbb{Z}Y_i @>>> \text{Pic}(X) @>>> \text{Pic}(X \setminus X_0) @>>> 0 \end{CD}$$

(where the rightmost vertical map is surjective by the principal \mathbb{G}_m -bundle argument). Hence \mathcal{L} can be written $\mathcal{O}_{X[1]}(\sum_{km} n_{km} H_{km}) \otimes \pi^*(\mathcal{M})$ for some line bundle \mathcal{M} on X . But the restriction of $\pi^*(\mathcal{M})$ to each \mathbb{P}_u^1 is trivial, hence also $\sum_{km} n_{km} H_{km}$ has positive degree on each \mathbb{P}_u^1 . We shall show that this imposes conditions on the coefficients n_{km} that are incompatible with the presence of a cycle in $\Gamma(X_0)$.

For each D_u , there is a corresponding arrow $[Y_j] \rightarrow [Y_i]$ in $\Gamma(X_0)$. Replace X with a Zariski local neighbourhood of D_u such that X_0 just consists of the two components Y_i and Y_j , with irreducible intersection D_u . This has the effect of replacing $X[1]$ with a Zariski open neighbourhood of \mathbb{P}_u^1 . Then $X[1]$ is the blow-up of $X \times_{\mathbb{A}^1} \mathbb{A}^2$ along the Weil divisor $Y_j \times_{\mathbb{A}^1} V(t_2)$. One can check, e.g. by a computation in local coordinates, that the total and strict transforms of $Y_j \times V(t_2)$ agree. Hence $H_{j,2}$, viewed as the inverse image of the blow-up centre, restricts to $\mathcal{O}_{\mathbb{P}_u^1}(-1)$ on \mathbb{P}_u^1 . Locally around \mathbb{P}_u^1 , the divisors $H_{i,1} + H_{j,1}$, $H_{i,2} + H_{j,2}$, $H_{i,1} + H_{i,2}$ and $H_{j,1} + H_{j,2}$ are all principal, given by $t_1 = 0$, $t_2 = 0$, a local equation for Y_i and a local equation for Y_j , respectively. So we have

$$\begin{aligned} \mathcal{O}_{\mathbb{P}_u^1}(H_{i,1}) &= \mathcal{O}_{\mathbb{P}_u^1}(-1) & \mathcal{O}_{\mathbb{P}_u^1}(H_{i,2}) &= \mathcal{O}_{\mathbb{P}_u^1}(1) \\ \mathcal{O}_{\mathbb{P}_u^1}(H_{j,1}) &= \mathcal{O}_{\mathbb{P}_u^1}(1) & \mathcal{O}_{\mathbb{P}_u^1}(H_{j,2}) &= \mathcal{O}_{\mathbb{P}_u^1}(-1) \end{aligned}$$

whereas all other $\mathcal{O}_{\mathbb{P}_u^1}(H_{k,m})$ are trivial, for $m = 1, 2$ and k anything but i and j . Thus, the condition for $\sum_{km} n_{km} H_{km}$ to have positive degree on \mathbb{P}_u^1 is

$$(n_{j,1} - n_{j,2}) + (n_{i,2} - n_{i,1}) > 0.$$

Now label the nodes $[Y_j]$ in a directed loop as $j = 1, \dots, r$. Then

$$\begin{aligned}(n_{1,1} - n_{1,2}) + (n_{2,2} - n_{2,1}) &> 0 \\(n_{2,1} - n_{2,2}) + (n_{3,2} - n_{3,1}) &> 0 \\(n_{3,1} - n_{3,2}) + (n_{4,2} - n_{4,1}) &> 0 \\&\vdots \\(n_{r,1} - n_{r,2}) + (n_{1,2} - n_{1,1}) &> 0\end{aligned}$$

and the sum of the left hand sides is zero; this is the required contradiction. \square

1.2.4 INVERSION OF ORIENTATION

The expanded degeneration $X[n] \rightarrow C[n]$ depends on a choice of orientation of the dual graph $\Gamma(X_0)$. We observe that the effect of reversing the orientation, i.e. reversing the direction of all arrows, is only to permute the coordinates in $C[n]$:

PROPOSITION 1.11. *Let $X[n] \rightarrow C[n]$ and $X[n]' \rightarrow C[n]$ be the two expanded degenerations associated with opposite orientations of the dual graph $\Gamma(X_0)$. Then there is a $G[n]$ -equivariant isomorphism $X[n] \cong X[n]'$ covering the involution ρ of $C[n] = C \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ induced by*

$$\mathbb{A}^{n+1} \rightarrow \mathbb{A}^{n+1}, \quad (t_1, t_2, \dots, t_{n+1}) \mapsto (t_{n+1}, t_n, \dots, t_1).$$

Proof. $X[n]$ and $X[n]'$ are both resolutions of $X \times_C C[n]$, so there is a unique birational map ϕ making the following diagram commute:

$$\begin{array}{ccc} X[n] & \overset{\phi}{\dashrightarrow} & X[n]' \\ \downarrow & & \downarrow \\ X \times_C C[n] & \xrightarrow{1_X \times \rho} & X \times_C C[n] \end{array}$$

We claim that ϕ is in fact biregular. This is an étale local claim over X , so it suffices to verify that ϕ is biregular in the situation of the local equations in Proposition 1.7 (i). This is immediate, since reversal of the orientation amounts to interchanging the roles of x and y in these equations. It is clear that ϕ is equivariant. \square

1.3 THE FIBRES OF $X[n] \rightarrow C[n]$

Fix a strict simple degeneration $X \rightarrow C$ and an orientation of the dual graph Γ . We shall introduce notation describing the fibres of $X[n] \rightarrow C[n]$ and how they are smoothed as coordinates in \mathbb{A}^{n+1} move from zero to nonzero.

1.3.1 EXPANDED GRAPHS

Let $I \subset [n + 1]$ be a subset and let $\mathbb{A}_I^{n+1} \subset \mathbb{A}^{n+1}$ be the locus where all the coordinates t_i vanish for $i \in I$. Let $C[n]_I = C \times_{\mathbb{A}^1} \mathbb{A}_I^{n+1}$ and let $X[n]_I \rightarrow C[n]_I$ be the restriction of $X[n] \rightarrow C[n]$. Let Γ be the dual graph of X_0 equipped with an orientation.

In view of future applications we will use the following notation. If $I \subset [n + 1]$ is a non-empty subset, then we denote its elements by

$$I = \{a_1, \dots, a_r\}, \quad a_1 < a_2 < \dots < a_r.$$

We construct an oriented graph Γ_I (associated to Γ) by replacing each arrow

$$\bullet \rightarrow \bullet$$

in Γ with $|I|$ arrows labelled by I in ascending order in the direction of the arrow:

$$\bullet \xrightarrow{a_1} \circ \xrightarrow{a_2} \circ \rightarrow \dots \xrightarrow{a_r} \bullet \tag{4}$$

It is useful to colour the old nodes black and the new ones white — so the valence of any white node is 2, and the valence of any black node is unchanged from Γ . Label the black nodes $[Y_I]$, where $[Y]$ is the corresponding node in Γ . Label the white nodes Δ_I^{D, a_i} , where a_i is the incoming arrow and $[D]$ is the corresponding arrow in Γ . We frequently suppress D and write $\Delta_I^{a_i} = \Delta_I^{D, a_i}$.

1.3.2 COMPONENTS OF $X[n]_I$

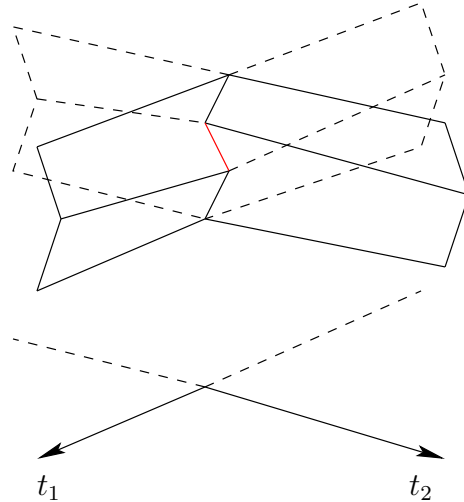
When $I \subset J$ are two non-empty subsets of $[n + 1]$ we may view Γ_I as constructed from Γ_J by deleting all arrows labelled by $J \setminus I$, and identifying the nodes at the ends of each deleted arrow. Thus the set of nodes in Γ_I is a quotient of the set of nodes in Γ_J , and we let

$$q = q_{J,I}: \Gamma_J \rightarrow \Gamma_I$$

denote the quotient map on nodes (it is not defined on arrows, despite the notation).

PROPOSITION 1.12. *Let $X \rightarrow C$ be a strict simple degeneration with oriented dual graph $\Gamma(X_0)$ and let $I \subset [n + 1]$ be non-empty.*

- a) $X[n]_I$ is a union of nonsingular components with normal crossings without triple intersections, i.e. it is étale locally isomorphic to the union of two hyperplanes in affine space. Furthermore, each component is flat over $C[n]_I$ and is a simple degeneration over the pointed curve $(\mathbb{A}^1, 0)$, via any coordinate $t_i: \mathbb{A}_I^{n+1} \rightarrow \mathbb{A}^1$ for $i \notin I$.
- b) There is a natural isomorphism between Γ_I and the dual graph of $X[n]_I$, uniquely determined by the following: each (black) node $[Y_I]$ in Γ_I corresponds to a component $Y_I \subset X[n]_I$ which is mapped birationally onto

Figure 1: $X[1]$ over \mathbb{A}^2

$Y \times_C C[n]_I$ by the natural birational map $X[n] \rightarrow X \times_C C[n]$, whereas the (white) nodes $[\Delta_I^{D,a_i}]$ correspond to components $\Delta_I^{D,a_i} \subset X[n]_I$ which are contracted onto $D \times_C C[n]_I$.

- c) When V is a component of $X[n]_I$ and $I \subset J$ are non-empty, the intersection $V \cap X[n]_J$ is the union of all components W of $X[n]_J$ such that $q([W]) = [V]$.

The Proposition can be seen as a detailed version of [LW11, Lemma 2.2] by Li–Wu and we only sketch a proof.

Proof. The base curve C plays no role so we let $C = \mathbb{A}^1$. Arguing via an appropriate Zariski open cover of X it suffices to treat the basic situation with central fibre $X_0 = Y \cup Y'$ having two irreducible components and irreducible intersection $D = Y \cap Y'$. Label the components such that the arrow $[D]$ in $\Gamma(X_0)$ points from $[Y']$ to $[Y]$.

Firstly the small resolution $\pi: X[1] \rightarrow X \times_{\mathbb{A}^1} \mathbb{A}^2$ is the blow-up along the Weil divisor $Y' \times (\mathbb{A}^1 \times \{0\})$. Let $E \subset X[1]$ denote the inverse image of $D \times \{(0, 0)\}$. It is straight forward to verify the following by computing the blow-up explicitly:

- The restriction of $X[1]$ to $\{0\} \times \mathbb{A}^1 \subset \mathbb{A}^2$ is a normal crossing union $X[1]_{\{1\}} = Y'_{\{1\}} \cup Y_{\{1\}}$, where π restricts to an isomorphism $Y'_{\{1\}} \rightarrow Y' \times (\{0\} \times \mathbb{A}^1)$ and a blow-up $Y_{\{1\}} \rightarrow Y \times (\{0\} \times \mathbb{A}^1)$ along $D \times \{(0, 0)\}$. The exceptional divisor of the blow-up is $E \subset Y_{\{1\}}$. The intersection $Y'_{\{1\}} \cap Y_{\{1\}}$ maps isomorphically to $D \times (\{0\} \times \mathbb{A}^1)$.
- The restriction of $X[1]$ to $\mathbb{A}^1 \times \{0\} \subset \mathbb{A}^2$ is similar with the roles of Y and Y' interchanged.

- The restriction of $X[1]$ to $(0, 0) \in \mathbb{A}^2$ is a normal crossing union $X[1]_{\{1,2\}} = Y'_{\{1,2\}} \cup \Delta^1_{\{1,2\}} \cup Y_{\{1,2\}}$, where $\Delta^1_{\{1,2\}} = E$ and π restricts to isomorphisms $Y'_{\{1,2\}} \rightarrow Y' \times \{(0, 0)\}$ and $Y_{\{1,2\}} \rightarrow Y \times \{(0, 0)\}$. Via these identifications, $Y'_{\{1,2\}} \cap E$ is $D \subset Y'$ and $Y_{\{1,2\}} \cap E$ is $D \subset Y$, whereas $Y'_{\{1,2\}}$ and $Y_{\{1,2\}}$ are disjoint.

The proposition follows for $X[1]$.

Before continuing it is useful to observe that by Proposition 1.11 inverting the orientation of $\Gamma(X_0)$ has the same effect as interchanging the coordinates on \mathbb{A}^2 . Thus $X[1]$ can equally well be obtained by blowing up $X \times_{\mathbb{A}^1} \mathbb{A}^2$ along $Y \times (\{0\} \times \mathbb{A}^1)$.

Inductively assume the proposition holds for $X[n - 1]$. Let $I \subset [n + 1]$ be an index set containing neither n nor $n + 1$ and denote by \bar{I} the same set considered as a subset of $[n]$.

Claim: There is a fibre diagram

$$\begin{array}{ccc} X[n]_I & \xrightarrow{\pi_I} & X[n - 1]_{\bar{I}} \times_{\mathbb{A}^1} \mathbb{A}^2 \\ \uparrow & & \uparrow \\ Y_I & \longrightarrow & Y_{\bar{I}} \times_{\mathbb{A}^1} \mathbb{A}^2 \end{array}$$

where

- π_I is the restriction of π to \mathbb{A}^{n+1}_I
- π_I is an isomorphism outside Y_I
- $Y_I = Y_{\bar{I}}[1]$ viewing $Y_{\bar{I}}$ as a simple degeneration over \mathbb{A}^1 via $t_n : \mathbb{A}^n \rightarrow \mathbb{A}^1$ with dual graph oriented such that its unique arrow points towards the node $[Y_{\bar{I} \cup \{n\}}]$.

To verify the claim, take advantage of the observation preceding it to view π as the blow-up of $X[n - 1] \times_{\mathbb{A}^1} \mathbb{A}^2$ along the Weil divisor $Y_{\{n\}} \times (\{0\} \times \mathbb{A}^1)$. This blow-up centre restricts to

$$(Y_{\{n\}} \times (\{0\} \times \mathbb{A}^1)) \cap (X[n - 1]_{\bar{I}} \times_{\mathbb{A}^1} \mathbb{A}^2) = Y_{\bar{I} \cup \{n\}} \times (\{0\} \times \mathbb{A}^1).$$

by part (c) for $X[n - 1]$. By a local computation one checks that the total and strict transforms of $Y_{\bar{I}} \times_{\mathbb{A}^1} \mathbb{A}^2$ agree, so that Y_I is in fact its blow-up along $Y_{\bar{I} \cup \{n\}} \times (\{0\} \times \mathbb{A}^1)$. Again this is $Y_{\bar{I}}[1]$. This proves the claim.

By the claim one can read off the components of $X[n]_I$ inductively from $X[n - 1]_{\bar{I}}$. Applying the $n = 1$ case to $Y_I = Y_{\bar{I}}[1]$ one also obtains the components of $X[n]_{I \cup \{n\}}$, $X[n]_{I \cup \{n+1\}}$ and $X[n]_{I \cup \{n, n+1\}}$. The verification of the proposition from the claim and the $n = 1$ case is then reduced to a book keeping exercise we refrain from writing out. \square

Figure 1 depicts $X[1]$ over the coordinate axes in \mathbb{A}^2 . With notation as in the proof the uppermost components are $Y'_{\{1\}}$ and $Y'_{\{2\}}$ whereas the lowermost components are $Y_{\{1\}}$ and $Y_{\{2\}}$. The fibre over the origin has the additional component $\Delta_{\{1,2\}}^1$.

1.3.3 IN LOCAL COORDINATES

Consider the étale local situation from Section 1.1.5, where $X[n]$ has the explicit open affine cover $\{W_k\}$. Let $W_{k,I} = W_k \cap X[n]_I$. Let (i, j) be a pair of consecutive elements in $I \cup \{0, n+2\}$. As one immediately verifies, each Δ_I^i is given by the local expressions

$$\begin{aligned} \Delta_I^i \cap W_{i,I} &= V\left(\frac{v_{i-1}}{u_{i-1}}\right), \\ \Delta_I^i \cap W_{k,I} &= W_{k,I} && \text{for } i < k < j, \\ \Delta_I^i \cap W_{j,I} &= V\left(\frac{u_j}{v_j}\right). \end{aligned}$$

These expressions include Y'_I and Y_I for $Y' = V(y)$ and $Y = V(x)$ as the extremal cases $i = 0$ and $i = \max I$.

1.4 LINEARIZATION

In this section we shall assume that $X \rightarrow C$ is a *projective* simple degeneration, and we moreover assume that the dual graph $\Gamma(X_0)$ is equipped with a *bipartite* orientation (see below for a formal definition). The main aim of this section is to exhibit a particular $G[n]$ -linearized line bundle on $X[n]$, which will then be used for our application of GIT to the relative Hilbert scheme of $X[n] \rightarrow C[n]$ in Section 2. The choice of linearization we make is not obvious. The one we have found has the advantage that it gives a well behaved semi-stable locus in the Hilbert scheme. The bipartite condition is indeed a crucial condition as we shall see in Section 2, in particular Example 2.11.

1.4.1 ÉTALE FUNCTORIALITY

As preparation, we observe that the construction $X \mapsto X[n]$ is functorial with respect to étale maps.

PROPOSITION 1.13. *Let $X \rightarrow C$ be a strict simple degeneration and let $f: X' \rightarrow X$ be an étale morphism. Orient the dual graphs such that the induced map $\Gamma(X'_0) \rightarrow \Gamma(X_0)$ is orientation preserving. Then there are induced étale morphisms $f[n]: X'[n] \rightarrow X[n]$ for all n , making the following diagram Cartesian:*

$$\begin{array}{ccc} X'[n] & \xrightarrow{f[n]} & X[n] \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X. \end{array}$$

Proof. Observe that for each component $Y' \subset X'_0$, the image $f(Y')$ is dense in some component $Y \subset X_0$. Similarly, if $D' \subset X'_0$ is a component of the singular locus, the image $f(D')$ is dense in some component D of the singular locus of X_0 . This defines the map $\Gamma(X'_0) \rightarrow \Gamma(X_0)$ on vertices and edges, respectively. Let $D \subset X_0$ be a component of the singular locus and let $U \subset X$ be a Zariski open neighbourhood of D , such that $U_0 = U \cap X_0$ has two components Y_1 and Y_2 with $D = Y_1 \cap Y_2$. Order the two components such that the arrow $[D]$ in $\Gamma(X_0)$ points from $[Y_1]$ to $[Y_2]$. Then $U[1] \rightarrow U \times_{\mathbb{A}^1} \mathbb{A}^2$ is the blow-up along $Y_1 \times V(t_2)$.

Let $U' = f^{-1}(U)$. It is a Zariski open subset of X' , and $U'_0 = U' \cap X'_0$ has the following structure: it has a number (possibly zero) of components $Y'_{1,i}$ mapping to Y_1 , a number of components $Y'_{2,i}$ mapping to Y_2 , the only non-empty intersections are of the form $Y'_{1,i} \cap Y'_{2,i}$, and all components of $Y'_{1,i} \cap Y'_{2,i}$ map to D . We abuse notation and write $[Y'_{j,i}]$ for the vertex in $\Gamma(X'_0)$ corresponding to the closure of $Y'_{j,i}$. Since the map on oriented graphs respects the orientation, all arrows in $\Gamma(X'_0)$ point from $[Y'_{1,i}]$ to $[Y'_{2,i}]$. Thus $U'[1] \subset X'[1]$ is obtained by blowing up all $Y'_{1,i} \times V(t_2) \subset U' \times_{\mathbb{A}^1} \mathbb{A}^2$, and as the $Y'_{1,i}$'s are disjoint, they may be blown up simultaneously. As f is étale, and hence flat, blow-up commutes with base change in the sense that the diagram

$$\begin{array}{ccc} U'[1] & \longrightarrow & U[1] \\ \downarrow & & \downarrow \\ U' \times_C C[1] & \longrightarrow & U \times_C C[1] \end{array}$$

is Cartesian. The topmost arrow defines $f[1]$ over $U'[1]$. Cover X by Zariski open neighbourhoods U of this form to define $f[1]$ everywhere. Being a pullback of the étale map f , the map $f[1]$ is also étale.

In view of the recursive construction, the procedure may be repeated: having defined the étale morphism $f[n-1]: X'[n-1] \rightarrow X[n-1]$, inducing an orientation preserving map on the oriented graphs of the respective fibres over $t_n = 0$, cover $X[n-1]$ by Zariski opens $U \subset X[n-1]$ as before and let $U' = f[n-1]^{-1}(U)$. Let $\tilde{U} \subset X[n]$ be the inverse image of $U \times_{\mathbb{A}^1} \mathbb{A}^2$ by the resolution $X[n] \rightarrow X[n-1] \times_{\mathbb{A}^1} \mathbb{A}^2$ and let $\tilde{U}' \subset X'[n]$ be defined analogously. The Cartesian diagram

$$\begin{array}{ccc} \tilde{U}' & \longrightarrow & \tilde{U} \\ \downarrow & & \downarrow \\ U' \times_{\mathbb{A}^1} \mathbb{A}^2 & \longrightarrow & U \times_{\mathbb{A}^1} \mathbb{A}^2 \end{array}$$

defines $f[n]$ over \tilde{U}' . □

REMARK 1.14. In the notation of Proposition 1.13, once an orientation on $\Gamma(X_0)$ has been chosen, there is a unique orientation on $\Gamma(X'_0)$ making the map of graphs $\Gamma(X'_0) \rightarrow \Gamma(X_0)$ orientation preserving.

REMARK 1.15. As is immediate from Proposition 1.13, if the morphism $X \rightarrow C$ carries an action by a group H which respects the orientation on $\Gamma(X_0)$, then $X[n] \rightarrow C[n]$ inherits an H -action.

1.4.2 BIPARTITE ORIENTATIONS

Let $X \rightarrow C$ be a projective simple degeneration. The following notion will be crucial for our construction.

DEFINITION 1.16. We say that the dual graph $\Gamma(X_0)$ is *bipartite* if its vertex set V can be written as a disjoint union $V = V^+ \cup V^-$, such that there are no edges between any pair of vertices in V^+ or in V^- . The choice of such a decomposition $V = V^+ \cup V^-$ induces an orientation of $\Gamma(X_0)$ with all arrows pointing from V^- to V^+ . We shall call orientations of this form *bipartite*.

Equivalently, an orientation is bipartite when every vertex is either a source or a sink. As is well known, a graph can be given a bipartite orientation if and only if it has no cycles of odd length, and when this holds, and the graph is connected, there are exactly two bipartite orientations, obtained from one another by reversing all arrows. Although the assumption that $\Gamma(X_0)$ is bipartite is a restriction, one can always produce this situation after a quadratic base change:

REMARK 1.17. Up to a quadratic extension in the base, we can always assume that $\Gamma(X_0)$ admits a bipartite orientation. Indeed, let $C' \rightarrow C$ be a base extension obtained by extracting a square root of a local parameter at $0 \in C$, and let $D \subset Y \cap Y'$ be any component of the double locus in X_0 . Then $X \times_C C'$ acquires a transversal A_1 -singularity, i.e. a cone over a conic, along $D \times \{0\}$. Blowing up this A_1 -singularity yields a projective simple degeneration $X' \rightarrow C'$, where the inverse image \tilde{Y}_D of D in X' is a \mathbb{P}^1 -bundle intersecting the strict transforms of Y and Y' in disjoint sections. Now one can orient $\Gamma(X'_0)$ by letting all edges point towards the exceptional components.

In a bipartite orientation there are in particular no directed cycles, so by Proposition 1.10, all $X[n] \rightarrow C[n]$ are projective.

1.4.3 EMBEDDING IN \mathbb{P}^1 -BUNDLES

As a convenient tool for describing line bundles on $X[n]$ we next exhibit an embedding of $X[n]$ into a product of \mathbb{P}^1 -bundles over $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$. This is a globalized version of the local equations in Proposition 1.7.

In Proposition 1.10 the desingularization

$$X[n] \rightarrow X[n-1] \times_{\mathbb{A}^1} \mathbb{A}^2$$

was shown to be given globally as a sequence of blow-ups: with Y running through the components of X_0 we blow up along the strict transforms of $Y \times V(t_{n+1}) \subset X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ using the orientation of $\Gamma(X_0)$ to determine the

blow-up order. Moreover the penultimate blow-up already resolves all singularities, so the very last blow-up, corresponding to sinks in $\Gamma(X_0)$, has a Cartier divisor as centre and thus has now effect. Thus, in the bipartite situation the above desingularization is a single blow-up: its centre is the strict transform of $Y_{(0)} \times V(t_{n+1}) \subset X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ where $Y_{(0)}$ is the disjoint union of all components in X_0 corresponding to source nodes in $\Gamma(X_0)$. By e.g. a computation in local coordinates one verifies that this strict transform coincides with the total transform, so the blow-up centre can be written

$$p_{n-1}^{-1}(Y_{(0)}) \times V(t_{n+1}) \subset X[n-1] \times_{\mathbb{A}^1} \times \mathbb{A}^2$$

where we use coordinates (t_n, t_{n+1}) on the last factor \mathbb{A}^2 and

$$p_{n-1}: X[n-1] \rightarrow X$$

is the composition of the resolution $X[n-1] \rightarrow X \times_{\mathbb{A}^1} \mathbb{A}^n$ with the projection $X \times_{\mathbb{A}^1} \mathbb{A}^n \rightarrow X$.

We shall realize the blow-up $X[n] \rightarrow X[n-1] \times_{\mathbb{A}^1} \mathbb{A}^2$ as the strict transform of $X[n-1] \times_{\mathbb{A}^1} \mathbb{A}^2$ under the blow-up of the product $X[n-1] \times \mathbb{A}^2$ along $p_{n-1}^{-1}(Y_{(0)}) \times V(t_{n+1})$. Let $\mathcal{I} \subset \mathcal{O}_{X[n-1] \times \mathbb{A}^2}$ be the ideal sheaf of the latter and define the rank two vector bundle

$$\mathcal{E} = \text{pr}_1^* \mathcal{O}_{X[n-1]}(-p_{n-1}^* Y_{(0)}) \oplus \text{pr}_2^* \mathcal{O}_{\mathbb{A}^2}(-V(t_{n+1}))$$

on $X[n-1] \times \mathbb{A}^2$. Let $y \in H^0(X, \mathcal{O}_X(Y_{(0)}))$ be a defining equation for $Y_{(0)}$. The surjection

$$\mathcal{E} \xrightarrow{\begin{pmatrix} p_{n-1}^* y \\ t_{n+1} \end{pmatrix}} \mathcal{I}$$

induces a closed embedding $\mathbb{P}(\mathcal{I}) \subset \mathbb{P}(\mathcal{E})$ and the blow-up of $X[n-1] \times \mathbb{A}^2$ further embeds into $\mathbb{P}(\mathcal{I})$. Thus the strict transform $X[n]$ of $j: X[n-1] \times_{\mathbb{A}^1} \mathbb{A}^2 \hookrightarrow X[n-1] \times \mathbb{A}^2$ inherits a closed embedding into $\mathbb{P}(j^* \mathcal{E})$. Moreover, let

$$\pi_n: X[n-1] \times_{\mathbb{A}^1} \mathbb{A}^2 \rightarrow (X \times_{\mathbb{A}^1} \mathbb{A}^n) \times_{\mathbb{A}^1} \mathbb{A}^2 \cong X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$$

be the canonical projection and define the vector bundle

$$\mathcal{F}_n = \text{pr}_1^* \mathcal{O}_X(-Y_{(0)}) \oplus \text{pr}_2^* \mathcal{O}_{\mathbb{A}^{n+1}}(-V(t_{n+1}))$$

on $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$. Then there is a canonical identification $j^* \mathcal{E} \cong \pi_n^* \mathcal{F}_n$ and thus we have arrived at a closed embedding

$$X[n] \subset \pi_n^* \mathbb{P}(\mathcal{F}_n)$$

over $X[n-1] \times_{\mathbb{A}^1} \mathbb{A}^2$. We claim that, by iteration, we obtain an embedding

$$X[n] \subset \prod_{i=1}^n P_i \tag{5}$$

where the product symbol denotes fibred product over $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$, and P_i is the \mathbb{P}^1 -bundle over $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ obtained by pulling back $\mathbb{P}(\mathcal{F}_i)$ over the map $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1} \rightarrow X \times_{\mathbb{A}^1} \times \mathbb{A}^{i+1}$ that multiplies together the last $n+1-i$ coordinates on \mathbb{A}^{n+1} . For $n=1$ there is nothing to prove, and for $n=2$ there is a commutative diagram

$$\begin{array}{ccc} X[2] \subset \pi_2^* P_2 & \longrightarrow & X[1] \times_{\mathbb{A}^1} \mathbb{A}^2 \subset P_1 \\ \downarrow & & \downarrow \pi_2 \swarrow \\ P_2 & \longrightarrow & X \times_{\mathbb{A}^1} \mathbb{A}^3 \end{array}$$

where the square is Cartesian. It follows formally from the diagram that there is an embedding $X[2] \subset P_1 \times P_2$, where the product is over $X \times_{\mathbb{A}^1} \mathbb{A}^3$. The general induction step in proving (5) is similar.

1.4.4 THE LINEARIZATION

Consider first the local situation in Proposition 1.7 and let \mathcal{L}_0 be the ample line bundle $\mathcal{O}_{(\mathbb{P}^1)^n}(1, \dots, 1)$ pulled back to $X[n]$. We shall write down a particular linearization of the tensor power \mathcal{L}_0^{n+1} . First let $G[n]'$ be a “second copy” of the group $G[n]$, acting on $X[n]$ via the $(n+1)$ 'st power map $G[n]' \rightarrow G[n]$, sending $\tilde{\tau} \in G[n]'$ to $\tau = \tilde{\tau}^{n+1}$ in $G[n]$. The induced $G[n]'$ -action on the i 'th factor \mathbb{P}^1 can be lifted to \mathbb{A}^2 in many ways; we pick the particular lifting that acts on $(u_i, v_i) \in \mathbb{A}^2$ by

$$(u_i, v_i) \xrightarrow{\tilde{\tau}} ((\tilde{\tau}_i)^i u_i, \tilde{\tau}_i^{i-(n+1)} v_i), \quad (6)$$

using the coordinates in Remark 1.8. We remark that our preference for this choice is not obvious at this point, but it will lead to a well behaved GIT stable locus in Section 2. The lifted $G[n]'$ -action on $(\mathbb{A}^2)^n$ gives rise to a $G[n]'$ -linearization of \mathcal{L}_0 . The kernel of $G[n]' \rightarrow G[n]$ acts trivially on \mathcal{L}_0^{n+1} , hence we have defined a $G[n]$ -linearization on $\mathcal{L} = \mathcal{L}_0^{n+1}$.

Now we globalize this construction. For the notation in statement (i) in the following lemma, we refer to Proposition 1.13 and Remark 1.14.

LEMMA 1.18. *Let $X \rightarrow C$ be a simple degeneration together with a bipartite orientation of the dual graph $\Gamma(X_0)$. Then there exists a particular $G[n]$ -linearized ample line bundle \mathcal{L} on $X[n]$ such that:*

- (i) *(Compatibility with étale maps:)* Let $f: X' \rightarrow X$ be an étale map and give $\Gamma(X'_0)$ the orientation induced by the one on $\Gamma(X_0)$. Then $\Gamma(X'_0)$ is bipartite, and if \mathcal{L}' denotes the corresponding $G[n]$ -linearized ample line bundle on $X'[n]$, then \mathcal{L}' is isomorphic to the pullback of \mathcal{L} along $f[n]: X'[n] \rightarrow X[n]$.
- (ii) *(Local description:)* In the local situation of Proposition 1.7, the line bundle \mathcal{L} is the $(n+1)$ 'st power of $\mathcal{O}_{(\mathbb{P}^1)^n}(1, \dots, 1)$ pulled back to $X[n]$, with $G[n]$ -linearization given by (6) as above.

Proof. We use the notation from Section 1.4.3. Consider the following diagram:

$$\begin{array}{ccccc}
 X[n] \subset \prod_{i=1}^n P_i & \xrightarrow{\text{pr}_i} & P_i & \longrightarrow & \mathbb{P}(\mathcal{F}_i) \\
 & \searrow & \downarrow & & \downarrow \\
 & & X \times_{\mathbb{A}^1} \mathbb{A}^{n+1} & \longrightarrow & X \times_{\mathbb{A}^1} \mathbb{A}^{i+1}
 \end{array}$$

The rightmost horizontal arrows are $G[n]$ -equivariant when we let $G[n]$ act on the objects to the right via the projection

$$G[n] \rightarrow G[i], \quad (\tau_1, \dots, \tau_i, \dots, \tau_n) \mapsto (\tau_1, \dots, \tau_i), \tag{7}$$

where we use the coordinates in Remark 1.8.

For each $i \leq n$, the divisor $V(t_{i+1}) \subset \mathbb{A}^{i+1}$ is invariant under the $G[i]$ -action, and hence under the $G[n]$ -action via (7). Hence the locally free sheaf

$$\mathcal{F}_i = \text{pr}_1^* \mathcal{O}_X(-Y_{(0)}) \oplus \text{pr}_2^* \mathcal{O}_{\mathbb{A}^{n+1}}(-V(t_{i+1}))$$

on $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ has a canonical $G[n]$ -linearization (trivial on the first summand). The induced $G[n]$ -action on $\prod_i P_i$ is compatible with the action on $X[n]$. Since \mathcal{F}_i itself is $G[n]$ -linearized, the $G[n]$ -action on $\mathbb{P}(\mathcal{F}_i)$ lifts to the geometric vector bundle $\mathbb{V}(\mathcal{F}_i)$, and hence comes with a canonical linearization with underlying line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F}_i)}(1)$. In the local situation of Proposition 1.7, the lifted action can be checked to be given in the fibres by

$$(u_i, v_i) \mapsto (u_i, \tau_i^{-1} v_i).$$

Guided by equation (6) we thus pick the $G[n]'$ -action (where again $G[n]' \rightarrow G[n]$ is the $(n + 1)$ 'st power map) on $\mathbb{V}(\mathcal{F}_i)$ given by the canonical action via $G[n]$ followed by scalar multiplication in the fibres of the vector bundle $\mathbb{V}(\mathcal{F}_i)$ by the factor $\tilde{\tau}_i^i$. This induces the required $G[n]$ -linearization of $\mathcal{O}(n + 1)$ on $\mathbb{P}(\mathcal{F}_i)$ for each i . Pull these back to $\prod_{i=1}^n P_i$ and form their tensor product. Restrict to $X[n]$ to obtain the required linearized line bundle \mathcal{L} . It fulfills (ii) by construction.

For the compatibility with étale maps $f: X' \rightarrow X$ note that the union $Y'_{(0)}$ of components in X'_0 being source nodes in $\Gamma(X'_0)$ is precisely the inverse image of the corresponding union $Y_{(0)}$ in X_0 . Thus the vector bundles \mathcal{F}_i over $X \times_{\mathbb{A}^1} \mathbb{A}^{i+1}$ pull back to the corresponding bundles \mathcal{F}'_i over $X' \times_{\mathbb{A}^1} \mathbb{A}^{i+1}$ and so the product $\prod_i P_i$ of \mathbb{P}^1 -bundles over $X \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ pulls back to the corresponding product $\prod_i P'_i$ over $X' \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$. It is thus enough to check that the embeddings of $X[n]'$ and $X[n]$ in their respective product bundles are compatible. We leave the details to the reader. \square

1.4.5 HILBERT–MUMFORD INVARIANTS

We first recall the definition of the Hilbert–Mumford invariants. Let G denote a linearly reductive group over k , which acts on a quasi-projective k -scheme Y .

Assume moreover that we are given an ample G -linearized invertible sheaf \mathcal{P} on Y . Let $\lambda: \mathbb{G}_m \rightarrow G$ be a one-parameter subgroup (for short 1-PS) of G and $y \in Y$ a point. If the limit y_0 of y as $\tau \in \mathbb{G}_m$ tends to zero exists in Y , then y_0 is a \mathbb{G}_m -fixed point and we define the value $\mu^{\mathcal{P}}(\lambda, y)$ to be the negative of the \mathbb{G}_m -weight on the fibre $\mathcal{P}(y_0)$. Otherwise, we put $\mu^{\mathcal{P}}(\lambda, y) = \infty$.

As preparation for the application of GIT in Section 2, we shall compute the Hilbert–Mumford invariants $\mu^{\mathcal{L}}(\lambda_{\mathbf{s}}, P)$ associated to arbitrary one parameter subgroups

$$\lambda_{\mathbf{s}}: \mathbb{G}_m \rightarrow G[n], \quad \tau \mapsto (\tau^{s_1}, \tau^{s_2}, \dots, \tau^{s_n}), \tag{8}$$

where $\mathbf{s} = (s_1, \dots, s_n)$ is an n -tuple of integers, P is a point in $X[n]$ and we use τ -coordinates on $G[n]$ as in Remark 1.8. Let

$$P_0 = \lim_{\tau \rightarrow 0} \lambda_{\mathbf{s}}(\tau) \cdot P \in X[n]$$

provided the limit exists.

We shall write $t_i(P)$ for the i 'th coordinate of the image of $P \in X[n]$ in \mathbb{A}^{n+1} . We use the notation Y_I and $\Delta_I^{D,i}$ introduced in Section 1.3, and to avoid writing out special cases it is convenient to define

$$\Delta_I^{D,0} = Y'_I, \quad \Delta_I^{D,\max I} = Y_I$$

whenever $[Y'] \xrightarrow{[D]} [Y]$ is an arrow in $\Gamma(X_0)$. For the same reason we let $s_0 = s_{n+1} = 0$.

PROPOSITION 1.19. *Let $P \in X[n]$ and let P_0 be its limit under a 1-PS (8) as above, provided it exists. Define*

$$I = \{i \mid t_i(P) = 0\}$$

so that $P \in X[n]_I$.

- (a) *The limit P_0 exists if and only if $s_{i-1} \leq s_i$ for all $i \notin I$. If this is the case, we have $t_i(P_0) = 0$ if and only if i is in*

$$J = I \cup \{i \mid s_{i-1} < s_i\},$$

so $P_0 \in X[n]_J$.

- (b) *Assume the limit P_0 exists and $X[n]_I$ is smooth at P , so that P is in a unique component $\Delta_I^{D,i}$ of $X[n]_I$. Let $j > i$ be the successor to i in $I \cup \{n+2\}$. By part (a), we have $s_i \leq s_{i+1} \leq \dots \leq s_{j-1}$.*

- (i) *Assume all $s_k \neq 0$ for $i \leq k < j$. Then $i \neq 0$ and $i \neq \max I$. Define a ($i \leq a \leq j$) by the property $s_k < 0$ if and only if $k < a$, for all $i \leq k < j$. Then $P_0 \in \Delta_J^{D,a'} \cap \Delta_J^{D,a}$, where $a' < a$ is the predecessor to a in $J \cup \{0\}$, and $\mu^{\mathcal{L}}(\lambda_{\mathbf{s}}, P)$ is the sum over all $k = 1, 2, \dots, n$ of contributions*

$$\begin{aligned} & -ks_k \quad \text{for } k < a, \\ & (n+1-k)s_k \quad \text{for } k \geq a. \end{aligned}$$

(ii) Assume at least one $s_k = 0$ for $i \leq k < j$. Define a ($i \leq a \leq j$) and b ($i \leq b \leq j$) by the property $s_k = 0$ if and only if $a \leq k < b$, for all $i \leq k < j$. Then $P_0 \in \Delta_J^{D,a}$, $X[n]_J$ is smooth at P_0 , and $\mu^{\mathcal{L}}(\lambda_s, P)$ is the sum over all $k = 1, 2, \dots, n$ of contributions

$$\begin{aligned} & -ks_k \quad \text{for } k < a, \\ (n + 1 - k)s_k \quad & \text{for } k \geq b. \end{aligned}$$

(c) Assume the limit P_0 exists and $X[n]_I$ is singular at P , so that $P \in \Delta_I^{D,i} \cap \Delta_I^{D,j}$ for a consecutive pair $i < j$ in $I \cup \{0, n + 1\}$. Then $P_0 \in \Delta_J^{D,i} \cap \Delta_J^{D,j}$ and $\mu^{\mathcal{L}}(\lambda_s, P)$ is the sum over all $k = 1, 2, \dots, n$ of contributions

$$\begin{aligned} & -ks_k \quad \text{for } k < j, \\ (n + 1 - k)s_k \quad & \text{for } k \geq j. \end{aligned}$$

REMARK 1.20. In case (b), the contribution to $\mu^{\mathcal{L}}(\lambda_s, P)$ for k in the range $i \leq k < j$ may be written

$$\left(\frac{n+1}{2} - k\right) s_k + \frac{n+1}{2} |s_k|$$

regardless of the values of a and b .

Proof. Since $\pi: X[n] \rightarrow C[n]$ is proper, existence of the limit for P is equivalent to the existence of the limit for $Q = \pi(P)$. The $G[n]$ -action on $C[n] = C \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$ is a pullback from \mathbb{A}^{n+1} , on which $\sigma \in G[n]$ acts by

$$(t_1, \dots, t_{n+1}) \mapsto (\sigma_1 t_1, \dots, \sigma_{n+1} t_{n+1}).$$

The 1-PS $\lambda_s: \mathbb{G}_m \rightarrow G[n]$ is given in σ -coordinates by $\sigma_i = \tau^{s_i - s_{i-1}}$. If t_i is nonzero, the limit of $\tau^{s_i - s_{i-1}} t_i$, as τ approaches zero, exists if and only if the exponent $s_i - s_{i-1}$ is nonnegative. More precisely, the limit equals t_i if $s_i = s_{i-1}$ and it is 0 if $s_i > s_{i-1}$. This proves (a).

In view of Lemma 1.18, the \mathbb{G}_m -weight can be computed in the étale local coordinates from Section 1.1.5. Let $i < j$ be consecutive elements in $I \cup \{0, n + 1\}$. In the étale local coordinates, as one easily verifies, the component Δ_I^i of $X[n]_I$ is given by the vanishing of t_k for $k \in I$, together with

$$\begin{aligned} (u_k : v_k) &= (1 : 0) && \text{for } k < i, \\ (u_k : v_k) &= (0 : 1) && \text{for } k \geq j, \end{aligned}$$

and (consequently) $\Delta_I^i \cap \Delta_I^j$ is given by

$$\begin{aligned} (u_k : v_k) &= (1 : 0) && \text{for } k < j, \\ (u_k : v_k) &= (0 : 1) && \text{for } k \geq j. \end{aligned}$$

Clearly $G[n]$ acts trivially on $(u_k : v_k) = (1 : 0)$ and $(u_k : v_k) = (0 : 1)$, so to locate the limit point P_0 it remains only to work out the action on

$(u_k : v_k)$ for the remaining range $i \leq k < j$. The action by the 1-PS is given by $(u_k : v_k) \mapsto (\tau^{s_k} u_k : v_k)$. In case (b), when P is on a unique component Δ_J^i , we have $(u_k : v_k) \neq (1 : 0)$ and $(u_k : v_k) \neq (0 : 1)$ for k in the range $i \leq k < j$. In case (b.i), we have

$$\underbrace{s_i \leq \cdots \leq s_{a-1}}_{<0} \leq \underbrace{s_a \cdots \leq s_{j-1}}_{>0}$$

and thus the limit point P_0 has coordinates

$$\begin{aligned} (u_k : v_k) &= (1 : 0) && \text{for } k < a \\ (u_k : v_k) &= (0 : 1) && \text{for } k \geq a \end{aligned}$$

which shows $P_0 \in \Delta_J^{a'} \cap \Delta_J^a$ as claimed.

In case (b.ii), we have

$$\underbrace{s_i \leq \cdots \leq s_{a-1}}_{<0} \leq \underbrace{s_a \leq \cdots \leq s_{b-1}}_{=0} \leq \underbrace{s_b \cdots \leq s_{j-1}}_{>0}$$

and thus the limit point P_0 has coordinates

$$\begin{aligned} (u_k : v_k) &= (1 : 0) && \text{for } k < a \\ (u_k : v_k) &= (0 : 1) && \text{for } k \geq b \end{aligned}$$

with the remaining $(u_k : v_k)$, for $a \leq k < b$ equal to those of P . Thus $P_0 \in \Delta_J^a$ and it is a smooth point in $X[n]_J$.

In case (c), P has coordinates

$$\begin{aligned} (u_k : v_k) &= (1 : 0) && \text{for } k < j \\ (u_k : v_k) &= (0 : 1) && \text{for } k \geq j \end{aligned}$$

and the \mathbb{G}_m -action does not change these, so P_0 has the same $(u_k : v_k)$ -coordinates: thus $P_0 \in \Delta_J^i \cap \Delta_J^j$.

It remains to write down the weights for the induced \mathbb{G}_m -action on $\mathcal{L}(P_0)$. Consider the 1-PS $\lambda_s^{n+1} : \mathbb{G}_m \rightarrow G[n]$ obtained by composing λ_s with the $(n + 1)$ 'st power map. By definition of the linearized line bundle $\mathcal{L} = \mathcal{L}_0^{n+1}$ in Section 1.4, the λ_s -weight on $\mathcal{L}(P_0)$ agrees with the λ_s^{n+1} -weight on $\mathcal{L}_0(P_0)$. Since \mathcal{L}_0 is the tensor product of the pullbacks of the tautological bundles $\mathcal{O}_{\mathbb{P}^1}(1)$ on each factor in $(\mathbb{P}^1)^n$, the total λ_s^{n+1} -weight on \mathcal{L}_0 is the sum of contributions of λ_s^{n+1} -weights on each factor \mathbb{P}^1 . On the k 'th factor, the λ_s^{n+1} -linearization is defined by the lifted action, from \mathbb{P}^1 to \mathbb{A}^2 , for which $\tilde{\tau} \in \mathbb{G}_m$ acts by

$$(u_k, v_k) \mapsto ((\tilde{\tau})^{ks_k} u_k, (\tilde{\tau})^{(k-(n+1))s_k} v_k).$$

The λ_s^{n+1} -fixed point P_0 necessarily has coordinates $(u_k : v_k)$ of the form $(1 : 0)$ or $(0 : 1)$ for all k with $s_k \neq 0$. The λ_s^{n+1} -weight is thus the sum of ks_k over all k for which $(u_k : v_k) = (1 : 0)$ and $(k - (n + 1))s_k$ over all k for which $(u_k : v_k) = (0 : 1)$. Reversing the signs gives the claimed expressions for $\mu^{\mathcal{L}}(\lambda_s, P)$. \square

2 GIT-ANALYSIS

The $G[n]$ -linearized invertible sheaf \mathcal{L} on $X[n]$ constructed in Lemma 1.18 gives rise to a certain ample linearized invertible sheaf \mathcal{M}_ℓ on $\mathbf{H}^n := \text{Hilb}^n(X[n]/C[n])$ (the integer $\ell \gg 0$ plays only a formal role). In this section we apply a relative version of the Hilbert–Mumford criterion to carry out a detailed analysis of (semi-)stability for points in \mathbf{H}^n , with respect to \mathcal{M}_ℓ . This leads to our main result in this section, Theorem 2.10, which provides a detailed combinatorial description of the (semi-)stable locus.

2.1 RELATIVE GIT

We first give a brief summary of how Mumford’s Geometric Invariant Theory [MFK94] can be carried out in a relative setting. For further details, we refer to [GHH15].

Let $S = \text{Spec } A$ be an affine scheme of finite type over k , and let $f: Y \rightarrow S$ be a projective morphism. Let G be an affine linearly reductive group over k . Assume that G acts on Y and S such that f is equivariant. Let \mathcal{P} be an ample G -linearized invertible sheaf on Y . Then one can define the set of stable points $Y^s(\mathcal{P})$ and the set of semi-stable points $Y^{ss}(\mathcal{P})$ in a similar fashion as in the absolute case. These sets are open and invariant. For the semi-stable locus, there exists a universally good quotient

$$\phi: Y^{ss}(\mathcal{P}) \rightarrow Z.$$

We shall often refer to Z as the *GIT quotient* of Y by G . Moreover, there is an open subscheme $\tilde{Z} \subset Z$ with $Y^s(\mathcal{P}) = \phi^{-1}(\tilde{Z})$, such that the restriction

$$Y^s(\mathcal{P}) \rightarrow \tilde{Z}$$

is a universally geometric quotient. For the applications in this paper, it is of particular importance to note that Z is relatively projective over the quotient $S/G = \text{Spec } A^G$.

The main tool we shall use in order to compute the (semi-)stable locus, is a relative version of the well-known Hilbert–Mumford numerical criterion. This can be formulated as follows [GHH15, Cor. 1.1] (recall our definition of the Hilbert–Mumford invariants in 1.4.5).

PROPOSITION 2.1. *Let $y \in Y$ be a point.*

1. *The point y is stable if and only if $\mu^{\mathcal{P}}(\lambda, y) > 0$ for every nontrivial 1-PS $\lambda: \mathbb{G}_m \rightarrow G$.*
2. *The point y is semi-stable if and only if $\mu^{\mathcal{P}}(\lambda, y) \geq 0$ for every 1-PS $\lambda: \mathbb{G}_m \rightarrow G$.*

2.2 NOTATION AND SETUP

Let $C = \text{Spec } A$ be a smooth, connected affine k -curve and let $X \rightarrow C$ be a *projective* simple degeneration. We assume that the dual graph $\Gamma := \Gamma(X_0)$ allows a bipartite orientation, and we keep fixed one of the two possible such orientations throughout this section.

Let $X[n] \rightarrow C[n]$ be the n -th expanded degeneration of $X \rightarrow C$ (with respect to the given orientation on Γ). By Proposition 1.10, the model $X[n]$ is again projective over $C[n]$. We denote by \mathcal{L} the $G[n]$ -linearized line bundle on $X[n]$ constructed in Lemma 1.18. Finally let λ_s be a one parameter subgroup of $G[n]$ as in (8).

2.2.1 THE DETERMINANT LINE BUNDLE

The relative Hilbert scheme $\mathbf{H}^n := \text{Hilb}^n(X[n]/C[n])$ is again projective over $C[n]$, and it inherits an action by $G[n]$ such that the structural map $\mathbf{H}^n \rightarrow C[n]$ is equivariant. Let $\mathbf{Z}^n \subset \mathbf{H}^n \times_{C[n]} X[n]$ be the universal family and denote by p and q the first and second projections, respectively. Then the line bundle

$$\mathcal{M}_\ell := \det p_* (q^* \mathcal{L}^{\otimes \ell} |_{\mathbf{Z}^n})$$

is relatively ample when $\ell \gg 0$ [HL10, Prop. 2.2.5], and it inherits a $G[n]$ -linearization from \mathcal{L} (cf. e.g. the discussion in [HL10, Page 90]). To simplify notation, we write \mathcal{M} instead of \mathcal{M}_1 .

2.2.2 REDUCTION TO SMOOTH SUBSCHEMES

Let us fix a 1-PS λ_s and a point $[Z] \in \mathbf{H}^n$. Assume that the limit of $\lambda_s(\tau) \cdot Z$ as τ goes to zero exists in \mathbf{H}^n ; we denote this limit by Z_0 . Then \mathbb{G}_m acts on the fibre of \mathcal{M}_ℓ at Z_0 , and we will now investigate this representation in some detail.

We decompose the limit as

$$Z_0 = \bigcup_P Z_{0,P},$$

with $Z_{0,P}$ a finite subscheme of length n_P supported in P . Now $\mathcal{O}_{Z_0} \otimes \mathcal{L}$ is trivial as a line bundle on Z_0 , but its \mathbb{G}_m -action is nontrivial. Writing $\mathcal{L}(P)$ for the fibre of \mathcal{L} at P , we have an isomorphism

$$H^0(\mathcal{O}_{Z_0} \otimes \mathcal{L}) = \bigoplus_P (H^0(\mathcal{O}_{Z_{0,P}}) \otimes \mathcal{L}(P))$$

as \mathbb{G}_m -representations. Taking determinants, we find

$$\begin{aligned} \wedge^n H^0(\mathcal{O}_{Z_0} \otimes \mathcal{L}) &= \bigotimes_P \wedge^{n_P} (H^0(\mathcal{O}_{Z_{0,P}}) \otimes \mathcal{L}(P)) \\ &= \bigotimes_P \wedge^{n_P} (H^0(\mathcal{O}_{Z_{0,P}})) \otimes \mathcal{L}(P)^{n_P} \\ &= \left(\wedge^n H^0(\mathcal{O}_{Z_0}) \right) \otimes \left(\bigotimes_P \mathcal{L}(P)^{n_P} \right). \end{aligned}$$

DEFINITION 2.2. We define the *bounded weight* $\mu_b^{\mathcal{M}}(\mathbf{s}, Z)$, resp. the *combinatorial weight* $\mu_c^{\mathcal{M}}(\mathbf{s}, Z)$, to be the negative of the \mathbb{G}_m -weight on $\wedge^n H^0(\mathcal{O}_{Z_0})$, resp. on $\bigotimes_P \mathcal{L}(P)^{n_P}$.

Having made this definition we can, accordingly, write the negative of the \mathbb{G}_m -weight on $\wedge^n H^0(\mathcal{O}_{Z_0} \otimes \mathcal{L})$ as a sum of $\mu_b^{\mathcal{M}}(\mathbf{s}, Z)$ and $\mu_c^{\mathcal{M}}(\mathbf{s}, Z)$. Clearly, if we replace \mathcal{L} by \mathcal{L}^ℓ in these expressions we find, for any \mathbf{s} , that

$$\mu_c^{\mathcal{M}^\ell}(\mathbf{s}, Z) = \ell \cdot \mu_c^{\mathcal{M}}(\mathbf{s}, Z)$$

and that

$$\mu_b^{\mathcal{M}^\ell}(\mathbf{s}, Z) = \mu_b^{\mathcal{M}}(\mathbf{s}, Z),$$

since the bounded weight only depends on the underlying limit subscheme Z_0 . Now if $\ell \gg 0$, we in fact have that

$$\mathcal{M}^\ell(Z_0) = \wedge^n H^0(\mathcal{O}_{Z_0} \otimes \mathcal{L}^\ell),$$

thus we obtain a sum-decomposition of the Hilbert–Mumford invariant attached to \mathcal{M}^ℓ , \mathbf{s} and Z :

$$\mu^{\mathcal{M}^\ell}(\mathbf{s}, Z) = \mu_b^{\mathcal{M}^\ell}(\mathbf{s}, Z) + \mu_c^{\mathcal{M}^\ell}(\mathbf{s}, Z). \tag{9}$$

Since the right hand side is defined for all $\ell \in \mathbb{N}$, we formally use the expression $\mu^{\mathcal{M}^\ell}(\mathbf{s}, Z)$ to denote the above sum in all cases.

Note that for every Z and \mathbf{s} in the situation above, the value $\mu_c^{\mathcal{M}}(\mathbf{s}, Z)$ only depends on the underlying cycle of Z , and not on its scheme structure. This fact is why we chose the terminology *combinatorial weight*. The terminology *bounded weight*, however, is explained by the following lemma.

LEMMA 2.3. *Let $[Z] \in \mathbf{H}^n$ and let $\mathbf{s} \in \mathbb{Z}^n$ be any element such that the limit of $\lambda_{\mathbf{s}}(\tau) \cdot Z$, as τ goes to zero, exists. Then there are integers $a_i = a_i(Z, \mathbf{s})$ such that*

$$\mu_b^{\mathcal{M}}(\mathbf{s}, Z) = \sum_{i=1}^n a_i s_i,$$

where $|a_i| \leq 2n^2$ for every i .

Proof. Let $q \in C[n]$ be the point such that the limit Z_0 of Z is contained in the fibre $X[n]_q$. Then q is a \mathbb{G}_m -fixpoint. Let $\tilde{D} \subset X_0$ be the singular locus and denote by $\tilde{\Delta} \subset X[n]$ the inverse image of $\tilde{D} \times_C C[n]$ under the $G[n]$ -equivariant map $X[n] \rightarrow X \times_C C[n]$. This map restricts to an isomorphism $X[n] \setminus \tilde{\Delta} \rightarrow (X \setminus \tilde{D}) \times_C C[n]$, and it follows that the \mathbb{G}_m -action on each $Z_{0,P}$ is trivial (so the weight is zero) unless $Z_{0,P}$ is supported on $\tilde{\Delta}$.

Now we consider the case where P is a point in $\tilde{\Delta}$. Because $Z_{0,P}$ is a finite local scheme, with P a \mathbb{G}_m -fixpoint, we can work étale locally and use the coordinates from Section 1.1.5. More precisely, locally at P , we can find an étale chart W_{j+1} with coordinates $t_1, \dots, t_{n+1}, \tilde{v}_j, \tilde{u}_{j+1}$, with relation $t_{j+1} = \tilde{v}_j \tilde{u}_{j+1}$, and, depending on the relative dimension of $X \rightarrow C$, additional coordinates $\{z_\alpha\}_\alpha$ (subject to no relations). Here we write, for simplicity, \tilde{v}_j and \tilde{u}_{j+1} instead of v_j/u_j and u_{j+1}/v_{j+1} , respectively. Since $P \in \tilde{\Delta}$ we can assume $t_{j+1}(P) = 0$, which implies that $\tilde{v}_j \tilde{u}_{j+1} = 0$ at P as well.

If $\tilde{u}_{j+1} \neq 0$ or $\tilde{v}_j \neq 0$ at P , then, by the fact that P is a \mathbb{G}_m -fixpoint, a direct computation using our coordinates shows that \mathbb{G}_m acts trivially in an étale neighbourhood of P in $X[n]_q$, and hence on $Z_{0,P}$.

If $\tilde{u}_{j+1} = \tilde{v}_j = 0$ at P , then the coordinate ring of $Z_{0,P}$ is spanned by n_P monomials $M_{P,r}$ in the variables $\tilde{v}_j, \tilde{u}_{j+1}$ and the z_α -s, with each monomial necessarily of degree at most n_P . As \tilde{u}_{j+1} and \tilde{v}_j are semi-invariant with weights s_{j+1} and $-s_j$, whereas the z_α -s are invariant, it follows that the \mathbb{G}_m -weight for each monomial $M_{P,r}$ is of the form $-c_{r,j}s_j + c_{r,j+1}s_{j+1}$, where $c_{r,j}$, resp. $c_{r,j+1}$, denotes the multiplicity of \tilde{v}_j , resp. \tilde{u}_{j+1} , in $M_{P,r}$. In particular, $c_{r,j}$ and $c_{r,j+1}$ are bounded by n_P .

Now we sum over all the points P in the support of Z_0 . Since the integers n_P sum up to n as P runs over the points in the support of Z_0 , we arrive at the asserted expression for the weight on $\wedge^n H^0(\mathcal{O}_{Z_0})$. \square

The following lemma states that, under certain conditions, the combinatorial weight will dominate the bounded weight, provided that we replace \mathcal{L} by a sufficiently high tensor power.

LEMMA 2.4. *Let $[Z] \in \mathbf{H}^n$ and let $\ell \gg 2n^2$ be an integer.*

1. *Assume, for every $\mathbf{s} \in \mathbb{Z}^n$ such that the limit of $\lambda_{\mathbf{s}}(\tau) \cdot Z$ as τ goes to zero exists, that there exist integers $b_i = b_i(\mathbf{s}, Z)$ such that*

$$\mu_c^{\mathcal{M}}(\mathbf{s}, Z) = \sum_{i=1}^n b_i s_i,$$

where $b_i s_i > 0$ if $s_i \neq 0$. Then $[Z] \in \mathbf{H}^n(\mathcal{M}_\ell)^s$.

2. *Let $\mathbf{s} \in \mathbb{Z}^n$ be a nonzero tuple such that the limit of $\lambda_{\mathbf{s}}(\tau) \cdot Z$ as τ goes to zero exists. Assume there exist integers $b_i = b_i(\mathbf{s}, Z)$ such that*

$$\mu_c^{\mathcal{M}}(\mathbf{s}, Z) = \sum_{i=1}^n b_i s_i,$$

where $b_i s_i < 0$ if $s_i \neq 0$. Then $[Z] \notin \mathbf{H}^n(\mathcal{M}_\ell)^{ss}$.

Proof. In both cases, using the decomposition in Equation (9) and replacing \mathcal{M} by \mathcal{M}_ℓ , we can write

$$\mu^{\mathcal{M}_\ell}(\mathbf{s}, Z) = \sum_{i=1}^n (a_i + \ell \cdot b_i) s_i.$$

Assume that $s_i \neq 0$. Then, by assumption, we have $b_i \neq 0$ and by Lemma 2.3 we know that $|a_i| \leq 2n^2$. Since $\ell > 2n^2$, it follows that $a_i + \ell \cdot b_i \neq 0$ as well, with the same sign as b_i .

In case (1), this means that $\mu^{\mathcal{M}_\ell}(\mathbf{s}, Z) > 0$ for any nontrivial 1-PS, so Z is a stable point by Proposition 2.1. In case (2), this means that the 1-PS corresponding to \mathbf{s} is destabilizing for Z . \square

Lemma 2.3 and Lemma 2.4 will be crucial tools when we analyse (semi-)stability for the $G[n]$ -action on \mathbf{H}^n . Equipped with these results, we will prove that, in order to show that a Hilbert point Z is either stable or unstable (but not strictly semi-stable), we may treat Z (as well as its limit Z_0) just as a 0-cycle and forget its finer scheme structure, provided we replace \mathcal{L} by a sufficiently large tensor power. What is more, we will also see that there are *no* strictly semi-stable points.

2.3 NUMERICAL SUPPORT AND COMBINATORIAL WEIGHT.

2.3.1 INDEX NOTATION

To any point $[Z] \in \mathbf{H}^n$ we can associate the subset

$$I_{[Z]} = \{i \mid t_i(Z) = 0\} \subset [n + 1],$$

where the t_i -s denote coordinates on \mathbb{A}^{n+1} as usual. As we have explained in Section 1.3, this subset determines completely the combinatorial structure of the fibre $X[n]_q$ of $X[n]$ in which Z sits as a subscheme. Indeed, by Proposition 1.12, the dual graph of $X[n]_q$ can be identified with the oriented graph $\Gamma_{I_{[Z]}}$. For our purposes, it is useful to represent subsets also in terms of certain tuples of positive integers. To do this, let us fix an integer $1 \leq r \leq n + 1$. Then any tuple

$$\mathbf{a} = (a_0, a_1, \dots, a_r, a_{r+1}) \in \mathbb{Z}^{r+2} \tag{10}$$

such that

$$1 = a_0 \leq a_1 < \dots < a_i < \dots < a_r \leq a_{r+1} = n + 1$$

determines the subset

$$I_{\mathbf{a}} := \{a_1, \dots, a_r\} \subset [n + 1].$$

The values a_0 and a_{n+1} have been added for computational convenience and play only a formal role.

2.3.2 SMOOTH SUPPORT

Let $[D]: [Y] \rightarrow [Y']$ be an arrow in the oriented graph Γ . As we have explained in Section 1.3, this arrow gets replaced in the expanded graph $\Gamma_{I_{\mathbf{a}}}$ by a chain of r arrows. The internal (“white”) nodes in this chain are denoted $[\Delta_{I_{\mathbf{a}}}^{D, a_i}]$. When the set $I_{\mathbf{a}}$ is understood, we shall denote by Δ^{a_i} the (disjoint) union of the components $\Delta_{I_{\mathbf{a}}}^{D, a_i}$ of $X[n]_{I_{\mathbf{a}}}$, as $[D]$ runs over the arrows in Γ . In order to get coherent notation, we also denote by Δ^{a_0} the union of the components $Y_{I_{\mathbf{a}}}$, where $[Y]$ runs over vertices in V^- , and by Δ^{a_r} the union of the components $Y'_{I_{\mathbf{a}}}$, where $[Y']$ runs over the vertices in V^+ . Consider a point $[Z] \in \mathbf{H}^n$ and assume that

$$I_{[Z]} = \{i \mid t_i(Z) = 0\} = I_{\mathbf{a}}.$$

This means that Z is a subscheme in a general fibre of $X[n]_{I_{\mathbf{a}}} \rightarrow C[n]_{I_{\mathbf{a}}}$. To be precise; by *general* we mean that no other coordinates t_j are zero. As usual, we decompose Z as a disjoint union $\cup_P Z_P$, where Z_P is supported in P and has length n_P .

DEFINITION 2.5. We say that Z has *smooth support* if each $P \in \text{Supp}(Z)$ belongs to a unique component of $X[n]_{I_{\mathbf{a}}}$.

Consequently, when Z has smooth support, there exists for each $P \in \text{Supp}(Z)$ a unique integer $0 \leq i(P) \leq r$ such that $P \in \Delta^{a_{i(P)}}$.

DEFINITION 2.6. If Z has smooth support, we define the *numerical support* of Z to be the tuple

$$\mathbf{v}(Z) = \sum_P n_P \cdot \mathbf{e}_{i(P)} \in \mathbb{Z}^{r+1},$$

where $\mathbf{e}_{i(P)}$ denotes the $i(P)$ -th standard basis vector of \mathbb{Z}^{r+1} .

In down to earth terms, the numerical support keeps track of the distribution of the underlying cycle of Z on the Δ^{a_i} -s, for $0 \leq i \leq r$.

2.3.3 REPACKAGING THE NUMERICAL SUPPORT

In order to work efficiently with the numerical support, we need to introduce some more notation. First, for fixed integers r and n with $1 \leq r \leq n+1$, we define the set

$$\mathcal{B} = \{\mathbf{b} = (b_i) \in \mathbb{Z}^{r+2} \mid 1 = b_0 \leq \dots \leq b_i \leq \dots \leq b_{r+1} = n+1\}.$$

We also define the set

$$\mathcal{V} = \{\mathbf{v} = (v_i) \in (\mathbb{Z}_{\geq 0})^{r+1} \mid \sum_{i=0}^r v_i = n\}.$$

Observe that there is an obvious bijection of sets $\mathcal{B} \rightarrow \mathcal{V}$ defined by

$$\mathbf{b} = (b_0, \dots, b_{r+1}) \mapsto \mathbf{v}_{\mathbf{b}} := (b_1 - b_0, \dots, b_{r+1} - b_r).$$

Hence, if $[Z] \in \mathbf{H}^n$ is such that $I_{[Z]}$ has cardinality r , then $I_{[Z]} = I_{\mathbf{a}}$ for a suitable element $\mathbf{a} \in \mathcal{B}$. If, moreover, Z has smooth support, the numerical support $\mathbf{v}(Z)$ is an element of \mathcal{V} . In this situation, we shall prove that Z is semi-stable if and only if $\mathbf{v}(Z)$ equals $\mathbf{v}_{\mathbf{a}}$.

2.3.4 COMBINATORIAL WEIGHTS

We will next explain how we can use the expressions given in Proposition 1.19, for the \mathbb{G}_m -weights for points $P \in X[n]$, to compute the combinatorial \mathbb{G}_m -weights of a point $[Z] \in \mathbf{H}^n$ with smooth support.

We fix an integer $1 \leq r \leq n + 1$, and a subset $I_{\mathbf{a}} \subset [n + 1]$ of cardinality r . We denote by $\mathbf{e}_i \in \mathbb{Z}^{r+1}$ the i -th standard basis vector. For each $k \in [n]$ and each $\mathbf{s} \in \mathbb{Z}^n$, we define the value $\omega_k(\mathbf{e}_i, \mathbf{s})$ by the following recipe:

$$\omega_k(\mathbf{e}_i, \mathbf{s}) = \begin{cases} -k \cdot s_k, & 1 \leq k < a_i \\ \left(\frac{n+1}{2} - k\right) \cdot s_k + \frac{n+1}{2} |s_k|, & a_i \leq k < a_{i+1} \\ (n + 1 - k) \cdot s_k, & a_{i+1} \leq k \leq n \end{cases} \quad (11)$$

Note that if $P \in X[n]_{I_{\mathbf{a}}}$ is a point which belongs to a unique Δ^{a_i} , then Proposition 1.19 asserts that

$$\mu^{\mathcal{L}}(\lambda_{\mathbf{s}}, P) = \sum_{k=1}^n \omega_k(\mathbf{e}_i, \mathbf{s}),$$

assuming the limit P_0 of P exists.

We next extend the above construction to define a function

$$\omega_k(-, \mathbf{s}): \mathcal{V} \rightarrow \mathbb{Z}$$

for each $k \in [n]$ and each $\mathbf{s} \in \mathbb{Z}^n$, by setting

$$\omega_k(\mathbf{v}, \mathbf{s}) = \sum_{i=0}^r v_i \cdot \omega_k(\mathbf{e}_i, \mathbf{s}).$$

Finally, we put

$$\omega(\mathbf{v}, \mathbf{s}) = \sum_{k=1}^n \omega_k(\mathbf{v}, \mathbf{s}). \quad (12)$$

Hence, if $[Z] \in \mathbf{H}^n$ is a point with smooth support, and if $I_{[Z]} = I_{\mathbf{a}}$, it is immediate from Proposition 1.19 that the equality

$$\mu_c^{\mathcal{M}^\ell}(\lambda_{\mathbf{s}}, [Z]) = \ell \cdot \omega(\mathbf{v}(Z), \mathbf{s})$$

holds for all $\ell \geq 1$. In other words, the combinatorial weight of Z only depends on its numerical support $\mathbf{v}(Z)$.

2.3.5 NUMERICAL COMPUTATIONS

We keep the notation and assumptions from Paragraph 2.3.4. In particular, we have fixed an element $\mathbf{a} = (a_0, \dots, a_r, a_{r+1}) \in \mathcal{B}$, corresponding to a subset $I_{\mathbf{a}}$.

LEMMA 2.7. *Let $\mathbf{b} = (b_0, \dots, b_r, b_{r+1})$ be an arbitrary element of \mathcal{B} and let $\mathbf{s} \in \mathbb{Z}^n$. Then, for each $j \in \{0, \dots, r\}$ and $a_j \leq k < a_{j+1}$, the following hold:*

1. *If $s_k \geq 0$, then*

$$\omega_k(\mathbf{v}_{\mathbf{b}}, \mathbf{s}) = -|s_k| \cdot ((k+1 - b_{j+1})(n+1) - k).$$

2. *If $s_k \leq 0$, then*

$$\omega_k(\mathbf{v}_{\mathbf{b}}, \mathbf{s}) = |s_k| \cdot ((k+1 - b_j)(n+1) - k).$$

Proof. For any element $\mathbf{v} = (v_0, \dots, v_r)$ of \mathcal{V} , a direct computation using Equation (11) shows that $\omega_k(\mathbf{v}, \mathbf{s})$ equals

$$\sum_{i=0}^{j-1} v_i \cdot (n+1-k)s_k + v_j \cdot \left(\left(\frac{n+1}{2} - k \right) s_k + \frac{n+1}{2} |s_k| \right) - \sum_{i=j+1}^r v_i \cdot k s_k.$$

Substituting $v_i = b_{i+1} - b_i$ for each $i \in \{0, \dots, r\}$ easily yields the expressions in case (1) and (2). \square

The following result is a key ingredient in analysing (semi-)stability for points $[Z]$ with smooth support. In particular, it implies that $\omega_k(\mathbf{v}_{\mathbf{a}}, \mathbf{s}) \geq 0$ for all $\mathbf{s} \in \mathbb{Z}^n$, with equality if and only if $s_k = 0$.

LEMMA 2.8. *Let $\mathbf{b} = (b_0, b_1, \dots, b_r, b_{r+1}) \in \mathcal{B}$ and assume, for all j and for all k with $a_j \leq k < a_{j+1}$, that the inequalities*

1. $(k+1 - b_{j+1})(n+1) - k \leq 0$
2. $(k+1 - b_j)(n+1) - k \geq 0$

are satisfied. Then \mathbf{b} is equal to the fixed element \mathbf{a} . Moreover, if this is the case, all inequalities are strict.

Proof. We first consider the case where $\mathbf{b} = \mathbf{a}$. Then the *strict* inequalities

$$(k+1 - a_{j+1})(n+1) - k < 0$$

and

$$(k+1 - a_j)(n+1) - k > 0$$

are immediate from the choice of k .

Now let \mathbf{b} be an element in \mathcal{B} , and assume that (1) and (2) both hold for all j and all $a_j \leq k < a_{j+1}$. We will show that this implies $\mathbf{b} = \mathbf{a}$. If we put $k = a_{j+1} - 1$ in (1), we find that

$$(a_{j+1} - b_{j+1})(n + 1) \leq a_{j+1} - 1$$

which can be rewritten as

$$a_{j+1} \leq b_{j+1} + \frac{b_{j+1} - 1}{n}.$$

But observe that either $b_{j+1} = n + 1$ or the inequality

$$0 \leq \frac{b_{j+1} - 1}{n} < 1$$

holds. In both cases, we get

$$a_{j+1} \leq b_{j+1}.$$

If we instead put $k = a_j$, then (2) yields

$$(a_j + 1 - b_j)(n + 1) \geq a_j$$

which can be rewritten as

$$a_j \geq b_j + \frac{b_j - 1}{n} - 1.$$

But either $b_j = 1$, or

$$-1 < \frac{b_j - 1}{n} - 1 \leq 0$$

holds. In both cases, it is true that

$$a_j \geq b_j.$$

It follows that $\mathbf{b} = \mathbf{a}$. □

We shall also need the following lemma, in order to analyse the combinatorial \mathbb{G}_m -weights of points $[Z] \in \mathbf{H}^n$ which do *not* have smooth support.

LEMMA 2.9. *Let $P \in X[n]_{I_{\mathbf{a}}}$ and assume that $P \in \Delta^{a_j}$ for some $j \in \{0, \dots, r\}$. If P is not a smooth point of $X[n]_{I_{\mathbf{a}}}$, the inequality*

$$\mu^{\mathcal{L}}(\lambda_{\mathbf{s}}, P) \leq \sum_{k=1}^n \omega_k(\mathbf{e}_j, \mathbf{s})$$

holds for every $\mathbf{s} \in \mathbb{Z}^n$.

Proof. By Proposition 1.19, we can write

$$\mu^{\mathcal{L}}(\lambda_{\mathbf{s}}, P) = \sum_{k=1}^n \tilde{\omega}_k(\mathbf{s}, P),$$

where $\tilde{\omega}_k(\mathbf{s}, P) = \omega_k(\mathbf{e}_j, \mathbf{s})$ unless $a_j \leq k < a_{j+1}$. For k in this range, one computes that $\tilde{\omega}_k(\mathbf{s}, P) = (n + 1 - k)s_k$ if $P \in \Delta^{a_{j-1}} \cap \Delta^{a_j}$, and that $\tilde{\omega}_k(\mathbf{s}, P) = -ks_k$ if $P \in \Delta^{a_j} \cap \Delta^{a_{j+1}}$. In both cases, the inequality $\tilde{\omega}_k(\mathbf{s}, P) \leq \omega_k(\mathbf{e}_j, \mathbf{s})$ holds, and the assertion follows. □

2.4 THE SEMI-STABLE LOCUS

We are now ready to present our main result in this section, namely a complete description of the (semi-)stable locus in \mathbf{H}^n with respect to the $G[n]$ -linearized sheaf \mathcal{M}_ℓ , for any integer $\ell \gg 2n^2$.

Let $[Z] \in \mathbf{H}^n$, and assume that the associated subset

$$I_{[Z]} \subset [n+1]$$

has cardinality r . We denote by $\mathbf{a} \in \mathcal{B}$ (where \mathcal{B} depends on the values n and r) the unique element such that $I_{[Z]} = I_{\mathbf{a}}$.

THEOREM 2.10. *Let $\ell \gg 2n^2$. The (semi-)stable locus in \mathbf{H}^n with respect to \mathcal{M}_ℓ can be described as follows:*

1. *If $[Z] \in \mathbf{H}^n$ has smooth support, then $[Z] \in \mathbf{H}^n(\mathcal{M}_\ell)^{ss}$ if and only if*

$$\mathbf{v}(Z) = \mathbf{v}_{\mathbf{a}}.$$

In this case, it also holds that $[Z] \in \mathbf{H}^n(\mathcal{M}_\ell)^s$.

2. *If $[Z] \in \mathbf{H}^n$ does not have smooth support, then $[Z] \notin \mathbf{H}^n(\mathcal{M}_\ell)^{ss}$.*

Proof. We consider first the case where Z has smooth support. If $\mathbf{v}(Z) = \mathbf{v}_{\mathbf{a}}$, Lemma 2.8 states that $\mu_c^{\mathcal{M}}(\lambda_{\mathbf{s}}, Z) > 0$ for every nontrivial 1-PS $\lambda_{\mathbf{s}}$ such that the limit of Z exists. This implies, by Lemma 2.4, that the same statement holds for $\mu^{\mathcal{M}_\ell}(\lambda_{\mathbf{s}}, Z)$. Thus $[Z] \in \mathbf{H}^n(\mathcal{M}_\ell)^s$ by Proposition 2.1.

Assume instead that $\mathbf{v}(Z) = \mathbf{v}_{\mathbf{b}}$ for some element $\mathbf{b} \in \mathcal{B}$ where $\mathbf{b} \neq \mathbf{a}$. In this case we will produce an explicit 1-PS which is destabilizing for the Hilbert point Z .

Assume first that $a_{j+1} > b_{j+1}$, and put $\kappa = a_{j+1} - 1$. Then

$$(\kappa + 1 - b_{j+1})(n+1) - \kappa = (a_{j+1} - b_{j+1})(n+1) - (a_{j+1} - 1) > 0.$$

For $d \gg 0$, we define $\mathbf{s} = \mathbf{s}(d) \in \mathbb{Z}^n$ as follows. We put $s_i = 0$, unless $a_j \leq i < a_{j+1}$. We moreover put $s_{a_{j+1}-1} = d$, and, unless $a_{j+1}-1 = a_j$, we put $s_{a_j} = 0$. Then we define, inductively, $s_k = s_{k-1} + 1$ for $a_j < k < a_{j+1}-1$. Now we find that the expression $\sum_{k=a_j}^{a_{j+1}-2} \omega_k(\mathbf{v}(Z), \mathbf{s})$ is bounded, independently of d . On the other hand,

$$\omega_{a_{j+1}-1}(\mathbf{v}(Z), \mathbf{s}) = -((a_{j+1} - b_{j+1})(n+1) - (a_{j+1} - 1)) \cdot d < 0.$$

Hence, choosing d sufficiently large yields the desired 1-PS.

Assume instead that $b_j > a_j$, and set $\kappa = a_j$. Then

$$(\kappa + 1 - b_j)(n+1) - \kappa = (a_j + 1 - b_j)(n+1) - a_j < 0.$$

For $d \ll 0$, we define $\mathbf{s} = \mathbf{s}(d) \in \mathbb{Z}^n$ as follows. Put $s_{a_j} = d \ll 0$. Unless $a_{j+1}-1 = a_j$, we put $s_{a_{j+1}-m} = -m$ whenever $1 \leq m \leq a_{j+1} - (a_j + 1)$. Set

all remaining $s_i = 0$. A similar argument as in the previous case shows that this yields a destabilizing 1-PS for Z .

It remains to consider the case where Z does not have smooth support. As usual, let $Z = \cup_P Z_P$ be the decomposition of Z into local subschemes of length n_P . We construct two distinct vectors \mathbf{v}' and \mathbf{v}'' in \mathcal{V} as follows.

If P belongs to a unique component Δ^{a_j} , we set $\mathbf{v}'_P = \mathbf{v}''_P = n_P \cdot \mathbf{e}_j$. Let j_{\min} be the smallest index in $\{0, \dots, r - 1\}$ such that there is at least one point in the support of Z belonging to the intersection of $\Delta^{a_{j_{\min}}}$ and $\Delta^{a_{j_{\min}+1}}$. For each such point P , we set $\mathbf{v}'_P = n_P \cdot \mathbf{e}_{j_{\min}}$ and $\mathbf{v}''_P = n_P \cdot \mathbf{e}_{j_{\min}+1}$. Finally, if P is a point in the intersection of two components Δ^{a_j} and $\Delta^{a_{j+1}}$ where $j > j_{\min}$, we set $\mathbf{v}'_P = \mathbf{v}''_P = n_P \cdot \mathbf{e}_j$.

We now define $\mathbf{v}' := \sum_P \mathbf{v}'_P$ and $\mathbf{v}'' := \sum_P \mathbf{v}''_P$, where the sum runs over all points in the support of Z . By Lemma 2.9, both the inequalities $\omega(\mathbf{v}(Z), \mathbf{s}) \leq \omega(\mathbf{v}', \mathbf{s})$ and $\omega(\mathbf{v}(Z), \mathbf{s}) \leq \omega(\mathbf{v}'', \mathbf{s})$ hold. Since $\mathbf{v}' \neq \mathbf{v}''$, at least one of them is different from \mathbf{v}_a . Hence we can construct a 1-PS such that Z has a limit Z_0 in $X[n]$, and which is destabilizing for Z , in the same fashion as above. \square

2.5 NECESSITY OF BIPARTITE ASSUMPTION

We conclude this section by exhibiting an example which shows that the bipartite condition is in fact crucial. When $\Gamma(X_0)$ has no directed cycles, but is not necessarily bipartite, the construction of $X[n]$ in Proposition 1.10 by blowing up (invariant) Weil divisors, immediately leads to (essentially canonical) linearized ample line bundles on $X[n]$. The ample line bundle we have constructed in the bipartite case is indeed of this form, but the $G[n]$ -action on it has been modified. This modified linearization only works in the bipartite situation. The following example shows that our set-up cannot be extended, at least not simply through a clever choice of linearization, beyond the bipartite situation.

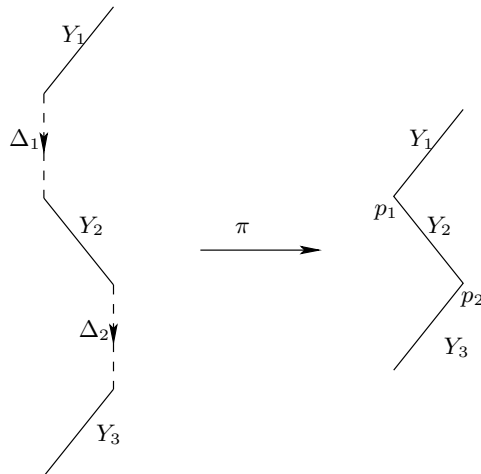
EXAMPLE 2.11. Let $X \rightarrow C$ be a curve degeneration with dual graph $\Gamma(X_0)$ of the form



and choose the (non bipartite) orientation shown. Consider the canonical map $\pi: X[1] \rightarrow X$. We claim: there is no linearization on $X[1]$ such that

- (i) the semi-stable locus $X[1]^{\text{ss}}$ is contained in the smooth locus $X[1]^{\text{sm}}$ over $C[1]$, and
- (ii) the image $\pi(X[1]^{\text{ss}}) \subset X$ contains the singular points of X .

Clearly, the latter condition is necessary if we also want to capture cycles supported on the singular locus of X_0 . To see this, consider Figure 2, showing the degenerate fibre $X[1]_0$ with its “old” components Y_1, Y_2, Y_3 , and the “new” components Δ_1 and Δ_2 , together with the canonical map to X_0 . The group

Figure 2: $X[1]$ for a non bipartite orientation

$G[1] = \mathbb{G}_m$ acts on Δ_i as indicated by the arrow, whereas Y_i are pointwise fixed. For each of the singular points p_i in X_0 , we have

$$\pi^{-1}(p_i) \cap X[1]^{\text{sm}} = \Delta_i^\circ$$

(where Δ_i° denotes the interior of Δ_i in $X[1]_0$). So for condition (ii) to hold, the orbits Δ_i° must be semi-stable. Now the \mathbb{G}_m -weight on any linearized line bundle is constant along the pointwise fixed component Y_2 , and it cannot be zero, since then $Y_2 \cap \Delta_i$ would be semi-stable, violating (i). By the Hilbert–Mumford criterion, Δ_1° is semi-stable only if that weight is nonpositive, and Δ_2° is semi-stable only if that weight is nonnegative. This is a contradiction.

3 THE QUOTIENTS

In this section, we introduce the stack quotient $\mathcal{I}_{X/C}^n$ and the GIT quotient $I_{X/C}^n$ of $\mathbf{H}^n(\mathcal{M}_\ell)^s$ by $G[n]$, where $\ell \gg 0$. We show in Theorem 3.2 that $\mathcal{I}_{X/C}^n$ is proper over C , with coarse moduli space $I_{X/C}^n$ (which is projective over C). We moreover demonstrate in Theorem 3.10 that $\mathcal{I}_{X/C}^n$ is isomorphic, as a DM stack over C , to the stack $\mathcal{I}_{\mathfrak{x}/\mathfrak{e}}^P$ introduced by Li and Wu (cf. e.g. [LW11]), when P is the constant Hilbert polynomial n .

3.1 STACK QUOTIENT AND GIT QUOTIENT

Let $X \rightarrow C$ denote a projective simple degeneration, where $C = \text{Spec } A$ is a smooth affine curve over k . We assume that $\Gamma(X_0)$ allows a bipartite orientation, and we fix one of the two possible such orientations. For any integer

$n > 0$, the expansion

$$X[n] \rightarrow C[n]$$

induces a $G[n]$ -equivariant morphism

$$\mathbf{H}^n = \text{Hilb}^n(X[n]/C[n]) \rightarrow C[n].$$

For any integer $\ell \gg 0$, we defined in 2.2.1 a $G[n]$ -linearized ample line bundle \mathcal{M}_ℓ on \mathbf{H}^n .

Theorem 2.10 provides, when $\ell \gg 2n^2$, an explicit description of the subset

$$\mathbf{H}^n(\mathcal{M}_\ell)^s = \mathbf{H}^n(\mathcal{M}_\ell)^{ss} \subset \mathbf{H}^n$$

of (semi-)stable points. As the (semi-)stable locus is independent of the choice of ℓ , we will in the sequel denote this set simply by $\mathbf{H}_{\text{GIT}}^n$.

DEFINITION 3.1. We define the following two quotients:

1. The *GIT quotient*

$$I_{X/C}^n = \mathbf{H}_{\text{GIT}}^n/G[n].$$

2. The *stack quotient*

$$\mathcal{I}_{X/C}^n = [\mathbf{H}_{\text{GIT}}^n/G[n]].$$

THEOREM 3.2. *The GIT quotient $I_{X/C}^n$ is projective over C . The stack $\mathcal{I}_{X/C}^n$ is a Deligne-Mumford stack, proper and of finite type over C , having $I_{X/C}^n$ as coarse moduli space.*

Proof. Since \mathcal{M}_ℓ is ample (by our assumption $\ell \gg 2n^2$), [GHH15, Prop. 2.6] asserts that $I_{X/C}^n$ is relatively projective over the quotient

$$C[n]/G[n] = \text{Spec}(A[n]^{G[n]}),$$

where

$$A[n] = A \otimes_{k[t]} k[t_1, \dots, t_{n+1}].$$

It is straightforward to check that $A[n]^{G[n]} = A$.

All stabilizers for the action of $G[n]$ on $\mathbf{H}_{\text{GIT}}^n$ are finite and reduced, hence, by [Vis89, (7.17)], $\mathcal{I}_{X/C}^n$ is a Deligne-Mumford stack. It is of finite type over C , as this holds for $\mathbf{H}_{\text{GIT}}^n$.

By [GHH15, Thm. 2.5], the quotient

$$\mathbf{H}_{\text{GIT}}^n \rightarrow I_{X/C}^n$$

is universally a geometric quotient. Therefore, [Vis89, (2.11)] asserts that $I_{X/C}^n$ is a coarse moduli space for $\mathcal{I}_{X/C}^n$. In particular, this means that there is a proper morphism

$$\mathcal{I}_{X/C}^n \rightarrow I_{X/C}^n.$$

Since $I_{X/C}^n \rightarrow C$ is projective, this implies that $\mathcal{I}_{X/C}^n$ is proper over C . □

We remark that it follows from Proposition 1.11 that these quotients do not depend on the choice of bipartite orientation of $\Gamma(X_0)$. It is moreover clear from the construction that both quotients $I_{X/C}^n$ and $\mathcal{I}_{X/C}^n$ are isomorphic, over $C^* = C \setminus \{0\}$, to the family $\text{Hilb}^n(X^*/C^*) \rightarrow C^*$.

REMARK 3.3. If a group H acts equivariantly on $X \rightarrow C$, and respecting the orientation on $\Gamma(X_0)$, one can show that there is an induced action on $I_{X/C}^n \rightarrow C$. This holds in particular in the situation described in Remark 1.15, meaning that the Galois group $\mathbb{Z}/2$ of the base extension C'/C acts naturally on $I_{X'/C'}^n \rightarrow C'$.

3.2 COMPARISON WITH LI–WU

We would now like to explain the relation between our construction and the results of Li and Wu. An important ingredient in their work is the so-called *stack of expanded degenerations* $\mathfrak{X}/\mathfrak{C}$. We will only explain the properties of this stack that are needed for our results in this section, for further details, we refer to [Li13, Ch. 2].

3.2.1 STANDARD EMBEDDINGS

First we recall some useful notation and facts, following [Li13, Ch. 2]. For any subset $I \subset [n+1]$, we let I° denote its complement in $[n+1]$. If $|I| = m+1$,

$$\iota_I: [m+1] \rightarrow I \subset [n+1]$$

denotes the unique order-preserving map.

We set

$$\mathbb{A}_{U(I)}^{n+1} = \{(t) \in \mathbb{A}^{n+1} \mid t_i \neq 0, i \in I^\circ\}.$$

Then there is a canonical isomorphism

$$\tilde{\tau}_I: \mathbb{A}^{m+1} \times G[n-m] \rightarrow \mathbb{A}_{U(I)}^{n+1},$$

defined by $(t'_1, \dots, t'_{m+1}; \sigma_1, \dots, \sigma_{n-m}) \mapsto (t_1, \dots, t_{n+1})$, where $t_k = t'_l$ if $k = \iota_I(l)$ and $t_k = \sigma_l$ if $k = \iota_I^\circ(l)$. Restricting $\tilde{\tau}_I$ to the identity element of $G[n-m]$ gives what Li calls the *standard embedding*

$$\tau_I: \mathbb{A}^{m+1} \rightarrow \mathbb{A}^{n+1}.$$

For each n , let $p_n: X[n] \rightarrow X$ be the canonical $G[n]$ -equivariant morphism. If $|I| = m+1$, then τ_I induces an isomorphism

$$(\tau_I^* X[n], \tau_I^* p_n) \cong (X[m], p_m).$$

over $C[m]$ [Li13, 2.14 + 2.15]. (We already encountered a special case of this in the proof of Proposition 1.9.)

3.2.2 THE STACK OF EXPANDED DEGENERATIONS

Returning to the stack of expanded degenerations, one can give the following useful description of the objects of this stack.

Let T be a C -scheme. An object (W, p) of $\mathfrak{X}(T)$, also called an *expanded degeneration* of X/C , is a family sitting in a commutative diagram [Li13, Def. 2.21, Prop. 2.22]

$$\begin{array}{ccc} W & \xrightarrow{p} & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & C \end{array}$$

where W/T is allowed to have *expansions* of X_0 [Li13, 2.2] as fibres, in addition to the original fibres of X .

More precisely, an *effective family* in $\mathfrak{X}(T)$ is simply the pullback $\xi^*X[m]$ through a C -morphism $\xi: T \rightarrow C[m]$, for some m , with projection induced by $X[m] \rightarrow X$. Two effective families are *effectively equivalent* if there are standard embeddings $\tau_i: C[m_i] \rightarrow C[m]$, $i \in \{1, 2\}$, and a T -valued point $\sigma: T \rightarrow G[m]$, such that

$$\tau_1 \circ \xi_1 = (\tau_2 \circ \xi_2)^\sigma.$$

In general, an expanded degeneration in $\mathfrak{X}(T)$ is a family $W \rightarrow T$ where T allows an étale cover $\cup T_i \rightarrow T$ such that $W \times_T T_i$ is effective, and such that the canonical isomorphism over $T_i \times_T T_j$ is induced by an effective equivalence. Finally, an arrow of two expanded degenerations (W, p) and (W', p') over T is a T -isomorphism $W \rightarrow W'$ which is locally an effective equivalence.

REMARK 3.4. Two objects ξ_1 and ξ_2 in $\mathfrak{X}(k)$ are equivalent if they can be embedded as fibres in the same expanded degeneration $X[n]$, for sufficiently large n , such that the fibre ξ_1 can be ‘translated’ to the fibre ξ_2 under the $G[n]$ -action. In particular, under this equivalence, any object ξ of $\mathfrak{X}(k)$ can be represented by a fibre $X[m]_0$, where $0 \in C[m]$ denotes the origin, for a suitable m .

3.2.3 THE LI–WU STACK

Li and Wu have defined a stack $\mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^P$ parametrizing *stable* ideal sheaves with fixed Hilbert polynomial P , which we will explain next. To do this, let J_Z be an ideal sheaf on $X[m]_0$, for some $m \geq 0$. Li and Wu call J_Z *admissible* [Li13, Def. 3.52] if, for every component D of the double locus, the natural homomorphism

$$J_Z \otimes \mathcal{O}_D \rightarrow \mathcal{O}_D$$

is injective. Then J_Z is *stable* if it is admissible and if $\text{Aut}_{\mathfrak{X}}(J_Z)$, the subgroup of elements $\sigma \in G[m]$ such that $\sigma^*J_Z = J_Z$, is finite. In this paper, we shall often call such ideal sheaves *Li–Wu stable*, in order to separate this notion of stability from GIT stability.

Now, for a C -scheme T , $\mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^P(T)$ consists of all triples (J_Z, W, p) , where $(W, p) \in \mathfrak{X}(T)$, and J_Z is a T -flat family of stable ideal sheaves on W with Hilbert polynomial P . Moreover, every morphism $T' \rightarrow T$ induces a pullback map $\mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^P(T) \rightarrow \mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^P(T')$.

We shall refer to $\mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^P$ as the *Li–Wu stack*. The following fundamental result has been proved by Li and Wu (cf. [LW11, Thm. 4.14] and [Li13, Thm. 3.54]).

THEOREM 3.5. *$\mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^P$ is a Deligne–Mumford stack, separated, proper and of finite type over C .*

We remark that [Li13, Thm. 3.54] is formulated under the assumption that $X_0 = Y \cup Y'$ with Y , Y' and $Y \cap Y'$ smooth and irreducible, whereas [LW11, Thm. 4.14] is formulated for a general simple degeneration.

3.2.4 LI–WU STABILITY

For the remainder of this section, we shall only consider the case where P is constant, in which case Li–Wu stability can be formulated in a simple way. In the statement, we shall use the following notation. For any $m \in \mathbb{N}$, and with $I = [m + 1]$, we denote by Δ^i the (disjoint) union of the components $\Delta_I^{D,i}$ of $X[m]_0$, where D runs over the edges in the oriented graph $\Gamma(X_0)$.

LEMMA 3.6. *Let $Z \subset X[m]_0$ be a subscheme of finite length. Then Z is Li–Wu stable if and only if the following properties hold:*

1. Z is supported on the smooth locus of $X[m]_0$.
2. Z has non-empty intersection with Δ^i , for all $i \in [m]$.

Proof. A straightforward computation shows that J_Z is admissible if and only if Z is supported on the smooth locus of $X[m]_0$. For (2), note that the i -th factor of $G[m]$ acts on Δ^i by multiplication in the fibres of the ruling. This means that the automorphism group is finite if and only if Z intersects every Δ^i nontrivially. \square

Note the similarity with the description of GIT stable subschemes given in Theorem 2.10. We shall next compare the locus \mathbf{H}_{LW}^n of Li–Wu stable points in $\text{Hilb}^n(X[n]/C[n])$ with the GIT stable locus $\mathbf{H}_{\text{GIT}}^n$. By [LW11, Lem. 4.3], \mathbf{H}_{LW}^n is an open subset, and it is clearly invariant. The same properties hold for $\mathbf{H}_{\text{GIT}}^n$.

LEMMA 3.7. *There is a $G[n]$ -equivariant open immersion*

$$\mathbf{H}_{\text{GIT}}^n \subset \mathbf{H}_{\text{LW}}^n$$

as subschemes in $\text{Hilb}^n(X[n]/C[n])$.

Proof. As $\mathbf{H}_{\text{GIT}}^n$ and \mathbf{H}_{LW}^n are both open and invariant, we only need to show that any GIT stable subscheme in a closed fibre of $X[n] \rightarrow C[n]$ is Li–Wu stable. This is clear from Lemma 3.6 and Theorem 2.10. \square

This inclusion is strict in general; by Theorem 2.10 (1), a Li–Wu stable subscheme Z will fail to be GIT stable if the numerical support $\mathbf{v}(Z)$ does not equal \mathbf{v}_a , where $I_{[Z]} = I_a$.

3.2.5 THE CANONICAL COMPARISON MORPHISM

There is an obvious morphism from our quotient $\mathcal{I}_{X/C}^n$ to the Li–Wu stack $\mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^n$. Indeed, the restriction to \mathbf{H}_{LW}^n of the universal family of the Hilbert scheme corresponds to a $G[n]$ -equivariant, surjective and smooth morphism

$$\psi: \mathbf{H}_{\text{LW}}^n \rightarrow \mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^n.$$

Restriction to the open subscheme H_{GIT}^n gives

$$\phi: \mathbf{H}_{\text{GIT}}^n \rightarrow \mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^n,$$

which is again equivariant and smooth. Hence ϕ factors through the quotient $\mathcal{I}_{X/C}^n$, giving a smooth morphism

$$f: \mathcal{I}_{X/C}^n \rightarrow \mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^n. \tag{13}$$

3.2.6 A CRITERION FOR ISOMORPHISM

If \mathfrak{Z} is an algebraic stack over k , we denote by $|\mathfrak{Z}(k)|$ the set of equivalence classes of objects in $\mathfrak{Z}(k)$.

LEMMA 3.8. *The following properties hold for f :*

1. $|f|: |\mathcal{I}_{X/C}^n(k)| \rightarrow |\mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^n(k)|$ is a bijection.
2. For every object ξ in $\mathcal{I}_{X/C}^n(k)$, f induces an isomorphism

$$\text{Aut}(\xi) \rightarrow \text{Aut}(f(\xi))$$

of automorphism groups.

Proof. By Remark 3.4, any point ξ' in $|\mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^n(k)|$ can be represented by a Li–Wu stable subscheme $Z \subset X[m]_0$ of length n , for some $m \leq n$. For any subset $I \subset [n+1]$ with $|I| = m+1$, we can, using the standard embedding τ_I , view $X[m]_0$ as the fibre $(\tau_I^* X[n])_0$ of $X[n]$, where $0 \in C[m]$. In the notation of 2.3.2, we then have

$$I = I_{[Z]} = \{i \mid t_i(Z) = 0\}.$$

On the other hand, as an element of $\mathcal{V} \subset \mathbb{Z}^{m+2}$, the numerical support $\mathbf{v}(Z)$ of Z is independent of the choice of I . Hence, by Theorem 2.10, there is a unique I for which Z is also GIT-stable, namely the subset I_a determined by the preimage \mathbf{a} of $\mathbf{v}(Z)$ in the bijection $\mathcal{B} \rightarrow \mathcal{V}$. Thus, the $G[n]$ -orbit of Z in $\mathbf{H}_{\text{GIT}}^n$ is the unique point $\xi \in |\mathcal{I}_{X/C}^n(k)|$ such that $f(\xi) = \xi'$, which proves (1). Clearly, the automorphism groups of ξ and its image $f(\xi)$ coincide as subgroups of $G[m]$ in the above construction, which shows (2). \square

In the proof of Theorem 3.10 below, we shall use the following standard technical result on stacks, whose proof we omit:

LEMMA 3.9. *Let \mathfrak{X} and \mathfrak{Y} be Deligne-Mumford stacks of finite type over an algebraically closed field k , and let*

$$f: \mathfrak{X} \rightarrow \mathfrak{Y}$$

be a representable étale morphism of finite type. Assume

1. $|f|: |\mathfrak{X}(k)| \rightarrow |\mathfrak{Y}(k)|$ is bijective.
2. For every $x \in \mathfrak{X}(k)$, f induces an isomorphism

$$\mathrm{Aut}_{\mathfrak{X}}(x) \rightarrow \mathrm{Aut}_{\mathfrak{Y}}(f(x)).$$

Then f is an isomorphism of stacks.

3.2.7 THE STACKS ARE ISOMORPHIC

To conclude, we prove that (13) above is an isomorphism.

THEOREM 3.10. *The morphism $f: \mathcal{I}_{X/C}^n \rightarrow \mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^n$ is an isomorphism of Deligne-Mumford stacks.*

Proof. First we observe that f is representable. Indeed, this follows from [AK13, Lem. 6], because $\mathcal{I}_{X/C}^n$ has finite inertia (being a separated DM-stack), and because f yields an isomorphism of automorphism groups for all geometric points. The second property is due to the fact that the formation of the standard models $X[n] \rightarrow C[n]$ commutes with base change to any algebraically closed overfield of k , together with a similar argument as in Lemma 3.8.

Moreover, f is of finite type and étale. Since we have already established that f is smooth, it suffices to prove that it is unramified. This can be checked on geometric points, and is a direct computation.

Since f is representable, it suffices to prove, for any étale atlas Y of $\mathcal{I}_{\mathfrak{X}/\mathfrak{C}}^n$, that the pullback f_Y of f is an isomorphism of schemes. We claim that f_Y is in fact a surjective open immersion. Indeed, this follows from Lemma 3.8 together with Lemma 3.9. \square

4 EXAMPLE

In this section we want to discuss one example in detail in order to demonstrate how our machinery works. We start with a simple degeneration $X \rightarrow C$ where the central fibre $X_0 = Y_1 \cup Y_2$ has two components intersecting along a smooth irreducible subvariety $D = Y_1 \cap Y_2$. We want to explain the geometry of the degenerate Hilbert scheme for n points. For most of this discussion the

dimension of the fibres will be irrelevant, so we will allow it to be arbitrary for the time being. In this case the dual graph $\Gamma = \Gamma(X_0)$ is simply

$$\bullet \xrightarrow{\gamma} \bullet \tag{14}$$

which is trivially a bipartite graph.

Recall the expanded degenerations $X[n] \rightarrow C[n]$. If $t: C \rightarrow \mathbb{A}^1$ is a local étale coordinate, then we obtain a map $(t_1, \dots, t_{n+1}): C[n] \rightarrow \mathbb{A}^{n+1}$. Let $I = \{a_1, \dots, a_r\} \subset [n + 1]$ and denote by $X[n]_I$ the locus of $X[n]$ which is the pre-image of the subscheme $C[n]_I$ where $t_{a_i} = 0, a_i \in I$. In Proposition 1.12 we analysed the components of $X[n]_I$ and found that they correspond to the vertices of a graph Γ_I which is derived from Γ by replacing each edge γ by new edges labelled $\gamma_{a_1}, \dots, \gamma_{a_r}$, arranged in increasing order, and inserting white vertices at the ends of $\gamma_{a_1}, \dots, \gamma_{a_{r-1}}$. Since in our case Γ only has one edge γ we can omit this from our notation and simply relabel the edges γ_{a_ℓ} by a_ℓ . The graph Γ_I thus becomes

$$\bullet \xrightarrow{a_1} \circ \xrightarrow{a_2} \circ \dots \circ \xrightarrow{a_r} \bullet \tag{15}$$

The extremal case is given by $I = I_{\max} = [n + 1]$, in which case we arrive at the graph $\Gamma_{I_{\max}}$ given by

$$\bullet \xrightarrow{1} \circ \xrightarrow{2} \circ \dots \circ \xrightarrow{n+1} \bullet \tag{16}$$

All other graphs Γ_I with $I \subset I_{\max}$ arise from $\Gamma_{I_{\max}}$ by deleting the arrows in $I_{\max} \setminus I$. By Proposition 1.12 we have a decomposition into irreducible components

$$X[n]_I = \Delta_I^0 \cup \dots \cup \Delta_I^{a_\ell} \cup \dots \cup \Delta_I^{a_r}$$

where Δ_I^0 and $\Delta_I^{a_r}$ belong to the black vertices of the graph (15) while the components $\Delta_I^{a_\ell}, \ell = 1, \dots, r - 1$ correspond to the white vertices. Note that since there is only one component D , we have dropped D from the notation and have thus set $\Delta_I^{D, a_\ell} = \Delta_I^{a_\ell}$. Under the natural projection $X[n] \rightarrow X \times_C C[n]$ the components Δ_I^0 and $\Delta_I^{a_r}$ are mapped birationally onto $Y_1 \times_C C[n]_I$ and $Y_2 \times_C C[n]_I$ respectively. The components $\Delta_I^{a_\ell}, \ell = 1, \dots, r - 1$ are contracted to $D \times_C C[n]_I$. The latter are the inserted components which have the structure of a \mathbb{P}^1 -bundle, whose fibres are contracted under the map to $D \times_C C[n]_I$.

There is another way of labelling the components of $X[n]_I$ which is sometimes helpful in geometric considerations. If $I = \{a_1, \dots, a_r\}$, then we decompose $I_{\max,0} = \{0\} \cup [n + 1]$ into $I_{\max,0} = I_0 \cup I_1 \cup \dots \cup I_r$ where $I_0 = [0, a_1 - 1]$, $I_\ell = [a_\ell, a_{\ell+1} - 1]$ for $1 \leq \ell \leq r - 1$ and $I_r = [a_r, n + 1]$. The components $\Delta_I^{a_\ell}$ then correspond to the first entry in each interval I_ℓ . We can understand the above graph (15) as a contraction of the maximal graph $\Gamma_{I_{\max}}$ given in (16) by identifying all the edges labelled in one of the sets I_ℓ in the partition $I_{\max,0} = I_0 \cup I_1 \cup \dots \cup I_r$. So we can symbolically think of the left hand bold vertex of (15) as

$$\bullet \xrightarrow{1} \circ \dots \circ \xrightarrow{a_1-1} \circ \xrightarrow{a_1} \circ \dots$$

the middle white vertices as

$$\dots \xrightarrow{a_\ell} \circ \underline{a_\ell+1} \circ \dots \circ \underline{a_{\ell+1}-1} \circ \xrightarrow{a_{\ell+1}} \dots$$

and finally the right hand bold vertex as

$$\dots \xrightarrow{a_r} \circ \underline{a_r+1} \circ \dots \circ \underline{n+1} \bullet.$$

This picture also helps us understand the smoothing or, in other words, the inclusion of the closure of the strata when we move from $t_{a_\ell} = 0$ to $t_{a_\ell} \neq 0$. This corresponds to removing a_ℓ from the set I or, equivalently, to replacing $I_{\ell-1}$ and I_ℓ by their union $I_{\ell-1} \cup I_\ell$.

Now consider a subscheme Z of length n representing a point in the relative Hilbert scheme $\mathbf{H}^n = \text{Hilb}^n(X[n]/C[n])$. Since \mathbf{H}^n is the relative Hilbert scheme, every subscheme Z lies in some fibre \mathbf{H}_q^n for a point $q \in C[n]$. Let I be the set of indices labelling the coordinates t_{a_i} which vanish at q . In Section 2 we developed a numerical criterion for stability. First of all recall that stability and semi-stability coincide. Moreover, all stable cycles have support in the smooth part $X[n]^\circ_I$ of $X[n]_I$, by which we mean that Z does not intersect the locus where different components of $X[n]_I$ meet. We shall denote the restriction of the smooth locus $X[n]^\circ_I$ to the components $\Delta_I^{a_\ell}$ by $\Delta_I^{a_\ell, \circ}$. We now claim that the numerical criterion of Theorem 2.10 is equivalent to

$$Z \subset X[n]_I \text{ is stable} \Leftrightarrow \text{length}(Z \cap \Delta_I^{a_\ell, \circ}) = |I_\ell \cap [n]| \quad \forall \ell. \tag{17}$$

Indeed, in the notation of Section 2 we have $\mathbf{a} = (1, a_1, \dots, a_r, n + 1)$ and thus $\mathbf{v}_\mathbf{a} = (a_1 - 1, a_2 - a_1, \dots, a_r - a_{r-1}, n + 1 - a_r)$. The stability condition of Theorem 2.10 for a cycle Z is $\mathbf{v}(Z) = \mathbf{v}_\mathbf{a}$ where $\mathbf{v}(Z)$ is the numerical support of Z , i.e. the length of the cycle restricted to the smooth part $\Delta_I^{a_\ell, \circ}$ of the components $\Delta_I^{a_\ell}$ of $X[n]_I$. The claim now follows since the entries of $\mathbf{v}_\mathbf{a}$ are exactly equal to the cardinality of the sets $I_\ell \cap [n]$.

Our aim is to understand the geometry of the GIT quotient $I_{X/C}^n = \mathbf{H}_{\text{GIT}}^n/G[n]$, in particular the geometry of the special fibre $(I_{X/C}^n)_0$. Since the Hilbert schemes of varieties of dimension greater than 2 are, in general, neither irreducible nor equi-dimensional, we will for the following discussion restrict the fibre dimension to $d \leq 2$. We first observe that the fibre $(I_{X/C}^n)_0$ is naturally stratified. As we have seen, any length r subset $I = \{a_1, \dots, a_r\} \subset [n + 1]$ defines a subscheme $X[n]_I$ of $X[n]$ and the stable n cycles supported on $X[n]_I$ give rise to a stratum $(I_{X/C}^n)_I$ of $(I_{X/C}^n)_0$, and it is the geometry of these strata and the inclusion relations of their closures which we want to describe here.

We start with the case where $I = \{a_1\}$ consists of one element. In this case I defines a partition of $I_{\text{max},0} = I_0 \cup I_1$ into two intervals, namely $I_0 = [0, a_1 - 1]$ and $I_1 = [a_1, n + 1]$. The graph Γ_I then becomes

$$\bullet \underline{1} \circ \dots \circ \underline{a_1-1} \circ \xrightarrow{a_1} \circ \underline{a_1+1} \circ \dots \circ \underline{n+1} \bullet$$

and we have no inserted components. The general fibre of $X[n]_I$ has two components, which are isomorphic to Y_1 and Y_2 respectively. Stability condition (17) then tells us that we must have $a_1 - 1$ points on Y_1 and $n + 1 - a_1$ points on Y_2 . In this case the group $G[n]$ acts freely on the base $C[n]_I$ of the fibration $X[n]_I \rightarrow C[n]_I$. Varying a_1 from 1 to $n + 1$ we thus obtain the strata $\text{Hilb}^{a_1 - 1}(Y_1^\circ) \times \text{Hilb}^{n + 1 - a_1}(Y_2^\circ)$ in the quotient, where Y_i° denotes open set away from the intersection $D = Y_1 \cap Y_2$.

Next we consider the other extremal case, namely where I is maximal, i.e. $I = I_{\max} = [n + 1]$. In this case $I_{\max,0}$ is partitioned into $n + 2$ subsets $\{\{0\}, \{1\}, \dots, \{n + 1\}\}$ and the associated graph is as in (16). Stability condition (17) then says that Z must have one point on each of the n inserted components, and consequently none on the components Y_1 or Y_2 . Recall that the fibres of every inserted component Δ_I^a , $a = 1, \dots, n$ are \mathbb{P}^1 -bundles over D and that the smooth locus $\Delta_I^{a,\circ}$ is a \mathbb{G}_m fibration, given by removing the 0-section and the ∞ -section of the \mathbb{P}^1 -bundle. Since stable cycles lie in the smooth part of $X[n]_I$ it follows that $Z = (P_1, \dots, P_n) \in \Delta_I^{1,\circ} \times \dots \times \Delta_I^{n,\circ}$ with $P_i \in \Delta_I^{i,\circ}$. Here the torus $G[n]$ acts trivially on C_I and transitively by multiplication on the product \mathbb{G}_m^n of the fibres of $\Delta_I^{1,\circ} \times \dots \times \Delta_I^{n,\circ}$ over a given point of D , see Section 1.1.5 for details. Hence the stable cycles in $X[n]_I$ map to an n -dimensional stratum D^n in $(I_{X/C}^n)_0$.

Now let us consider the general case $I = \{a_1, \dots, a_r\}$. In this case we have $r - 1$ inserted components $\Delta_I^{a_\ell}$, $\ell = 1, \dots, r - 1$. By the calculations of 1.1.5 the group $G[n]$ has a subgroup $G[k]$ which acts trivially on $C[n]_I$ and transitively by multiplication on the fibres of $\Delta_{I_1}^\circ \times \dots \times \Delta_{I_k}^\circ$, whose product, over each point in D , is isomorphic to \mathbb{G}_m^k . In this case we obtain quotients of products of the form $\text{Hilb}^{a_1 - 1}(Y_1^\circ) \times \text{Hilb}^{a_2 - a_1}(\Delta_I^{a_1,\circ}) \times \dots \times \text{Hilb}^{a_r - a_{r-1}}(\Delta_I^{a_{r-1},\circ}) \times \text{Hilb}^{n + 1 - a_r}(Y_2^\circ)$ by the group $G[k]$.

The above description provides a natural stratification of $(I_{X/C}^n)_0$ into locally closed subsets $(I_{X/C}^n)_I$ indexed by the subsets $I \subset I_{\max,0}$. Moreover, we can also describe how these strata are related with respect to inclusion, namely

$$(I_{X/C}^n)_J \subset \overline{(I_{X/C}^n)_I} \Leftrightarrow I \subset J.$$

It is natural to encode this information about the strata of $(I_{X/C}^n)_0$, together with the incidence relation of their closures, in a dual complex. In our example the situation is very simple: the k -simplices are in 1 : 1 correspondence to the subsets $I \subset I_{\max}$ of length $k + 1$ and the simplex corresponding to I is contained in the simplex corresponding to J if and only if $I \subset J$. Hence the resulting dual complex is the standard n -simplex. The maximal n -dimensional cell corresponds to the smallest stratum, which is isomorphic to D^n , and the 0-vertices correspond to the maximal-dimensional strata $\text{Hilb}^{a-1}(Y_1^\circ) \times \text{Hilb}^{n+1-a}(Y_2^\circ)$, $a = 1, \dots, n + 1$.

It is interesting to ask which dual complexes one obtains for more general degenerations. Given a degeneration graph Γ for a degeneration of curves or surfaces, one can indeed define a suitable Δ -complex, see [RS71], and describe

its combinatorial properties. We are planning to return to this in a future paper. Similarly, one can ask the same question for higher d -dimensional degenerations. As long as the degree $n \leq 3$, the Hilbert scheme is irreducible and smooth of dimension dn and one can hope for an interesting combinatorial object. For arbitrary dimension d and degree n the situation will become much more complicated as the Hilbert schemes, even of smooth varieties, are in general neither irreducible nor even equi-dimensional.

Finally, we want to say a few words about the singularities of the total space $I_{X/C}^n$ and, for the case of simplicity, we will restrict ourselves to degree 2 Hilbert schemes, and we will thus allow the dimension d of the fibres to be arbitrary again. Since $X[2]$ is smooth and all semi-stable points are stable, the quotient is also smooth at orbits where $G[2]$ acts freely. This is an easy consequence of Luna's slice theorem, see [Dr 04, Proposition 5.8]. In order to understand the set of stable points with nontrivial stabilizer we look at the various strata $X[2]_I$. Clearly $G[2]$ acts freely at points of $C[2]$, and hence also at points of $X[2]$, where all $t_{a_i} \neq 0$. The same is true if exactly one $t_{a_i} = 0$, i.e. if $|I| = 1$. If $|I| = 3$, then our above discussion shows that all stable points are of the form $Z = (P_1, P_2) \in \Delta_I^{1,\circ} \times \Delta_I^{2,\circ}$. Moreover, by Section 1.1.5 we know that $G[2]$ acts transitively and freely by multiplication on each fibre \mathbb{G}_m^2 of $\Delta_I^{1,\circ} \times \Delta_I^{2,\circ}$ over a given point of D .

It thus remains to consider the case where $|I| = 2$. We first consider $I = \{2, 3\}$. Then we have the partition $\{\{0, 1\}, \{2\}, \{3\}\}$ and one inserted component Δ_I^2 . By the stability condition (17) every stable cycle Z must contain a point in Δ_I^2 . The stabilizer of points in $C[2]$ with $t_2 = t_3 = 0$ and $t_1 \neq 0$ is the rank 1 subtorus $G[1] \subset G[2]$ given by $\sigma_1 = 1$. However, by Proposition 1.7 this stabilizer acts on the fibres of Δ_I^2 by $(u_2 : v_2) \mapsto (\sigma_2 u_2 : v_2)$. Hence $G[2]$ acts freely on the stable cycles supported on $X[n]_I$. A similar argument applies to $I = \{1, 2\}$ and it thus remains to consider $I = \{1, 3\}$. In this case we have one inserted component Δ_I^1 and by the stability condition every stable 2-cycle Z is supported on it. To study the non-free locus and the action of the stabilizer we work on the chart W_2 from Remark 1.8. where we have the coordinates $(t_1, t_2, t_3, x_2, \dots, x_d, u_1/v_1, u_2/v_2)$ and the relation $t_2 = (u_2/v_2) \cdot (v_1/u_1)$. Since stable cycles are supported on the smooth locus $\Delta_I^{1,\circ}$ we have $u_1/v_1 \neq 0$ and we can thus eliminate u_2/v_2 as a coordinate working with $(t_1, t_2, t_3, x_2, \dots, x_d, u_1/v_1)$. Here x_2, \dots, x_d are coordinates on D and the group $G[2]$ acts trivially on these coordinates. For simplicity we write $U = u_1/v_1$. Thus the action on our coordinates is given by

$$(t_1, t_2, t_3, x_2, \dots, x_d, U) \mapsto (\sigma_1 t_1, \sigma_2 t_2, (\sigma_1 \sigma_2)^{-1} t_3, x_2, \dots, x_d, \sigma_1 U).$$

Since $t_2 \neq 0$, any element in a nontrivial stabilizer must necessarily have $\sigma_2 = 1$. In particular, any nontrivial stabilizer group must lie in the rank 1 torus $G_1[2] = \langle \sigma_1 \rangle \subset G[2]$. This group acts freely on $\Delta_I^{1,\circ}$. Hence the only points in the relative degree 2 Hilbert schemes which can possibly have nontrivial stabilizers must be pairs of points $\{(x_2, \dots, x_d, U), (x_2, \dots, x_d, V)\}$ with $\sigma_1 U = V$ and $\sigma_1 V = U$. This implies $\sigma_1 = \pm 1$ and $U + V = 0$. In particular, the

corresponding point in the degree 2 Hilbert scheme is represented by a reduced 2-cycle and thus, when analysing the action of the stabilizer group, we can work with the relative second symmetric product rather than the Hilbert scheme. In order to describe this in coordinates we introduce a second set of fibre coordinates (y_2, \dots, y_d, V) . Forming the relative second symmetric product means factorizing by the involution which interchanges x_i and y_i as well as U and V . The invariants under this involution are generated by the linear invariant forms $A_i = x_i + y_i, i = 2, \dots, d$ and $B = U + V$ as well as the quadratic forms $C_{ij} = (x_i - y_i)(x_j - y_j), 2 \leq i, j \leq d, D_j = (x_j - y_j)(U - V), j = 2, \dots, d$ and $E = (U - V)^2$. The relations among these are generated by $C_{ij}E = D_i D_j$. The fixed points lie on $U + V = 0$, so we can assume that $E \neq 0$ near the fixed points. Thus we can eliminate C_{ij} and work with the coordinates given by A_i, B, D_j, E where $i, j = 2, \dots, d$. On these coordinates the torus $G_1[2]$ acts as

$$(t_1, t_2, t_3, A_i, B, D_j, E) \mapsto (\sigma_1 t_1, t_2, \sigma_1^{-1} t_3, A_i, \sigma_1 B, \sigma_1 D_j, \sigma_1^2 E).$$

From this we see immediately that the differential of involution given by $\{\pm 1\} \subset G[2]$ is a diagonal matrix with $3 + d - 1 = d + 2$ entries -1 and $d + 1$ entries 1 . It then follows from Luna's slice theorem [Dr 04, Theorem 5.4] that the quotient $I_{X/C}^n$ has a transversal singularity along D of type $\frac{1}{2}(1, \dots, 1)$ where we have $d + 2$ entries 1 . This singularity is the cone over the Veronese embedding of \mathbb{P}^{d+1} embedded by the linear system $|\mathcal{O}_{\mathbb{P}^{d+1}}(2)|$. We also note that in the case $d = 1$ we mistakenly labelled this an A_1 -singularity in [GHH15, Example 6.2].

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