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Creating a bridge between cardinal Br-spline fundamental functions for interpolation and subdivision

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Abstract

This paper presents innovative contributions to the fields of cardinal spline interpolation and subdivision. In particular, it unifies cardinal Br-spline fundamental functions for interpolation that are made of $r = M^{L+1}$ ($L \in \mathbb{N} \cup \{0\}$) distinct pieces between each pair of interpolation nodes and are featured by the properties of C^{2M-2} smoothness, approximation order $2M$ and support width $\frac{2M(r+1)}{r}$, with the basic limit functions of a special class of non-stationary subdivision schemes of arity M.
After introducing a general result, we focus our attention on the subclass of fourth-order accurate, C^2 smooth Br-splines with maximum width of the compact support 6. The binary subdivision scheme yielding these fundamental functions outperforms the existing interpolatory schemes and seems to be the most adequate starting point to obtain compactly supported fundamental (spline) functions for local interpolation over quadrilateral and triangular meshes.

Keywords: Cardinal splines; Subdivision; Exponential polynomials; Interpolation; Generalized Bezout Equation

1. Introduction

Subdivision schemes are efficient computational methods for generating functions (as well as curves, surfaces and volumes) from discrete data by repeated refinements. Their applications are indeed very broad and their usefulness is already well established in contexts like geometric modeling and computer graphics (see, e.g., [17, 34, 37]), biomedical imaging (see, e.g., [2, 13, 35]) and isogeometric analysis (see, e.g., [7, 25, 38]). In order to construct smooth curves and surfaces passing through a given set of data points, two different types of subdivision schemes can be used: the so-called "natural" interpolatory subdivision schemes and the ones known as interpolatory schemes "in the limit". Natural interpolatory schemes were first introduced by Dyn et al. in [18]. Later on, many generalizations of natural interpolatory subdivision methods (see, e.g., [3, 4, 10, 15, 24, 28] and references therein) were presented. A main limitation of natural interpolatory subdivision schemes lies in the fact that their limit curves/surfaces cannot be explicitly expressed by known mathematical functions. Moreover, it is also difficult to achieve high-order continuity while keeping the support of the basic limit function small. For instance, using binary "natural" interpolatory schemes of Dubuc-Deslauriers type [15] it is not possible to get a fourth-order accurate C^2 basic limit function supported in $[-3, 3]$. On the other hand, the so-called interpolatory schemes "in the limit" are not step-wise interpolants, i.e. they do not satisfy the interpolation property in each step of the subdivision process but only in the limit [29, 30, 32]. Their strength is in their capability of providing basic limit functions that compare favorably with the ones obtained by natural interpolatory schemes both in terms of support width and smoothness order. Moreover, if suitably defined, they can also provide limit functions with a closed-form mathematical expression, as already shown in [29].

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1.1. Contributions of this work

A recently appeared paper has shown that exponential polynomials are the only analytic functions that can be generated by level-dependent (non-stationary) subdivision schemes with finitely supported masks [6]. In this work we show that there exists a special class of non-stationary subdivision schemes with finitely supported masks that is capable of generating, in the limit, compactly supported cardinal Br-spline fundamental functions for interpolation that consist of r distinct, smoothly joined exponential-polynomial pieces between each pair of interpolation nodes. In particular, we show that by means of a suitably defined non-stationary subdivision scheme of arity $M \in \mathbb{N}\backslash\{1\}$, we can obtain C^{2M-2}-continuous cardinal Br-splines having $r = M^{L+1}$ ($L \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$) exponential-polynomial pieces between each pair of interpolation nodes and $2M(M^{L+1}+1)$ exponential-polynomial pieces in their entire compact support $[-M-M^{-L}, M+M^{-L}]$. We would like to point out that, before this work, it was never disclosed whether and how it was possible to obtain an exponential (or even a polynomial) Br-spline for cardinal interpolation, with $r > 2$ pieces, by means of a subdivision process. In fact, the current knowledge of subdivision schemes with spline limits is restricted to:

- polynomial and exponential B-splines (see, e.g., [5, 8, 17] and [9, 20, 23, 28, 36, 37], respectively),
- polynomial and exponential B2-splines (see [29] and [35], respectively),
- polynomial and exponential interpolating Hermite splines (see [21, 27, 33] and [13], respectively).

Thus, our general result (which includes as a special case the subdivision method recovering the cubic polynomial Br-splines for cardinal interpolation investigated in [1, 14, 26, 29] and the order-4 exponential B2-spline for cardinal interpolation presented in [35]) fills a knowledge gap in the related fields of subdivision and spline theory.

In addition, in this work, we show that the family of binary non-stationary subdivision schemes obtained when $M = 2$ outperforms the Dubuc-Deslauriers interpolatory 4-point scheme since it is able to reduce the support width of its basic limit function and to increase its smoothness up to C^2 without losing in approximation order. In particular, the basic limit functions of the binary non-stationary subdivision schemes obtained when $M = 2$ are C^2-continuous Br-splines for cardinal interpolation that are made of an overall number of $2^{L+3}+4$ pieces in their compact support $[-2-2^{-L}, 2+2^{-L}] \subseteq [-3, 3]$, and are featured by approximation order 4. This subclass of univariate binary schemes (which belongs to the class of interpolatory schemes "in the limit") is also generalizable to the bivariate case and lays the foundations for the construction of compactly supported basis functions (meeting the properties required in most applications) for interpolating quadrilateral and triangular meshes. With respect to the basic limit functions of Kobbelt's [22] and Butterfly [19] subdivision schemes (built-upon the Dubuc-Deslauriers interpolatory 4-point scheme), we expect the generalizations of the univariate fundamental functions proposed in this work to achieve the same approximation order, but to compare favorably both in terms of support width and smoothness order. In light of the above, the results contained in this paper could be considered a good starting point for developing a bivariate subdivision scheme capable of generating limit surfaces that, besides interpolating the vertices of a given mesh, turn out to meet the additional requirements concerning smoothness and local support that usually arise in high demanding application contexts like Computer-Aided Design, where the quality of surfaces is more important than in animated Computer Graphics.

1.2. Organization of the paper

The rest of the paper is organized as follows. Section 2, after fixing the notation and recalling basic notions about non-stationary subdivision schemes, introduces the subdivision method for generating, in the limit, exponential Br-splines for cardinal interpolation. The main properties of the resulting family of fundamental functions for local interpolation are investigated and then, in Section 3, a special subcase of the general result that leads to a subdivision method for generating cardinal polynomial Br-splines for local interpolation is discussed.

2. Non-stationary subdivision schemes generating cardinal exponential Br-splines

While polynomial Br-splines are piecewise polynomial functions, the so-called exponential Br-splines are piecewise exponential-polynomial functions with segments in more general function spaces that allow to represent hyperbolic and trigonometric functions as well. In this section we focus on the subclass of cardinal exponential Br-splines where r is a power of M, and show that they can be obtained as basic limit functions of a non-stationary subdivision scheme of arity M that relies on the refinability properties of the order-$2M$ cardinal exponential B-spline.

2.1. A short overview of cardinal exponential B-splines

Let $\alpha_\ell \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$, $\ell = 1, \dots, 2M$ denote the entries of the vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{2M})$, which are assumed to be either 0 or to come in pairs with opposite signs. If $\boldsymbol{\alpha}$ contains n_d distinct values, we denote them as $\alpha_{(1)}, \dots, \alpha_{(n_d)}$ and use the notation $\mu_{(1)}, \dots, \mu_{(n_d)}$ to refer to their multiplicities satisfying the constraint $\sum_{j=1}^{n_d} \mu_{(j)} = 2M$. Now let I and $D = \frac{d}{dx}$ denote the identity and derivative operators, respectively. The C^{2M-2}-continuous cardinal exponential B-spline of order $2M$ having support $[-M,\, M]$ and being associated with the differential operator

$$L_{\boldsymbol{\alpha}} = (D - \alpha_1 I)\ \dots\ (D - \alpha_{2M} I), \tag{2.1}$$

is known to be capable of generating functions from the null space of $L_{\boldsymbol{\alpha}}$ [35], i.e., from the space of exponential polynomials given by

$$\mathcal{E}_{2M} = span \left\{ x^{n-1}\, e^{\alpha_{(j)} x}, \quad j = 1, \dots, n_d,\ n = 1, \dots, \mu_{(j)} \quad \text{with} \quad \sum_{j=1}^{n_d} \mu_{(j)} = 2M \right\}.$$

In the following we assume $n_d = 3$ and $\alpha_{(1)} = 0$, $\alpha_{(2)} = \sigma$, $\alpha_{(3)} = -\sigma$ with $\sigma \in \mathbb{R}_+ \cup \mathrm{i}(0, \pi)$ and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. Moreover, we assume $\mu_{(3)} = \mu_{(2)}$. Then we denote by $\mathcal{B}_{2M,\sigma}$ the C^{2M-2}-continuous cardinal exponential B-spline of order $2M$ having support $[-M,\, M]$ and being associated with the differential operator in (2.1) where

$$\boldsymbol{\alpha} = (\underbrace{0}_{\mu_{(1)}}, \underbrace{\sigma}_{\mu_{(2)}}, \underbrace{-\sigma}_{\mu_{(2)}}), \qquad \mu_{(1)} + 2\mu_{(2)} = 2M. \tag{2.2}$$

Example 2.1. By varying the multiplicities of $\alpha_{(1)}$ and $\alpha_{(2)}, \alpha_{(3)}$, different exponential-polynomial spaces can be generated.

- If $\mu_{(1)} = 2$ and $\mu_{(2)} = 1$, then $\boldsymbol{\alpha} = (0, 0, \sigma, -\sigma)$, $2M = 4$ and $\mathcal{E}_4 = span\,\{1, x, e^{\sigma x}, e^{-\sigma x}\}$.
- If $\mu_{(1)} = 4$ and $\mu_{(2)} = 1$, then $\boldsymbol{\alpha} = (0, 0, 0, 0, \sigma, -\sigma)$, $2M = 6$ and $\mathcal{E}_6 = span\,\{1, x, x^2, x^3, e^{\sigma x}, e^{-\sigma x}\}$.
- If $\mu_{(1)} = 2$ and $\mu_{(2)} = 2$, then $\boldsymbol{\alpha} = (0, 0, \sigma, \sigma, -\sigma, -\sigma)$, $2M = 6$ and $\mathcal{E}_6 = span\,\{1, x, e^{\pm\sigma x}, xe^{\pm\sigma x}\}$.

Note that, for all $M > 2$, there is not a unique vector $\boldsymbol{\alpha}$ that matches the conditions in (2.2), and thus more than one exponential-polynomial space $\mathcal{E}_{2M}$ with its associated cardinal exponential B-spline $\mathcal{B}_{2M,\sigma}$ can be constructed.

2.2. A short overview of non-stationary subdivision schemes

A subdivision scheme is a two-scale process using data at one refinement level to compute denser data at the next refinement level. In the following we denote the refinement level with $k \in \mathbb{N}_0$, and sometimes we simply write $k \geq 0$ omitting the trivial information that the refinement level is always assumed to be an integer. Moreover, we equivalently identify a subdivision scheme $\mathcal{S}$ by a sequence of subdivision operators $\{S_{m^{(k)}},\, k \geq 0\}$, a sequence of subdivision masks $\{\mathbf{m}^{(k)},\, k \geq 0\}$ or a sequence of subdivision symbols $\{m^{(k)}(z),\, k \geq 0\}$, $z \in \mathbb{C} \setminus \{0\}$ (see, e.g., [9]).

The result of the application of $N+1$ subdivision steps of a certain scheme $\mathcal{S}$ of arity M to the data $\mathbf{P}^{(0)} = \{P_i^{(0)} \in \mathbb{R},\, i \in \mathbb{Z}\}$ can be represented in terms of subdivision symbols as

$$\begin{aligned} P^{(N+1)}(z) &= m^{(N)}(z)\, P^{(N)}(z^M) \\ &= m^{(N)}(z)\, m^{(N-1)}(z^M)\, P^{(N-1)}(z^{M^2}) \\ &= \dots \\ &= m^{(N)}(z)\, m^{(N-1)}(z^M) \dots m^{(0)}(z^{M^N})\, P^{(0)}(z^{M^{N+1}}) \;=\; \left(\prod_{j=0}^{N} m^{(j)}(z^{M^{N-j}})\right) P^{(0)}(z^{M^{N+1}}) \end{aligned} \tag{2.3}$$

and in terms of subdivision operators as

$$\mathbf{P}^{(N+1)} = S_{m^{(N)}}\, S_{m^{(N-1)}} \,\dots\, S_{m^{(0)}}\, \mathbf{P}^{(0)}.$$

For $\mathbf{P}^{(N+1)} = \{P_i^{(N+1)} \in \mathbb{R},\, i \in \mathbb{Z}\}$ denoting the refined data set obtained from the initial data $\mathbf{P}^{(0)}$ after $N+1$ subdivision steps, $P^{(N+1)}(z) = \sum_{i\in\mathbb{Z}} P_i^{(N+1)} z^i$ in (2.3) is exactly the associated subdivision symbol. Hereafter we additionally denote by

$$\phi := \lim_{k\to+\infty} S_{m^{(k)}} \,\dots\, S_{m^{(0)}}\, \boldsymbol{\delta}$$

the *basic limit function* of a convergent subdivision scheme $\mathcal{S}$ obtained by recursively refining the initial sequence $\boldsymbol{\delta} = \{\delta_{i,0},\, i \in \mathbb{Z}\}$, usually called the *delta*-sequence. If the masks $\mathbf{m}^{(k)}$ of $\mathcal{S}$ have supports $[\ell(k), r(k)]$ (i.e., are such that $m_i^{(k)} = 0$ for all $i < \ell(k)$ and $i > r(k)$), then it can be easily shown (see, e.g., [11]) that

$$supp(\phi) = [\mathcal{L}, \mathcal{R}] \qquad \text{with} \qquad \mathcal{L} = \sum_{k=0}^{+\infty} M^{-k-1}\ell(k), \quad \mathcal{R} = \sum_{k=0}^{+\infty} M^{-k-1} r(k). \tag{2.4}$$

2.3. The general case and the main result

We denote with $B_{2M,\sigma}^{(k)}(z)$ the k-level symbol of the non-stationary subdivision scheme of arity M having, as basic limit function, the exponential B-spline $\mathcal{B}_{2M,\sigma}$ supported on $[-M, M]$ (for details see [8, 12, 28, 37]). In other words,

$$\mathcal{B}_{2M,\sigma} = \lim_{k\to+\infty} S_{B_{2M,\sigma}^{(k)}} \,\dots\, S_{B_{2M,\sigma}^{(0)}}\, \boldsymbol{\delta}$$

where $S_{B_{2M,\sigma}^{(k)}}$ denotes the k-level subdivision operator associated with the symbol $B_{2M,\sigma}^{(k)}(z)$.
Our main result is given in the following theorem where, for the sake of shortness, we use the notation

$$p_{M,\sigma,L}(z) := \prod_{j=0}^{L} B_{2M,\sigma}^{(j)}(z^{M^{L-j}}) \tag{2.5}$$

and

$$\bar{B}_{M,\sigma}^{(\infty)}(z) := \sum_{i\in\mathbb{Z}} \mathcal{B}_{2M,\sigma}(i)\, z^i. \tag{2.6}$$

Knowing that the support of $\mathcal{B}_{2M,\sigma}$ is $[-M, M]$, we can deduce that $p_{M,\sigma,L}(z)$ is a Laurent polynomial. Indeed, its non-zero coefficients turn out to be $\left(p_{M,\sigma,L}\right)_j$, $j = -M(r-1), \dots, M(r-1)$ with $r = M^{L+1}$. Instead, $\bar{B}_{M,\sigma}^{(\infty)}(z)$ is a Laurent polynomial with $2M-1$ non-zero coefficients only, since $\mathcal{B}_{2M,\sigma}(i)$, $i = -(M-1), \dots, M-1$ are the only non-zero values attained by the exponential B-spline $\mathcal{B}_{2M,\sigma}$ at the integers.

Theorem 2.2. *Let $M \in \mathbb{N} \setminus \{1\}$, $L \in \mathbb{N}_0$ and $r = M^{L+1}$. If $q_{M,\sigma,L}(z) = \sum_{j=-M}^{M} \left(q_{M,\sigma,L}\right)_j z^j$ is a Laurent polynomial such that the coefficient sequence $\left\{ \left(\bar{m}_{M,\sigma,L}^{(\infty)}\right)_\ell,\ \ell = -M(r+1)+1, \dots, M(r+1)-1 \right\}$ of the product polynomial*

$$\bar{m}_{M,\sigma,L}^{(\infty)}(z) := q_{M,\sigma,L}(z)\, p_{M,\sigma,L}(z)\, \bar{B}_{M,\sigma}^{(\infty)}(z)$$

satisfies the condition

$$\left(\bar{m}^{(\infty)}_{M,\sigma,L}\right)_{ri} = \delta_{i,0}, \quad i=-M,\ldots,M,$$

then the non-stationary subdivision scheme $\mathcal{S}_{M,\sigma,L}$ *of arity* M *having symbols*

$$m^{(k)}_{M,\sigma,L}(z) = \begin{cases} B^{(k)}_{2M,\sigma}(z) \quad \text{when} \quad k \neq L, \\ q_{M,\sigma,L}(z)\, B^{(L)}_{2M,\sigma}(z) \quad \text{when} \quad k = L, \end{cases}$$

converges to a C^{2M-2}*-continuous Br-spline fundamental function for cardinal interpolation having compact support* $\left[-\frac{M(r+1)}{r}, \frac{M(r+1)}{r}\right] \subseteq [-M-1, M+1]$ *for all* $L \in \mathbb{N}_0$.

Proof. First we show that the non-stationary subdivision scheme $\{m^{(k)}_{M,\sigma,L}(z),\ k \geq 0\}$ is convergent and then we provide a closed-form expression for its basic limit function $\phi_{M,\sigma,L}$.
Recalling the definition of basic limit function we have that

$$\begin{aligned} \phi_{M,\sigma,L} &= \lim_{k\to+\infty} S_{m^{(k)}_{M,\sigma,L}} \ldots S_{m^{(0)}_{M,\sigma,L}}\, \boldsymbol{\delta} \\ &= \lim_{j\to+\infty} S_{m^{(L+j)}_{M,\sigma,L}} \ldots S_{m^{(L+1)}_{M,\sigma,L}} S_{m^{(L)}_{M,\sigma,L}} \ldots S_{m^{(0)}_{M,\sigma,L}}\, \boldsymbol{\delta} \\ &= \lim_{j\to+\infty} S_{m^{(L+j)}_{M,\sigma,L}} \ldots S_{m^{(L+1)}_{M,\sigma,L}}\, \mathbf{P}^{(L+1)} \\ &= \lim_{j\to+\infty} S_{B^{(L+j)}_{2M,\sigma}} \ldots S_{B^{(L+1)}_{2M,\sigma}}\, \mathbf{P}^{(L+1)} \end{aligned}$$

with

$$\mathbf{P}^{(L+1)} := S_{c_{M,\sigma,L}}\, \boldsymbol{\delta} \tag{2.7}$$

and

$$S_{c_{M,\sigma,L}} := S_{m^{(L)}_{M,\sigma,L}} S_{m^{(L-1)}_{M,\sigma,L}} \ldots S_{m^{(0)}_{M,\sigma,L}} = S_{q_{M,\sigma,L} B^{(L)}_{2M,\sigma}} S_{B^{(L-1)}_{2M,\sigma}} \ldots S_{B^{(0)}_{2M,\sigma}}. \tag{2.8}$$

Thus, in view of the fact that the non-stationary subdivision scheme based on the operators $S_{B^{(k)}_{2M,\sigma}}$ is convergent, and converges to a C^{2M-2}-continuous basic limit function, we can conclude that the non-stationary subdivision scheme $\{S_{m^{(k)}_{M,\sigma,L}}, k \geq 0\}$ is also convergent and its basic limit function

$$\phi_{M,\sigma,L} = \lim_{j\to+\infty} S_{B^{(L+j)}_{2M,\sigma}} \ldots S_{B^{(L+1)}_{2M,\sigma}}\, \mathbf{P}^{(L+1)} \tag{2.9}$$

is C^{2M-2}-continuous as well. In terms of symbols, (2.7) and (2.8) respectively read as

$$P^{(L+1)}(z) = c_{M,\sigma,L}(z)\, \delta(z^{M^{L+1}}) \tag{2.10}$$

and

$$c_{M,\sigma,L}(z) = q_{M,\sigma,L}(z)\, B^{(L)}_{2M,\sigma}(z)\, B^{(L-1)}_{2M,\sigma}(z^M) \ldots B^{(0)}_{2M,\sigma}(z^{M^L}) = q_{M,\sigma,L}(z)\, p_{M,\sigma,L}(z) \tag{2.11}$$

with $p_{M,\sigma,L}(z)$ in (2.5). Taking into account that

$$\delta(z^{M^{L+1}}) = \sum_{i\in\mathbb{Z}} \delta_{i,0} \left(z^{M^{L+1}}\right)^i = 1,$$

then (2.10) implies

$$P^{(L+1)}(z) = c_{M,\sigma,L}(z), \quad \forall z$$

and, consequently,

$$P^{(L+1)}_j = \left(c_{M,\sigma,L}\right)_j, \quad \forall j=-Mr,\ldots,Mr \tag{2.12}$$

with $\left(c_{M,\sigma,L}\right)_j$, $j=-Mr,\ldots,Mr$ denoting all the non-zero coefficients (associated with the powers z^j) of the Laurent polynomial $c_{M,\sigma,L}(z)$ in (2.11).

Finally, recalling that the non-stationary subdivision scheme $\{B^{(k)}_{2M,\sigma}(z),\, k \geq 0\}$ converges to the C^{2M-2}-continuous exponential B-spline of order-$2M$ supported on $[-M, M]$, in view of (2.9) we can write

$$\phi_{M,\sigma,L}(x) = \sum_{j\in\mathbb{Z}} P_j^{(L+1)}\, \mathcal{B}_{2M,\sigma}(M^{L+1}x - j)$$

and then, recalling (2.12), we arrive at

$$\phi_{M,\sigma,L}(x) = \sum_{j=-Mr}^{Mr} (c_{M,\sigma,L})_j \; \mathcal{B}_{2M,\sigma}(rx - j). \tag{2.13}$$

Thus, $supp(\phi_{M,\sigma,L}) = \left[-\frac{M(r+1)}{r}, \frac{M(r+1)}{r}\right] = \left[-M - M^{-L}, M + M^{-L}\right] \subseteq [-M-1, M+1]$ for all $L \in \mathbb{N}_0$. Now, introducing the notation

$$\bar{\mathbf{m}}^{(\infty)}_{M,\sigma,L} = \left\{ \left(\bar{m}^{(\infty)}_{M,\sigma,L}\right)_\ell := \phi_{M,\sigma,L}\left(\frac{\ell}{r}\right),\, \ell \in \mathbb{Z} \right\},$$

and recalling (2.13), we can write

$$\left(\bar{m}^{(\infty)}_{M,\sigma,L}\right)_\ell = \phi_{M,\sigma,L}\left(\frac{\ell}{r}\right) = \sum_{j\in\mathbb{Z}} (c_{M,\sigma,L})_j \; \mathcal{B}_{2M,\sigma}(\ell - j), \qquad \ell \in \mathbb{Z}.$$

The latter equation can be equivalently rewritten in terms of Laurent polynomials as

$$\bar{m}^{(\infty)}_{M,\sigma,L}(z) = c_{M,\sigma,L}(z)\, \bar{B}^{(\infty)}_{M,\sigma}(z).$$

If the unknown factor $q_{M,\sigma,L}(z) = \sum_{j=-M}^{M} \left(q_{M,\sigma,L}\right)_j z^j$ defining $c_{M,\sigma,L}(z)$ in (2.11) is determined in such a way that

$$\left(\bar{m}^{(\infty)}_{M,\sigma,L}\right)_{ri} = \delta_{i,0}, \quad i = -M, \ldots, M,$$

then in light of the equivalence

$$\phi_{M,\sigma,L}(i) = \left(\bar{m}^{(\infty)}_{M,\sigma,L}\right)_{ri} = \delta_{i,0}, \quad i = -M, \ldots, M,$$

and of the fact that $supp(\phi_{M,\sigma,L}) = [-M - M^{-L}, M + M^{-L}] \subseteq [-M-1, M+1]$, we can conclude that $\phi_{M,\sigma,L}$ is a fundamental function for cardinal interpolation.
In view of (2.13) we also obtain that, for $x \in [h, h+1] \subset supp(\phi_{M,\sigma,L})$ and $h \in \mathbb{Z}$ (i.e., for $h = -M - \delta_{L,0}, \ldots, M - 1 + \delta_{L,0}$), the function $\phi_{M,\sigma,L}(x)$ is made of r distinct exponential-polynomial pieces defined on the subintervals $\left[h + \frac{\ell-1}{r}, h + \frac{\ell}{r}\right]$, $\ell = 1, \ldots, r$. □

Corollary 2.3. *The values attained by the basic limit function $\phi_{M,\sigma,L}$ of $\mathcal{S}_{M,\sigma,L}$ at $\frac{1}{r}\mathbb{Z} \cap supp(\phi_{M,\sigma,L})$ ($r = M^{L+1}$) are the coefficients of the Laurent polynomial $\bar{m}^{(\infty)}_{M,\sigma,L}(z)$, namely*

$$\phi_{M,\sigma,L}\left(\frac{\ell}{r}\right) = \left(\bar{m}^{(\infty)}_{M,\sigma,L}\right)_\ell, \qquad \ell = -M(r+1)+1, \ldots, M(r+1)-1. \tag{2.14}$$

Having proven Theorem 2.2, the computational challenge is now to find a closed-form expression of the Laurent polynomial $q_{M,\sigma,L}(z)$ that provides the fundamental function $\phi_{M,\sigma,L}$. Since this cannot be done for arbitrary M (due to the lack of a closed-form expression for $p_{M,\sigma,L}(z)$ and $\bar{B}^{(\infty)}_{M,\sigma}(z)$ when $M > 2$), from here on we focus on the case $M = 2$ and show all computational details that lead to a closed-form expression of $q_{2,\sigma,L}(z)$. The case $M = 2$ is indeed the one that meets the most common set of requirements appearing in applications.

2.4. The subclass of binary, 4-th order, C^2 subdivision methods defining the fundamental functions $\phi_{2,\sigma,L}$

Recalling the results in [23, 31], the Laurent polynomial

$$B_{4,\sigma}^{(k)}(z) := \frac{1}{2}(z+1)^2\,\frac{z^2+2v_\sigma^{(k)}z+1}{2(v_\sigma^{(k)}+1)}\,z^{-2} \qquad \text{with} \quad v_\sigma^{(k)} := \frac{1}{2}\left(e^{\frac{\sigma}{2^{k+1}}}+e^{\frac{-\sigma}{2^{k+1}}}\right) = \cosh\left(\frac{\sigma}{2^{k+1}}\right), \quad k \ge 0 \tag{2.15}$$

is well-known to be the k-level symbol of the binary subdivision scheme whose basic limit function is the order-4 exponential B-spline $\mathcal{B}_{4,\sigma}$ generating the space

$$\mathcal{E}_4 = span\{1, x, e^{\sigma x}, e^{-\sigma x}\}.$$

For later use we introduce the abbreviations

$$v_\sigma^{(-1)} := \cosh(\sigma), \qquad \rho_\sigma^{(-1)} := \frac{\sinh(\sigma)}{\sigma} \qquad \text{and} \qquad \rho_\sigma^{(k)} := \frac{\sinh\left(\frac{\sigma}{2^{k+1}}\right)}{\frac{\sigma}{2^{k+1}}},\; k \ge 0 \tag{2.16}$$

as well as the notation

$$\Gamma_\sigma := \frac{\sigma\cosh(\sigma)-\sinh(\sigma)}{\sigma(\cosh(\sigma)-1)} = \frac{v_\sigma^{(-1)}-\rho_\sigma^{(-1)}}{v_\sigma^{(-1)}-1}. \tag{2.17}$$

Moreover, we also observe that, since $supp(\mathcal{B}_{4,\sigma}) = [-2,2]$ and the only non-zero values attained by $\mathcal{B}_{4,\sigma}$ at the integers are (see [31])

$$\mathcal{B}_{4,\sigma}(-1) = \frac{1-\Gamma_\sigma}{2}, \quad \mathcal{B}_{4,\sigma}(0) = \Gamma_\sigma, \quad \mathcal{B}_{4,\sigma}(1) = \frac{1-\Gamma_\sigma}{2},$$

then (2.6) simplifies as

$$\bar{B}_{2,\sigma}^{(\infty)}(z) = \frac{1-\Gamma_\sigma}{2}\,z^{-1} + \Gamma_\sigma + \frac{1-\Gamma_\sigma}{2}\,z. \tag{2.18}$$

Remark 2.4. When $\sigma = 0$ the expression of the Laurent polynomial in (2.15) does not depend on k anymore. Thus we drop the superscript (k) and denote it as

$$B_{4,0}(z) := \frac{1}{8}(z+1)^4\,z^{-2}.$$

The latter is indeed the symbol of the cubic polynomial B-spline generating the space

$$span\{1, x, x^2, x^3\} =: \Pi_3$$

(see, e.g, [23, 31]).

Now, let $L \in \mathbb{N}_0$. Exploiting the notation previously introduced we define the level-dependent, binary subdivision scheme $\mathcal{S}_{2,\sigma,L}$ having symbols $\{m_{2,\sigma,L}^{(k)}(z),\, k \ge 0\}$, where

$$m_{2,\sigma,L}^{(k)}(z) := \begin{cases} B_{4,\sigma}^{(k)}(z) & \text{when} \quad k \ne L, \\ q_{2,\sigma,L}(z)\,B_{4,\sigma}^{(L)}(z) & \text{when} \quad k = L. \end{cases} \tag{2.19}$$

In the second case of (2.19)

$$q_{2,\sigma,L}(z) := \left(a_{\sigma,L}z^4 + b_{\sigma,L}z^3 + (1-2a_{\sigma,L}-2b_{\sigma,L})z^2 + b_{\sigma,L}z + a_{\sigma,L}\right)z^{-2} \tag{2.20}$$

is a Laurent polynomial defined by the coefficients

$$a_{\sigma,L} := \frac{(1-\Gamma_\sigma)\left(v_\sigma^{(L)}(1-\Gamma_\sigma)+\Lambda_{\sigma,L}\right)}{4\Gamma_\sigma\,v_\sigma^{(L)}\left(v_\sigma^{(L)}(1-\Gamma_\sigma)+\Gamma_\sigma\right)}, \qquad b_{\sigma,L} := -\frac{\left(v_\sigma^{(L)}(1-\Gamma_\sigma)+1\right)\left(v_\sigma^{(L)}(1-\Gamma_\sigma)+\Lambda_{\sigma,L}\right)}{2\Gamma_\sigma\,v_\sigma^{(L)}\left(v_\sigma^{(L)}(1-\Gamma_\sigma)+\Gamma_\sigma\right)}$$

with

$$\Lambda_{\sigma,L} := \frac{\rho_\sigma^{(-1)} v_\sigma^{(L)} - \rho_\sigma^{(L-1)}}{\rho_\sigma^{(-1)}(v_\sigma^{(L)} - 1)},$$

$\rho_\sigma^{(-1)}$, $\rho_\sigma^{(L-1)}$ as in (2.16) and Γ_σ given in (2.17).

Remark 2.5. For later use we observe that $\Lambda_{\sigma,0} = 1$ and

$$\lim_{\sigma\to 0} \Gamma_\sigma = \frac{2}{3}, \quad \lim_{\sigma\to 0} \Lambda_{\sigma,L} = \frac{2^{2L+2}-1}{3}, \quad \lim_{\sigma\to 0} a_{\sigma,L} = \frac{2^{2L-1}}{3} =: a_{0,L}, \quad \lim_{\sigma\to 0} b_{\sigma,L} = -\frac{2^{2L+2}}{3} =: b_{0,L}.$$

A unifying representation for the k-level mask $\mathbf{m}^{(k)}_{2,\sigma,L}$ associated with the symbol in (2.19) is:

$$\mathbf{m}^{(k)}_{2,\sigma,L} = \Bigg\{ w^{(k)}_{\sigma,L},\ \frac{4w^{(k)}_{\sigma,L}(v_\sigma^{(k)})^2 + u^{(k)}_{\sigma,L}}{2v_\sigma^{(k)}},\ u^{(k)}_{\sigma,L} + \frac{1}{4(v_\sigma^{(k)}+1)},\ \frac{1}{2} - \frac{4w^{(k)}_{\sigma,L}(v_\sigma^{(k)})^2 + u^{(k)}_{\sigma,L}}{2v_\sigma^{(k)}},$$
$$\frac{2v_\sigma^{(k)}+1}{2(v_\sigma^{(k)}+1)} - 2(w^{(k)}_{\sigma,L} + u^{(k)}_{\sigma,L}),\ \frac{1}{2} - \frac{4w^{(k)}_{\sigma,L}(v_\sigma^{(k)})^2 + u^{(k)}_{\sigma,L}}{2v_\sigma^{(k)}},\ u^{(k)}_{\sigma,L} + \frac{1}{4(v_\sigma^{(k)}+1)},\ \frac{4w^{(k)}_{\sigma,L}(v_\sigma^{(k)})^2 + u^{(k)}_{\sigma,L}}{2v_\sigma^{(k)}},\ w^{(k)}_{\sigma,L} \Bigg\}$$

with

$$w^{(k)}_{\sigma,L} := \frac{a^{(k)}_{\sigma,L}}{4(v_\sigma^{(k)}+1)}, \qquad u^{(k)}_{\sigma,L} := \frac{v_\sigma^{(k)}(2a^{(k)}_{\sigma,L} + b^{(k)}_{\sigma,L})}{2(v_\sigma^{(k)}+1)}$$

and

$$a^{(k)}_{\sigma,L} := \delta_{k,L} a_{\sigma,L} = \begin{cases} 0 & \text{if } k \neq L \\ a_{\sigma,L} & \text{if } k = L, \end{cases} \qquad b^{(k)}_{\sigma,L} := \delta_{k,L} b_{\sigma,L} = \begin{cases} 0 & \text{if } k \neq L \\ b_{\sigma,L} & \text{if } k = L. \end{cases}$$

The latter is associated with the binary subdivision scheme denoted by $\mathcal{S}_{2,\sigma,L}$, which is equivalently identified by the symbol $m^{(k)}_{2,\sigma,L}(z)$, the mask $\mathbf{m}^{(k)}_{2,\sigma,L}$ or the k-level refinement rules

$$\begin{cases} P^{(k+1)}_{2i} = w^{(k)}_{\sigma,L}(P^{(k)}_{i-2} + P^{(k)}_{i+2}) + \left(u^{(k)}_{\sigma,L} + \dfrac{1}{4(v_\sigma^{(k)}+1)}\right)(P^{(k)}_{i-1} + P^{(k)}_{i+1}) + \left(\dfrac{2v_\sigma^{(k)}+1}{2(v_\sigma^{(k)}+1)} - 2(w^{(k)}_{\sigma,L} + u^{(k)}_{\sigma,L})\right) P^{(k)}_i \\ P^{(k+1)}_{2i+1} = \left(\dfrac{4w^{(k)}_{\sigma,L}(v_\sigma^{(k)})^2 + u^{(k)}_{\sigma,L}}{2v_\sigma^{(k)}}\right)(P^{(k)}_{i-1} + P^{(k)}_{i+2}) + \left(\dfrac{1}{2} - \dfrac{4w^{(k)}_{\sigma,L}(v_\sigma^{(k)})^2 + u^{(k)}_{\sigma,L}}{2v_\sigma^{(k)}}\right)(P^{(k)}_i + P^{(k)}_{i+1}). \end{cases} \tag{2.21}$$

The just defined non-stationary subdivision scheme $\mathcal{S}_{2,\sigma,L}$ has the following properties.

Proposition 2.6 (Convergence and Smoothness)**.** *The level-dependent subdivision scheme $\mathcal{S}_{2,\sigma,L}$ with symbols in* (2.19) *(or refinement rules in* (2.21)*) converges to a C^2-continuous basic limit function $\phi_{2,\sigma,L}$.*

Proof. The claimed result is a straightforward consequence of Theorem 2.2 and of the fact that the exponential B-spline scheme used to define $\mathcal{S}_{2,\sigma,L}$ is C^2-convergent. □

Remark 2.7. The basic limit function $\phi_{2,\sigma,0}$ was also investigated in [35], but a subdivision scheme as in (2.21) was never proposed so far for its generation.

Remark 2.8. A subdivision scheme as the one in (2.21) allows the user to efficiently generate a C^2-continuous curve $\mathcal{C}(x) = \sum_{i=1}^{\mathcal{N}^{(0)}} P_i^{(0)} \phi_{2,\sigma,L}(x-i)$, $x \in \mathcal{I} \subset \mathbb{R}$ that interpolates the vertices $\{P_i^{(0)}, i = 1, \ldots, \mathcal{N}^{(0)}\}$ of a given polygon by repeated refinement of such a polygon. At each iteration a new polygon having twice the vertices of the previous one is generated by computing linear combinations of nearby vertices. More precisely, the local linear combinations used by the subdivision scheme $\mathcal{S}_{2,\sigma,L}$ in the L-th iteration involve at most five points from the $(L-1)$-th level, whereas in all other iterations the points involved in each local linear combination are at most three. Since less than ten subdivision iterations are definitely more than enough for a good discrete representation of the interpolating curve $\mathcal{C}$ on the screen, from a computational point of view using the subdivision scheme turns out to be more efficient than constructing the piecewise-form of $\mathcal{C}$ and evaluating each of its exponential-polynomial pieces at the corresponding grid of parameter values obtained by discretizing the parameter domain $\mathcal{I}$.

In the following we analyze the properties of the basic limit function $\phi_{2,\sigma,L}$, $L \in \mathbb{N}_0$ generated by the level-dependent subdivision scheme in (2.21).

Proposition 2.9 (Support width). *The basic limit function $\phi_{2,\sigma,L}$ of the binary subdivision scheme $\mathcal{S}_{2,\sigma,L}$ has compact support $[\mathcal{L}_L, \mathcal{R}_L]$ with*

$$\mathcal{L}_L = -2 - 2^{-L} \quad \text{and} \quad \mathcal{R}_L = 2 + 2^{-L}.$$

Proof. Since

$$\ell(k) = \begin{cases} -2 & \text{if } k \neq L \\ -4 & \text{if } k = L \end{cases} \qquad \text{and} \qquad r(k) = \begin{cases} 2 & \text{if } k \neq L \\ 4 & \text{if } k = L \end{cases}$$

then, in light of (2.4), we have

$$\mathcal{L}_L = \left(\sum_{k=0}^{L-1} 2^{-k-1} (-2) \right) + 2^{-L-1} (-4) + \left(\sum_{k=L+1}^{+\infty} 2^{-k-1} (-2) \right) = -2 - 2^{-L}$$

and

$$\mathcal{R}_L = \left(\sum_{k=0}^{L-1} 2^{-k-1} (2) \right) + 2^{-L-1} (4) + \left(\sum_{k=L+1}^{+\infty} 2^{-k-1} (2) \right) = 2 + 2^{-L}.$$

□

Remark 2.10. The free parameter $L \in \mathbb{N}_0$ allows the user to control the support width of the fundamental function. The larger is L the smaller is the support width of $\phi_{2,\sigma,L}$. Moreover, for any arbitrary L, it is verified that $\mathcal{R}_L - \mathcal{L}_L = \frac{4(2^{L+1}+1)}{2^{L+1}}$ and thus $4 < \mathcal{R}_L - \mathcal{L}_L \leq 6$.

We now introduce a preliminary lemma that is needed to show the result in Proposition 2.12.

Lemma 2.11. *The Laurent polynomial in* (2.5) *can be explicitly written as*

$$p_{2,\sigma,L}(z) = \frac{1}{2^{L+1}} \left(\frac{1 - z^{2^{L+1}}}{1 - z} \right)^2 \frac{\left(\dfrac{1 - e^{\sigma} z^{2^{L+1}}}{1 - e^{\sigma/2^{L+1}} z} \right) \left(\dfrac{1 - e^{-\sigma} z^{2^{L+1}}}{1 - e^{-\sigma/2^{L+1}} z} \right)}{\left(\dfrac{1 - e^{\sigma}}{1 - e^{\sigma/2^{L+1}}} \right) \left(\dfrac{1 - e^{-\sigma}}{1 - e^{-\sigma/2^{L+1}}} \right)} z^{2(1-2^{L+1})}. \tag{2.22}$$

Proof. The claimed result is proven by induction on L. First we show that

$$\begin{aligned} p_{2,\sigma,0}(z) &= B_{4,\sigma}^{(0)}(z) = \frac{1}{2}(z+1)^2 \frac{z^2 + 2v_\sigma^{(0)} z + 1}{2(v_\sigma^{(0)} + 1)} z^{-2} = \frac{1}{2} (z+1)^2 \frac{z^2 + (e^{\sigma/2} + e^{-\sigma/2}) z + 1}{e^{\sigma/2} + e^{-\sigma/2} + 2} z^{-2} \\ &= \frac{1}{2} \left(\frac{1 - z^2}{1 - z} \right)^2 \frac{(1 + e^{\sigma/2} z)(1 + e^{-\sigma/2} z)}{(1 + e^{\sigma/2})(1 + e^{-\sigma/2})} z^{-2} = \frac{1}{2} \left(\frac{1 - z^2}{1 - z} \right)^2 \frac{\left(\dfrac{1 - e^{\sigma} z^2}{1 - e^{\sigma/2} z} \right) \left(\dfrac{1 - e^{-\sigma} z^2}{1 - e^{-\sigma/2} z} \right)}{\left(\dfrac{1 - e^{\sigma}}{1 - e^{\sigma/2}} \right) \left(\dfrac{1 - e^{-\sigma}}{1 - e^{-\sigma/2}} \right)} z^{-2} \end{aligned}$$

and thus $p_{2,\sigma,0}(z)$ satisfies equation (2.22) with $L = 0$.
Now we assume that $p_{2,\sigma,L-1}(z)$ fulfills the inductive hypothesis and show that

$$p_{2,\sigma,L}(z) = B_{4,\sigma}^{(L)}(z)\, p_{2,\sigma,L-1}(z^2)$$

fulfills exactly (2.22). In fact, recalling that

$$\begin{aligned}
B_{4,\sigma}^{(L)}(z) &= \frac{1}{2}(z+1)^2 \frac{z^2 + 2v_\sigma^{(L)} z + 1}{2(v_\sigma^{(L)}+1)} z^{-2} = \frac{1}{2}(z+1)^2 \frac{z^2 + (e^{\sigma/2^{L+1}} + e^{-\sigma/2^{L+1}})z + 1}{e^{\sigma/2^{L+1}} + e^{-\sigma/2^{L+1}} + 2} z^{-2} \\
&= \frac{1}{2}\left(\frac{1-z^2}{1-z}\right)^2 \frac{(1+e^{\sigma/2^{L+1}} z)(1+e^{-\sigma/2^{L+1}} z)}{(1+e^{\sigma/2^{L+1}})(1+e^{-\sigma/2^{L+1}})} z^{-2} \\
&= \frac{1}{2}\left(\frac{1-z^2}{1-z}\right)^2 \frac{\left(\frac{1-e^{\sigma/2^L} z^2}{1-e^{\sigma/2^{L+1}} z}\right)\left(\frac{1-e^{-\sigma/2^L} z^2}{1-e^{-\sigma/2^{L+1}} z}\right)}{\left(\frac{1-e^{\sigma/2^L}}{1-e^{\sigma/2^{L+1}}}\right)\left(\frac{1-e^{-\sigma/2^L}}{1-e^{-\sigma/2^{L+1}}}\right)} z^{-2}
\end{aligned}$$

and that, due to the inductive hypothesis,

$$p_{2,\sigma,L-1}(z) = \frac{1}{2^L}\left(\frac{1-z^{2^L}}{1-z}\right)^2 \frac{\left(\frac{1-e^{\sigma} z^{2^L}}{1-e^{\sigma/2^L} z}\right)\left(\frac{1-e^{-\sigma} z^{2^L}}{1-e^{-\sigma/2^L} z}\right)}{\left(\frac{1-e^{\sigma}}{1-e^{\sigma/2^L}}\right)\left(\frac{1-e^{-\sigma}}{1-e^{-\sigma/2^L}}\right)} z^{2(1-2^L)}$$

we indeed obtain

$$\begin{aligned}
p_{2,\sigma,L}(z) &= \frac{1}{2}\left(\frac{1-z^2}{1-z}\right)^2 \frac{\left(\frac{1-e^{\sigma/2^L} z^2}{1-e^{\sigma/2^{L+1}} z}\right)\left(\frac{1-e^{-\sigma/2^L} z^2}{1-e^{-\sigma/2^{L+1}} z}\right)}{\left(\frac{1-e^{\sigma/2^L}}{1-e^{\sigma/2^{L+1}}}\right)\left(\frac{1-e^{-\sigma/2^L}}{1-e^{-\sigma/2^{L+1}}}\right)} z^{-2} \quad \cdot \\
&\quad \cdot \quad \frac{1}{2^L}\left(\frac{1-z^{2^{L+1}}}{1-z^2}\right)^2 \frac{\left(\frac{1-e^{\sigma} z^{2^{L+1}}}{1-e^{\sigma/2^L} z^2}\right)\left(\frac{1-e^{-\sigma} z^{2^{L+1}}}{1-e^{-\sigma/2^L} z^2}\right)}{\left(\frac{1-e^{\sigma}}{1-e^{\sigma/2^L}}\right)\left(\frac{1-e^{-\sigma}}{1-e^{-\sigma/2^L}}\right)} z^{4-2^{L+2}} \\
&= \frac{1}{2^{L+1}}\left(\frac{1-z^{2^{L+1}}}{1-z}\right)^2 \frac{\left(\frac{1-e^{\sigma} z^{2^{L+1}}}{1-e^{\sigma/2^{L+1}} z}\right)\left(\frac{1-e^{-\sigma} z^{2^{L+1}}}{1-e^{-\sigma/2^{L+1}} z}\right)}{\left(\frac{1-e^{\sigma}}{1-e^{\sigma/2^{L+1}}}\right)\left(\frac{1-e^{-\sigma}}{1-e^{-\sigma/2^{L+1}}}\right)} z^{2(1-2^{L+1})}
\end{aligned}$$

so concluding the proof. □

Proposition 2.12. *The Laurent polynomial $q_{2,\sigma,L}(z)$ in (2.20) is such that the coefficient sequence $\left\{ \left(\bar{m}_{2,\sigma,L}^{(\infty)}\right)_\ell, \ \ell = -2^{L+2}-1, \ldots, 2^{L+2}+1 \right\}$ of the product polynomial*

$$\bar{m}_{2,\sigma,L}^{(\infty)}(z) := q_{2,\sigma,L}(z)\, p_{2,\sigma,L}(z)\, \bar{B}_{2,\sigma}^{(\infty)}(z)$$

satisfies the condition

$$\left(\bar{m}_{2,\sigma,L}^{(\infty)}\right)_{2^{L+1}i} = \delta_{i,0}, \quad i = -2, \ldots, 2.$$

Proof. Recalling the closed-form expression of $p_{2,\sigma,L}(z)$ in (2.22), it can be easily checked that $p_{2,\sigma,L}(z)$ is the symbol of an approximating scheme (the exponential B-spline scheme) of arity $r = 2^{L+1}$. Thus, $t_{2,\sigma,L}(z) := q_{2,\sigma,L}(z)\, \bar{B}^{(\infty)}_{2,\sigma}(z)$ can be viewed as the so-called Laurent correction (see [10]) yielding the symbol

$$\bar{m}^{(\infty)}_{2,\sigma,L}(z) = p_{2,\sigma,L}(z)\, t_{2,\sigma,L}(z).$$

According to the theoretical results in [10], in order to show that, for $r = 2^{L+1}$,

$$\left(\bar{m}^{(\infty)}_{2,\sigma,L}\right)_{ri} = \delta_{i,0}, \quad i \in \mathbb{Z}$$

or, equivalently,

$$\sum_{i\in\mathbb{Z}} \left(\bar{m}^{(\infty)}_{2,\sigma,L}\right)_{ri} z^{ri} = 1,$$

we need to verify that

$$\sum_{\ell=0}^{r-1} \bar{m}^{(\infty)}_{2,\sigma,L}(\zeta_\ell\, z) = r \qquad \text{with} \quad \zeta_\ell = e^{\frac{2\pi \mathrm{i}}{r}\ell},\ \ell = 0, \ldots, r-1 \quad \text{and} \quad r = 2^{L+1}.$$

The latter means that the Laurent correction $t_{2,\sigma,L}(z)$ solves the (generalized) Bezout equation

$$p_{2,\sigma,L}(z)\, t_{2,\sigma,L}(z) + p_{2,\sigma,L}(\zeta_1\, z)\, t_{2,\sigma,L}(\zeta_1\, z) + \ldots + p_{2,\sigma,L}(\zeta_{r-1}\, z)\, t_{2,\sigma,L}(\zeta_{r-1}\, z) = r. \tag{2.23}$$

To construct $t_{2,\sigma,L}(z)$ that fulfills (2.23), we first look for the minimal support solution

$$h_{\sigma,L}(z) := h_{-1}z^{-1} + h_0 z^0 + h_1 z^1$$

that satisfies

$$p_{2,\sigma,L}(z)\, h_{\sigma,L}(z) + p_{2,\sigma,L}(\zeta_1\, z)\, h_{\sigma,L}(\zeta_1\, z) + \ldots + p_{2,\sigma,L}(\zeta_{r-1}\, z)\, h_{\sigma,L}(\zeta_{r-1}\, z) = r;$$

then we build a "kernel" Laurent polynomial $\kappa_{\sigma,L}(z)$ such that

$$p_{2,\sigma,L}(z)\, \kappa_{\sigma,L}(z) + p_{2,\sigma,L}(\zeta_1\, z)\, \kappa_{\sigma,L}(\zeta_1\, z) + \ldots + p_{2,\sigma,L}(\zeta_{r-1}\, z)\, \kappa_{\sigma,L}(\zeta_{r-1}\, z) = 0,$$

and finally we write $t_{2,\sigma,L}(z)$ as

$$t_{2,\sigma,L}(z) := h_{\sigma,L}(z) + \kappa_{\sigma,L}(z).$$

For the construction of $h_{\sigma,L}(z)$ we follow the procedure proposed in [10]. Precisely, the coefficient vector of $h_{\sigma,L}(z)$ is obtained by taking the coefficients of the central (second) row of the inverse of

$$\mathcal{H}^{(r-2)} = \left(\mathcal{H}^{(r-2)}_{i,j}\right)_{1\le i,j\le 3}, \qquad \mathcal{H}^{(r-2)}_{i,j} := \left(p_{2,\sigma,L}\right)_{(j-2)r-(i-2)}, \quad 1 \le i,j \le 3,\ r = 2^{L+1}$$

which turns out to be

$$[h_{-1},\, h_0,\, h_1] = \left[-\frac{\Lambda_{\sigma,L}}{2v^{(L)}_\sigma},\, \frac{\Lambda_{\sigma,L} + v^{(L)}_\sigma}{v^{(L)}_\sigma},\, -\frac{\Lambda_{\sigma,L}}{2v^{(L)}_\sigma} \right],$$

so providing

$$h_{\sigma,L}(z) = -\frac{\Lambda_{\sigma,L}}{2v^{(L)}_\sigma}\, z^{-1} + \frac{\Lambda_{\sigma,L} + v^{(L)}_\sigma}{v^{(L)}_\sigma} - \frac{\Lambda_{\sigma,L}}{2v^{(L)}_\sigma}\, z. \tag{2.24}$$

As to the kernel polynomial, we choose

$$\kappa_{\sigma,L}(z) = \frac{1-\Gamma_\sigma}{2}\, a_{\sigma,L}\, (z^2+1)\,(z-1)^2 \left(z^2 - 2v^{(L)}_\sigma z + 1\right) z^{-3},$$

so obtaining for $t_{2,\sigma,L}(z)$ the closed-form expression

$$\begin{aligned} t_{2,\sigma,L}(z) &= a_{\sigma,L}\frac{1-\Gamma_\sigma}{2}\,z^{-3} - a_{\sigma,L}(1-\Gamma_\sigma)(v_\sigma^{(L)}+1)\,z^{-2} + \left(a_{\sigma,L}\frac{1-\Gamma_\sigma}{2}(4v_\sigma^{(L)}+3) - \frac{\Lambda_{\sigma,L}}{2v_\sigma^{(L)}}\right)z^{-1} \\ &+ \left(1-2a_{\sigma,L}(1-\Gamma_\sigma)(v_\sigma^{(L)}+1) + \frac{\Lambda_{\sigma,L}}{v_\sigma^{(L)}}\right) \\ &+ \left(a_{\sigma,L}\frac{1-\Gamma_\sigma}{2}(4v_\sigma^{(L)}+3) - \frac{\Lambda_{\sigma,L}}{2v_\sigma^{(L)}}\right)z - a_{\sigma,L}(1-\Gamma_\sigma)(v_\sigma^{(L)}+1)z^2 + a_{\sigma,L}\frac{1-\Gamma_\sigma}{2}\,z^3. \end{aligned} \tag{2.25}$$

Now, dividing the latter by $\bar{B}_{2,\sigma}^{(\infty)}(z)$ in (2.18), we get exactly the sought expression of the Laurent polynomial $q_{2,\sigma,L}(z)$ in (2.20). □

Corollary 2.13 (Interpolation)**.** *The basic limit function $\phi_{2,\sigma,L}$ of the binary subdivision scheme $\mathcal{S}_{2,\sigma,L}$ is a fundamental function for interpolation, i.e., it satisfies*

$$\phi_{2,\sigma,L}(i) = \delta_{i,0}, \quad i \in \mathbb{Z}.$$

Proof. It is a straightforward consequence of Proposition 2.12 and Theorem 2.2. □

Corollary 2.14 (Linear Independence)**.** *The integer shifts of $\phi_{2,\sigma,L}$ are linearly independent.*

Proof. It follows immediately from Corollary 2.13. □

Remark 2.15. If, instead of defining $t_{2,\sigma,L}(z)$ as in (2.25) we had simply used the Laurent correction $h_{\sigma,L}(z)$ in (2.24), then the coefficient sequence of the Laurent polynomial $\hat{m}_{2,\sigma,L}^{(\infty)}(z) := p_{2,\sigma,L}(z)\,h_{\sigma,L}(z)$ would have still satisfied the interpolatory condition $\left(\hat{m}_{2,\sigma,L}^{(\infty)}\right)_{ri} = \delta_{i,0}$, $i \in \mathbb{Z}$. For example, when $L = 0$,

$$\begin{aligned} \hat{\mathbf{m}}_{2,\sigma,0}^{(\infty)} &= \left\{\left(\hat{m}_{2,\sigma,0}^{(\infty)}\right)_\ell,\ \ell=-5,\ldots,5\right\} \\ &= \left\{0,\,0,\,-\frac{1}{8v_\sigma^{(0)}(v_\sigma^{(0)}+1)},\,0,\,\frac{\left(2v_\sigma^{(0)}+1\right)^2}{8v_\sigma^{(0)}(v_\sigma^{(0)}+1)},\,1,\,\frac{\left(2v_\sigma^{(0)}+1\right)^2}{8v_\sigma^{(0)}(v_\sigma^{(0)}+1)},\,0,\,-\frac{1}{8v_\sigma^{(0)}(v_\sigma^{(0)}+1)},\,0,\,0\right\} \end{aligned}$$

whereas, when $L = 1$,

$$\begin{aligned} \hat{\mathbf{m}}_{2,\sigma,1}^{(\infty)} &= \left\{\left(\hat{m}_{2,\sigma,1}^{(\infty)}\right)_\ell,\ \ell=-9,\ldots,9\right\} \\ &= \Bigg\{0,\,0,\,-\frac{\Lambda_{\sigma,1}}{32(v_\sigma^{(0)}+1)v_\sigma^{(1)}(v_\sigma^{(1)}+1)},\,-\frac{1}{8v_\sigma^{(0)}(v_\sigma^{(0)}+1)},\,-\frac{\Lambda_{\sigma,1}+2}{32(v_\sigma^{(0)}+1)v_\sigma^{(1)}(v_\sigma^{(1)}+1)},\,0, \\ &\quad \frac{1}{4}+\frac{\Lambda_{\sigma,1}-2}{32(v_\sigma^{(0)}+1)v_\sigma^{(1)}(v_\sigma^{(1)}+1)},\,\frac{\left(2v_\sigma^{(0)}+1\right)^2}{8v_\sigma^{(0)}(v_\sigma^{(0)}+1)},\,\frac{3}{4}+\frac{\Lambda_{\sigma,1}+4}{32(v_\sigma^{(0)}+1)v_\sigma^{(1)}(v_\sigma^{(1)}+1)},\,1, \\ &\quad \frac{3}{4}+\frac{\Lambda_{\sigma,1}+4}{32(v_\sigma^{(0)}+1)v_\sigma^{(1)}(v_\sigma^{(1)}+1)},\,\frac{\left(2v_\sigma^{(0)}+1\right)^2}{8v_\sigma^{(0)}(v_\sigma^{(0)}+1)},\,\frac{1}{4}+\frac{\Lambda_{\sigma,1}-2}{32(v_\sigma^{(0)}+1)v_\sigma^{(1)}(v_\sigma^{(1)}+1)}, \\ &\quad 0,\,-\frac{\Lambda_{\sigma,1}+2}{32(v_\sigma^{(0)}+1)v_\sigma^{(1)}(v_\sigma^{(1)}+1)},\,-\frac{1}{8v_\sigma^{(0)}(v_\sigma^{(0)}+1)},\,-\frac{\Lambda_{\sigma,1}}{32(v_\sigma^{(0)}+1)v_\sigma^{(1)}(v_\sigma^{(1)}+1)},\,0,\,0\Bigg\}. \end{aligned}$$

It is interesting to observe that both $\hat{\mathbf{m}}_{2,\sigma,0}^{(\infty)}$ and the subsequence $\left\{\left(\hat{m}_{2,\sigma,1}^{(\infty)}\right)_{2i},\ i \in \mathbb{Z}\right\}$ are exactly the first (i.e., 0-level) subdivision mask of the non-stationary interpolatory 4-point scheme in [3]. Moreover, $h_{\sigma,0}(z)$ is exactly the 0-level link polynomial introduced in [4, 28] to convert the order-4 exponential B-spline symbol $B_{4,\sigma}^{(0)}(z) = p_{2,\sigma,0}(z)$ into the interpolatory symbol $\hat{m}_{2,\sigma,0}^{(\infty)}(z)$.

The following examples show the closed-form expressions of $\bar{\mathbf{m}}_{2,\sigma,0}^{(\infty)}$ and $\bar{\mathbf{m}}_{2,\sigma,1}^{(\infty)}$ that, according to Corollary 2.3, are nothing but the sequences of values attained by $\phi_{2,\sigma,0}$ and $\phi_{2,\sigma,1}$ at $\mathbb{Z}/2$ and $\mathbb{Z}/4$, respectively.

Example 2.16. When $L = 0$, from Proposition 2.12 we find that $\left(\bar{m}_{2,\sigma,0}^{(\infty)}\right)_{2i} = \delta_{i,0}$, $i = -2, \ldots, 2$ and the non-zero values attained by $\phi_{2,\sigma,0}$ at $\mathbb{Z}/2 \setminus \mathbb{Z}$ are

$$\begin{aligned}
\left(\bar{m}_{2,\sigma,0}^{(\infty)}\right)_{\pm 5} &= \frac{a_{\sigma,0}(1-\Gamma_\sigma)}{8(v_\sigma^{(0)}+1)},\\
\left(\bar{m}_{2,\sigma,0}^{(\infty)}\right)_{\pm 3} &= \frac{a_{\sigma,0}(1-\Gamma_\sigma)v_\sigma^{(0)}\left(1-4(v_\sigma^{(0)})^2\right)-1}{8v_\sigma^{(0)}(v_\sigma^{(0)}+1)},\\
\left(\bar{m}_{2,\sigma,0}^{(\infty)}\right)_{\pm 1} &= \frac{2a_{\sigma,0}(1-\Gamma_\sigma)v_\sigma^{(0)}\left(2(v_\sigma^{(0)})^2-1\right)+\left(2v_\sigma^{(0)}+1\right)^2}{8v_\sigma^{(0)}(v_\sigma^{(0)}+1)}.
\end{aligned}$$

Example 2.17. When $L = 1$, from Proposition 2.12 we find that $\left(\bar{m}_{2,\sigma,1}^{(\infty)}\right)_{4i} = \delta_{i,0}$, $i = -2, \ldots, 2$ and the non-zero values attained by $\phi_{2,\sigma,1}$ at $\mathbb{Z}/4 \setminus \mathbb{Z}$ are

$$\begin{aligned}
\left(\bar{m}_{2,\sigma,1}^{(\infty)}\right)_{\pm 9} &= \frac{a_{\sigma,1}(1-\Gamma_\sigma)}{32(v_\sigma^{(0)}+1)(v_\sigma^{(1)}+1)},\\
\left(\bar{m}_{2,\sigma,1}^{(\infty)}\right)_{\pm 7} &= \frac{a_{\sigma,1}(1-\Gamma_\sigma)v_\sigma^{(1)}-\Lambda_{\sigma,1}}{32(v_\sigma^{(0)}+1)v_\sigma^{(1)}(v_\sigma^{(1)}+1)},\\
\left(\bar{m}_{2,\sigma,1}^{(\infty)}\right)_{\pm 6} &= -\frac{1}{8v_\sigma^{(0)}(v_\sigma^{(0)}+1)},\\
\left(\bar{m}_{2,\sigma,1}^{(\infty)}\right)_{\pm 5} &= -\frac{4a_{\sigma,1}(1-\Gamma_\sigma)\left(v_\sigma^{(0)}\right)^2v_\sigma^{(1)}+\Lambda_{\sigma,1}+2}{32(v_\sigma^{(0)}+1)v_\sigma^{(1)}(v_\sigma^{(1)}+1)},\\
\left(\bar{m}_{2,\sigma,1}^{(\infty)}\right)_{\pm 3} &= \frac{1}{4}-\frac{4a_{\sigma,1}(1-\Gamma_\sigma)\left(v_\sigma^{(0)}\right)^2v_\sigma^{(1)}-\Lambda_{\sigma,1}+2}{32(v_\sigma^{(0)}+1)v_\sigma^{(1)}(v_\sigma^{(1)}+1)},\\
\left(\bar{m}_{2,\sigma,1}^{(\infty)}\right)_{\pm 2} &= \frac{\left(2v_\sigma^{(0)}+1\right)^2}{8v_\sigma^{(0)}(v_\sigma^{(0)}+1)},\\
\left(\bar{m}_{2,\sigma,1}^{(\infty)}\right)_{\pm 1} &= \frac{3}{4}+\frac{2a_{\sigma,1}(1-\Gamma_\sigma)\left(4\left(v_\sigma^{(0)}\right)^2-1\right)v_\sigma^{(1)}+\Lambda_{\sigma,1}+4}{32(v_\sigma^{(0)}+1)v_\sigma^{(1)}(v_\sigma^{(1)}+1)}.
\end{aligned}$$

Proposition 2.18 (Exponential-Polynomial Generation)**.** *The subdivision scheme $\mathcal{S}_{2,\sigma,L}$ with symbols in* (2.19) *(or refinement rules in* (2.21)*) generates functions from the exponential-polynomial space $\mathcal{E}_4 = span\{1, x, e^{\sigma x}, e^{-\sigma x}\}$, $\sigma \in \mathbb{R}_+ \cup \mathrm{i}(0, \pi)$.*

Proof. Since the subdivision symbols (2.19) that are used in all steps $k \geq 0$ contain the factor $B_{4,\sigma}^{(k)}(z)$, we can conclude that the subdivision scheme $\mathcal{S}_{2,\sigma,L}$ generates the exponential-polynomial space $\mathcal{E}_4$ [12]. In fact, in view of such a factor, for all $k \geq 0$ the subdivision symbol $m_{2,\sigma,L}^{(k)}(z)$ satisfies

$$m_{2,\sigma,L}^{(k)}(-1) = 0, \quad (m_{2,\sigma,L}^{(k)})'(-1) = 0, \quad m_{2,\sigma,L}^{(k)}(-e^{\frac{\pm\sigma}{2^{k+1}}}) = 0.$$

Recalling [12, Proposition 1], the latter conditions imply the generation of the 4-dimensional space of exponential polynomials $\mathcal{E}_4$. □

Corollary 2.19 (Exponential-Polynomial Reproduction)**.** *The subdivision scheme* $\mathcal{S}_{2,\sigma,L}$ *with symbols in* (2.19) *reproduces the exponential-polynomial space* $\mathcal{E}_4$ *[12].*

Proof. It is a straightforward consequence of Corollary 2.13 and Proposition 2.18. $\square$

Proposition 2.20 (Piecewise exponential-polynomial structure)**.** *The basic limit function* $\phi_{2,\sigma,L}$ *of the subdivision scheme* $\mathcal{S}_{2,\sigma,L}$ *with symbols in* (2.19) *has piecewise exponential-polynomial structure and, for* $x \geq 0$*, the pieces are*

$$\phi^{\ell}_{2,\sigma,L}(x) = \phi_{2,\sigma,L}(x)\Big|_{x\in\left[\frac{\ell-1}{2^{L+1}},\frac{\ell}{2^{L+1}}\right]}, \quad \ell = 1,\ldots,2^{L+2}+2.$$

Proof. Taking into account that $M = 2$ and $r = 2^{L+1}$, we have that $(c_{2,\sigma,L})_j = 0$ for all $j < -2^{L+2}$ and $j > 2^{L+2}$. Then (2.13) simplifies as

$$\phi_{2,\sigma,L}(x) = \sum_{j=-2^{L+2}}^{2^{L+2}} (c_{2,\sigma,L})_j \; \mathcal{B}_{4,\sigma}(2^{L+1}x - j).$$

There follows that, for $x \in [0,1]$, the function $\phi_{2,\sigma,L}(x)$ is made of 2^{L+1} distinct exponential-polynomial pieces which are defined for $x \in \left[\frac{\ell-1}{2^{L+1}}, \frac{\ell}{2^{L+1}}\right]$, $\ell = 1,\ldots,2^{L+1}$. Thus, recalling that the right endpoint of the support of $\phi_{2,\sigma,L}$ is $\mathcal{R}_L = 2+2^{-L}$, we can conclude that the total number of exponential-polynomial pieces of $\phi_{2,\sigma,L}(x)$ for $x \in [0,\mathcal{R}_L]$ is $(2+2^{-L})2^{L+1} = 2^{L+2}+2$. $\square$

The following corollary summarizes the key properties of the basic limit function $\phi_{2,\sigma,L}$ of the binary subdivision scheme $\mathcal{S}_{2,\sigma,L}$. In light of the previous results these are a natural consequence of the fact that $\phi_{2,\sigma,L}$ is recovered by a suitable linear combination of order-4, shifted exponential B-splines on the grid $\mathbb{Z}/2^{L+1}$.

Corollary 2.21. *The binary, non-stationary subdivision scheme* $\mathcal{S}_{2,\sigma,L}$ *generates a basic limit function* $\phi_{2,\sigma,L}$ *that satisfies all the following properties for any arbitrary choice of* $L \in \mathbb{N}_0$ *and* $\sigma \in \mathbb{R}_+ \cup \mathrm{i}(0,\pi)$*:*

- $\phi_{2,\sigma,L}(i) = \delta_{i,0}$, $i \in \mathbb{Z}$*;*
- *supp*$(\phi_{2,\sigma,L}) = [-2-2^{-L}, 2+2^{-L}] = \left[-\frac{2(r+1)}{r}, \frac{2(r+1)}{r}\right]$ *for* $r = 2^{L+1}$*;*
- $\phi_{2,\sigma,L}$ *is a piecewise exponential-polynomial function made of* $4(2^{L+1}+1) = 4(r+1)$ *pieces (*$r = 2^{L+1}$ *between each pair of interpolation nodes);*
- $\phi_{2,\sigma,L} \in C^2(\mathbb{R})$*;*
- $\{\phi_{2,\sigma,L}(\cdot - i)\}_{i\in\mathbb{Z}}$ *are linearly independent and reproduce the exponential-polynomial space* $\mathcal{E}_4 =$ *span*$\{1, x, e^{\sigma x}, e^{-\sigma x}\}$*, thus ensuring approximation order* 4*.*

In light of the above, $\phi_{2,\sigma,L}$*,* $L \in \mathbb{N}_0$ *are fourth-order accurate,* C^2 *cardinal exponential Br-spline (with* $r = 2^{L+1}$*) fundamental functions for interpolation.*

We conclude this section by providing first a graphical illustration of $\phi_{2,\sigma,L}$ for different values of L and σ (see Figures 1 and 2), and then some application examples of the subdivision schemes $\mathcal{S}_{2,\sigma,L}$. In the first example we fix the value of L and show how the interpolating curves change by selecting different values of σ. In the second example we fix the value of σ and show the advantages that can be obtained by increasing the value of L.

Example 2.22. We consider the problem of interpolating the vertices of a given polygon by using the binary subdivision scheme $\mathcal{S}_{2,\sigma,L}$ with a selected value of L and several choices of σ. In Figure 3 we illustrate the results obtained for $L = 0$ and different values of σ in $\mathrm{i}(0,\pi)$ and $\mathbb{R}_+$.

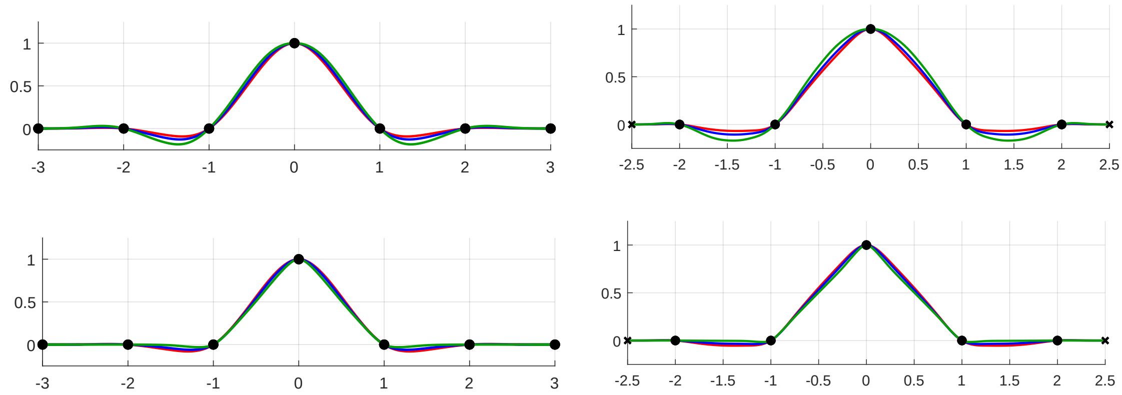

Figure 1: Illustration of $\phi_{2,\sigma,0}$ ($L = 0$, first column) and $\phi_{2,\sigma,1}$ ($L = 1$, second column) for different values of σ. Top row: $\sigma = \mathrm{i}\frac{\pi}{6}, \mathrm{i}\frac{\pi}{2}, \mathrm{i}\frac{2\pi}{3}$ (red, blue, green respectively). Bottom row: $\sigma = 1, 2, 5$ (red, blue, green respectively). (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

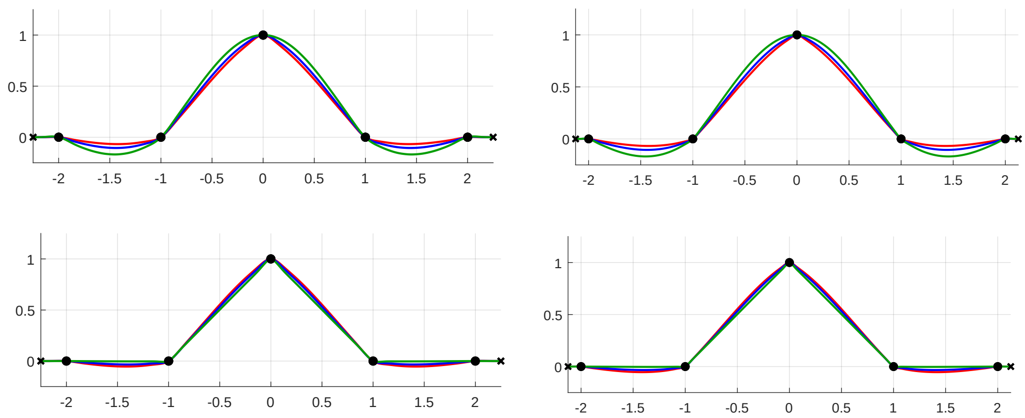

Figure 2: Illustration of $\phi_{2,\sigma,2}$ ($L = 2$, first column) and $\phi_{2,\sigma,3}$ ($L = 3$, second column) for different values of σ. Top row: $\sigma = \mathrm{i}\frac{\pi}{6}, \mathrm{i}\frac{\pi}{2}, \mathrm{i}\frac{2\pi}{3}$ (red, blue, green respectively). Bottom row: $\sigma = 1, 2, 5$ (red, blue, green respectively). (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

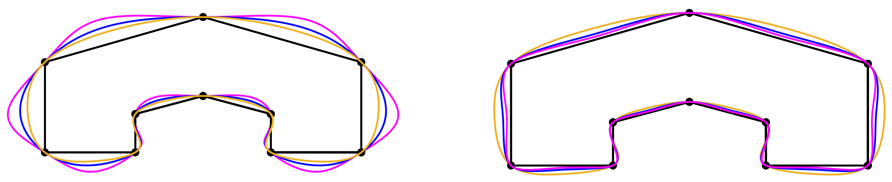

Figure 3: Interpolation data ($\bullet$) and interpolating curves obtained by the subdivision scheme $\mathcal{S}_{2,\sigma,0}$ with (left) $\sigma = \mathrm{i}\frac{\pi}{6}, \mathrm{i}\frac{\pi}{2}, \mathrm{i}\frac{2\pi}{3}$ (yellow, blue, magenta respectively) and (right) $\sigma = 1, 3, 5$ (yellow, blue, magenta respectively). (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

Example 2.23. The goal of this example is to show that, if the initial data are sampled from four touching circles as illustrated in Figure 4, then by increasing the value of L it is possible to get a progressive reduction of the error between the limit curve pieces and the underlying circle arc. For instance, concerning the initial polygon displayed in Figure 4 it is easy to see that, while the circle arc between p_6 and p_7 is exactly reproduced by any scheme with $L \geq 0$, the circle arc between p_{12} and p_{13} does not. Table 1 shows the max errors made by the interpolatory 4-point scheme in [3] and by the six binary schemes $\mathcal{S}_{2,\sigma,L}$ with $L = 0, 1, 2, 3, 4, 5$, in the reconstruction of the circle arc between p_{12} and p_{13} after 10 subdivision steps. In light of the shrinking of the support width of $\phi_{2,\sigma,L}$ when L is increased (see Remark 2.10), the reconstruction error of the circle arc between p_{12} and p_{13} is approximately reduced by a factor of 2 by passing from L to $L+1$.

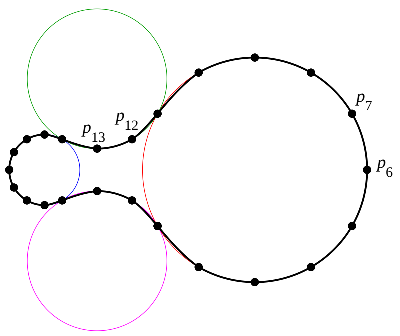

Figure 4: Limit curve (black line) obtained by the subdivision scheme $\mathcal{S}_{2,\sigma,L}$ with $L = 3$ and $\sigma = \mathrm{i}\frac{\pi}{6}$ starting from the data marked by black bullets.

	max error
4-point [3]	2.8107e-03
$\mathcal{S}_{2,\sigma,0}$ $(\sigma = \mathrm{i}\frac{\pi}{6})$	5.1583e-03
$\mathcal{S}_{2,\sigma,1}$ $(\sigma = \mathrm{i}\frac{\pi}{6})$	2.5613e-03
$\mathcal{S}_{2,\sigma,2}$ $(\sigma = \mathrm{i}\frac{\pi}{6})$	1.2794e-03
$\mathcal{S}_{2,\sigma,3}$ $(\sigma = \mathrm{i}\frac{\pi}{6})$	6.3952e-04
$\mathcal{S}_{2,\sigma,4}$ $(\sigma = \mathrm{i}\frac{\pi}{6})$	3.1978e-04
$\mathcal{S}_{2,\sigma,5}$ $(\sigma = \mathrm{i}\frac{\pi}{6})$	1.6036e-04

Table 1: Max errors made by the interpolatory 4-point scheme in [3] and by the six schemes $\mathcal{S}_{2,\sigma,L}$, $L = 0, 1, 2, 3, 4, 5$, in the reconstruction of the circle arc between p_{12} and p_{13} after 10 subdivision steps.

3. A special subcase: a subdivision method for cardinal polynomial Br-splines

In the case $\sigma = 0$, Theorem 2.2 shows that C^{2M-2}-continuous, piecewise-polynomial fundamental functions for cardinal interpolation, that are made of M^{L+1} pieces between each pair of interpolation nodes, can be generated by a *quasi-stationary* subdivision scheme of arity M (for short, $\mathcal{S}_{M,0,L}$) that exploits the symbol

$$B_{2M,0}(z) = \frac{1}{M^{2M-1}} \left(\frac{1 - z^M}{1 - z} \right)^{2M} z^{-M(M-1)}$$

of the order-$2M$ polynomial B-spline in all subdivision levels except the Lth one. In the Lth subdivision step the scheme $\mathcal{S}_{M,0,L}$ indeed uses a subdivision symbol of the form $q_{M,0,L}(z)\, B_{2M,0}(z)$ where $q_{M,0,L}(z)$ is a Laurent polynomial to be suitably defined in such a way that the coefficient sequence $\left\{ \left(\bar{m}^{(\infty)}_{M,0,L} \right)_\ell, \ \ell = -M(r+1)+1, \ldots, M(r+1)-1 \right\}$ of the product polynomial

$$\bar{m}^{(\infty)}_{M,0,L}(z) := q_{M,0,L}(z)\, p_{M,0,L}(z)\, \bar{B}^{(\infty)}_{M,0}(z)$$

satisfies the condition

$$\left(\bar{m}^{(\infty)}_{M,0,L} \right)_{M^{L+1}i} = \delta_{i,0}, \quad i = -M, \ldots, M.$$

In the case $\sigma = 0$ the two known factors of $\bar{m}^{(\infty)}_{M,0,L}(z)$ assume the very simple expressions

$$p_{M,0,L}(z) = \frac{1}{r^{2M-1}} \left(\frac{1 - z^r}{1 - z} \right)^{2M} z^{-M(r-1)} \quad \text{with} \quad r = M^{L+1}$$

and

$$\bar{B}^{(\infty)}_{M,0}(z) = \sum_{i=-(M-1)}^{M-1} \mathcal{B}_{2M,0}(i)\ z^i$$

with

$$\mathcal{B}_{2M,0}(i) := \frac{1}{(2M-1)!} \sum_{h=i+M}^{2M} (-1)^{h-i-M} \binom{2M}{h-i-M} (2M-h)^{2M-1}.$$

Example 3.1. When $M=3$ and $L=0$, the ternary subdivision scheme $\mathcal{S}_{3,0,0}$ is equivalently identified by the k-level symbol

$$m^{(k)}_{3,0,0}(z)=\begin{cases}\frac{1}{243z^6}(z^2+z+1)^6\,q_{3,0,0}(z) & \text{when} \quad k=0,\\ \frac{1}{243z^6}(z^2+z+1)^6 & \text{when} \quad k\neq 0,\end{cases}$$

with

$$q_{3,0,0}(z)=\Big(-\frac{131}{110}z^6+\frac{8829}{880}z^5-\frac{3483}{110}z^4+\frac{20523}{440}z^3-\frac{3483}{110}z^2+\frac{8829}{880}z-\frac{131}{110}\Big)\,z^{-3},$$

and by the k-level refinement rules

$$\begin{cases} P^{(k+1)}_{3i} &=& -\frac{131}{330}d^{(k)}\left(P^{(k)}_{i-3}+P^{(k)}_{i+3}\right)+\left(\frac{1028}{453}d^{(k)}+\frac{1}{243}\right)\left(P^{(k)}_{i-2}+P^{(k)}_{i+2}\right)\\ &+& \left(\frac{50}{243}-\frac{2833}{110}d^{(k)}\right)\left(P^{(k)}_{i-1}+P^{(k)}_{i+1}\right)+\left(\frac{8311}{174}d^{(k)}+\frac{141}{243}\right)P^{(k)}_{i}\\ P^{(k+1)}_{3i+1} &=& \frac{47}{40}d^{(k)}\,P^{(k)}_{i-2}+\left(\frac{7}{81}-\frac{1379}{80}d^{(k)}\right)P^{(k)}_{i-1}+\left(\frac{469}{20}d^{(k)}+\frac{14}{27}\right)P^{(k)}_{i}\\ &+& \left(\frac{43}{40}d^{(k)}+\frac{10}{27}\right)P^{(k)}_{i+1}+\left(\frac{2}{81}-\frac{377}{40}d^{(k)}\right)P^{(k)}_{i+2}+\frac{77}{80}d^{(k)}\,P^{(k)}_{i+3}\\ P^{(k+1)}_{3i+2} &=& \frac{77}{80}d^{(k)}\,P^{(k)}_{i-2}+\left(\frac{2}{81}-\frac{377}{40}d^{(k)}\right)P^{(k)}_{i-1}+\left(\frac{43}{40}d^{(k)}+\frac{10}{27}\right)P^{(k)}_{i}\\ &+& \left(\frac{469}{20}d^{(k)}+\frac{14}{27}\right)P^{(k)}_{i+1}+\left(\frac{7}{81}-\frac{1379}{80}d^{(k)}\right)P^{(k)}_{i+2}+\frac{47}{40}d^{(k)}\,P^{(k)}_{i+3}\end{cases}$$

where

$$d^{(k)}=\frac{1}{81}\,\delta_{k,0}=\begin{cases}\frac{1}{81} & \text{when } k=0\\ 0 & \text{when } k\neq 0.\end{cases}$$

The basic limit function $\phi_{3,0,0}$ of $\mathcal{S}_{3,0,0}$ is the B3 fundamental spline function having polynomial pieces of degree 5, order of continuity 4, degree of polynomial reproduction 5 and compact support $[-4,4]$. In [1] this was denoted as B3 $D^5C^4P^5S^8$. Its piecewise-polynomial representation is

$$\phi_{3,0,0}(x)=\begin{cases}\phi^{\ell}_{3,0,0}(|x|), & \frac{\ell-1}{3}\le|x|<\frac{\ell}{3},\ \ell=1,\dots,12\\ 0, & |x|\ge 4\end{cases}$$

with

$$\begin{aligned}
\phi^{1}_{3,0,0}(|x|)&=-\tfrac{1}{5280}\Big(21309|x|^5-23570|x|^4+11960|x|^2-5280\Big), && 0\le|x|<\tfrac{1}{3},\\
\phi^{2}_{3,0,0}(|x|)&=\tfrac{1}{63360}\Big(158121|x|^5-406875|x|^4+459810|x|^3-296790|x|^2+25545|x|+61657\Big), && \tfrac{1}{3}\le|x|<\tfrac{2}{3},\\
\phi^{3}_{3,0,0}(|x|)&=-\tfrac{1}{21120}(|x|-1)\Big(66201|x|^4-194534|x|^3+180676|x|^2-72714|x|+36211\Big), && \tfrac{2}{3}\le|x|<1,\\
\phi^{4}_{3,0,0}(|x|)&=\tfrac{1}{105600}(|x|-1)\Big(311933|x|^4-1599082|x|^3+2954248|x|^2-2208182|x|+461883\Big), && 1\le|x|<\tfrac{4}{3},\\
\phi^{5}_{3,0,0}(|x|)&=-\tfrac{1}{105600}\Big(159001|x|^5-1228545|x|^4+3818830|x|^3-6000450|x|^2+4771855|x|-1522629\Big), && \tfrac{4}{3}\le|x|<\tfrac{5}{3},\\
\phi^{6}_{3,0,0}(|x|)&=\tfrac{1}{52800}(|x|-2)\Big(70309|x|^4-493522|x|^3+1264916|x|^2-1405568|x|+582624\Big), && \tfrac{5}{3}\le|x|<2,\\
\phi^{7}_{3,0,0}(|x|)&=-\tfrac{1}{52800}(|x|-2)(61213|x|^4-558654|x|^3+1891612|x|^2-2803136|x|+1521728\Big), && 2\le|x|<\tfrac{7}{3},\\
\phi^{8}_{3,0,0}(|x|)&=\tfrac{1}{316800}\Big(160761|x|^5-2073975|x|^4+10695270|x|^3-27562350|x|^2+35492595|x|-18260875\Big), && \tfrac{7}{3}\le|x|<\tfrac{8}{3},\\
\phi^{9}_{3,0,0}(|x|)&=-\tfrac{1}{105600}(|x|-3)\Big(27251|x|^4-304762|x|^3+1269104|x|^2-2334518|x|+1604621\Big), && \tfrac{8}{3}\le|x|<3,\\
\phi^{10}_{3,0,0}(|x|)&=\tfrac{1}{105600}(|x|-3)\Big(20083|x|^4-263246|x|^3+1286932|x|^2-2777554|x|+2229433\Big), && 3\le|x|<\tfrac{10}{3},\\
\phi^{11}_{3,0,0}(|x|)&=-\tfrac{1}{316800}\Big(23343|x|^5-422715|x|^4+3057990|x|^3-11044950|x|^2+19913715|x|-14335103\Big), && \tfrac{10}{3}\le|x|<\tfrac{11}{3},\\
\phi^{12}_{3,0,0}(|x|)&=\tfrac{131}{13200}(|x|-4)^5, && \tfrac{11}{3}\le|x|<4,
\end{aligned}$$

(see Figure 5).

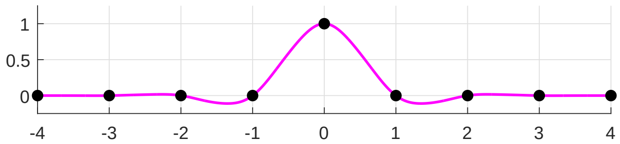

Figure 5: Illustration of $\phi_{3,0,0}$.

In the following we focus on the case $M = 2$ and show the expression of $q_{M,0,L}(z)$ that satisfies Theorem 2.2 and leads to the family of Br fundamental spline functions having polynomial pieces of degree 3, order of continuity 2, degree of polynomial reproduction 3 and support width $\frac{4(r+1)}{r}$ with $r = 2^{L+1}$. Thus, using the acronym proposed in [1] we recover the class of functions $\{\phi_{2,0,L},\ L \in \mathbb{N}_0\}$ identified as:

$$\mathrm{B}r\,\mathrm{D}^3\mathrm{C}^2\mathrm{P}^3\mathrm{S}^{\frac{4(r+1)}{r}},\ r = 2^{L+1},\ L \in \mathbb{N}_0.$$

As we will point out later on, this subclass contains two members that can be obtained using also the constructive approach proposed in [1]: the fundamental functions B2 $\mathrm{D}^3\mathrm{C}^2\mathrm{P}^3\mathrm{S}^6$ and B4 $\mathrm{D}^3\mathrm{C}^2\mathrm{P}^3\mathrm{S}^5$, respectively.

3.1. The subclass of binary, 4-th order, C^2 subdivision methods defining the fundamental functions $\phi_{2,0,L}$

Fixing $M = 2$ and recalling Remarks 2.4 and 2.5, the binary subdivision scheme for the case $\sigma = 0$ turns out to be described by the k-level symbol

$$m^{(k)}_{2,0,L}(z) = \begin{cases} \frac{1}{8z^2}(z+1)^4 & \text{when} \quad k \neq L, \\ \frac{1}{8z^2}(z+1)^4\, q_{2,0,L}(z) & \text{when} \quad k = L, \end{cases} \tag{3.1}$$

with

$$q_{2,0,L}(z) = \left(a_{0,L}z^4 + b_{0,L}z^3 + (1 - 2a_{0,L} - 2b_{0,L})z^2 + b_{0,L}z + a_{0,L}\right) z^{-2}$$

and

$$a_{0,L} = \frac{2^{2L-1}}{3}, \qquad b_{0,L} = -\frac{2^{2L+2}}{3}.$$

Remark 3.2. According to the result in Proposition 2.12, $q_{2,0,L}(z)$ has been defined in such a way that the Laurent polynomial $t_{2,0,L}(z) = h_{0,L}(z) + \kappa_{0,L}(z)$ with

$$h_{0,L}(z) = -\frac{2^{2L+2}-1}{6}\, z^{-1} + \frac{2^{2L+2}+2}{3} - \frac{2^{2L+2}-1}{6}\, z$$

and

$$\kappa_{0,L}(z) = \frac{2^{2L-2}}{9}\,(z^2+1)(z-1)^4\, z^{-3}$$

solves the generalized Bezout equation which guarantees that the coefficient sequence $\left\{ \left(\bar{m}^{(\infty)}_{2,0,L}\right)_\ell,\ \ell = -2^{L+2}-1, \ldots, 2^{L+2}+1 \right\}$ of the product polynomial

$$\bar{m}^{(\infty)}_{2,0,L}(z) := q_{2,0,L}(z)\, p_{2,0,L}(z)\, \bar{B}^{(\infty)}_{2,0}(z)$$

satisfies the condition

$$\left(\bar{m}^{(\infty)}_{2,0,L}\right)_{2^{L+1}i} = \delta_{i,0}, \quad i = -2, \ldots, 2$$

yielding the cardinal interpolation property of $\phi_{2,0,L}$.

A unifying representation for the k-level mask $\mathbf{m}_{2,0,L}^{(k)}$ is:

$$\mathbf{m}_{2,0,L}^{(k)} = \Big\{\frac{a_{0,L}^{(k)}}{8}, \frac{4a_{0,L}^{(k)}+b_{0,L}^{(k)}}{8}, \frac{2a_{0,L}^{(k)}+b_{0,L}^{(k)}}{4}+\frac{1}{8}, \frac{1}{2}-\frac{4a_{0,L}^{(k)}+b_{0,L}^{(k)}}{8}, \frac{3}{4}-\frac{5a_{0,L}^{(k)}+2b_{0,L}^{(k)}}{4}, \frac{1}{2}-\frac{4a_{0,L}^{(k)}+b_{0,L}^{(k)}}{8}, \frac{2a_{0,L}^{(k)}+b_{0,L}^{(k)}}{4}+\frac{1}{8}, \frac{4a_{0,L}^{(k)}+b_{0,L}^{(k)}}{8}, \frac{a_{0,L}^{(k)}}{8}\Big\}$$

with

$$a_{0,L}^{(k)} = \delta_{k,L}\, a_{0,L} = \begin{cases} 0 & \text{when } k \neq L \\ a_{0,L} & \text{when } k = L \end{cases} \quad \text{and} \quad b_{0,L}^{(k)} = \delta_{k,L}\, b_{0,L} = \begin{cases} 0 & \text{when } k \neq L \\ b_{0,L} & \text{when } k = L. \end{cases}$$

The resulting binary subdivision scheme $\mathcal{S}_{2,0,L}$ is identified by the k-level refinement rules

$$\begin{cases} P_{2i}^{(k+1)} = \frac{a_{0,L}^{(k)}}{8}(P_{i-2}^{(k)}+P_{i+2}^{(k)}) + \left(\frac{2a_{0,L}^{(k)}+b_{0,L}^{(k)}}{4}+\frac{1}{8}\right)(P_{i-1}^{(k)}+P_{i+1}^{(k)}) + \left(\frac{3}{4}-\frac{5a_{0,L}^{(k)}+2b_{0,L}^{(k)}}{4}\right)P_i^{(k)} \\ P_{2i+1}^{(k+1)} = \frac{4a_{0,L}^{(k)}+b_{0,L}^{(k)}}{8}(P_{i-1}^{(k)}+P_{i+2}^{(k)}) + \left(\frac{1}{2}-\frac{4a_{0,L}^{(k)}+b_{0,L}^{(k)}}{8}\right)(P_i^{(k)}+P_{i+1}^{(k)}). \end{cases} \tag{3.2}$$

Equations (3.2) can be read as a relaxation of the Dubuc-Deslauriers interpolatory 4-point scheme [15], which is indeed recovered by the parameter setting $a_{0,L}^{(k)}=0$, $b_{0,L}^{(k)}=-\frac{1}{2}$ for all $L,k\geq 0$.
In light of the results in the previous section, the binary scheme $\mathcal{S}_{2,0,L}$ provides a basic limit function $\phi_{2,0,L}$ having compact support $[-2-2^{-L}, 2+2^{-L}]$, which guarantees reproduction of the polynomial space Π_3 for all $L\geq 0$. In the following we point out the piecewise-polynomial expression of $\phi_{2,0,L}$ and the values it attains at $\mathbb{Z}/2^{L+1}$ when $L=0$ and $L=1$.

3.1.1. The fundamental function $\phi_{2,0,0}$ (also known as B2 $D^3C^2P^3S^6$)

The choice $L=0$ yields the cubic B2-spline $\phi_{2,0,0}$ supported on $[-3,3]$. It was proposed in [14, 26] and generalized by the introduction of a shape parameter in [29]. Its piecewise-polynomial representation is

$$\phi_{2,0,0}(x) = \begin{cases} \phi_{2,0,0}^{\ell}(|x|), & \frac{\ell-1}{2}\leq |x| < \frac{\ell}{2},\ \ell=1,\dots,6 \\ 0, & |x|\geq 3 \end{cases}$$

with

$$\begin{aligned}
\phi_{2,0,0}^{1}(|x|) &= \tfrac{14}{9}|x|^3 - \tfrac{5}{2}|x|^2 + 1, & 0\leq |x| < \tfrac{1}{2},\\
\phi_{2,0,0}^{2}(|x|) &= \tfrac{(|x|-1)}{18}\big(20|x|^2 - 13|x| - 19\big), & \tfrac{1}{2}\leq |x| < 1,\\
\phi_{2,0,0}^{3}(|x|) &= \tfrac{(|x|-1)}{24}\big(-22|x|^2 + 80|x| - 74\big), & 1\leq |x| < \tfrac{3}{2},\\
\phi_{2,0,0}^{4}(|x|) &= -\tfrac{(|x|-2)}{24}\big(6|x|^2 - 18|x| + 10\big), & \tfrac{3}{2}\leq |x| < 2,\\
\phi_{2,0,0}^{5}(|x|) &= \tfrac{(|x|-2)}{36}\big(7|x|^2 - 37|x| + 49\big), & 2\leq |x| < \tfrac{5}{2},\\
\phi_{2,0,0}^{6}(|x|) &= -\tfrac{1}{36}\,(|x|-3)^3, & \tfrac{5}{2}\leq |x| < 3,
\end{aligned}$$

(see Figure 6 left). The values attained by $\phi_{2,0,0}$ at $\mathbb{Z}/2$ are:

$$\bar{\mathbf{m}}_{2,0,0}^{(\infty)} = \left\{0, \tfrac{1}{288}, 0, -\tfrac{7}{96}, 0, \tfrac{41}{72}, 1, \tfrac{41}{72}, 0, -\tfrac{7}{96}, 0, \tfrac{1}{288}, 0\right\}.$$

3.1.2. The fundamental function $\phi_{2,0,1}$ (also known as B4 $D^3C^2P^3S^5$)

The choice $L=1$ yields the cubic B4-spline $\phi_{2,0,1}$ supported on $[-\frac{5}{2},\frac{5}{2}]$. It was introduced in [1] and is described by the piecewise-polynomial representation

$$\phi_{2,0,1}(x) = \begin{cases} \phi_{2,0,1}^{\ell}(|x|), & \frac{\ell-1}{4}\leq |x| < \frac{\ell}{4},\ \ell=1,\dots,10 \\ 0, & |x|\geq \frac{5}{2} \end{cases}$$

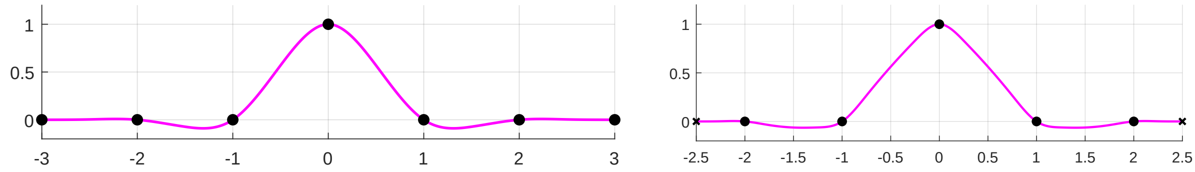

Figure 6: Illustration of $\phi_{2,0,L}$ with $L=0$ (left) and $L=1$ (right).

with

$$\begin{array}{ll}
\phi^1_{2,0,1}(|x|) = \frac{31}{6}|x|^3 - 4|x|^2 + 1, & 0 \le |x| < \frac{1}{4}, \\
\phi^2_{2,0,1}(|x|) = -\frac{1}{6}|x|^3 - |x| + \frac{13}{12}, & \frac{1}{4} \le |x| < \frac{1}{2}, \\
\phi^3_{2,0,1}(|x|) = \frac{1}{18}|x|^3 - \frac{1}{3}|x|^2 - \frac{5}{6}|x| + \frac{19}{18}, & \frac{1}{2} \le |x| < \frac{3}{4}, \\
\phi^4_{2,0,1}(|x|) = \frac{1}{18}(|x|-1)(65|x|^2 - 85|x| + 8), & \frac{3}{4} \le |x| < 1, \\
\phi^5_{2,0,1}(|x|) = -\frac{1}{18}(|x|-1)(59|x|^2 - 163|x| + 116), & 1 \le |x| < \frac{5}{4}, \\
\phi^6_{2,0,1}(|x|) = \frac{5}{18}|x|^3 - |x|^2 + \frac{7}{6}|x| - \frac{1}{2}, & \frac{5}{4} \le |x| < \frac{3}{2}, \\
\phi^7_{2,0,1}(|x|) = -\frac{1}{18}|x|^3 + \frac{1}{2}|x|^2 - \frac{13}{12}|x| + \frac{5}{8}, & \frac{3}{2} \le |x| < \frac{7}{4}, \\
\phi^8_{2,0,1}(|x|) = -\frac{1}{36}(|x|-2)(34|x|^2 - 118|x| + 97), & \frac{7}{4} \le |x| < 2, \\
\phi^9_{2,0,1}(|x|) = \frac{1}{36}(|x|-2)(28|x|^2 - 130|x| + 151), & 2 \le |x| < \frac{9}{4}, \\
\phi^{10}_{2,0,1}(|x|) = -\frac{1}{72}(2|x|-5)^3, & \frac{9}{4} \le |x| < \frac{5}{2},
\end{array}$$

(see Figure 6 right). The values attained by $\phi_{2,0,1}$ at $\mathbb{Z}/4$ are:

$$\bar{\mathbf{m}}^{(\infty)}_{2,0,1} = \left\{0, \frac{1}{576}, 0, -\frac{43}{1152}, -\frac{1}{16}, -\frac{71}{1152}, 0, \frac{307}{1152}, \frac{9}{16}, \frac{319}{384}, 1, \frac{319}{384}, \frac{9}{16}, \frac{307}{1152}, 0, -\frac{71}{1152}, -\frac{1}{16}, -\frac{43}{1152}, 0, \frac{1}{576}, 0\right\}.$$

It is interesting to observe that the subsequence $\left\{\left(\bar{m}^{(\infty)}_{2,0,1}\right)_{2i}, \ i = -5, \dots, 5\right\}$ is exactly the subdivision mask of the Dubuc-Deslauriers interpolatory 4-point scheme [15] and provides the values attained by its basic limit function at $\mathbb{Z}/2$.

Remark 3.3. For the sake of conciseness, we do not report the piecewise-polynomial expressions of $\phi_{2,0,2}$ and $\phi_{2,0,3}$, but we simply display them in Figure 7. However, we believe of interest to point out that the values attained by $\phi_{2,0,2}$ at $\mathbb{Z}/8$ are

$$\begin{aligned}
\bar{\mathbf{m}}^{(\infty)}_{2,0,2} = \Big\{&0, \frac{1}{1152}, 0, -\frac{181}{9216}, -\frac{5}{128}, -\frac{55}{1024}, -\frac{1}{16}, -\frac{65}{1024}, -\frac{7}{128}, -\frac{127}{3373}, 0, \frac{415}{3233}, \frac{35}{128}, \frac{429}{1024}, \frac{9}{16}, \frac{715}{1024}, \frac{105}{128}, \frac{2851}{3072}, \\
&1, \frac{2851}{3072}, \frac{105}{128}, \frac{715}{1024}, \frac{9}{16}, \frac{429}{1024}, \frac{35}{128}, \frac{415}{3233}, 0, -\frac{127}{3373}, -\frac{7}{128}, -\frac{65}{1024}, -\frac{1}{16}, -\frac{55}{1024}, -\frac{5}{128}, -\frac{181}{9216}, 0, \frac{1}{1152}, 0\Big\}
\end{aligned}$$

and the subsequence $\left\{\left(\bar{m}^{(\infty)}_{2,0,2}\right)_{4i}, \ i = -4, \dots, 4\right\}$ provides exactly the subdivision mask of the Dubuc-Deslauriers interpolatory 4-point scheme [15] whereas the subsequence $\left\{\left(\bar{m}^{(\infty)}_{2,0,2}\right)_{4i+2}, \ i = -4, \dots, 4\right\}$ is nothing but the subdivision mask of the dual 4-point scheme [16].

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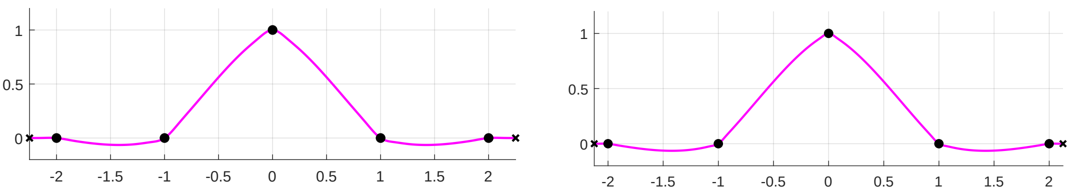

Figure 7: Illustration of $\phi_{2,0,L}$ with $L = 2$ (left) and $L = 3$ (right).

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