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(Article begins on next page)

On the attainment of the maximum sustainable yield in the Verhulst-Lotka-Volterra model

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Abstract

We reformulate the Verhulst-Lotka-Volterra model of natural resource extraction under the alternative assumptions of Cournot behaviour and perfect competition, to revisit the tragedy of commons vs the possibility of sustainable harvesting. We stress the different impact of demand elasticity on the regulator's possibility of driving industry harvest to the maximum sustainable yield in the two settings. The presence of a flat demand function offers the authority a fully effective regulatory tool in the form of the exogenous price faced by perfectly competitive firms, to drive their collective harvest rate to the maximum sustainable yield. The same cannot happen under Cournot competition, as in this case the price is endogenous and the regulator's policy is confined to limiting access to the common pool.

Key words: Tragedy of commons; Sustainability; Resource extraction; Differential games; Open-loop controls.

1 Introduction

Since Gordon (1954) and Hardin (1968), a *leitmotiv* of the discussion about the tragedy of commons is our perception of the impact of market power (or the lack thereof) on the preservation of renewable resources or natural species. In a nutshell, considering free access and perfect competition as equivalent, one is induced to think that the negative impact of perfect competition on a renewable asset's preservation is larger than the impact of any less-than-perfectly competitive industry.

We revisit this issue using the logistic growth model of Verhulst (1838), Lotka (1925) and Volterra (1931) (VLV henceforth), under both Cournot behaviour and perfect competition. Due to the nonlinear dynamics characterising the VLV model, it has been investigated in detail under perfect competition (Pearce and Turner, 1989), while the current literature modelling firms' strategic behaviour in the tragedy of commons has adopted either a piecewise linear approximation of the original state equation (Benchekroun, 2003, 2008; Colombo and Labrecciosa, 2013a, 2015) or a linear approximation of it (Fujiwara, 2008; Colombo and Labrecciosa, 2013b; Lambertini and Leitmann, 2013; Lambertini and Mantovani, 2014, 2016; and Lambertini, 2016). When the piecewise linear approximation is adopted, the maximum sustainable yield (MSY) corresponds to the non-differentiable point of the piecewise-linear function appearing in the state equation. Both linear and nonlinear feedback strategies can be characterised. In particular, if the resource is relatively scarce (abundant), the stable solution is generated by a linear feedback strategy which is increasing (flat) w.r.t. the stock, and the flat solution indeed coincides with open-loop one.

The use of the original VLV formulation prevents one to analyse the feedback problem but engenders the Ramsey rule, whereby harvest is dictated solely by discounting and the resource growth rate. To this aim we reconstruct the analysis appearing in Cellini and Lambertini (2008), showing, amongst other things, that if the market-driven harvest is lower that that associated with the Ramsey rule, then the former is a saddle point while the latter is unstable. The focus of our analysis is the attainment of the MSY, the ideal target of a public authority interested in a sustainable exploitation pattern. This choice is not obvious, as while Pearce and Turner (1989) discuss the conditions under which a monopolist (or a cartel) or a perfectly competitive industry may perpetually harvest at the MSY, [1] there also exists a strand of research

[1] A related but not equivalent stream of literature discusses

focussing on the optimal number of firms in the commons (see Mason et al., 1988; Mason and Polasky, 1997), where *optimal* means *welfare-maximising*, and this, in general, does not imply harvesting at the MSY.

Our approach is based on the strong time consistency of the open-loop solution of the VLV model. We prove that the game is state-redundant and admits the market-driven harvest as a degenerate feedback control, irrespective of the nature or intensity of competition characterising the market. In this scenario, the Ramsey rule disappears and the model produces two steady states located symmetrically to the left and right of the MSY. In both cases, that lying to the right is stable, while the other is not.

If market demand is downward sloping and firms behave *à la* Cournot, there exists a parameter region in which the regulator may limit access to the common pool so as to drive industry harvest as close as possible to the maximum sustainable level, without reaching it. If instead perfect competition prevails and demand is flat, firms' lack of control on price offers the policy maker an additional tool to achieve the MSY for any number of firms in the industry. Hence, what really matters is not free vs regulated access to the commons, but rather the elasticity of demand: an infinitely elastic market demand yields the possibility of regulating the exogenously given price to attain the goal of sustainability.

2 Setup and Cournot competition

Consider a market existing over continuous time $t \in [0, \infty)$, being supplied by $n \geq 1$ identical firms exploiting a renewable resource $X(t)$ to produce a homogeneous final good sold to consumers. The state dynamics is as in the VLV model,

$$\dot{X}(t) = \delta X(t)[1 - \beta X(t)] - Q(t) \tag{1}$$

in which β and δ are positive constants, and $Q(t) = \sum_{j=1}^{n} q_j(t)$ is the sum of the n firms' individual harvest at any instant. With an appropriate choice of measure, $q_i(t)$ and $Q(t)$ are also the instantaneous individual and industry output levels. Note that, if $\beta = 0$, then (1) collapses to the linear state dynamics used in part of the aforementioned literature, where it is responsible of the instability of the open-loop solution.

Let the instantaneous demand and individual cost function be $p(t) = a - Q(t)$ and $C_i(t) = cq_i^2(t)$, respectively. The choice of a quadratic cost function is motivated by the intent of including the possibility of decreasing returns and the need of generating the equilibrium under perfect competition with an infinitely elastic demand. Parameters $a, c > 0$ measure the choke price at which demand is nil and the steepness of the cost function, respectively. Firm i's profit function is $\pi_i(t) = [p(t) - cq_i(t)]\, q_i(t)$. Firm i has to choose harvest $q_i(t)$ so as to maximise the value of discounted profit $\Pi_i(t) = \int_0^\infty \pi_i(t)\, e^{-\rho t} dt$ under the constraint (1). Firm i's current value Hamiltonian function is

$$\mathcal{H}_i(t) = [p(t) - cq_i(t)]\, q_i(t) + \lambda_i(t)\, \dot{X}(t) \tag{2}$$

to be maximised w.r.t. $q_i(t)$, the initial condition being $X_0 = X(0) > 0$.

Suppose firms operate under open-loop information. The first order condition (FOC) taken w.r.t. q_i is (henceforth, the time argument is omitted):

$$\frac{\partial \mathcal{H}_i}{\partial q_i} = a - 2(1 + c)\, q_i - \sum_{j \neq i} q_j - \lambda_i = 0, \tag{3}$$

and the costate equation is

$$\dot{\lambda}_i = [\delta(2\beta X - 1) + \rho]\, \lambda_i \tag{4}$$

We may impose symmetry across firms (thereby dropping index i) and then, replicating the procedure illustrated in Cellini and Lambertini (2008), take one of two possible routes: either (a) solving (3) to obtain the expression of the optimal λ and proceed to the construction of the control equation; or (b) noting that (4) is a differential equation in separable variables, admitting the solution $\lambda_i = 0$ at any time, for all $i = 1, 2, ...n$.

In case (a), we have $\lambda^* = a - (n + 1 + 2c)\, q$ and $\dot{q} = -\dot{\lambda}/(n + 1 + 2c)$, which, using (4) and λ^*, becomes

$$\dot{q} = \frac{[a - (n + 1 + 2c)\, q]\, [\delta(1 - 2\beta\delta X) - \rho]}{n + 1 + 2c} \tag{5}$$

and, together with (1), constitutes the state-control system. The three candidate steady state points are

$$X^R = \frac{\delta - \rho}{2\beta\delta}; Q^R = \frac{\delta^2 - \rho^2}{4\beta\delta} \tag{6}$$

$$X_{\pm}^{CN} = \frac{\delta(n + 1 + 2c) \pm \sqrt{\Upsilon}}{2\beta\delta(n + 1 + 2c)}; Q^{CN} = \frac{na}{n + 1 + 2c} \tag{7}$$

where $\Upsilon \equiv \delta(n + 1 + 2c)[\delta(n + 1 + 2c) - 4\beta an]$. Hence, $X_{\pm}^{CN} \in \mathbb{R}^+ \forall \beta \in (0, \delta(n + 1 + 2c)/(4an)]$. It is also worth stressing that $Q^{CN} = nq^{CN}$ monotonically increases in n while the pair (X^R, q^R) engendered

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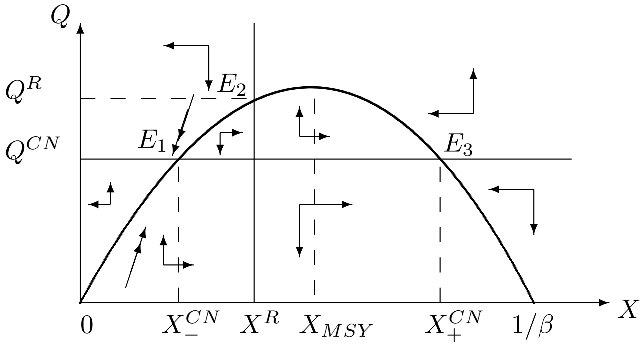
the possible arising of efficient equilibria, possibly by means of cooperation among agents. Again, efficiency does not in general imply the attainment of the MSY (see, e.g., Dockner and Kaitala, 1989; Ehtamo and Hämäläinen, 1993; Martin-Herran and Rincon-Zapatero, 2005).

by the *Ramsey rule* is independent of n. Moreover, $X^R < X_{MSY}$ everywhere, and $X^R \gtreqless 0$ for all $\delta \gtreqless \rho$. Scenario (a) produces five different regimes, as in Cellini and Lambertini (2008): (I) there exist three steady state points, E_1, E_2 and E_3, with $X_-^{CN} < X^R < X_+^{CN}$ (as in Figure 1); (II) there exist two steady state points, $E_1 \equiv E_2$ and E_3, with $X_-^{CN} = X^R < X_+^{CN}$; (III) there exist three steady state points, E_2, E_1 and E_3, with $X^R < X_-^{CN} < X_+^{CN}$; (IV) there exist two steady state points, E_2 and $E_1 \equiv E_3$, with $X^R < X_-^{CN} = X_+^{CN}$. This portrays the tangency solution, on which we will say more below; (V) there exists a unique steady state equilibrium point, $\left(X^R, Q^R\right)$. This happens when Q^{CN} lies above the parabola. In (I), E_1 and E_3 are saddle points, while E_2 is an unstable focus. In (II), both steady state points are saddles: the saddle path approaches $E_1 \equiv E_2$ from the left only, while E_3 is reached along the horizontal line. In (III), E_2 and E_3 are saddle points and E_1 is an unstable focus; in (IV), E_2 is a saddle, while the tangency solution $E_1 = E_3$ is half-stable. In (V), there exists a unique saddle point at E_2.

Figure 1 The phase diagram under Cournot competition, $\delta > \rho$

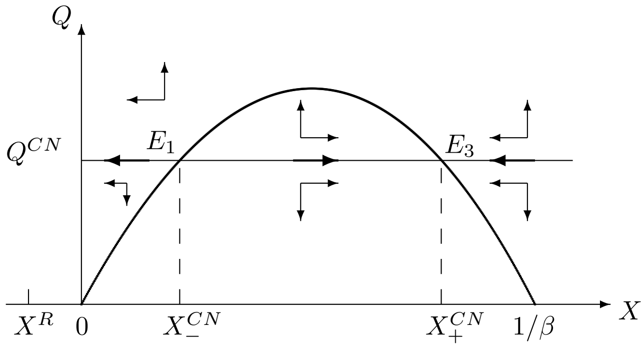


The phase diagram associated with $\rho > \delta$ is in Figure 2, where $X^R < 0$. Impatience rules out the Ramsey solution, firms behaving as quasi-static Cournot agents. E_1 is unstable, and firms approach the saddle point E_3 along the flat line at Q^{CN} as shown by the horizontal arrows, since $\dot{X} \gtreqless 0$ for all $\delta X\left(1 - \beta X\right) \gtreqless Q^{CN}$.

This situation is observationally (but not conceptually) equivalent to the case where $\lambda = 0$ at all times: if $\lambda \neq 0$, the Ramsey rule becomes immaterial when ρ is too high but the behaviour of the open-loop control in (3) still depends on X, while posing $\lambda = 0$ in (3) means that the shadow price of a resource unit is always nil (and therefore time discounting, appearing only in (4), plays no role). This yields a subgame perfect (or strongly time consistent) open-loop control, although the game is not state-linear. Indeed,

Lemma 1. The VLV Cournot game admits a state redundant solution.

Figure 2 The phase diagram under Cournot competition, $\delta < \rho$



This property of the VLV model was originally pointed out by Goh *et al.* (1974).[2] Relying on Lemma 1, we may proceed with the characterisation of the scenario associated with the subgame perfect Cournot-Nash output $q^{CN} = a/\left(n + 1 + 2c\right)$, invariant in the resource stock. The relevant phase diagram is the same as in Figure 2, except for the absence of X^R. Since $\partial Q^{CN}/\partial n > 0$, we have $\partial X_-^{CN}/\partial n > 0$ and $\partial X_+^{CN}/\partial n < 0$. More importantly, if $\delta\left(n + 1 + 2c\right) - 4\beta a n = 0$, the locus Q^{CN} is tangent to the concave locus describing the undisturbed growth rate of the resource. Should this happen (and it may do so in correspondence of infinitely many values of the parameter set $\{\beta, \delta, a, c, n\}$), then industry harvest would correspond to the maximum sustainable yield $X_{MSY} \overset{def}{=} 1/\left(2\beta\right)$. The tangency solution is half-stable, as Figure 3 shows: X_{MSY} is attracting for $X_0 > X_{MSY}$, while it is repelling for $X_0 \in \left(0, X_{MSY}\right)$. However, note that $X_\pm^{CN} = X_{MSY}$ at $n_{MSY} = \delta\left(1 + 2c\right)/\left(4\beta a - \delta\right) > 0$ for all $a > \delta/\left(4\beta\right)$, and $n_{MSY} \geq 1$ for all $a \leq \left(1 + c\right)\delta/\left(2\beta\right)$, with $\left(1 + c\right)\delta/\left(2\beta\right) > \delta/\left(4\beta\right)$.

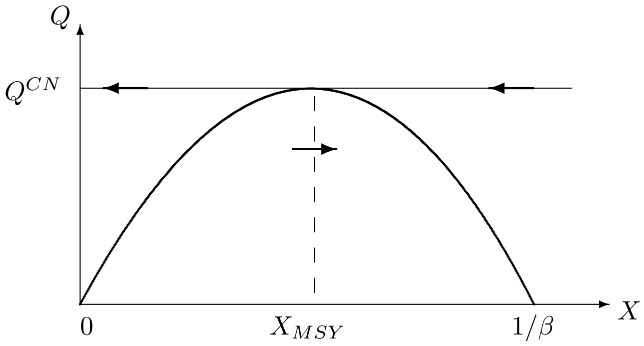
Accordingly, we may formulate the following

Proposition 2. Let $\widetilde{n}$ be the largest integer which does not exceed n_{MSY}, and let Q_{MSY} be the harvest rate corresponding to X_{MSY}. Then, for all $a \in \left(\delta/\left(4\beta\right), \left(1 + c\right)\delta/\left(2\beta\right)\right)$, (i) the industry harvest rate $Q^{CN}|_{n=\widetilde{n}}$ is as close as possible to Q_{MSY}, and (ii) the resulting steady state stock give by $X_+^{CN}|_{n=\widetilde{n}}$ is stable.

Hence, the regulator may drive the industry harvest close to the MSY by limiting access to the commons at $\widetilde{n}$, which, if n_{MSY} is an integer, will be equal to $n_{MSY} - 1$. The above proposition says that there exists an intermediate range of values of the choke price a such that the maximum sustainable yield is attained by an admissible industry structure (i.e., at least in monopoly).

[2] See also Leitmann (1973); for more on state-redundant games, see Dockner *et al.*, (2000).

Figure 3 The maximum sustainable yield under Cournot competition



Before proceeding to the analysis of the impact of perfect competition, it is worth observing that the Cournot oligopoly reproduces a perfectly competitive industry in the limit, as n tends to infinity under free entry, but does so with a downward sloping demand function - which is not infinitely elastic - and an endogenous market price. These are typically not the assumptions adopted to describe a perfectly competitive industry in which firms exert no control whatsoever on price. Moreover, if the number of firms becomes infinitely large, the Cournot-Nash equilibrium tends to coincide with the evolutionary stable equilibrium (see, e.g., Weibull, 1995). However, in both cases, $\lim_{n\to\infty} Q^{CN} = a$, which implies that the horizontal line at Q^{CN} lies above the concave locus and therefore the resource is bound to extinction if indeed perfect competition obtains as the limit of a Cournot game where firms' strategies determine market price.

3 Perfect competition

Now we turn to the case in which the n firms are perfectly competitive and behave as price-takers, which is the scenario examined in the debate from Smith (1969) to Berck and Perloff (1984) and summarised in Pearce and Turner (1989). The demand function is infinitely elastic and the price $p > 0$ is exogenous and time-invariant. Consequently, firm i's instantaneous profit function is $\pi_i(t) = [p - cq_i(t)]\, q_i(t)$. The current value Hamiltonian function is defined as in (2), but the presence of a constant market price modifies the FOC taken w.r.t. individual harvest as follows:

$$\frac{\partial \mathcal{H}_i}{\partial q_i} = p - 2cq_i - \lambda_i = 0 \qquad (8)$$

From a qualitative standpoint, the analysis replicates the one outlined in section 2, whereby, if λ_i solves (8), we have the Ramsey rule and a market-driven solution. For brevity, we shall skip the details of this case, which can be easily replicated, to focus on the scenario in which $\lambda_i = 0$ always. As a result, the perfectly competitive individual harvest solving (8) is $q^{PC} = p/(2c)$. This generates competitive equilibrium profits $\pi^{pc} = p^2/(4c)$ at all times, independently of the number of firms, provided the resource stock remains positive forever. The steady state stock associated with q^{PC} is

$$X_{\pm}^{pc} = \frac{\sqrt{\delta c} \pm \sqrt{\delta c - 2\beta np}}{2\beta\sqrt{\delta c}} \in \mathbb{R}^{+}\, \forall\, p < \frac{c\delta}{2\beta n} \qquad (9)$$

This condition intuitively says that the resource will not extinguish if market price is not excessively high. The phase diagram is analogous to that in Figure 2. Moreover, as under Cournot behaviour, the steady state characterised by X_{+}^{pc} is stable, while the other is not, for the same reasons. And, once again, $\partial X_{-}^{pc}/\partial n > 0$ and $\partial X_{+}^{pc}/\partial n < 0$.

Looking at (9), it is evident that $X_{\pm}^{pc} = X_{MSY}$ iff $\delta c - 2\beta np = 0$. This gives the regulator an additional degree of freedom, as the price can be maneuvered to drive X_{+}^{pc} arbitrarily close to X_{MSY}, still preserving the stability of the resulting steady state, *for any number of firms*. Indeed, from the policy maker's standpoint, the problem boils down to regulating np (that is, either access to the common pool or price, or both) to minimise the difference between X_{+}^{pc} and X_{MSY}. While in the Cournot setting the integer problem must be explicitly accounted for, here the additional tool offered by the exogenous price opens the possibility of reaching any $X_{+}^{pc} = X_{MSY} + \varepsilon$, with ε positive and arbitrarily small. In this respect, it can be added that, to the best of our knowledge, the earliest contribution studying the possibility of regulating either access (i.e., n) or price in a competitive model is that of Smith (1969), and this aspect has been largely neglected in the following literature. Moreover, regulating price has a similar flavour as taxing emissions produced by the same industry when the problem is posed by pollution instead of resource extraction. The difference lies in the fact that here price can be manipulated by a public authority to force firms to get as close as possible to X_{MSY}, while the Pigouvian tax induces firms to efficiently internalise the marginal cost associated with the environmental externality. [3]

The foregoing discussion can be summarised in the following:

Proposition 3. Under perfect competition, there exist infinitely many pairs (n, p) satisfying $np = \delta c/(2b)$, such that industry harvest equals X_{MSY}.

The joint assessment of Propositions 2-3 deserves some additional remarks. First, one has to consider that, under Cournot competition, the integer problem must explicitly be accounted for, while the presence of an arbitrarily large number of firms in a perfectly competitive

[3] A tax on extraction is used by Karp (1992) as a regulatory tool in an oligopoly exploting a common nonrenewable pool.

industry poses no issue in view of the fact that the public authority can take n as given and just regulate market price. Second, Proposition 2 says that, by regulating access to the common pool, the regulator may drive industry harvest close to the MSY without reaching it. Put differently, the integer problem causes the Cournot-Nash industry harvest Q^{CN} (even under regulated access) to be strictly lower than that corresponding to the MSY. This is not the case under perfect competition, as the integer problem is no obstacle to the regulator's action when the demand is perfectly elastic.

This reveals a flaw in our perception of the impact of competition (or, conversely, market power) on the resource stock in the long run, which has long been debated since Smith (1969). The point is that *free access* and *perfect competition* have been usually taken as synonymous in the literature on common pools' exploitation, while they are not. Free access to the commons implies that there is no upper limit to the number of firms extracting the resource, other than their profit incentives. If we conceive perfect competition as a scenario in which firms have no control on price and demand is flat, then it clearly appears that free access and perfect competition are not the same thing. Indeed, free access may also characterise an oligopoly game whose limit properties under free entry include marginal cost pricing, but this does not apply to the equilibrium configuration of such an industry for any finite number of firms. The foregoing analysis shows that the ultimate consequence of the tragedy of commons, namely, resource exhaustion (or, the extinction of species) can be more easily avoided under perfect competition than in any other less competitive situation in which some population of agents do have a degree of market power and therefore endogenously determine price along a negatively sloped demand function.

4 Extensions

The analysis carried out in section 2 relies on a standard oligopoly model based upon linear demand and quadratic cost functions, but its main results stretch beyond the limits of these assumptions. Relying, *inter alia*, on Novshek (1980) and Dixit (1986), suppose the inverse demand function is $p(Q)$, with $\partial p(Q)/\partial Q < 0$, and the individual cost function is $C_i(q_i)$, with $\partial C_i(q_i)/\partial q_i \geq 0$ and $\partial^2 C_i(q_i)/\partial q_i^2 \geq 0$. Thus, firm i's profit function is $\pi_i = p(Q)q_i - C_i(q_i)$, in which case it is easily proved that the state-redundancy property holds, since $\lambda_i = 0$ for all i at all times.

A specific example can be figured out by looking at the second order condition for the concavity of π_i:

$$2\frac{\partial p(Q)}{\partial q_i} + q_i \frac{\partial^2 p(Q)}{\partial q_i^2} \leq \frac{\partial^2 C_i(q_i)}{\partial q_i^2} \tag{10}$$

This admits the case of a strongly convex inverse demand in each output, for which $\partial^2 p(Q)/\partial q_i^2 > 0$. For instance, if demand is hyperbolic, with $p = a/Q$, and the cost function is the same as above, (10) is satisfied everywhere and the individual Cournot-Nash output is $q^{CN} = \sqrt{a(n-1)/(2cn)}$ which, as is known from the debate on oligopoly models with hyperbolic demand, implies excluding the case $n = 1$ because the monopoly optimum is not determined if the demand function is isoelastic (see, e.g., Lambertini, 2010). The corresponding stable steady state level of the stock is

$$X_+^{CN} = \frac{c\delta + \sqrt{c\delta\left(c\delta - 2\beta\sqrt{2ac(n-1)}\right)}}{2c\beta\delta} \tag{11}$$

with $X_+^{CN} \in \mathbb{R}^+ \,\forall\, a \leq c\delta^2/\left[8(n-1)\beta^2\right]$. When met at the margin, this condition ensures the attainment of $X_+^{CN} = X_{MSY}$ by $n_{MSY} = 1 + c\delta^2/\left(8a\beta^2\right)$ firms, and suffices for $n_{MSY} \geq 2$. In the aforementioned parameter range, the regulator may control access to the commons in such a way that profit-seeking behaviour brings this oligopoly as close as possible to X_{MSY}.

As for perfect competition, what matters is again that in such a case demand becomes infinitely elastic. Hence, the results derived in section 3 hold whenever $\partial \pi_i(\cdot)/\partial X = 0$ for all $i = 1, 2, ...n$ and are qualitatively robust to the adoption of any cost function $C_i(q_i)$, as long as it is independent of the state; that is, the sufficient condition for the above results to obtain is $\partial C_i(\cdot)/\partial X = 0$ for all $i = 1, 2, ...n$, provided second order conditions are met.

5 Concluding remarks

Our reformulation of the VLV model has shown that properly distinguishing between free access to the commons and perfect competition offers the possibility of identifying a policy based on price regulation inducing a perfectly competitive industry to harvesting arbitrarily close to the MSY. The analogous approach to the same problem under Cournot competition has highlighted that the integer problem matters in a strategic oligopoly, preventing the regulator to replicate the same outcome. This is due to the endogeneity of price when firms have market power, obliging the public authority to explicitly regulate access to the commons.

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