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Invariance and Controlled Invariance in Switching Structured Systems with Application to Disturbance Decoupling

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Abstract. In this paper, we consider dynamical systems where the graph of the relations between state, input and output variables switches between different configurations, according to the action of a switching time signal. Moreover, in each configuration the relations between the variables are known only for being zero or nonzero. Switching structured systems of this kind are described by families of simple, directed graphs. They can be used to model complex networks of systems as well as systems of systems for which the only available information consists in the patterns that the set of the interconnections between the components, or agents, may assume in different situations. Using an approach that is conceptually similar to the geometric approach developed for linear time-invariant systems, suitable notions of invariance and controlled invariance are introduced and related to the action of feedback. These notions are used to provide general solvability conditions for the disturbance decoupling problem expressed in graph-theoretic terms.

1. Introduction

In dealing with dynamical systems, one is often interested in properties that depend on the existence of relations between internal and/or external variables, namely on the structure of the system, rather than in properties that depend on the actual values that specify the relations, namely on the coefficients of the systems. In that case, one speaks of *structured system* and represents them by using simple, directed graphs, whose vertices symbolize the state variables of the system and whose edges symbolize nonzero relations between them. The system properties that are expressed by graph-theoretic characteristics are called *structural* or, since they hold for any value of the nonzero parameters of the system, *generic*. Complex dynamical objects, like networks of systems and systems of systems, are conveniently modeled as structured systems by associating the vertices of the graph to the agents of the network or to the components of the overall system. Thus, features are pointed out that depend on the influence that each agent or component exerts on the others, rather than on the individual dynamics of each of them.

The study of linear structured systems was initiated in [1] and, since then, it has received contributions by many authors. A comprehensive survey of classic results dates back to the early 2000s [2]. How to model and to study complex systems, like networked systems and



systems of systems, by means of structured systems is the object of later papers, like [3], [4], [5], [6], [7], [8], [9], [10], [11] and the references therein.

In a recent paper [12], the authors introduced the notions of controlled invariance and conditioned invariance for a class of structured systems characterized by the fact that each input channel, as well as each output channel, only affects one single vertex of the representing graph. Structured systems of that kind are considered, e.g., in [13], [14], [15], [16], [17] and in some examples of [3]. In the paper mentioned above [12], the novel definition of *essential* state feedback has allowed the notion of controlled invariance for the considered class of structured systems to be characterized in terms of dynamics. Consequently, necessary and sufficient conditions for solvability of the disturbance decoupling problem, either by state or by output feedback, have been established.

Here, we consider structured systems whose structure can vary over the time, thus causing, for instance, a nonzero relation between two variables to vanish or a new relation between previously unrelated variables to appear, as specified by a switching time signal. A system of this kind, that we call *switching structured system*, is modeled by a finite indexed family of structured systems (called *modes*), whose graphs have the same set of vertices, and by a switching time signal, which is assumed to be a piece-wise constant function from \mathbb{R}^+ to the set of indices of the family of graphs. Switching structured systems can be used to describe and to study complex dynamical behaviours that are due, for instance, to the action of switches in electrical circuits, to modifications in the topology of communication networks, to variations of the team in multi-agent systems. Examples are given at the end of Section 2.

By introducing a suitable, novel notion of *switching controlled invariance*, we develop a graph-theoretic approach to analysis and synthesis problems for switching structured systems, conceptually akin to the geometric approach for classical linear systems of [19] and [20]. In particular, we show that controlled invariance can be interpreted in terms of feedbacks whose action modifies the modes of the switching structured system at issue. This fact leads to the main theoretic contribution of this paper: the notion of *switching essential state feedback*, defined in graph-theoretic terms.

The developed approach can be used to study a number of control problems that, in particular, include disturbance decoupling problems by means of state feedback. In particular, using controlled invariance, we can state necessary and sufficient solvability conditions that are analogous to those provided in the framework of classical linear systems by the geometric approach. This characterization of the solvability of the disturbance decoupling problem, together with the procedure to construct a solution (if it exists), is the second contribution of this work.

The paper is organized as follows. In Section 2, we introduce the class of switching structured systems considered and we describe their graph representation. In Section 3, we introduce the fundamental notions of switching invariance and switching controlled invariance in graph-theoretic terms. Then, we define the notion of essential switching feedback and we show how it can be used to characterize switching controlled invariance. In Section 4, we study the disturbance decoupling problem by means of state feedback and we characterize its solvability by means of necessary and sufficient conditions. Proofs of our results will appear in complete form elsewhere. Section 5 contains conclusions and description of future work.

Notation Given two sets A and B , we will denote by $A \setminus B$ the set defined by $A \setminus B = \{a \in A, \text{ such that } a \notin B\}$ and by $A \parallel B$ the set defined by $A \parallel B = (A \cup B) \setminus (A \cap B)$.

2. Preliminaries

Let $H = \{1, \dots, \bar{h}\}$ be a finite set of indices and consider the set $\Gamma = \{(G, \mathcal{E}_h)\}_{h \in H}$, where, for any $h \in H$, (G, \mathcal{E}_h) is a simple directed graph (i.e. a directed graph without multiple edges and without auto-loops) with a finite set of vertices $G = \{v_1, \dots, v_n\}$ and set of edges $\mathcal{E}_h \subseteq G \times G$.

Note that all graphs in Γ have the same set of vertices and different sets of edges. If $(v_j, v_i) \in \mathcal{E}_h$, for some $h \in H$, we say that v_j is the tail and v_i is the head of the edge $(v_j, v_i) \in \mathcal{E}_h$. A path P in (G, \mathcal{E}_h) is an ordered finite sequence of edges (e_1, \dots, e_m) , all belonging to \mathcal{E}_h , in which the head of the edge e_k coincides with the tail of the edge e_{k+1} . The tail of the first edge in a path P is called the tail of the path and the head of the last edge is called the head of the path.

To each of the graphs $(G, \mathcal{E}_h) \in \Gamma$, we associate an $n \times n$ matrix $A_h = [a_{ij}]$ whose entries are real, mutually independent parameters that satisfy the following conditions

- $a_{ij}^h \neq 0$ for $i \neq j$ if and only if $(v_j, v_i) \in \mathcal{E}_h$ (i.e. there is an edge from v_j to v_i in (G, \mathcal{E}_h)).

Note that no condition is imposed on a_{ij}^h for $i = j$.

Letting $G_h^{in} = \{v_{h1}, \dots, v_{hm_h}\} \subseteq G$, for any $h \in H$, be a subset of vertices, we associate to the pair $((G, \mathcal{E}_h), G_h^{in})$ an $n \times m_h$ matrix $B_h = [b_{ij}^h]$ whose entries are real, mutually independent parameters that satisfy the following conditions

- $b_{ij}^h = 1$ (or more generally $b_{ij}^h \neq 0$) if $v_{hj} = v_i$ (that is: if the j -th element v_{hj} of G_h^{in} is equal to the i -th element v_i of G)
- $b_{ij}^h = 0$ otherwise.

Note that in any column of B_h there is just one entry different from 0, while in any row of B_h there is at most one entry different from 0.

Letting $G_h^{out} = \{v_{h1}, \dots, v_{hp_h}\} \subseteq G$, for any $h \in H$, be a subset of vertices, we associate to the pair $((G, \mathcal{E}_h), G_h^{out})$ a $p_h \times n$ matrix $C_h = [c_{ij}^h]$ whose entries are real, mutually independent parameters that satisfy the following conditions

- $c_{ij}^h = 1$ (or more generally $c_{ij}^h \neq 0$) if $v_{hj} = v_i$ (that is: if the j -th element v_{hj} of G_h^{out} is equal to the i -th element v_i of G)
- $c_{ij}^h = 0$ otherwise.

Note that in any row of C_h there is just one entry different from 0, while in any column of C_h there is at most one entry different from 0.

In representing graphically the triplet $((G, \mathcal{E}_h), G_h^{in}, G_h^{out})$ for a given $h \in H$, we use arrows between vertices to indicate edges belonging to \mathcal{E}_h and we mark the elements of G_h^{in} by ingoing arrows and the elements of G_h^{out} by outgoing arrows, as in Figure 1 (where $G = \{v_1, \dots, v_5\}$, $\mathcal{E}_h = \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_3, v_2), (v_4, v_3), (v_4, v_5)\}$, $G_h^{in} = \{v_1, v_2\}$, $G_h^{out} = \{v_5\}$).

Let \mathcal{S} denote the set of piece-wise constant, left-continuous functions $\sigma : \mathbb{R}^+ \rightarrow H$ with a finite number of discontinuities in any interval. The **switching structured system** $\Sigma_\sigma((G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{out})$ associated to the set of triplets $\{(G, \mathcal{E}_h), G_h^{in}, G_h^{out}\}$, $h \in H$ is the switching linear time-invariant system described in parametric state-space form by the equation

$$\Sigma_\sigma \equiv \begin{cases} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) \end{cases} \quad (1)$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $y \in \mathbb{R}^p$. The structured linear systems

$$\Sigma_h \equiv \begin{cases} \dot{x}(t) &= A_hx(t) + B_hu(t) \\ y(t) &= C_hx(t) \end{cases} \quad \text{with } h \in H$$

are the modes of the switching structured system Σ_σ and $\sigma(t)$ represents the time signal that governs the switching from one mode to another. Generally, in dealing with structured switching systems, we are interested in features and properties that hold for all $\sigma(t) \in \mathcal{S}$.

Switched structured systems can be used to model complex dynamical structures, like networks of systems or systems of systems, that can switch between different configurations. In that case, the vertices of the graph G represent the systems, or dynamical agents, of the network and the edge $(v_j, v_i) \in \mathcal{E}_h$ represents a link, whose weight is expressed by the parameters a_{ij}^h , that account

for the way in which the (state of the) agent v_j affects the (dynamics of the) agent v_i when the network is in the configuration corresponding to the index h . The dynamical structure modeled by Σ_σ switches between different configurations according to the time signal $\sigma(t)$ and, as a consequence, some links may disappear and new ones may appear, accounting for a modification of the relationship between agents. The vertices in G_h^{in} and those in G_h^{out} correspond to agents that, respectively, can be influenced by inputs coming from the external environment or that send outputs to the external environment when the system is in the configuration corresponding to the index h .

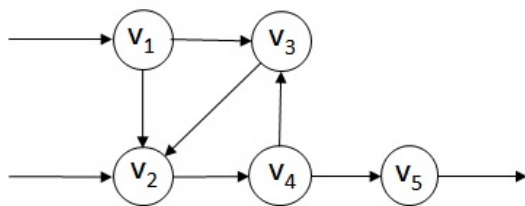


Figure 1: Graphical representation of a triplet $((G, \mathcal{E}_h), G_h^{in}, G_h^{out})$.

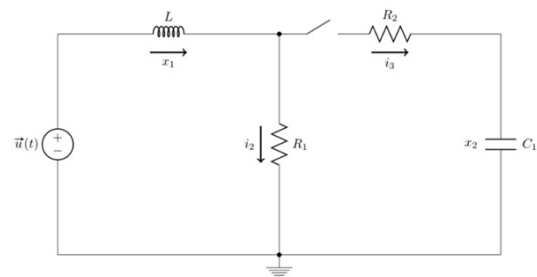


Figure 2: An electrical circuit with one switch.

Example 1 Consider the electrical circuit of Figure 2, whose configuration varies according to the action of the switch. Choosing the current through the inductor L and the voltage across the capacitor C_1 as the state variables $x_1(t)$ and $x_2(t)$, respectively, and the output $y(t)$ equal to the current through the inductor L if the switch is open and equal to the voltage across the capacitor C_1 if the switch is closed, we get the following equations for the two configurations:

$$\Sigma_1 \equiv \begin{cases} \dot{x}_1(t) &= \frac{-R_1}{L}x_1(t) + \frac{1}{L}u(t) \\ \dot{x}_2(t) &= 0 \\ y(t) &= x_1(t) \end{cases} \quad \Sigma_2 \equiv \begin{cases} \dot{x}_1(t) &= \frac{-R_1}{L(R_1+R_2)}(R_2x_1(t) + x_2(t)) + \frac{1}{L}u(t) \\ \dot{x}_2(t) &= \frac{1}{C_1(R_1+R_2)}(R_1x_1(t) - x_2(t)) \\ y(t) &= x_2(t) \end{cases}$$

Correspondingly, we have the switching structured system $\Sigma_\sigma((G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{in})$, whose two modes have the graphical representation given in Figure 3

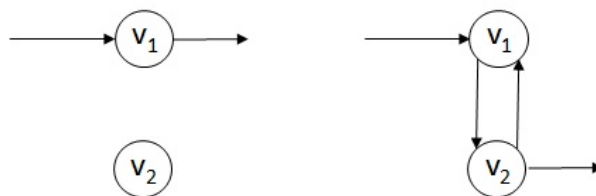


Figure 3: Modes of the system $\Sigma_\sigma((G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{in})$ that model the circuit of Figure 2.

Example 2 Consider a cyber-physical system consisting of agents that influence each other by exchanging information through a communication network. If some links of the communication network may be open or closed over a time interval, the system takes different configurations and, without specifying its parameters, it can be conveniently modeled by a switching structured system. Take, for instance, a set of three agents, whose dynamics is one-dimensional, and assume that there are one-directional communication links from the agent v_1 to the agents

v_2 and v_3 and from the agent v_2 to the agent v_3 . Assume that only one of the last two communication links can be disabled at one time, otherwise they are both enabled; that only agent v_1 get inputs from the external environment and only agent v_2 sends outputs to the external environment. Then, the overall system can be described as $\Sigma_\sigma(((G, \mathcal{E}_\sigma), G_h \sigma in, G_\sigma^{out}))$, where $H = \{1, 2, 3\}$, $G = \{v_1, v_2, v_3\}$, $\mathcal{E}_1 = \{(v_1, v_2), (v_1, v_3), (v_3, v_2)\}$, $\mathcal{E}_2 = \{(v_1, v_2), (v_3, v_2)\}$, $\mathcal{E}_3 = \{(v_1, v_2), (v_1, v_3)\}$, $G_1^{in} = G_2^{in} = G_3^{in} = \{v_1\}$, $G_1^{out} = G_2^{out} = G_3^{out} = \{v_2\}$. The switching signal σ determines if both the communication links between v_1 and v_3 and between v_3 and v_2 are enabled (e.g. $\sigma(t) = 1$), if only the first one is disabled (e.g. $\sigma(t) = 2$) or if only the second one is disabled (e.g. $\sigma(t) = 3$). The triplets $((G, \mathcal{E}_h), G_h^{in}, G_h^{out})$ for $h = 1, 2, 3$ are represented in Figure 4. The modes Σ_1, Σ_2 and Σ_3 of the systems have, respectively, the following equations:

$$\Sigma_1 \equiv \begin{cases} \dot{x}_1(t) &= a_{11}^1 x_1 + b_{11}^1 u_1(t) \\ \dot{x}_2(t) &= a_{21}^1 x_1 + a_{22}^1 x_2(t) + a_{32}^1 x_3(t) \\ \dot{x}_3(t) &= a_{31}^1 x_1 + a_{33}^1 x_3(t) \\ y(t) &= c_{11}^1 x_3(t) \end{cases} \quad \Sigma_2 \equiv \begin{cases} \dot{x}_1(t) &= a_{11}^2 x_1 + b_{11}^2 u_1(t) \\ \dot{x}_2(t) &= a_{21}^2 x_1 + a_{22}^2 x_2(t) + a_{23}^2 x_3(t) \\ \dot{x}_3(t) &= a_{33}^2 x_3(t) \\ y(t) &= c_{11}^2 x_3(t) \end{cases} \quad \Sigma_3 \equiv \begin{cases} \dot{x}_1(t) &= a_{11}^3 x_1 + b_{11}^3 u_1(t) \\ \dot{x}_2(t) &= a_{21}^3 x_1 + a_{22}^3 x_2(t) \\ \dot{x}_3(t) &= a_{31}^3 x_1 + a_{33}^3 x_3(t) \\ y(t) &= c_{11}^3 x_3(t) \end{cases}$$

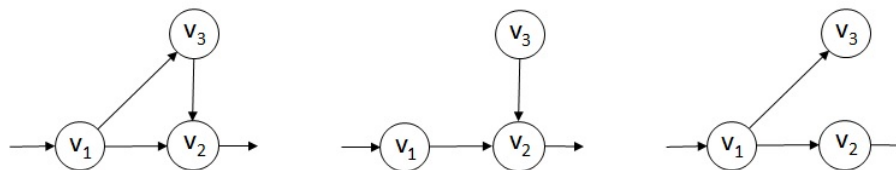


Figure 4: Graphical representation of the triplets $((G, \mathcal{E}_h), G_h^{in}, G_h^{out})$ for $h = 1, 2, 3$.

Example 3 The glucose metabolism in the human body can be represented as a switching structured system which is triggered by the external signal consisting of ingestion of carbohydrates as meals and snacks and which aims at reaching a target glycemmic level. As depicted in Figure 5, the core of glucose homeostasis consists in the interactions between the liver and the pancreas. The liver acts a glucose reservoir which releases glucose in blood plasma through some glycogenesis process during fasting periods, like overnight periods. During prandial and postprandial periods, the liver "refills" the glucose reservoir as the full amount of carbohydrates is not consumed in real-time. The pancreas produces essentially either insulin during prandial and postprandial periods, or glucagon during fasting periods to regulate the glucose concentration level in blood plasma. The interaction between the pancreas and the liver can be schematically illustrated as in Figure 6, where the arrows describe how the blood glucose concentration \mathbf{G} , the state of the liver \mathbf{L} , the blood plasma insulinemia \mathbf{I} , the glucagon concentration in the blood plasma \mathbf{A} , the status of the pancreatic alpha cells α and the status of the pancreatic beta cells β influence each other. The input CHO represents the disturbance caused by carbohydrates ingestion. As shown by the basic glucose-insulin-glucagon model of [18], the overall regulation mechanism splits into two actions, which are effective either during the prandial and postprandial periods, when the beta cells produce a higher rate of insulin, or during fasting periods, when alpha cells produce glucagon to stimulate glycogen release by the liver. We have therefore the two modes graphically described in Figure 7: in case of hypoglycemia (G is below the target glycemmic level), the glucagon secretion is maximal, with an insulin secretion close to zero (Figure 7 on the left); in case of hyperglycemia (G exceeds the target glycemmic level) after ingestion of carbohydrates, the insulin secretion will be maximal, with a glucagon secretion close to zero (Figure 7 on the right).

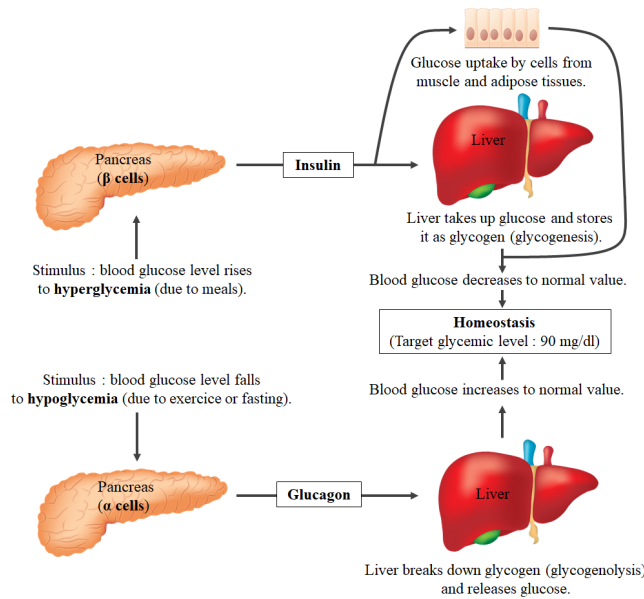


Figure 5: Glucose metabolism.

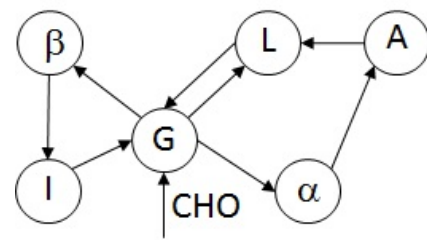


Figure 6: Functional scheme of the glucose metabolism.

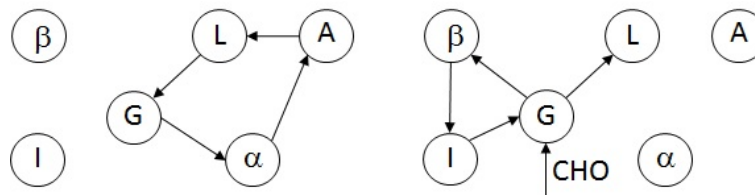


Figure 7: Modes of the system representing the glucose metabolism: hypoglycemia case (left) and hyperglycemia case (right).

3. Invariance and controlled invariance

In this section, we extend to switching structured system the geometric notions that, in accordance with the approach developed for linear systems in [19] and [20], characterize invariance and controlled invariance for structured system (see [2], [12]).

In order to proceed, we introduce the following notions. Given a switching time signal $\sigma(t) \in \mathcal{S}$, we denote by $|\sigma(t)| = (h_1, h_2, \dots)$, with $h_k \in H$ and $h_k \neq h_{k+1}$, the ordered sequence of different values that are assumed by $\sigma(t)$ as t goes from 0 to $+\infty$. In practice, $|\sigma(t)|$ describes the sequence of modes that the system $\Sigma_\sigma((G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{out})$ goes across when the implemented switching time signal is $\sigma(t)$.

Definition 1 Given a structured switching system $\Sigma_\sigma((G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{out})$, a **switching path** P_s in (G, \mathcal{E}_σ) is an ordered finite sequence of edges $(e_{h_1}, e_{h_2}, \dots, e_{h_m}) \in \mathcal{E}_{h_1} \times \dots \times \mathcal{E}_{h_m}$, in which the head of the edge e_{h_k} coincides with the tail of the edge $e_{h_{k+1}}$. The tail of the first edge in a switching path P_s is called the tail of the switching path and the head of the last edge is called the head of the switching path.

A switching path P_s in (G, \mathcal{E}_σ) , with $P_s = (e_{h_1}, e_{h_2}, \dots, e_{h_m}) \in \mathcal{E}_{h_1} \times \dots \times \mathcal{E}_{h_m}$, indicates that the agent represented by the tail of P_s , say v_1 , influences the dynamics of the agent represented by the head of P_s , say v_m , if the switching of Σ_σ is governed by a time signal $\bar{\sigma}$ such that $|\bar{\sigma}| = (\dots, h_1, h_2, \dots, h_m, \dots)$.

In a way conceptually similar to that followed in [12], we can introduce in our framework a suitable notions of **switching invariance** and **switching controlled invariance** as follows.

Definition 2 Given a switching structured system $\Sigma_\sigma((G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{out})$, a subset $V \subseteq G$ of vertices is said to be **switching invariant** for Σ_σ if, for any $h \in H$, $(v_j, v_i) \in \mathcal{E}_h$ with $v_j \in V$ implies $v_i \in V$.

Definition 3 Given a structured system $\Sigma_\sigma((G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{out})$, a subset $V \subseteq G$ of vertices is said to be **switching controlled invariant** for Σ_σ if, for any $h \in H$, $(v_j, v_i) \in \mathcal{E}_h$ with $v_j \in V$ implies $v_i \in V \cup G_h^{in}$.

The structural notion of controlled invariance can be characterized in dynamical terms by using a suitable notion of feedbacks. Following the approach developed in [12], we focus on switching state feedbacks that, acting on each mode of a given switching structured system $\Sigma_\sigma(G, \mathcal{E}_h), G_h^{in}, G_h^{out}$, modify the dynamics expressed by its underlying graph. Remarking that the action of a state feedback on a mode $\Sigma_h((G, \mathcal{E}_h), G_h^{in}, G_h^{out})$ may only involve the relationship between any component of the state, i.e any element of G , and those components whose dynamics is directly affected by the inputs, i.e any element of G_h^{in} , we recall the definition below from [12].

Definition 4 Given the structured system $\Sigma_h((G, \mathcal{E}_h), G_h^{in}, G_h^{out})$, an **essential state feedback** consists of a subset $\mathcal{F}_h \subset G \times G_h^{in}$ such that $(v_j, v_i) \in \mathcal{F}_h$ implies $i \neq j$. The action of an essential state feedback \mathcal{F}_h on the structured system $\Sigma_h((G, \mathcal{E}_h), G_h^{in}, G_h^{out})$ gives rise to the compensated switching structured system $\Sigma_h^{comp}((G, \mathcal{E}_h \parallel \mathcal{F}_h), G_h^{in}, G_h^{out})$, where $\mathcal{E}_h \parallel \mathcal{F}_h = (\mathcal{E}_h \cup \mathcal{F}_h) \setminus (\mathcal{E}_h \cap \mathcal{F}_h)$.

Now, we can give the following novel definition.

Definition 5 Given the switching structured system $\Sigma_\sigma(G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{out}$, an **essential switching state feedback** \mathcal{F}_σ consists of a set $\{\mathcal{F}_h\}_{h \in H}$, where \mathcal{F}_h is an essential state feedback for $\Sigma_h((G, \mathcal{E}_h), G_h^{in}, G_h^{out})$ for any $h \in H$.

Note that applying an essential switching state feedback \mathcal{F}_σ to $\Sigma_\sigma((G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{out})$ means to modify \mathcal{E}_h (unless $\mathcal{F}_h = \{\emptyset\}$), either by adding new elements of the form (v_j, v_i) with $v_i \in G_h^{in}$ to it or by removing elements of the same form, if present, from it. This modifies the set of switching paths and hence the way in which agents influences each other as the system switches. For any $h \in H$, we associate to the essential switching state feedback \mathcal{F}_h a linear relation of the form $u = F_h x$, where $F_h = [f_{ij}^h]$ is an $m_h \times n$ matrix whose entries are real, mutually independent parameters that satisfy specific conditions as described below. First, let us remark that if (v_j, v_i) belongs to $\mathcal{E}_h \cap \mathcal{F}_h$, then we have that v_i is, for some k , the k -th element of G_h^{in} , i.e. $v_i = v_{hk} \in G_h^{in}$, and, hence, $b_{ik}^h \neq 0$ in the matrix B_h . Now, let us take the parameters f_{ij}^h as follows

- $f_{ij}^h \neq 0$ if and only if $(v_j, v_i) \in \mathcal{F}_h$
- $f_{ij}^h = -a_{ij}^h / b_{ik}^h$ if $(v_j, v_i) \in \mathcal{F}_h$ with $v_i = v_{ik} \in G_h^{in}$ and a_{ij}^h is different from 0.

Note that no condition is imposed on f_{ij}^h if $i = j$ and if, for $i \neq j$ and $(v_j, v_i) \in \mathcal{F}_h$ with $v_i = v_{hk} \in G_h^{in}$, one has $a_{ik}^h = 0$.

With the above choice, the compensated system $\Sigma_\sigma^{comp}((G, \mathcal{E}_\sigma \parallel \mathcal{F}_\sigma), G_\sigma^{in}, G_\sigma^{out})$ turns out to be defined in parametric form by the following set of equations

$$\Sigma_\sigma^{comp} \equiv \begin{cases} \dot{x}(t) &= (A_\sigma + B_\sigma F_\sigma)x(t) + B_\sigma u(t) \\ y(t) &= C_\sigma x(t) \end{cases} \quad (2)$$

Example 4 In the same vein of Example 2, let us consider the switching structured system $\Sigma_\sigma(G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{out}$, where $H = \{1, 2\}$, whose modes are defined by the triplets represented in Figure 8. By applying the switching essential state feedback \mathcal{F}_σ with $\mathcal{F}_1 = \{(v_3, v_1)\}$ and $\mathcal{F}_2 = \{(v_4, v_1)\}$, we obtain the compensated system $\Sigma_\sigma^{comp}(G, \mathcal{E}_\sigma \parallel \mathcal{F}_\sigma), G_\sigma^{in}, G_\sigma^{out}$ whose modes are defined by the triplets represented in Figure 9.

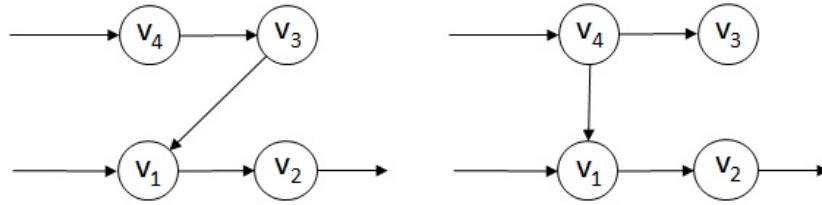


Figure 8: Graphical representation of the triplets $((G, \mathcal{E}_h), G_h^{in}, G_h^{out})$ for $h = 1, 2$.

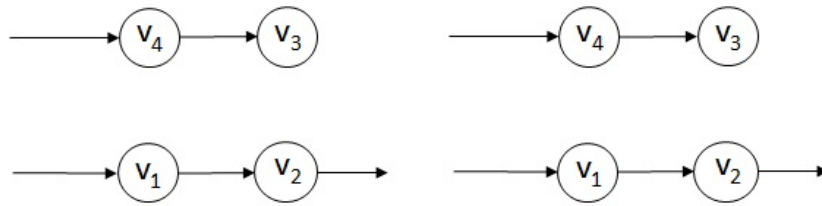


Figure 9: Graphical representation of the triplets $((G, \mathcal{E}_h \parallel \mathcal{F}_h), G_h^{in}, G_h^{out})$ for $h = 1, 2$.

We can now give a basic dynamical characterization of switching controlled invariance that turns out to be formally analogous to the one given for the corresponding notion in the classical linear framework (compare with [19], [20]).

Proposition 1 *Given a switching structured system $\Sigma_\sigma((G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{out})$, a subset of vertices $V \subseteq G$ is switching controlled invariant for Σ_σ if and only if there exists an essential switching state feedback \mathcal{F}_σ such that V is switching invariant for the structured switching compensated system $\Sigma_\sigma^{comp}((G, \mathcal{E}_\sigma \parallel \mathcal{F}_\sigma), G_\sigma^{in}, G_\sigma^{out})$.*

Any essential switching state feedback \mathcal{F}_σ that has the property of making V switching invariant in $\Sigma_\sigma^{comp}((G, \mathcal{E}_\sigma \parallel \mathcal{F}_\sigma), G_\sigma^{in}, G_\sigma^{out})$ is called a **friend** of V .

As switching controlled invariant subspaces in the framework of linear systems, switching controlled invariant subsets of vertices form a semi-lattice with respect to union and inclusion of sets. This property implies the result of the following proposition.

Proposition 2 *Given a switching structured system $\Sigma_\sigma((G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{out})$ and a subset $K \subseteq G$, there exists a maximal subset of vertices V such that $V \subseteq K$ and V is switching controlled invariant for Σ_σ . We denote such subset by $V^*(\mathcal{E}_\sigma, G_\sigma^{in}, K)$ or simply by V^* if no confusion arises.*

Given a structured switching system $\Sigma_\sigma((G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{out})$ and a subset $K \subseteq G$, it is possible to construct $V^*(\mathcal{E}_\sigma, G_\sigma^{in}, K)$ by considering the sequence of subset $V_k \subseteq G$ defined recursively by

$$\begin{aligned} V_0 &= K \\ V_{k+1} &= V_k \setminus \bigcup_{h \in H} \{v_j, \text{ such that } (v_j, v_i) \in \mathcal{E}_h \text{ and } v_i \notin V_k \cup G_h^{in}\}. \end{aligned} \tag{3}$$

The sequence V_k converges to $V^*(\mathcal{E}_\sigma, G_\sigma^{in}, K)$ in at most $r = \text{card}(K)$ steps.

4. Disturbance decoupling by state feedback

A structured switching system subject to disturbance is a system $\Sigma_\sigma((G, \mathcal{E}_\sigma), G_\sigma^{in}, G_\sigma^{out})$ in which G_h^{in} is partitioned as $G_h^{in} = G_h^c \cup G_h^d$ (possibly with $G_c \cap G_d = \emptyset$) for any $h \in H$. Interpreting Σ_σ as a network of agents, G_h^d describes the set of agents that are influenced by a disturbance

input when the network is in the configuration corresponding to the index h , while, in the same situation, G_h^c describes the set of agents that are influenced by a control input. For a structured switching system subject to disturbance we can consider the problem of finding a feedback such that, in the compensated system, the disturbance cannot influence the output. In order to analyze this problem, let us remark that the disturbance input influences the output if and only if there is a switching path P_s in (G, \mathcal{E}_σ) with tail in G_h^d and head in $G_{h'}^{out}$ for some $h, h' \in H$. Accordingly, we state the disturbance decoupling problem in the following way.

Problem 1 *Given a disturbed structured switching system $\Sigma_\sigma((G, \mathcal{E}_\sigma), (G_\sigma^c \cup G_\sigma^d), G_\sigma^{out})$, the **Disturbance Decoupling Problem by State Feedback (DDPSF)** consists in finding an essential switching state feedback \mathcal{F}_σ , if any exists, such that in the compensated system $\Sigma_\sigma^{comp}((G, \mathcal{E}_\sigma \parallel \mathcal{F}_\sigma), (G_\sigma^c \cup G_\sigma^d), G_\sigma^{out})$ there are no switching paths in $(G, \mathcal{E}_\sigma \parallel \mathcal{F}_\sigma)$ with tail in G_h^d and head in $G_{h'}^{out}$ for any $h, h' \in H$.*

Applying the procedure that has been used to derive the system of equations (1), we get the following representation in parametric terms of the disturbed system $\Sigma_\sigma((G, \mathcal{E}_\sigma), (G_\sigma^c \cup G_\sigma^d), G_\sigma^{out})$

$$\Sigma_\sigma \equiv \begin{cases} \dot{x}(t) &= A_\sigma x(t) + B_\sigma u(t) + D_\sigma d(t) \\ y(t) &= C_\sigma x(t) \end{cases} \quad (4)$$

where, the matrices $B_h = [b_{ij}^h]$ and $D_h = [d_{ij}^h]$ have dimensions, respectively, $n \times c_h$ and $n \times d_h$, with $c_h = \text{card}(G_h^c)$ and $d_h = \text{card}(G_h^d)$ and, letting $G_h^c = \{v_{h1}, \dots, v_{hc_h}\}$ and $G_h^d = \{v_{hd_1}, \dots, v_{hd_{d_h}}\}$, their entries satisfies the following conditions

- $b_{ij}^h = 1$ (more generally $b_{ij} \neq 0$) if $v_{hj} = v_i \in$ (i.e. the j -th element of G_h^c is equal to v_i)
- $d_{ij}^h = 1$ (more generally $d_{ij} \neq 0$) if $v_{hj} = v_i \in$ (i.e. the j -th element of G_h^d is equal to v_i)
- $b_{ij} = 0$ and $d_{ij} = 0$ otherwise

with control input $u \in \mathbb{R}^{c_h}$ and disturbance input $d \in \mathbb{R}^{d_h}$.

Solvability of the DDPSF stated above in the framework of structured switching systems means solvability of the problem for all values of the parameters which appear in (4). The following theorem gives necessary and sufficient condition for the solvability of the DDPSF and it indicates how to construct a switching feedback \mathcal{F}_σ that solves it, if any exists.

Theorem 1 *Given a structured switching disturbed system $\Sigma_\sigma((G, \mathcal{E}_\sigma), (G_\sigma^c \cup G_\sigma^d), G_\sigma^{out})$, the associated DDPSF is solvable if and only if the condition $G_h^d \subseteq V^*(\mathcal{E}_\sigma, G_\sigma^c, G \setminus \cup_{i \in I} G_i^{out})$ is satisfied for all $h \in H$.*

Hint of proof Sufficiency. Let \mathcal{F}_σ be a friend of $V^*(\mathcal{E}_\sigma, G_\sigma^c, G \setminus \cup_{i \in I} G_i^{out})$, so that $V^*(\mathcal{E}_\sigma, G_\sigma^c, G \setminus \cup_{i \in I} G_i^{out})$ is switching invariant in $\Sigma((G, \mathcal{E}_\sigma \parallel \mathcal{F}_\sigma), (G_\sigma^c \cup G_\sigma^d), G_\sigma^{out})$. The above condition implies that, for any $h \in H$, any edge in $\mathcal{E}_\sigma \parallel \mathcal{F}_\sigma$ with tail in G_h^d has its head in $V^*(\mathcal{E}_\sigma, G_\sigma^c, G \setminus \cup_{i \in I} G_i^{out})$ and, therefore, in $G \setminus \cup_{i \in I} G_i^{out}$. This implies that there are no switching paths in $(G, \mathcal{E}_\sigma \parallel \mathcal{F}_\sigma)$ with tail in G_h^d and head in $G_{h'}^{out}$ for any $h, h' \in H$.

Necessity. Let \mathcal{F}_σ be a solution of the DDPSF and consider the largest switching invariant $V \subseteq G \setminus \cup_{i \in I} G_i^{out}$ for the compensated system $\Sigma((G, \mathcal{E}_\sigma \parallel \mathcal{F}_\sigma), (G_\sigma^c \cup G_\sigma^d), G_\sigma^{out})$. Clearly, $G_h^d \subseteq V$ for all $h \in H$ and V is switching controlled invariant for $\Sigma_\sigma((G, \mathcal{E}_\sigma), (G_\sigma^c \cup G_\sigma^d), G_\sigma^{out})$. The conclusion follows by maximality of $V^*(\mathcal{E}_\sigma, G_\sigma^c, G \setminus \cup_{i \in I} G_i^{out})$. \square

Example 5 *Consider the simple example provided by the disturbed structured switching systems $\Sigma_\sigma((G, \mathcal{E}_\sigma), (G_\sigma^c \cup G_\sigma^d), G_\sigma^{out})$, with $H = \{1, 2\}$, whose modes are defined by the triplets represented in Figure 10. Computations show that $G_h^d \subseteq V^*(\mathcal{E}_\sigma, G_\sigma^c, G \setminus \cup_{i \in I} G_i^{out})$. Therefore, the condition of Theorem 1 is satisfied. The switching state feedback $\mathcal{F}_\sigma = \{F_1, F_2\}$, with $F_1 = \{(v_1, v_c)\}$ and $F_2 = \{(v_d, v_c)\}$, gives rise to the compensated systems $\Sigma_\sigma^{comp}((G, \mathcal{E}_\sigma \parallel \mathcal{F}_\sigma), (G_\sigma^c \cup G_\sigma^d), G_\sigma^{out})$, whose modes are defined by the triplets represented in Figure 11 and it solves the DDPSF.*

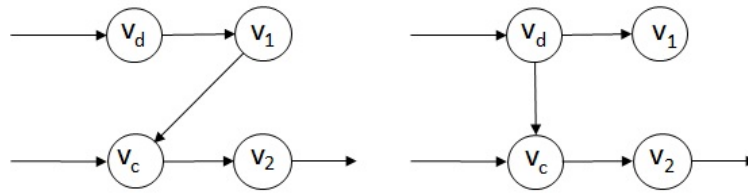


Figure 10: The modes of $\Sigma_\sigma((G, \mathcal{E}_\sigma), (G_\sigma^c \cup G_\sigma^d), G_\sigma^{out})$ for $h = 1, 2$.

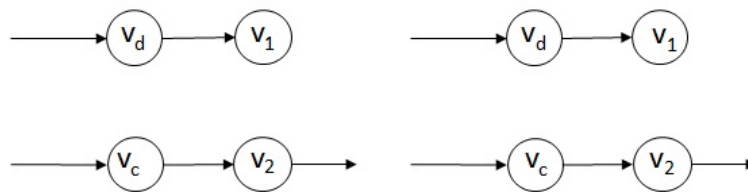


Figure 11: The modes of $\Sigma_\sigma^{comp}((G, \mathcal{E}_\sigma \parallel \mathcal{F}_\sigma), (G_\sigma^c \cup G_\sigma^d), G_\sigma^{out})$ for $h = 1, 2$.

5. Conclusions

A novel approach based on the notion of controlled invariance and of essential feedbacks has been developed for the class of structured systems whose structure varies according to a switching time signal. This approach makes possible to characterize solvability of classical disturbance decoupling problems and to construct solutions. Future work will aim at exploiting this approach in other non-interacting control problems for switching structured systems and in developing specific applications to complex networked systems and systems of systems.

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