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(Article begins on next page)

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Abstract

Several properties of stationary subdivision schemes are nowadays well understood. In particular, it is known that the polynomial generation and reproduction capability of a stationary subdivision scheme is strongly connected with sum rules, its convergence, smoothness and approximation order. The aim of this paper is to show that, in the non-stationary case, exponential polynomials and approximate sum rules play an analogous role of polynomials and sum rules in the stationary case. Indeed, in the non-stationary univariate case we are able to show the following important facts: i) reproduction of N exponential polynomials implies approximate sum rules of order N; ii) generation of N exponential polynomials implies approximate sum rules of order N, under the additional assumption of asymptotical similarity and reproduction of one exponential polynomial; iii) reproduction of an N-dimensional space of exponential polynomials and asymptotical similarity imply approximation order N; iv) the sequence of basic limit functions of a non-stationary scheme reproducing one exponential polynomial converges uniformly to the basic limit function of the asymptotically similar stationary scheme.

Keywords: Subdivision schemes; exponential polynomial generation and reproduction; asymptotical similarity; approximate sum rules; approximation order.

AMS (MOS) Subject Classification. 65D17, 65D15, 41A25, 41A10, 41A30

1. Introduction

In this paper we investigate theoretical properties of non-stationary subdivision schemes. In particular, we study their approximation order and the role played by approximate sum rules, a non-stationary extension of the well-known notion of sum rules. The obtained results allow us to point out similarities and differences between the stationary and the non-stationary cases.

For unfamiliar readers, we briefly introduce subdivision schemes as efficient tools to design smooth curves and surfaces out of sequences of initial points. As a matter of fact, a subdivision curve or surface is obtained as the limit of an iterative procedure based on the repeated application of local refinement rules generating denser and denser sets of points starting from a coarse initial set roughly describing the desired limit shape [3, 20, 42]. In practical use, however, only a limited number of iterations are needed. As a consequence, subdivision schemes are very efficient if compared with traditional parametric curve and surface representations. They also stand out for ease of implementation and versatility in building free-form surfaces of arbitrary topological genus. All these advantages are the reasons for the overwhelming development of subdivision methods and their increasing use in many application areas such as computer–aided geometric design [20, 42], curve and surface reconstruction [35], wavelets and multiresolution analysis [16], signal/image processing [15, 39],

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computer games and animation [17]. Within the variety of subdivision methods studied in the literature, the class of non-stationary subdivision schemes is currently receiving great attention. This is due to the fact that non-stationary subdivision schemes are general and flexible enough to overcome the restricted capabilities of stationary subdivision schemes. As an example, we can think of the fact that stationary subdivision schemes are not capable of representing conic sections or, in general, exponential polynomials. On the contrary, non-stationary schemes can also generate exponential polynomials or exponential B-splines, that is piecewisely defined exponential polynomials [1, 6, 11, 21, 26, 30, 33, 34, 37, 38]. Reproduction of piecewise exponential polynomials is important in several applications, e.g., in biomedical imaging, in geometric design and in isogeometric analysis. Moreover, non-stationary subdivision schemes include Hermite subdivision schemes. Hermite subdivision schemes are iterative methods mapping, at each iteration, a set of vector data consisting of functional values and associated derivatives, to a denser set of vector data of the same type [13, 32]. They are applied in geometric modeling for the construction of curves and surfaces out of points and directional derivatives, and have recently found application in other contexts such as, for example, in the design of one-step numerical methods for the numerical solution of ODE Initial Value Problems.

The main goal of this paper is to investigate the approximation order of non-stationary subdivision schemes and the role played by approximate sum rules, a non-stationary extension of the well-known notion of sum rules. Approximate sum rules allow us to link the response of non-stationary subdivision schemes to specific types of initial data (precisely data sampled from exponential polynomials) with the approximation and smoothness orders. With some extent we find that exponential polynomials and approximate sum rules play an analogous role of polynomials and sum rules in the stationary case. However, important differences unexpectedly arise.

In stationary subdivision, it is well-known that:

- The property of *reproducing polynomials of degree* $N-1$ (*i.e.*, the capability of a stationary subdivision scheme to reproduce in the limit exactly the same degree-$(N-1)$ polynomial from which the data are sampled) implies sum rules of order N (see [4]). Moreover, it implies *approximation order* N (see [31]).

- The property of *generating polynomials of degree* $N-1$ (*i.e.*, the capability of a stationary subdivision scheme to provide polynomials of degree $N-1$ as limit functions) is equivalent to the fulfillment of sum rules of order N (see [4]). Moreover, it guarantees the existence of difference operators whose spectral properties characterize convergence and regularity of the subdivision scheme (see [20]).

- *Sum rules of order* N are a necessary condition for convergence and for C^{N-1}-continuity of stationary subdivision schemes (see, e.g., [2, 7, 22–24] and references therein).

In contrast, in the non-stationary setting, although the notions of generation and reproduction of polynomials are straightforwardly replaced by the notions of generation and reproduction of exponential polynomials, the situation is not so clear. For this reason, in this paper, we prove the following important results, which allow us to point out similarities and differences between the stationary and non-stationary cases:

- As in the stationary case, the property of *reproducing N exponential polynomials* implies approximate sum rules of order N. Moreover, we are able to show that it implies *approximation order* N if asymptotical similarity to a convergent stationary scheme is assumed.

- The property of *generating N exponential polynomials* implies *approximate sum rules of order N* if asymptotical similarity to a convergent stationary scheme and reproduction of one exponential polynomial are assumed. Moreover, as in the stationary case, the property of generating exponential polynomials guarantees the existence of *difference operators*.

- *Approximate sum rules of order* N and asymptotical similarity to a stationary C^{N-1} subdivision scheme provide sufficient conditions for C^{N-1} regularity of non-stationary subdivision schemes.

We emphasize that, in order to maintain the notation and the proofs of the results as simple as possible, we deliberately focus the discussion on univariate non-stationary subdivision schemes only. Note also that, motivated by the fact that primal and dual subdivision schemes are essentially the ones of interest in applications, we restrict our attention to these two subclasses of subdivision schemes, which are known to include odd and even symmetric subdivision schemes.

The remainder of this paper is organized as follows. In Section 2, after providing some basic definitions, we recall known results concerning exponential polynomial generation and reproduction. Then, in Section 3 we study the link between exponential polynomial generation/reproduction and approximate sum rules. Section 4 focuses on the uniform convergence of the sequence of basic limit functions of a non-stationary scheme to the basic limit function of the asymptotically similar stationary scheme. This extends an existing result in [19], which resorts to the assumption of asymptotical equivalence between the two schemes. Finally, under the weaker assumption of asymptotical similarity rather than asymptotical equivalence to a stationary scheme, in Section 5 we also investigate the approximation order of a non-stationary subdivision scheme reproducing a space of exponential polynomials. All the results of this paper are summarized in Section 6.

2. Symbols, exponential polynomial generation and reproduction

Given an initial set of discrete data $\mathbf{f}^{[0]} = \{\mathrm{f}_i^{[0]}, \ i \in \mathbb{Z}\}$, a univariate, non-stationary subdivision scheme constructs the sequence of refined data $\{\mathbf{f}^{[k]}, \ k > 0\}$ via the repeated application of the subdivision operators $S_{\mathbf{a}^{[0]}}, \cdots, S_{\mathbf{a}^{[k-1]}}$ associated with the finitely supported sequences of coefficients $\{\mathbf{a}^{[k-1]}, \ k > 0\}$, $\mathbf{a}^{[k]} := \{\mathrm{a}_i^{[k]} \in \mathbb{R}, \ i \in \mathbb{Z}\}$, named the *subdivision masks*. We assume that all masks have the same support $\{-M, \cdots, M\}$, $M \in \mathbb{N}$, *i.e.*, for all $k \geq 0$, $\mathrm{a}_i^{[k]} = 0$ if $i < -M$ or $i > M$. To generate the refined data sequence $\mathbf{f}^{[k]} = \{\mathrm{f}_i^{[k]}, i \in \mathbb{Z}\}$, $k > 0$, the subdivision operator $S_{\mathbf{a}^{[k-1]}}$ is applied to the data sequence $\mathbf{f}^{[k-1]} = S_{\mathbf{a}^{[k-2]}} \cdots S_{\mathbf{a}^{[0]}} \mathbf{f}^{[0]}$ acting as

$$\mathrm{f}_i^{[k]} := \left(S_{\mathbf{a}^{[k-1]}} \mathbf{f}^{[k-1]} \right)_i := \sum_{j \in \mathbb{Z}} \mathrm{a}_{i-2j}^{[k-1]} \mathrm{f}_j^{[k-1]}, \quad i \in \mathbb{Z}. \tag{2.1}$$

A non-stationary subdivision scheme is thus identified by the collection of subdivision operators and therefore denoted as $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$. When all subdivision operators are the same, the subdivision scheme is called *stationary* and is simply denoted as $\{S_{\mathbf{a}}\}$.

Definition 1. *A subdivision scheme is said to be C^ℓ-convergent if, for any initial sequence $\mathbf{f}^{[0]} \in \ell_\infty(\mathbb{Z})$, there exists a function $g_{\mathbf{f}^{[0]}} \in C^\ell(\mathbb{R})$ such that*

$$\lim_{k \to \infty} \sup_{i \in \mathbb{Z}} |g_{\mathbf{f}^{[0]}}(i2^{-k}) - (S_{\mathbf{a}^{[k-1]}} \cdots S_{\mathbf{a}^{[0]}} \mathbf{f}^{[0]})_i| = 0 \,.$$

If $\ell = 0$, the scheme is simply said to be convergent.

For the practical use of a convergent subdivision scheme it is important to link the properties of the limit function $g_{\mathbf{f}^{[0]}}$ with the properties of the initial sequence $\mathbf{f}^{[0]}$. This is particularly true in the case when $\mathbf{f}^{[0]}$ consists of samples of special types of functions: polynomial, trigonometric and hyperbolic functions or, more generally, exponential polynomial functions. Indeed, the response of the subdivision scheme to these types of starting data is not only important for the design of shapes of practical interest in applications (see, e.g., [1, 36, 38, 42]), but is also strongly connected to the following key properties of the subdivision scheme: its smoothness and its approximation order (see [2, 5, 7, 8, 22–24, 27–29] and Section 3 as well as [25] and Section 5, respectively).

Throughout this paper we use the term *generation* to refer to the capability of a subdivision scheme of providing specific types of limit functions. On the contrary, by the term *reproduction* we refer to the capability of a subdivision scheme of reproducing in the limit exactly the same function from which the data are sampled.

To study the capabilities of a non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ of generating/reproducing exponential polynomial functions, we need the following definition.

Definition 2. *Let $x \in \mathbb{R}$ and $\eta, N \in \mathbb{N}$. Assume also $\lambda_n \in \mathbb{C}$ $n = 1, \cdots, \eta$, and $\mu_n \in \mathbb{N}$ $n = 1, \cdots, \eta$, to be such that $\sum_{n=1}^{\eta} \mu_n = N$. The N-dimensional space spanned by N linearly independent exponential polynomials is*

$$\Phi_N := \mathrm{span}\{x^\beta e^{\lambda_n x}, \ \beta = 0, \cdots, \mu_n - 1, \ n = 1, \cdots, \eta\}.$$

For simplicity, in the following we write Φ_N as $\Phi_N = \mathrm{span}\{\varphi_0(x), \cdots, \varphi_{N-1}(x)\}$.

The next definition stresses the difference between the notions of generation and reproduction of the space of exponential polynomials Φ_N.

Definition 3. *Let $t_i^{[0]}$, $i \in \mathbb{Z}$, be ordered values on the real axis such that $t_{i+1}^{[0]} - t_i^{[0]} = 1$ for all $i \in \mathbb{Z}$. A convergent, non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ is said to be*

i) Φ_N-*generating, if for all initial sequences* $\mathbf{f}^{[0]} := \{f(t_i^{[0]}), \ i \in \mathbb{Z}\}$, $f \in \Phi_N$, *we get*

$$\lim_{k \to \infty} S_{\mathbf{a}^{[k]}} S_{\mathbf{a}^{[k-1]}} \cdots S_{\mathbf{a}^{[0]}} \mathbf{f}^{[0]} \in \Phi_N;$$

ii) Φ_N-*reproducing, if for all initial sequences* $\mathbf{f}^{[0]} := \{f(t_i^{[0]}), \ i \in \mathbb{Z}\}$, $f \in \Phi_N$, *we get*

$$\lim_{k \to \infty} S_{\mathbf{a}^{[k]}} S_{\mathbf{a}^{[k-1]}} \cdots S_{\mathbf{a}^{[0]}} \mathbf{f}^{[0]} = f.$$

Remark 4. *It is easy to see that the space of polynomials of degree at most $N - 1$, can be obtained from Definition 2 by choosing $\lambda_1 = 0$ and $\beta = 0, \cdots, N-1$. As a consequence, Definition 3 applies also in the stationary case, namely when $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ is simply $\{S_{\mathbf{a}}\}$ and Φ_N reduces to $\Pi_{N-1} := \mathrm{span}\{1, x, \cdots, x^{N-1}\}$.*

For a given non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$, its generation and reproduction properties are encoded in the so-called subdivision symbols

$$a^{[k]}(z) := \sum_{i \in \mathbb{Z}} \mathrm{a}_i^{[k]} z^i, \quad z \in \mathbb{C} \setminus \{0\}, \quad k \geq 0,$$

associated to the subdivision masks

$$\mathbf{a}^{[k]} := \{\mathrm{a}_i^{[k]} \in \mathbb{R}, i \in \mathbb{Z}\}, \quad k \geq 0.$$

Then (2.1) can be written as

$$f^{[k+1]}(z) = a^{[k]}(z) f^{[k]}(z^2), \quad k \geq 0 \tag{2.2}$$

where

$$f^{[k]}(z) = \sum_{i \in \mathbb{Z}} \mathrm{f}_i^{[k]} z^i.$$

To simplify the presentation of the algebraic conditions on non-stationary subdivision symbols that guarantee certain generation and reproduction properties of the corresponding subdivision scheme, we start from reviewing the results of the stationary case.

It is well-known that a stationary subdivision scheme $\{S_{\mathbf{a}}\}$ is Π_{N-1}-generating if and only if its symbol $a(z)$ can be written in the form

$$a(z) = (1 + z)^N b(z), \tag{2.3}$$

for some Laurent polynomial $b(z)$ (see [18, Theorem 3.2]). In [18, Theorems 4.6, 4.7] the authors also showed that for a Π_{N-1}-generating subdivision scheme $\{S_{\mathbf{a}}\}$ with symbol $a(z)$, a necessary and sufficient condition for reproducing Π_{N-1} is given by the existence of a Laurent polynomial $c(z)$ such that

$$2 - a(z^{1+\nu}) z^\nu = (1 - z)^N c(z) \qquad \text{with} \quad \nu \in \{0, 1\}. \tag{2.4}$$

To better understand the role of the parameter $\nu \in \{0,1\}$ in (2.4), we need to recall the concept of *parametrization* of a subdivision scheme, i.e. the choice of the grid points $t_i^{[k]}$, $i \in \mathbb{Z}$, to which the k-level data $\mathrm{f}_i^{[k]}$, $i \in \mathbb{Z}$, are attached. The general definition of the grid points $t_i^{[k]}$, $i \in \mathbb{Z}$, is

$$t_i^{[k]} := \frac{i+p}{2^k}, \qquad p := -\left(\tau + \frac{\nu}{2}\right), \quad \tau \in \mathbb{Z}, \quad \nu \in \{0,1\}, \tag{2.5}$$

where the number p is called the *shift parameter* (see [12]). When $\nu = 0$, then $p \in \mathbb{Z}$ and the sets of parameters $\mathbf{t}^{[k]} = \{t_i^{[k]}, i \in \mathbb{Z}\}$, $k \geq 0$, provide the so-called *primal* parametrization. Subdivision schemes based on this choice are thus called *primal* subdivision schemes. In contrast, when $\nu = 1$, then $p \in \frac{1}{2}\mathbb{Z}$ and the parametrization is called *dual* as well as the corresponding subdivision schemes. To simplify the notation, it is convenient to assume $p = -\frac{\nu}{2}$, $\nu \in \{0,1\}$, namely $p \in \{0, -\frac{1}{2}\}$. In fact, if a subdivision scheme is Π_{N-1}-reproducing with respect to the shift parameter p, we can always multiply its symbol by z^τ to make it Π_{N-1}-reproducing with respect to the shift parameter $p + \tau$, namely to $-\frac{\nu}{2}$.

In the non-stationary case, conditions on the subdivision symbols that guarantee exponential polynomial generation were first given in [41]. In case of primal and dual schemes only, a complete set of algebraic conditions to be satisfied by the subdivision symbols of a non-stationary scheme in order to guarantee exponential polynomial generation and reproduction, can be found both in [14] and in [25]. In the following theorem we recall the set of conditions given in the second reference. They are the non-stationary counterparts of (2.3) and (2.4), and are stated in (2.6) and (2.7), respectively.

Theorem 5. *Let Φ_N denote the N-dimensional space of exponential polynomials given in Definition 2. If the subdivision scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ is Φ_N-generating, then its symbols $\{a^{[k]}(z), k \geq 0\}$ satisfy*

$$a^{[k]}(z) = b^{[k]}(z) \prod_{n=1}^{\eta} (1 + e^{\lambda_n 2^{-k-1}} z)^{\mu_n}, \quad k \geq 0, \tag{2.6}$$

for some Laurent polynomials $\{b^{[k]}(z), k \geq 0\}$. Moreover, if $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ is Φ_N-reproducing with respect to the parametrization in (2.5), then

$$2 - a^{[k]}(z^{1+\nu})z^\nu = c^{[k]}(z) \prod_{n=1}^{\eta} (1 - e^{\lambda_n 2^{-k-1-\nu}} z)^{\mu_n}, \quad \nu \in \{0,1\}, \qquad k \geq 0, \tag{2.7}$$

for some Laurent polynomials $\{c^{[k]}(z), k \geq 0\}$.

In the stationary case, conditions (2.3) and (2.4) simply express the fact that the symbols $a(z)$ and $2 - a(z^{1+\nu})z^\nu$ contain the factors $(1+z)^N$ and $(1-z)^N$, respectively. The analogous conditions for the non-stationary case, stated in equations (2.6) and (2.7), are nothing but the requirement that the k-level symbols $a^{[k]}(z)$ and $2 - a^{[k]}(z^{1+\nu})z^\nu$ contain the factors $\prod_{n=1}^{\eta}(1+e^{\lambda_n 2^{-k-1}}z)^{\mu_n}$ and $\prod_{n=1}^{\eta}(1-e^{\lambda_n 2^{-k-1-\nu}}z)^{\mu_n}$, respectively.

Remark 6. *Condition (2.7) was proven in [25] under the assumption that $a^{[k]}(z)$ is a symmetric symbol. However, (2.7) is valid also for primal/dual subdivision schemes that are not symmetric. In fact, the proof in [25] actually does not make use of the symmetry assumption. As an example, consider the not symmetric primal subdivision scheme with k-level symbol*

$$a^{[k]}(z) = (1+z)^2(z+r_k)(z+r_k^{-1})\left(1 + b_1^{[k]}z + b_2^{[k]}z^2 + b_3^{[k]}z^3\right),$$

where $r_k = e^{\lambda 2^{-k-1}}$ and

$$b_1^{[k]} = -\frac{4r_k^4 + 6r_k^3 + 9r_k^2 + 6r_k + 4}{2(r_k^4 + 2r_k^3 + 2r_k^2 + 2r_k + 1)}, \quad b_2^{[k]} = \frac{2r_k^2 + r_k + 2}{2(r_k^2 + 1)}, \quad b_3^{[k]} = -\frac{r_k(2r_k^2 + r_k + 2)}{2(r_k^2 + 1)(r_k + 1)^2}.$$

It is not difficult to show that it is convergent and reproduces $\Phi_4 = \mathrm{span}\{1, x, e^{\pm\lambda x}\}$ with respect to the primal parametrization $\{t_i^{[k]} = \frac{i}{2^k},\ i \in \mathbb{Z}\}$, thus it indeed satisfies (2.6) and (2.7) with $\nu = 0$.

Similarly, we can also easily see that the not symmetric dual subdivision scheme with k-level symbol

$$a^{[k]}(z) = (1+z)(z+r_k)(z+r_k^{-1})\big(b_1^{[k]} + b_2^{[k]} z + b_3^{[k]} z^2\big),$$

where $r_k = e^{\lambda 2^{-k-1}}$ and

$$b_1^{[k]} = \frac{r_k^{13/2} - r_k^4 - r_k^2 + r_k^{-1/2}}{(r_k^2-1)^2(r_k^2+1)}, \qquad b_2^{[k]} = -\frac{r_k^{13/2} + r_k^{11/2} - r_k^5 - 2r_k^3 - r_k + r_k^{1/2} + r_k^{-1/2}}{(r_k^2-1)^2(r_k^2+1)}, \qquad b_3^{[k]} = \frac{r_k(r_k^{9/2} - r_k^3 - r_k + r_k^{-1/2})}{(r_k^2-1)^2(r_k^2+1)},$$

reproduces $\Phi_3 = \mathrm{span}\{1, e^{\pm\lambda x}\}$ with respect to the dual parametrization $\{t_i^{[k]} = \frac{i-\frac{1}{2}}{2^k},\ i \in \mathbb{Z}\}$ and indeed satisfies (2.6) and (2.7) with $\nu = 1$.

3. Exponential polynomials and approximate sum rules

The notion of *approximate sum rules* was first introduced in [5] as a possible generalization of the notion of *sum rules* to the non-stationary setting. For the reader's convenience both definitions are recalled here.

Definition 7. *Let $N \in \mathbb{N}$. The sequence of symbols $\{a^{[k]}(z),\ \ k \geq 0\}$ is said to satisfy* sum rules of order N *if for all $k \geq 0$ it is verified that*

$$a^{[k]}(1) = 2 \qquad and \qquad \frac{d^\beta a^{[k]}(-1)}{dz^\beta} = 0, \quad \beta = 0, \ldots, N-1. \tag{3.1}$$

Definition 8. *Let $N \in \mathbb{N}$. The sequence of symbols $\{a^{[k]}(z),\ \ k \geq 0\}$ is said to satisfy* approximate sum rules of order N *if it fulfills*

$$\sum_{k=0}^{\infty} |a^{[k]}(1) - 2| < \infty \qquad and \qquad \sum_{k=0}^{\infty} 2^{k(N-1)} \sigma_k < \infty, \quad for \quad \sigma_k := \max_{\beta=0,\ldots,N-1} 2^{-k\beta} \left| \frac{d^\beta a^{[k]}(-1)}{dz^\beta} \right|. \tag{3.2}$$

In stationary subdivision, it is well-known that generation of Π_{N-1} is equivalent to the fact that the subdivision symbol satisfies sum rules of order N, while reproduction of Π_{N-1} just implies sum rules of order N (see, e.g., [2, 8, 10, 23, 27]).

The goal of this section is to study the link between generation/reproduction of exponential polynomials and approximate sum rules. In particular, in Theorem 10 we show that reproduction of the exponential polynomial space Φ_N implies approximate sum rules of order N, if the $N \times N$ Wronskian matrix of Φ_N defined by

$$\mathcal{W}_{\Phi_N}(x) := \left(\frac{1}{\beta!} \frac{d^\beta \varphi_\alpha(x)}{dx^\beta},\ \alpha, \beta = 0, \ldots, N-1 \right), \tag{3.3}$$

is invertible for all x in a neighborhood of zero. Moreover, in Theorem 13, we prove that generation of Φ_N and reproduction of just one exponential polynomial in Φ_N imply approximate sum rules of order N if, besides the invertibility of $\mathcal{W}_{\Phi_N}(x)$ for all x in a neighborhood of zero, we additionally assume asymptotical similarity of the non-stationary scheme to some convergent stationary scheme.

Thus, for the sake of completeness, before formulating Theorem 13 we provide the definition of the notion of asymptotical similarity, originally introduced in [9] to weaken the notion of asymptotical equivalence proposed earlier in [19].

Definition 9. *A non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ and a stationary one $\{S_{\mathbf{a}}\}$ are said to be* asymptotically similar, *respectively* asymptotically equivalent, *if the associated sequence of subdivision masks $\{\mathbf{a}^{[k]},\ k \geq 0\}$ and the subdivision mask $\{\mathbf{a}\}$ satisfy*

$$\lim_{k\to\infty} \|\mathbf{a}^{[k]} - \mathbf{a}\| = 0, \quad respectively \quad \sum_{k=0}^{\infty} \|\mathbf{a}^{[k]} - \mathbf{a}\| < \infty.$$

6

Note that, here and in the sequel, $\|\cdot\|$ stands for the infinity norm of subdivision operators, sequences or functions, *i.e.*, $\|S_{\mathbf{a}^{[k]}}\| := \max\left\{\sum_{i\in\mathbb{Z}}|\mathrm{a}_{2i}^{[k]}|, \ \sum_{i\in\mathbb{Z}}|\mathrm{a}_{2i+1}^{[k]}|\right\}$, $\|\mathbf{f}\| := \sup_{i\in\mathbb{Z}}|\mathrm{f}_i|$ and $\|F\| := \sup_{x\in\mathbb{R}}|F(x)|$.

Theorem 10. *Let Φ_N denote the N-dimensional space of exponential polynomials given in Definition 2 and let $\{a^{[k]}(z), k \geq 0\}$ be the symbols of a Φ_N-reproducing non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$. If the Wronskian matrix of Φ_N, $\mathcal{W}_{\Phi_N}(x)$, is invertible for all x in a neighborhood of zero, then for $\beta = 0, \cdots, N-1$ it is verified that*

$$|a^{[k]}(1) - 2| = O(2^{-kN}) \qquad and \qquad \left|\frac{d^\beta a^{[k]}(-1)}{dz^\beta}\right| = O(2^{-k(N-\beta)}), \qquad k \to \infty. \tag{3.4}$$

Proof. We first note that the β-th derivative of $a^{[k]}(z)$ evaluated at $z = -1$ can be expressed as

$$\frac{d^\beta a^{[k]}(-1)}{dz^\beta} = \sum_{\ell=0}^\beta \theta_{\beta,\ell} \sum_{i\in\mathbb{Z}} \mathrm{a}_i^{[k]} i^\ell (-1)^i, \tag{3.5}$$

for some constants $\theta_{\beta,\ell}$, $\ell = 0, \cdots, \beta$ such that $\theta_{\beta,0} = \delta_{\beta,0}$. Thus, to verify (3.4), it suffices to show that

$$\left|\sum_{i\in\mathbb{Z}} \mathrm{a}_i^{[k]} - 2\right| = O(2^{-kN}) \qquad and \qquad \left|\sum_{i\in\mathbb{Z}}(-1)^i i^\beta \mathrm{a}_i^{[k]}\right| = O(2^{-k(N-\beta)}), \qquad k \to \infty. \tag{3.6}$$

Our approach for this task is to separate the summation in equations (3.6) into two parts, and then estimate them separately. To do this, for each $\beta = 0, \cdots, N-1$, we define the pair of functions

$$\mathcal{P}_{j,\beta}(x) := \sum_{n=0}^{N-1} m_{j,\beta,n}^{[k]} \varphi_n(x), \quad j = 0, 1, \tag{3.7}$$

where the coefficient vector $\mathbf{m}_{j,\beta}^{[k]} = (m_{j,\beta,n}^{[k]}, \ n = 0, \cdots, N-1)$ is obtained by solving the Hermite interpolation problem

$$\frac{d^\ell \mathcal{P}_{j,\beta}(j2^{-k-1})}{dx^\ell} = \delta_{\beta,\ell}(-1)^\ell \ell!, \quad \ell = 0, \ldots, N-1, \quad j = 0, 1. \tag{3.8}$$

Denoting by $\delta_{\beta,\ell}$ the Kronecker symbol, equations (3.8) translate into the linear systems

$$\mathcal{W}_{\Phi_N}(j2^{-k-1}) \cdot (\mathbf{m}_{j,\beta}^{[k]})^T = \mathbf{c}_\beta^T, \quad j = 0, 1 \tag{3.9}$$

with $\mathbf{c}_\beta := (\delta_{\beta,\ell}(-1)^\ell \ell!, \ \ell = 0, \cdots, N-1)$. Due to the assumption that $\mathcal{W}_{\Phi_N}(x)$ is invertible for all x in a neighborhood of zero, such a linear system clearly has a unique solution. Next we define

$$Q_{j,\beta} := \sum_{i\in\mathbb{Z}} \mathrm{a}_{j-2i}^{[k]} \left((j2^{-1} - i)2^{-k}\right)^\beta - \mathcal{P}_{j,\beta}(j2^{-k-1}), \quad j = 0, 1, \tag{3.10}$$

and note that, in view of (3.8),

$$\mathcal{P}_{0,\beta}(0) = \mathcal{P}_{1,\beta}(2^{-k-1}) = \delta_{\beta,0}. \tag{3.11}$$

Thus, noting that $\mathcal{P}_{0,0}(0) = \mathcal{P}_{1,0}(2^{-k-1}) = 1$, we get

$$\sum_{i\in\mathbb{Z}} \mathrm{a}_i^{[k]} - 2 = Q_{0,0} + Q_{1,0}. \tag{3.12}$$

Moreover, due to (3.11), we have

$$2^{-\beta(k+1)}\sum_{i\in\mathbb{Z}}(-1)^i i^\beta \mathrm{a}_i^{[k]} = \sum_{i\in\mathbb{Z}} \mathrm{a}_{-2i}^{[k]}(-i2^{-k})^\beta - \sum_{i\in\mathbb{Z}} \mathrm{a}_{1-2i}^{[k]}\left((2^{-1} - i)2^{-k}\right)^\beta = Q_{0,\beta} - Q_{1,\beta}. \tag{3.13}$$

7

We first estimate $Q_{0,\beta}$. The other term $Q_{1,\beta}$ can be handled analogously. Since $\mathcal{P}_{0,\beta}$ is a linear combination of exponential polynomials in Φ_N and the non-stationary scheme $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ reproduces Φ_N, we get the identity $\mathcal{P}_{0,\beta}(0) = \sum_{i \in \mathbb{Z}} \mathrm{a}_{-2i}^{[k]} \mathcal{P}_{0,\beta}(i2^{-k})$. Plugging this into (3.10) for $j = 0$ leads to

$$Q_{0,\beta} = \sum_{i \in \mathbb{Z}} \mathrm{a}_{-2i}^{[k]} \left((-i2^{-k})^\beta - \mathcal{P}_{0,\beta}(i2^{-k}) \right). \tag{3.14}$$

Here, we will use the arguments of Taylor expansion for $\mathcal{P}_{0,\beta}$. Precisely, let $T_{\mathcal{P}_{0,\beta}}^N$ be the degree-$(N-1)$ Taylor polynomial of the function $\mathcal{P}_{0,\beta}$ around 0, that is, $T_{\mathcal{P}_{0,\beta}}^N(x) := \sum_{\ell=0}^{N-1} \frac{x^\ell}{\ell!} \frac{d^\ell \mathcal{P}_{0,\beta}(0)}{dx^\ell}$. Then, we replace $\mathcal{P}_{0,\beta}$ in $Q_{0,\beta}$ by its Taylor polynomial $T_{\mathcal{P}_{0,\beta}}^N$ plus the remainder term, say $R_{\mathcal{P}_{0,\beta}}^N$, such that we have the form $\mathcal{P}_{0,\beta}(i2^{-k}) = T_{\mathcal{P}_{0,\beta}}^N(i2^{-k}) + R_{\mathcal{P}_{0,\beta}}^N(i2^{-k})$. In fact, from the Hermite interpolation conditions in (3.8), we find that $T_{\mathcal{P}_{0,\beta}}^N(i2^{-k}) = (-i2^{-k})^\beta$. Hence, it leads to the equations

$$Q_{0,\beta} = -\sum_{i \in \mathbb{Z}} \mathrm{a}_{-2i}^{[k]} R_{\mathcal{P}_{0,\beta}}^N(i2^{-k}). \tag{3.15}$$

On the other hand, by (3.9), $(\mathbf{m}_{0,\beta}^{[k]})^T = \mathcal{W}_{\Phi_N}(0)^{-1} \mathbf{c}_\beta^T$ such that for a given β, the coefficient vector $\mathbf{m}_{0,\beta}^{[k]}$ for $\mathcal{P}_{0,\beta}$ in (3.7) can be bounded independently of $k \geq 0$. It implies that the α-th derivative of $\mathcal{P}_{0,\beta}$ for each $\alpha = 0, \ldots, N-1$, is uniformly bounded around the origin. Consequently, we get $|R_{\mathcal{P}_{0,\beta}}^N(i2^{-k})| = O(2^{-kN})$ and hence, by (3.15), $|Q_{0,\beta}| = O(2^{-kN})$ as $k \to \infty$. Similarly, we can prove the same convergence rate for $Q_{1,\beta}$, namely $|Q_{1,\beta}| = O(2^{-kN})$ as $k \to \infty$. Combining these two convergence properties and applying equations (3.12) and (3.13), we finally get (3.6). Thus, referring back to the identity in (3.5), the proof is completed. $\qquad\square$

Corollary 11. *Let $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ be a Φ_N-reproducing non-stationary subdivision scheme with $\mathcal{W}_{\Phi_N}(x)$ invertible for all x in a neighborhood of zero. Then $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ satisfies approximate sum rules of order N.*

Remark 12. *It is easy to see that conditions (3.4) are not sufficient for the reproduction of any exponential polynomial. As a counterexample, consider the level-dependent perturbation of quadratic B-splines (still a dual scheme with $\nu = 1$) given by the k-level symbol*

$$a^{[k]}(z) = \frac{1}{4} + 2^{-k} + \left(\frac{3}{4} - 2^{-k} \right) z + \left(\frac{3}{4} - 2^{-k} \right) z^2 + \left(\frac{1}{4} - 2^{-k} \right) z^3, \tag{3.16}$$

which satisfies $a^{[k]}(1) = 2(1 - 2^{-k})$ and $a^{[k]}(-1) = 2^{-k+1}$.

Although (3.4) are satisfied with $N = 1$, no exponential polynomials can be reproduced by this scheme. To see it, it suffices to observe that, for $k = 1$ and $\lambda \in \mathbb{C}$, the complex function $2 - a^{[k]}((e^{-\lambda/2^{k+2}})^2) e^{-\lambda/2^{k+2}}$ is always different from 0.

In the next theorem we replace the assumption that $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ is Φ_N-reproducing with the weaker assumption that it is Φ_N-generating and reproduces only one exponential polynomial in Φ_N; to compensate the latter weaker conditions, we additionally require asymptotical similarity of $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ to a convergent stationary scheme $\{S_{\mathbf{a}}\}$. Therefore, we consider subdivision schemes $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ satisfying the first condition in Definition 9.

Theorem 13. *Let Φ_N denote the N-dimensional space of exponential polynomials given in Definition 2 and let $\{a^{[k]}(z), k \geq 0\}$ be the symbols of a non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ which is Φ_N-generating and reproduces one exponential polynomial in Φ_N. If $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ is asymptotically similar to a convergent stationary subdivision scheme $\{S_{\mathbf{a}}\}$, and the Wronskian matrix $\mathcal{W}_{\Phi_N}(x)$, defined in (3.3), is invertible for all x in a neighborhood of zero, then for $\beta = 0, \cdots, N-1$*

$$|a^{[k]}(1) - 2| = O(2^{-k}) \qquad and \qquad \left| \frac{d^\beta a^{[k]}(-1)}{dz^\beta} \right| = O(2^{-k(N-\beta)}), \qquad k \to \infty. \tag{3.17}$$

Proof. Since $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ is Φ_N-generating, in view of Theorem 5 we know that equation (2.6) holds true for all $k \geq 0$. By the asymptotical similarity assumption, as $k \to \infty$ the left hand side of (2.6) tends to $a(z)$ (the symbol of the stationary subdivision scheme $\{S_{\mathbf{a}}\}$) and the term $\prod_{n=1}^{\eta}(1 + e^{\lambda_n 2^{-k-1}}z)^{\mu_n}$, appearing in the right hand side, tends to $(1+z)^N$. Therefore, we can conclude that $\lim_{k \to \infty} b^{[k]}(z) = b(z)$ for a suitable $b(z)$. The latter means that, for k large enough, $\|\mathbf{b}^{[k]}\| < C$ where $C > 0$ is a constant independent of k. Next, we observe that each $\frac{d^\beta a^{[k]}(z)}{dz^\beta}$ contains at least $N - \beta$ factors of the form $(1 + e^{\lambda_n 2^{-k-1}}z)$ and, since $1 - e^{\lambda_n 2^{-k-1}} = O(2^{-k})$, we conclude that $\frac{d^\beta a^{[k]}(-1)}{dz^\beta} = O(2^{-k(N-\beta)})$ as $k \to \infty$. Thus the second part of the claim is proven. Since the Wronskian matrix $\mathcal{W}_{\Phi_N}(x)$ is invertible for all x in a neighborhood of zero, the first part of the claim is obtained from Theorem 10 with $N = 1$, so completing the proof. $\qquad\square$

Corollary 14. *Let $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ be a Φ_N-generating non-stationary subdivision scheme which reproduces at least one exponential polynomial in Φ_N with $\mathcal{W}_{\Phi_N}(x)$ invertible for all x in a neighborhood of zero. If $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ is asymptotically similar to a convergent stationary scheme, then it satisfies approximate sum rules of order N.*

Remark 15. *Note that, the assumption that $\mathcal{W}_{\Phi_N}(x)$ is invertible for all x in a neighborhood of zero, is needed to prove only the first of conditions (3.17). We additionally emphasize that the assumptions in Theorem 13 do not guarantee asymptotical equivalence between $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ and $\{S_{\mathbf{a}}\}$. For example, the k-level subdivision mask*

$$\mathbf{a}^{[k]} = \Big\{0,\cdots,0, \frac{1}{(r_k^{-1}+r_k)(r_k^{-\frac{1}{2}}+r_k^{\frac{1}{2}})} - \frac{1}{k}, \frac{(r_k^{-\frac{1}{2}}+r_k^{\frac{1}{2}})^2-1}{(r_k^{-1}+r_k)(r_k^{-\frac{1}{2}}+r_k^{\frac{1}{2}})} - \frac{r_k}{k},$$
$$\frac{(r_k^{-\frac{1}{2}}+r_k^{\frac{1}{2}})^2-1}{(r_k^{-1}+r_k)(r_k^{-\frac{1}{2}}+r_k^{\frac{1}{2}})} + \frac{1}{k\,r_k^2}, \frac{1}{(r_k^{-1}+r_k)(r_k^{-\frac{1}{2}}+r_k^{\frac{1}{2}})} + \frac{1}{k\,r_k}, 0,\cdots,0\Big\}, \qquad r_k = e^{\lambda 2^{-k-1}},$$

is such that $\lim_{k \to \infty} \|\mathbf{a}^{[k]} - \mathbf{a}\| = 0$ with $\mathbf{a}$ denoting the mask of the quadratic B-spline scheme having symbol $a(z) = \frac{1}{4z^2}(1+z)^3$. The associated symbols satisfy $a^{[k]}(-r_k) = a^{[k]}(-r_k^{-1}) = 0$ as well as $a^{[k]}(r_k^{-1}) = 2r_k^{\frac{1}{2}}$ for all $k \geq 0$, so that $e^{\pm\lambda x}$ are generated whereas only $e^{\lambda x}$ is reproduced with respect to the dual parametrization $\mathbf{t}^{[k]} = \{t_i^{[k]} = \frac{i-\frac{1}{2}}{2^k},\ i \in \mathbb{Z}\}$. Thus, in view of Theorem 13, conditions $|a^{[k]}(1) - 2| = O(2^{-k})$, $|a^{[k]}(-1)| = O(2^{-2k})$, $\left|\frac{da^{[k]}(-1)}{dz}\right| = O(2^{-k})$, are all satisfied for $k \to \infty$. However, since $\sum_{k=0}^{\infty}\left|\frac{1}{(r_k^{-1}+r_k)(r_k^{-\frac{1}{2}}+r_k^{\frac{1}{2}})} - \frac{1}{k} - \frac{1}{4}\right|$ is not convergent, $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ and $\{S_{\mathbf{a}}\}$ are not asymptotically equivalent.

We conclude this section by recalling that, in stationary subdivision, sum rules of order N are known to be necessary conditions for C^{N-1}-continuity [2, 8, 20, 23, 27]. As a consequence of the results in [5], in the non-stationary setting, approximate sum rules of order N and asymptotical similarity to a stationary C^{N-1} subdivision scheme provide sufficient conditions for C^{N-1} regularity of non-stationary subdivision schemes. More precisely, in view of [5, Corollary 1], we can state the following.

Corollary 16. *Let $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ be a Φ_N-generating non-stationary subdivision scheme which reproduces at least one exponential polynomial in Φ_N. Moreover, let $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ be asymptotically similar to a C^ℓ-convergent subdivision scheme, $\ell \in \mathbb{N}_0$ and assume that $\mathcal{W}_{\Phi_N}(x)$ is invertible for all x in a neighborhood of zero. Then, setting $\rho := \min\{\ell, N-1\}$, it follows that $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ is at least C^ρ-convergent.*

4. Asymptotic behavior of basic limit functions

It is well-known that a convergent non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ defines a family of basic limit functions. For $\boldsymbol{\delta} := \{\delta_{i,0},\ i \in \mathbb{Z}\}$, they are

$$\phi_m := \lim_{\ell \to \infty} S_{\mathbf{a}^{[m+\ell]}}\cdots S_{\mathbf{a}^{[m]}}\boldsymbol{\delta}, \qquad m \geq 0. \tag{4.1}$$

The goal of this section is to weaken the assumptions used in [19, Lemma 15] to prove that, as $m \to \infty$, the sequence of basic limit functions $\{\phi_m, \ m \geq 0\}$ of the non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ converges uniformly to the basic limit function ϕ of a C^0 stationary scheme $\{S_{\mathbf{a}}\}$. Precisely, in Theorem 20 we prove that $\lim_{m \to \infty} \|\phi_m - \phi\| = 0$ without requiring that $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ is asymptotically equivalent to $\{S_{\mathbf{a}}\}$. Differently, we just assume that $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ is a non-stationary subdivision scheme reproducing one exponential polynomial in Φ_N, say $e^{\lambda x}$, $\lambda \in \mathbb{C}$, and it is asymptotically similar to $\{S_{\mathbf{a}}\}$. We remark that, the assumption that $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ reproduces $e^{\lambda x}$, $\lambda \in \mathbb{C}$, is not restrictive. In fact, if $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ reproduces a higher order exponential polynomial $x^r e^{\lambda x}$, $r \in \mathbb{N}$, then it obviously reproduces $e^{\lambda x}$ as well. Moreover, if $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ reproduces constants, with the choice $\lambda = 0$ we can also recover this case. For the proof of Theorem 20 we also recall that, accordingly to [14, Theorem 1], a non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ reproduces $e^{\lambda x}$, $\lambda \in \mathbb{C}$ with respect to the parametrization $\mathbf{t}^{[k]} = \{t_i^{[k]} = \frac{i+p}{2^k}, \ i \in \mathbb{Z}\}$, $p \in \{0, -\frac{1}{2}\}$, if its symbol $a^{[k]}(z)$ is such that $a^{[k]}(-r_k^{-1}) = 0$ and $a^{[k]}(r_k^{-1}) = 2r_k^{-p}$, for all $k \geq 0$, with $r_k = e^{\lambda 2^{-k-1}}$.

Before proving Theorem 20, we additionally need the following auxiliary results.

Proposition 17. *For $p \in \{0, \ -\frac{1}{2}\}$, let $\{S_{\mathbf{h}_p^{[k]}}, k \geq 0\}$ be the non-stationary subdivision scheme with k-level symbol*

$$h_p^{[k]}(z) := \frac{r_k^{-p}}{(r_k^{-1} + r_k)z}(1 + r_k^{-1}z)(1 + r_k z), \qquad r_k = e^{\lambda 2^{-k-1}}, \quad \lambda \in \mathbb{C}. \qquad (4.2)$$

Then

(a) *for $p \in \{0, \ -\frac{1}{2}\}$, $\{S_{\mathbf{h}_p^{[k]}}, k \geq 0\}$ is C^0 and stable (in the sense of [19]). Moreover, the scheme is interpolatory when $p = 0$ while approximating when $p = -\frac{1}{2}$.*

(b) *for $p \in \{0, \ -\frac{1}{2}\}$, the basic limit functions*

$$H_{m,p} := \lim_{\ell \to \infty} S_{\mathbf{h}_p^{[m+\ell]}} \dots S_{\mathbf{h}_p^{[m]}} \boldsymbol{\delta}, \quad m \geq 0 \qquad (4.3)$$

of $\{S_{\mathbf{h}_p^{[k]}}, k \geq 0\}$ are such that $\lim_{m \to \infty} H_{m,p} = H$, where H is the linear B-spline supported on $[-1, 1]$. Moreover, $H_{m,0}(0) = 1$ while $H_{m,-\frac{1}{2}}(0) \to 1$ as $m \to \infty$.

Proof. a) The observation that $|1 - r_k| \leq C2^{-k}$ for some constant $C > 0$, implies that the non-stationary subdivision scheme $\{S_{\mathbf{h}_p^{[k]}}, k \geq 0\}$ is asymptotically equivalent to the linear B-spline scheme, which is C^0 and stable (in the sense of [19]). Accordingly, the scheme $\{S_{\mathbf{h}_p^{[k]}}, k \geq 0\}$ is also C^0 and stable. Moreover, by definition of $h_p^{[k]}(z)$ in (4.2), $h_p^{[k]}(-r_k^{-1}) = 0$ and $h_p^{[k]}(r_k^{-1}) = 2r_k^{-p}$, $p \in \{0, \ -\frac{1}{2}\}$. Since by definition (4.2) we can immediately write that $h_p^{[k]}(-z) + h_p^{[k]}(z) = 2r_k^{-p}$, the claim follows.

b) Since the basic limit function of the linear B-spline scheme is the degree-1 B-spline H supported on $[-1, 1]$, in view of [19, Lemma 15] we have that $H_{m,p} \to H$ uniformly as $m \to \infty$. Moreover, since the mask of the scheme $\{S_{\mathbf{h}_p^{[k]}}, k \geq 0\}$ has the same support as the one of the linear B-spline scheme, we have that for all $m \geq 0$ the basic limit function $H_{m,p}$ of $\{S_{\mathbf{h}_p^{[k]}}, k \geq 0\}$ is supported on $[-1, 1]$ and is such that $H_{m,p}(0) = (e^{\lambda 2^{-m}})^{-p}$. Hence, we can conclude that, $H_{m,0}(0) = 1$ implying that $H_{m,-\frac{1}{2}}(0) \to 1$ as $m \to \infty$. $\qquad \square$

We continue by providing an additional intermediate result that will be exploited in the proof of Theorem 20. This result is in Theorem 19. It extends [9, Theorem 11] by replacing the assumption that $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ reproduces constants with the reproduction of at least one exponential polynomial $e^{\lambda x}$, $\lambda \in \mathbb{C}$. For the proof of Theorem 19, we need to replace the classical notion of backward difference operator $(\Delta^k \mathbf{f}^{[k]})_i := \mathbf{f}_i^{[k]} - \mathbf{f}_{i-1}^{[k]}$ with

$$(\Delta_\lambda^k \mathbf{f}^{[k]})_i := \mathbf{f}_i^{[k]} - r_{k-1} \mathbf{f}_{i-1}^{[k]},$$

where $r_{k-1} := e^{\lambda 2^{-k}}$, $\lambda \in \mathbb{C}$, $k \geq 0$; see also [40] for the details of this difference operator. Obviously, Δ_λ^k reduces to Δ^k if $\lambda = 0$. The following lemma is useful for our further analysis.

Note that, from now on, C will be used to denote any generic positive constant.

Lemma 18. *Let $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ be a non-stationary subdivision scheme reproducing $e^{\lambda x}$, $\lambda \in \mathbb{C}$ with respect to the shift parameter $p \in \{0, -\frac{1}{2}\}$, and let $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ be asymptotically similar to a convergent stationary scheme $\{S_{\mathbf{a}}\}$. For all $m \geq 0$, let*

$$\mathbf{f}_m^{[k+1]} := S_{\mathbf{a}^{[m+k]}} \cdots S_{\mathbf{a}^{[m]}} \mathbf{f}^{[0]}. \tag{4.4}$$

Then, there exist a constant $C > 0$, $\mu \in (0,1)$ and K large enough such that

$$\|\Delta_\lambda^{[m+k+1]} \mathbf{f}_m^{[k+1]}\| \leq C\mu^k, \quad \forall k \geq K. \tag{4.5}$$

Proof. The proof of this result is very similar to the proof of [9, Theorem 3]. However, there are some crucial different points that we need to put in evidence. From the generation properties of $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ we know that $a^{[k]}(z)$ factorizes as $a^{[k]}(z) = (1 + r_k z) b^{[k]}(z)$ for some Laurent polynomial $b^{[k]}(z)$. In view of (2.2) and (4.4), we have $f_m^{[k+1]}(z) = a^{[k+m]}(z) f_m^{[k]}(z^2)$ with $f_m^{[k]}(z) := \sum_{i \in \mathbb{Z}} (\mathbf{f}_m^{[k]})_i z^i$. Multiplying both sides of this equation by $1 - r_{k+m}z$ and using the relation $r_{k+m}^2 = r_{k+m-1}$, we arrive at

$$\begin{aligned}
\Delta_\lambda^{k+m+1} f_m^{[k+1]}(z) = (1 - r_{k+m}z) f_m^{[k+1]}(z) &= (1 - r_{k+m}^2 z^2) b^{[k+m]}(z) f_m^{[k]}(z^2) \\
&= (1 - r_{k+m-1}z^2) b^{[k+m]}(z) f_m^{[k]}(z^2) = b^{[k+m]}(z) \Delta_\lambda^{k+m} f_m^{[k]}(z^2).
\end{aligned}$$

The latter means that, under the assumption that the non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ generates $e^{\lambda x}$, there exists a non-stationary subdivision scheme $\{S_{\mathbf{b}^{[k]}}, k \geq 0\}$ such that for the sequence $\{\mathbf{f}_m^{[k+1]} := S_{\mathbf{a}^{[k+m]}} \mathbf{f}_m^{[k]}, k \geq 0\}$, it is verified that

$$\Delta_\lambda^{k+m+1} \mathbf{f}_m^{[k+1]} = S_{\mathbf{b}^{[k+m]}} \Delta_\lambda^{k+m} \mathbf{f}_m^{[k]}, \quad k \geq 0 \tag{4.6}$$

where $\mathbf{f}_m^{[0]} := \mathbf{f}^{[0]}$. We next show that there exist $\mu \in (0,1)$ and K large enough such that, for all $k \geq K$,

$$\|\Delta_\lambda^{[k+m+1]} \mathbf{f}_m^{[k+1]}\| \leq C\mu^k,$$

with a constant $C > 0$.
From the assumption that $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ reproduces $e^{\lambda x}$, we know that, in addition to $a^{[k]}(z) = (1 + r_k z) b^{[k]}(z)$, $a^{[k]}(z)$ verifies the condition $a^{[k]}(r_k^{-1}) = 2r_k^{-p}$, $p \in \{0, -\frac{1}{2}\}$. Therefore, we can conclude that $b^{[k]}(r_k^{-1}) = r_k^{-p}$. Moreover, since $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ is asymptotically similar to a convergent scheme $\{S_{\mathbf{a}}\}$, we have that $a(z) = (1 + z) b(z)$ with $\lim_{k \to \infty} \|\mathbf{b}^{[k]} - \mathbf{b}\| = 0$, and there exists a positive integer L such that $\|S_{\mathbf{b}}^L\| < 1$. Writing the symbol $a^{[k]}(z)$ as $a^{[k]}(z) = \sum_{j=-M}^M \mathrm{a}_j^{[k]} z^j$, for the Laurent polynomial $b^{[k]}(z) = \sum_{j=-M}^{M-1} \mathrm{b}_j^{[k]} z^j$ we have that its coefficients $\mathrm{b}_j^{[k]}$, $j = -M, \cdots, M-1$ satisfy

$$\mathrm{b}_{-M}^{[k]} = \mathrm{a}_{-M}^{[k]}, \quad \mathrm{b}_j^{[k]} = \sum_{i=0}^{j-M+1} (-1)^i (r_k^{-1})^{i+1} \mathrm{b}_{j+1-i}^{[k]}, \quad j = -M+1, \ldots, M-1. \tag{4.7}$$

Due to $r_k^{-1} \leq \max(1, e^{-\lambda})$, we have

$$\|\mathbf{b}^{[k]}\| \leq 2MC \|\mathbf{a}^{[k]}\| \tag{4.8}$$

with a suitable constant $C > 0$. Since $\lim_{k \to \infty} \|\mathbf{b}^{[k]} - \mathbf{b}\| = 0$, for all $\epsilon > 0$, there exists a positive integer K_ϵ such that for $k > K_\epsilon$

$$\|S_{\mathbf{b}^{[k]}}\| \leq \|S_{\mathbf{b}^{[k]}} - S_{\mathbf{b}}\| + \|S_{\mathbf{b}}\| < \|S_{\mathbf{b}}\| + \epsilon. \tag{4.9}$$

Since for all $k > K_\epsilon$

$$\|S_{\mathbf{b}^{[k+L-1]}} \cdots S_{\mathbf{b}^{[k]}} - S_{\mathbf{b}}^L\| \leq \sum_{\ell=1}^{L} \|S_{\mathbf{b}^{[k+L-1]}}\| \cdots \|S_{\mathbf{b}^{[k+L-\ell+1]}}\| \|S_{\mathbf{b}^{[k+L-\ell]}} - S_{\mathbf{b}}\| \|S_{\mathbf{b}}^{L-\ell}\|,$$

due to (4.9) we get

$$\|S_{\mathbf{b}^{[k+L-1]}} \cdots S_{\mathbf{b}^{[k]}} - S_{\mathbf{b}}^L\| \leq L\epsilon \left(\|S_{\mathbf{b}}\| + \epsilon\right)^{\ell-1} \|S_{\mathbf{b}}\|^{L-\ell} < C\epsilon,$$

for a suitable constant $C > 0$. Therefore,

$$\|S_{\mathbf{b}^{[k+L-1]}} \cdots S_{\mathbf{b}^{[k]}}\| \leq \|S_{\mathbf{b}}^L\| + C\epsilon, \qquad \forall k > K_\epsilon. \tag{4.10}$$

Further, since $\|S_{\mathbf{b}}^L\| < 1$, we especially choose a number $\tilde{\epsilon} > 0$ such that

$$\tilde{\mu} := \|S_{\mathbf{b}}^L\| + C\tilde{\epsilon} < 1.$$

Letting $K_{\tilde{\epsilon}}$ be the corresponding number satisfying (4.9), put $K = \max(K_{\tilde{\epsilon}}, m)$. Then, from (4.6), we have that for an arbitrary integer $n > 0$,

$$\Delta_\lambda^{[K+m+n+1]} \mathbf{f}_m^{[K+n+1]} = S_{\mathbf{b}^{[K+m+n]}} \cdots S_{\mathbf{b}^{[K+1]}} \left(S_{\mathbf{b}^{[K]}} \cdots S_{\mathbf{b}^{[m]}}\right) \Delta_\lambda^{[m]} \mathbf{f}^{[0]}.$$

Hence, applying (4.10) and using the value of $\tilde{\mu}$

$$\|\Delta_\lambda^{[K+m+n+1]} \mathbf{f}_m^{[K+n+1]}\| \leq \widetilde{C} \tilde{\mu}^{\lfloor \frac{n+m}{L} \rfloor} \leq \widetilde{C} \tilde{\mu}^{\frac{n+m}{L}-1}$$

with

$$\widetilde{C} = \max_{r=0,\cdots,L-1} \left(\prod_{j=m}^{K+r} \|S_{\mathbf{b}^{[j]}}\|\right) \|\Delta_\lambda^{[m]} \mathbf{f}^{[0]}\|.$$

Moreover, introducing the notation $C := \widetilde{C} \tilde{\mu}^{\frac{m-K}{L}-1}$ and $\mu := \tilde{\mu}^{\frac{1}{L}}$, we arrive at the bound

$$\|\Delta_\lambda^{[K+m+n+1]} \mathbf{f}_m^{[K+n+1]}\| \leq C\mu^{K+n}.$$

As a conclusion, there exists a constant $C > 0$ such that $\|\Delta_\lambda^{[m+k+1]} \mathbf{f}_m^{[k+1]}\| \leq C\mu^k$ for $\mu \in (0,1)$. $\qquad \square$

Theorem 19. *Let $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ be a non-stationary subdivision scheme reproducing $e^{\lambda x}$, $\lambda \in \mathbb{C}$ with respect to the shift parameter $p \in \{0, -\frac{1}{2}\}$, and let $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ be asymptotically similar to a convergent stationary scheme $\{S_{\mathbf{a}}\}$. Then, for all $m \geq 0$, there exist a constant $C > 0$, $\mu \in (0,1)$ and K large enough such that*

$$\left\| \lim_{\ell \to \infty} \mathbf{f}_m^{[\ell]} - F_{m,p}^k \right\| \leq C\mu^k \qquad \forall \, k \geq K, \tag{4.11}$$

where $\mathbf{f}_m^{[k]}$ is in (4.4) and $F_{m,p}^k$ is the function defined by

$$F_{m,p}^k := \sum_{i \in \mathbb{Z}} (\mathbf{f}_m^{[k]})_i \, H_{m+k,p}(2^k \cdot -i), \tag{4.12}$$

with $H_{m+k,p}$ in (4.3).

Proof. Let $\mathbf{d}^{[k]} = \mathbf{a}^{[k]} - \mathbf{h}_p^{[k]}$ with $\mathbf{h}_p^{[k]}$ the subdivision mask associated with the symbol in (4.2). Since $d^{[k]}(\pm r_k^{-1}) = 0$ we can obviously write it as

$$d^{[k]}(z) = (1 - r_k^2 z^2) e^{[k]}(z), \tag{4.13}$$

12

for a suitable Laurent polynomial $e^{[k]}(z)$. With the same arguments as in (4.7) and (4.8) and using the fact that $\lim_{k\to\infty}\|\mathbf{h}_p^{[k]}-\mathbf{h}\|=0$ in view of Proposition 17-(b), we obtain

$$\|\mathbf{e}^{[k]}\|\leq 2M\|\mathbf{d}^{[k]}\|\leq 2M\left(\|\mathbf{a}^{[k]}\|+\|\mathbf{h}\|\right)=2M\left(\|\mathbf{a}^{[k]}\|+1\right).$$

Moreover, since $\lim_{k\to\infty}\|\mathbf{a}^{[k]}-\mathbf{a}\|=0$, the sequence $\{\mathbf{a}^{[k]},\,k\geq 0\}$ is uniformly bounded independent of k, so that we conclude that

$$\|\mathbf{e}^{[k]}\|\leq C,\tag{4.14}$$

with C a positive constant.

The rest of the proof mimics [9, Theorem 11] where Δ_λ replaces Δ and $H_{m,p}$ replaces the hat function H. By using standard arguments (see, e.g., [20, Theorem 4.11]), to show (4.11) it suffices to show that, for the sequence of approximating functions $\{F_{m,p}^k,k\geq 0\}$ defined in (4.12), there exists $K>0$ such that for all $k\geq K$, $\|F_{m,p}^{k+1}-F_{m,p}^k\|\leq C\mu^k$ with $\mu\in(0,1)$ and C a positive constant. For this purpose we write $F_{m,p}^k$ as

$$F_{m,p}^k=\sum_{i\in\mathbb{Z}}\left(S_{\mathbf{h}_p^{[k+m]}}\mathbf{f}_m^{[k]}\right)_i H_{m+k+1,p}(2^{k+1}\cdot-i)\,.$$

Hence, it is clear from the relation $\mathbf{f}_m^{[k+1]}=S_{\mathbf{a}^{[k+m]}}\mathbf{f}_m^{[k]}$ that

$$F_{m,p}^{k+1}-F_{m,p}^k=\sum_{i\in\mathbb{Z}}\left(\left(S_{\mathbf{a}^{[k+m]}}-S_{\mathbf{h}_p^{[k+m]}}\right)\mathbf{f}_m^{[k]}\right)_i H_{m+k+1,p}(2^{k+1}\cdot-i)\,.$$

Next, we define the sequence $\mathbf{g}_m^{[k+1]}$ as

$$\mathbf{g}_m^{[k+1]}:=\left(S_{\mathbf{a}^{[k+m]}}-S_{\mathbf{h}_p^{[k+m]}}\right)\mathbf{f}_m^{[k]}=S_{\mathbf{d}^{[k+m]}}\mathbf{f}_m^{[k]}\,.\tag{4.15}$$

By the expression $g_m^{[k+1]}(z):=\sum_{i\in\mathbb{Z}}(\mathbf{g}_m^{[k+1]})_i z^i$, we write (4.15) as $g_m^{[k+1]}(z)=d^{[k+m]}(z)f_m^{[k]}(z^2)$, where $f_m^{[k]}(z):=\sum_{i\in\mathbb{Z}}(\mathbf{f}_m^{[k]})_i z^i$. Then, using (4.13) we have

$$g_m^{[k+1]}(z)=(1-r_{k+m}^2 z^2)e^{[k+m]}(z)f_m^{[k]}(z^2)=e^{[k+m]}(z)\sum_{i\in\mathbb{Z}}\left((\mathbf{f}_m^{[k]})_i-r_{k+m-1}(\mathbf{f}_m^{[k]})_{i-1}\right)z^{2i}\,,$$

or equivalently

$$(\mathbf{g}_m^{[k+1]})_i=\sum_{j\in\mathbb{Z}}\mathrm{e}_{i-2j}^{[k+m]}\left((\mathbf{f}_m^{[k]})_j-r_{k+m-1}(\mathbf{f}_m^{[k]})_{j-1}\right)\,,$$

implying that

$$|(\mathbf{g}_m^{[k+1]})_i|\leq\sum_{j\in\mathbb{Z}}|\mathrm{e}_{i-2j}^{[k+m]}|\,\left|(\mathbf{f}_m^{[k]})_j-r_{k+m-1}(\mathbf{f}_m^{[k]})_{j-1}\right|\leq 2M\max_j|\mathrm{e}_j^{[k+m]}|\max_j\left|\left(\Delta_\lambda^{[k+m]}\mathbf{f}_m^{[k]}\right)_j\right|\,.$$

Hence, recalling (4.5) and (4.14), there exists $K>0$ such that for all $k\geq K$ we have $\|\mathbf{g}_m^{[k+1]}\|\leq C\mu^k$ with $\mu\in(0,1)$ and C a positive constant, resulting in that for the sequence of functions $\{F_{m,p}^k,k\geq 0\}$ $\|F_{m,p}^{k+1}-F_{m,p}^k\|\leq C\mu^k$ for all $k\geq K$ with $\mu\in(0,1)$ and C a positive constant. Thus we can obtain that for any positive integer ℓ,

$$\|F_{m,p}^{k+\ell}-F_{m,p}^k\|\leq\sum_{j=0}^{\ell-1}\|F_{m,p}^{k+j+1}-F_{m,p}^{k+j}\|\leq\sum_{j=0}^{\ell-1}C\mu^{k+j}\leq C\frac{\mu^k}{1-\mu},\tag{4.16}$$

from which it follows that $\{F_{m,p}^{k+1},k\geq 0\}$ is a Cauchy sequence in the L_∞-norm and therefore converges uniformly to a continuous limit. To show that such limit is exactly $\lim_{\ell\to\infty}\mathbf{f}_m^{[\ell]}$ we use standard arguments (see

[19, Lemma 14]): since $\{H_{m+k,p}, \ k \geq 0\}$ is a stable sequence of continuous, compactly supported functions which approximate partition of unity uniformly, the uniform convergence of the sequence $\{F_{m,p}^{k+1}, \ k \geq 0\}$ implies the uniform convergence of the subdivision scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ and also

$$\lim_{\ell \to \infty} F_{m,p}^{k+\ell} = \lim_{\ell \to \infty} \sum_{i \in \mathbb{Z}} \left(\mathbf{f}_m^{[k+\ell]}\right)_i H_{m+k+\ell,p}(2^{k+\ell} \cdot -i) = \lim_{\ell \to \infty} \mathbf{f}_m^{[\ell]},$$

so concluding the proof. $\qquad \square$

We are finally ready to prove the main result of this section.

Theorem 20. *Let $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ be a non-stationary subdivision scheme reproducing $e^{\lambda x}$, $\lambda \in \mathbb{C}$ with respect to the shift parameter $p \in \{0, -\frac{1}{2}\}$. Let $\{\phi_m, \ m \geq 0\}$ be the associated sequence of basic limit functions. Assume further that $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ is asymptotically similar to a convergent stationary scheme $\{S_{\mathbf{a}}\}$ with stable basic limit function ϕ having Hölder continuity $\alpha \in (0,1)$. Then $\lim_{m \to \infty} \|\phi_m - \phi\| = 0$.*

Proof. Let $k \in \mathbb{N}_0$ be a non-negative integer, $\boldsymbol{\delta}_m^{[k]} := S_{\mathbf{a}^{[m+k-1]}} \cdots S_{\mathbf{a}^{[m]}} \boldsymbol{\delta}$ and $\boldsymbol{\phi}_m^{[k]} := \{\phi_m(i2^{-k}), \ i \in \mathbb{Z}\}$. Then

$$S_{\mathbf{a}^{[m+k]}} \cdots S_{\mathbf{a}^{[m]}} \boldsymbol{\delta} - S_{\mathbf{a}}^{k+1} \boldsymbol{\delta} = \sum_{j=0}^{k-1} S_{\mathbf{a}}^j \left(S_{\mathbf{a}^{[m+k-j]}} - S_{\mathbf{a}}\right) \boldsymbol{\delta}_m^{[k-j]} + S_{\mathbf{a}}^k (S_{\mathbf{a}^{[m]}} - S_{\mathbf{a}}) \boldsymbol{\delta}. \qquad (4.17)$$

The last term on the right hand side of the above equation can be estimated as

$$\|S_{\mathbf{a}}^k (S_{\mathbf{a}^{[m]}} - S_{\mathbf{a}}) \boldsymbol{\delta}\| \leq \|S_{\mathbf{a}}^k\| \, \|\mathbf{a}^{[m]} - \mathbf{a}\|.$$

Due to the convergence assumption of $\{S_{\mathbf{a}}\}$, there exists a constant $C > 0$ such that for all k, $\|S_{\mathbf{a}}^k\| < C$ (see [19, Section 2]). Hence, from the asymptotical similarity of $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ and $\{S_{\mathbf{a}}\}$, it is immediate that

$$\lim_{m \to \infty} \lim_{k \to \infty} \|S_{\mathbf{a}}^k (S_{\mathbf{a}^{[m]}} - S_{\mathbf{a}}) \boldsymbol{\delta}\| = 0.$$

Now, to estimate the other summands in (4.17) note that

$$((S_{\mathbf{a}^{[m+k-j]}} - S_{\mathbf{a}}) \boldsymbol{\delta}_m^{[k-j]})_i = \sum_{\ell \in \mathbb{Z}} (\mathbf{a}_{i-2\ell}^{[m+k-j]} - \mathbf{a}_{i-2\ell})(\boldsymbol{\delta}_m^{[k-j]})_\ell, \quad i \in \mathbb{Z}, \quad j = 0, \cdots, k-1. \qquad (4.18)$$

To estimate the expression in (4.18) we approximate $\boldsymbol{\delta}_m^{[k-j]}$ by using $\boldsymbol{\phi}_m^{[k-j]}$, *i.e.* the values of the basic limit functions ϕ_m on the grids $2^{-(k-j)}\mathbb{Z}$. For each $\ell \in \mathbb{Z}$ we write

$$(\boldsymbol{\delta}_m^{[k-j]})_\ell = \left((\boldsymbol{\delta}_m^{[k-j]})_\ell - (\boldsymbol{\phi}_m^{[k-j]})_\ell\right) + \left((\boldsymbol{\phi}_m^{[k-j]})_\ell - (\boldsymbol{\phi}_m^{[k-j]})_i\right) + (\boldsymbol{\phi}_m^{[k-j]})_i. \qquad (4.19)$$

In view of (4.18), since the masks $\mathbf{a}^{[k]}$ and $\mathbf{a}$ have the same finite support, we need to consider ℓ only around i, say $|i - \ell| \leq 2M$ for some $M > 0$. Now, to estimate the first term in the right hand side of (4.19), since $a^{[k]}(r_k^{-1}) = 2r_k^{-p}$ with $r_k = e^{\lambda 2^{-k-1}}$, we consider the sequence of functions $\{\mathcal{F}_{m,p}^k, k \geq 0\}$ defined by

$$\mathcal{F}_{m,p}^k := \sum_{i \in \mathbb{Z}} (\boldsymbol{\delta}_m^{[k]})_i H_{m+k,p}(2^k \cdot -i), \qquad (4.20)$$

where $H_{m+k,p}$ is the basic limit function of the non-stationary scheme with s-level symbol $h_p^{[s]}(z)$, $s \geq m+k$ in (4.2). By Proposition 17-(b) we get, for a suitable constant C,

$$\left|(\boldsymbol{\delta}_m^{[k-j]})_\ell - \mathcal{F}_{m,p}^{k-j}(\ell 2^{-(k-j)})\right| = \left|(1 - e^{-p\lambda 2^{-m-(k-j)}})(\boldsymbol{\delta}_m^{[k-j]})_\ell\right| \leq C 2^{-m-(k-j)} \|\boldsymbol{\delta}_m^{[k-j]}\|.$$

This estimate and Theorem 19 yield the bound

$$\left|(\boldsymbol{\delta}_m^{[k-j]})_\ell - \phi_m(\ell 2^{-(k-j)})\right| \leq C(2^{-(k-j)} + \mu^{k-j}), \quad \mu \in (0,1), \qquad (4.21)$$

14

for some other constant $C > 0$. Moreover, to estimate $(\phi_m^{[k-j]})_\ell - (\phi_m^{[k-j]})_i$ with $|i - \ell| \leq 2M$ for some $M > 0$, we exploit the Hölder continuity of ϕ_m which follows by [5, Theorem 3] in view of the fact that approximate sum rules of order one are satisfied. Hence we get the bound

$$\left| \phi_m(\ell 2^{-(k-j)}) - \phi_m(i2^{-(k-j)}) \right| \leq (2M)^\alpha 2^{-\alpha(k-j)}, \quad \alpha \in (0, 1). \tag{4.22}$$

Setting $\theta_j := \left(\mu^j + 2^{-\alpha j} \right)$ we thus clearly have that $\{ \theta_j, j \in \mathbb{N} \}$ is an absolutely summable sequence. Then, combining (4.19), (4.21) and (4.22) with (4.18) and applying the fact that $\|\phi_m^{[k-j]}\| \leq \|\phi_m\|$, we have

$$\left| ((S_{\mathbf{a}^{[m+k-j]}} - S_{\mathbf{a}}) \boldsymbol{\delta}_m^{[k-j]})_i \right| \leq C\theta_{k-j} \sum_{\ell \in \mathbb{Z}} \left| a_{i-2\ell}^{[m+k-j]} - a_{i-2\ell} \right| + \left| (\phi_m^{[k-j]})_i \sum_{\ell \in \mathbb{Z}} (a_{i-2\ell}^{[m+k-j]} - a_{i-2\ell}) \right|$$

$$\leq C\|\mathbf{a}^{[m+k-j]} - \mathbf{a}\| \theta_{k-j} + \|\phi_m\| \left| \sum_{\ell \in \mathbb{Z}} \left(a_{i-2\ell}^{[m+k-j]} - a_{i-2\ell} \right) \right| \tag{4.23}$$

for some constant $C > 0$. Now, to bound the term $\left| \sum_{\ell \in \mathbb{Z}} \left(a_{i-2\ell}^{[m+k-j]} - a_{i-2\ell} \right) \right|$ we proceed as follows. Recalling the assumption that $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ reproduces $e^{\lambda x}$ with $\lambda \in \mathbb{C}$, we can apply Theorem 10 in the case $N = 1$, to write that

$$\left| a^{[k]}(1) - 2 \right| = \left| \sum_{\ell \in \mathbb{Z}} a_{2\ell}^{[k]} + \sum_{\ell \in \mathbb{Z}} a_{2\ell+1}^{[k]} - 2 \right| = O(2^{-k}), \quad k \to \infty. \tag{4.24}$$

Then, using the fact that $\{S_{\mathbf{a}}\}$ is a convergent stationary scheme, we also have that $\sum_{\ell \in \mathbb{Z}} a_{2\ell} + \sum_{\ell \in \mathbb{Z}} a_{2\ell+1} = 2$. As a consequence, from (4.24) we easily obtain that

$$\left| \sum_{\ell \in \mathbb{Z}} \left(a_{2\ell}^{[k]} - a_{2\ell} \right) + \sum_{\ell \in \mathbb{Z}} \left(a_{2\ell+1}^{[k]} - a_{2\ell+1} \right) \right| = O(2^{-k}), \quad k \to \infty.$$

Similarly, due to the fact that $a^{[k]}(-1) = O(2^{-k})$ as $k \to \infty$ (see Theorem 10), we get

$$\left| \sum_{\ell \in \mathbb{Z}} \left(a_{2\ell}^{[k]} - a_{2\ell} \right) - \sum_{\ell \in \mathbb{Z}} \left(a_{2\ell+1}^{[k]} - a_{2\ell+1} \right) \right| = O(2^{-k}), \quad k \to \infty,$$

implying that, for all $i \in \mathbb{Z}$,

$$\left| \sum_{\ell \in \mathbb{Z}} \left(a_{i-2\ell}^{[k]} - a_{i-2\ell} \right) \right| = O(2^{-k}), \quad k \to \infty.$$

Hence, from the latter equation the bound $\left| \sum_{\ell \in \mathbb{Z}} (a_{i-2\ell}^{[m+k-j]} - a_{i-2\ell}) \right| \leq C 2^{-(m+k-j)}$ is obtained straightforwardly. Moreover, since we also have the bound $\|S_{\mathbf{a}}^j\| < C$ for all j, for the summation in (4.17) we are finally able to write from (4.23) that

$$\sum_{j=0}^{k-1} \left| (S_{\mathbf{a}}^j (S_{\mathbf{a}^{[m+k-j]}} - S_{\mathbf{a}}) \boldsymbol{\delta}_m^{[k-j]})_i \right| \leq C \sum_{j=0}^{k-1} \left(\|\mathbf{a}^{[m+j+1]} - \mathbf{a}\| \theta_j + 2^{-(m+j+1)} \right) \tag{4.25}$$

with a suitable constant $C > 0$. Now we first let $k \to \infty$ and then $m \to \infty$. From the asymptotical similarity of $\{\mathbf{a}^{[k]}, k \geq 0\}$ and $\{\mathbf{a}\}$, we know that for all $\epsilon > 0$ there exists $\bar{m}$ such that for $m + j > \bar{m}$ we have $\|\mathbf{a}^{[m+j]} - \mathbf{a}\| \leq \epsilon$. Therefore we can conclude from (4.25) that $\lim_{m \to \infty} \lim_{k \to \infty} \|S_{\mathbf{a}^{[m+k]}} \cdots S_{\mathbf{a}^{[m]}} \boldsymbol{\delta} - S_{\mathbf{a}}^{k+1} \boldsymbol{\delta}\| = 0$. $\qquad \square$

5. Approximation order

In this section we estimate the approximation order of a Φ_N-reproducing non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ in case the initial data are sampled from a function in the Sobolev space $W_\infty^n(\mathbb{R})$,

15

$n \in \mathbb{N}$, $n \leq N$. The latter is defined to be the set of all functions f in $L_\infty(\mathbb{R})$ that have derivative $\frac{d^\ell f}{du^\ell} \in L_\infty(\mathbb{R})$ for all $0 \leq \ell \leq n$. We recall that, for any $f \in W_\infty^n(\mathbb{R})$, the associated norm is defined by

$$\|f\|_{n,\infty} := \sum_{\ell=0}^{n} \left\| \frac{d^\ell f}{du^\ell} \right\|_{L_\infty(\mathbb{R})}. \tag{5.1}$$

Let $f \in W_\infty^n(\mathbb{R})$ and let the initial data be of the form $\mathbf{f}^{[m]} := \{ \mathrm{f}_i^{[m]} = f(2^{-m}i), \ i \in \mathbb{Z} \}$ for some $m \geq 0$. In the following theorem we estimate the convergence order of the error $\|f - g_{\mathbf{f}^{[m]}}\|_{L_\infty(\mathbb{R})}$ as $m \to \infty$, where $g_{\mathbf{f}^{[m]}}$ is the limit of the subdivision scheme obtained from the initial data $\mathbf{f}^{[m]}$. We emphasize that Theorem 21 extends the result in [25, Theorem 2.4], where the more restrictive assumption of asymptotical equivalence rather than asymptotical similarity is assumed.

Theorem 21. *Assume that the non-stationary scheme $\{S_{\mathbf{a}^{[k]}}, \ k \geq 0\}$ is Φ_N-reproducing and is asymptotically similar to a convergent stationary scheme $\{S_{\mathbf{a}}\}$. Assume further that the initial data are of the form $\mathbf{f}^{[m]} := \{ \mathrm{f}_i^{[m]} = f(2^{-m}i), \ i \in \mathbb{Z} \}$ for some fixed $m \geq 0$ and for some function $f \in W_\infty^\gamma(\mathbb{R})$ where $\gamma \in \mathbb{N}$, $\gamma \leq N$. If the Wronskian matrix $\mathcal{W}_{\Phi_\gamma}(0)$ of $\Phi_\gamma \subseteq \Phi_N$ is invertible, then*

$$\|g_{\mathbf{f}^{[m]}} - f\|_{L_\infty(\mathbb{R})} \leq C_f 2^{-\gamma m} \quad m \geq 0$$

with a constant $C_f > 0$ depending only on f.

Proof. For the Sobolev exponent γ, let $\Phi_\gamma := \{\varphi_0, \ldots, \varphi_{\gamma-1}\}$, $\gamma \leq N$ and let x be a fixed point in $\mathbb{R}$. Our proof employs an auxiliary function ψ_x, which depends on x and is defined by

$$\psi(u) := \psi_x(u) := \sum_{n=0}^{\gamma-1} d_n \varphi_n(u - x), \quad u \in \mathbb{R}$$

where the entries of the coefficient vector $\mathbf{d} := (d_n, \ n = 0, \cdots, \gamma - 1)$ are obtained by solving the linear system

$$\frac{d^r \psi(x)}{du^r} = \frac{d^r f(x)}{du^r}, \quad r = 0, \ldots, \gamma - 1. \tag{5.2}$$

In this way, ψ equals f at x, but not on $\mathbb{R}$. Equations in (5.2) can be equivalently written in the matrix form

$$\mathcal{W}_{\Phi_\gamma}(0) \cdot \mathbf{d}^T = \mathbf{f}^T, \tag{5.3}$$

with $\mathbf{f} := (\frac{d^r f(x)}{du^r}, \ r = 0, \cdots, \gamma - 1)$. Due to the invertibility of the Wronskian matrix $\mathcal{W}_{\Phi_\gamma}(0)$, the uniqueness of the solution of this linear system is guaranteed. Moreover, since the function ψ belongs to Φ_N and the non-stationary scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ is Φ_N-reproducing, we can write

$$\psi(u) = \sum_{i \in \mathbb{Z}} \psi(2^{-m}i) \phi_m(2^m u - i), \quad u \in \mathbb{R},$$

with $\{\phi_m, \ m \geq 0\}$ denoting the basic limit functions of $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ defined in (4.1). By assumption, $\mathrm{f}_i^{[m]} = f(2^{-m}i)$, $i \in \mathbb{Z}$ and, in view of the linearity of the subdivision operators, we can write

$$g_{\mathbf{f}^{[m]}}(u) = \sum_{i \in \mathbb{Z}} \mathrm{f}_i^{[m]} \phi_m(2^m u - i), \quad u \in \mathbb{R}. \tag{5.4}$$

Thus, using the expression of $g_{\mathbf{f}^{[m]}}$ in (5.4) and observing that $f(x) = \psi(x)$ due to the construction of ψ in (5.2), we can rewrite $f(x) - g_{\mathbf{f}^{[m]}}(x)$ in the following way:

$$f(x) - g_{\mathbf{f}^{[m]}}(x) = \psi(x) - \sum_{i \in \mathbb{Z}} f(2^{-m}i) \phi_m(2^m x - i) = \sum_{i \in \mathbb{Z}} \left(\psi(2^{-m}i) - f(2^{-m}i) \right) \phi_m(2^m x - i)$$

16

Now, let $T_g^\gamma(u) := \sum_{\ell=0}^{\gamma-1}(u-x)^\ell \frac{d^\ell g(x)}{du^\ell}/\ell!$ be the degree-$(\gamma-1)$ Taylor polynomial of a function $g \in C^{\gamma-1}(\mathbb{R})$ around x. Then, consider the Taylor expansions T_ψ^γ and T_f^γ of the functions ψ and f, respectively. Due to the condition in (5.2), it is obvious that $T_\psi^\gamma(2^{-m}i) = T_f^\gamma(2^{-m}i)$. Hence, applying the remainder form of the Taylor expansion, we get

$$|f(x) - g_{\mathbf{f}[m]}(x)| \le \frac{2^{-m\gamma}}{\gamma!} \sum_{i\in\mathbb{Z}} \left| (i - 2^m x)^\gamma \frac{d^\gamma(f-\psi)(\xi_i)}{du^\gamma} \phi_m(2^m x - i) \right|,$$

for some ξ_i between x and $i2^{-m}$. By (5.2) and (5.3), we are able to write $|\frac{d^\gamma\psi(\xi_i)}{du^\gamma}| \le C_\gamma \|f\|_{\gamma,\infty}$ for some positive constant C_γ independent of x and $i2^{-m}$, with $\|\cdot\|_{\gamma,\infty}$ in (5.1). Thus, it is immediate that

$$|f(x) - g_{\mathbf{f}[m]}(x)| \le C_\gamma \frac{2^{-m\gamma}}{\gamma!} \|f\|_{\gamma,\infty} \sum_{i\in\mathbb{Z}} |\phi_m(2^m x - i)(i - 2^m x)^\gamma|. \tag{5.5}$$

Finally, by Theorem 20, ϕ_m is uniformly bounded independent of m. Moreover, since ϕ_m is compactly supported, $\#\{i \in \mathbb{Z} : \phi_m(2^m x - i) \neq 0\} \le C$ for any fixed x and m. Therefore the claim follows from (5.5). $\qquad\square$

6. Conclusions

In this paper we have shown that for primal or dual non-stationary subdivision schemes (essentially the ones of interest in applications) the reproduction of N exponential polynomials implies approximate sum rules of order N. This mimics the stationary case where the reproduction of N polynomials implies sum rules of order N. Furthermore, also in analogy to the stationary case where generation of N polynomials implies sum rules of order N, we have shown that generation of N exponential polynomials implies approximate sum rules of order N if asymptotical similarity to a convergent stationary scheme is assumed together with reproduction of a single exponential polynomial.

We additionally considered the non-stationary counterpart of the well-known result asserting that the reproduction of an N-dimensional space of polynomials is sufficient for the subdivision scheme to have approximation order N. In particular, for a non-stationary subdivision scheme, the reproduction of an N-dimensional space of exponential polynomials, jointly with asymptotical similarity, has been shown to imply approximation order N. The proof of the latter required us to show also the uniform convergence of the sequence of basic limit functions of a non-stationary scheme, reproducing one exponential polynomial, to the basic limit function of the asymptotically similar stationary scheme.

We finally remark that, since asymptotical similarity is needed to get several of our results, we cannot claim that approximate sum rules are the complete satisfactory notion to replace sum rules when moving from the stationary to the non-stationary setting. However, this notion is definitely very helpful in almost all practical cases.

Acknowledgements

17

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19