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ANALYSIS OF TRUSS-LIKE CRACKED STRUCTURES WITH UNCERTAIN-BUT-BOUNDED DEPTHS

Roberta Santoro[1], **and Cristina Gentilini**[2]

[1] Department of Engineering, University of Messina

Villaggio S. Agata, 98166 Messina, Italy

e-mail: roberta.santoro@unime.it

[2] Department of Architecture, University of Bologna

Viale del Risorgimento 2, 40136 Bologna, Italy

cristina.gentilini@unibo.it

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Abstract. *In this paper, an approach that deals with the dynamic response of linear elastic truss structures with cracked members is presented. Crack depths are modeled as uncertain-but-bounded variables. The objective of the study is the evaluation of the time-varying upper and lower bounds of the response of a truss structure with multiple cracks with uncertain interval depths subjected to a deterministic excitation. The proposed procedure is validated through numerical tests on truss-like structures. The accuracy is evidenced by the excellent agreement between the response bounds calculated by the present approach compared with the exact bounds derived via a combinatorial procedure.*

INTRODUCTION

The present paper deals with the dynamic response of linear elastic structures with cracked members [1]. For a realistic prediction of the mechanical response of such structures, the unavoidable uncertainty affecting the cracked members has to be taken into account. To this aim, the crack depths are modeled as uncertain-but-bounded variables [2]. The objective of the study is the evaluation of the time-varying upper and lower bounds of the response of a truss structure with multiple cracks with uncertain interval depths subjected to a deterministic excitation. The analysis is performed by adopting a finite element approach where the crack in a member is modeled by introducing an additional local compliance that produces a discontinuity of the displacement in correspondence of the cracked section [3,4]. The compliance of the cracked member is determined by simply adding the compliance of the intact element to the overall compliance due to the crack. In order to provide the bounds of the response, an approach originating from [5-9] is followed. It requires the derivation of the bounds of the interval eigenvalues evaluated as solution of two suitable deterministic eigenvalue problems. The proposed procedure is validated through numerical tests on truss-like structures. The accuracy is evidenced by the excellent agreement between the response bounds calculated by the present approach compared with the exact bounds derived via a combinatorial procedure (vertex method) [10]. The application of the method can be straightforwardly extended to frame-like structures with uncertain damage.

FORMULATION OF THE PROBLEM

1.1 Damaged member with uncertain-but-bounded crack depth

Consider a linear elastic truss structure subjected to a dynamic load. The 1 × 4 displacement vector for the *i-th* bar element of length L_i, mass density ρ, elastic modulus E and cross-sectional area A_i, which connects node 1 to node 2 is $\mathbf{u}_i(t) = \left[u_1(t) \quad v_1(t) \quad u_2(t) \quad v_2(t) \right]$ (see Fig. 1).

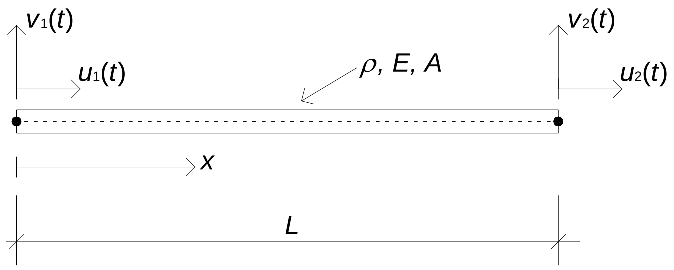

Figure 1: Bar element.

We consider that the *i-th* element of the truss structure is damaged with a crack of depth a_i. The presence of the crack increases the axial compliance of the *i-th* bar

$$c_i = c_i(a_i) \tag{1}$$

and it is taken into account introducing for the *i*-th damaged element an additional axial compliance, named c_i^{crack}, to the intact axial compliance, c_i^{intact}. Parameter c_i^{intact} for an intact, homogeneous bar with a constant cross-section is given by

$$c_i^{intact} = L_i / (EA_i) \tag{2}$$

The additional compliance c_i^{crack} can be expressed by the following expression:

$$c_i^{crack}\left(a_i\right)=\frac{2\left(1-\nu^2\right)}{E}\int_0^{A_{crack}}\left(\frac{K_{IN}}{N}\right)^2 \mathrm{d}A_{crack} \tag{3}$$

where A_{crack} is the cracked area, K_{IN} is the Mode I stress intensity factor [1], ν is the Poisson ratio and N is the axial force. The total axial compliance of the *i*-th cracked element results equal to

$$c_i\left(a_i\right)=c_i^{intact}+c_i^{crack}\left(a_i\right) \tag{4}$$

If the depth of the crack a_i is modeled as an uncertain-but-bounded variable $a_i^I=[\underline{a}_i,\overline{a}_i]$ in terms of its lower and upper bounds , it can be expressed following the interval analysis symbolism [2] as

$$a_i^I=a_{0,i}\left(1+\alpha_i^I\right)=a_{0,i}\left(1+\Delta\alpha_i\, e_i^I\right) \tag{5}$$

with $a_{0,i}$ the mean crack depth (midpoint), $\Delta\alpha_i$ representing the symmetric dimensionless fluctuation of the crack depth and $e_i^I=\left[-1,1\right]$ the unitary interval. The superscript *I* stands for the interval value. Substituting $e_i^I=-1$ in Eq. (5), the lower bound $\underline{a}_i$ of the crack depth is obtained, while substituting $e_i^I=1$ in Eq. (5), the upper bound depth $\overline{a}_i$ is derived. In presence of *r* cracked bars, the midpoint vector $\mathbf{a}_0$ of order $r\times 1$ collects the midpoints $a_{0,i}$ while the vector $\boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right]$ collects the fluctuations of the uncertain crack depths around their mean values. Following a standard assembling procedure, the stiffness matrix of the truss structure with uncertain damage can be derived as:

$$\mathbf{K}(\mathbf{a}_0,\boldsymbol{\alpha}^I)=\mathbf{G}^T\mathbf{C}^{-1}(\mathbf{a}_0,\boldsymbol{\alpha}^I)\mathbf{G} \tag{6}$$

where $\mathbf{G}$ and $\mathbf{G}^T$ are the compatibility and equilibrium matrices, respectively, and $\mathbf{C}(\mathbf{a}_0,\boldsymbol{\alpha}^I)$ is the uncertain compliance matrix. Superscript *T* denotes the transpose of the matrix.

1.2 Governing equation for a truss structure with cracked members

Based on the previous considerations and assumptions, the governing equation of motion of a quiescent cracked truss structure with *r* damaged members under an external excitation $\mathbf{f}(t)$ takes the following form:

$$\mathbf{M}\,\ddot{\mathbf{u}}(\mathbf{a}_0,\boldsymbol{\alpha}^I,t)+\mathbf{D}(\mathbf{a}_0,\boldsymbol{\alpha}^I)\,\dot{\mathbf{u}}(\mathbf{a}_0,\boldsymbol{\alpha}^I,t)+\mathbf{K}(\mathbf{a}_0,\boldsymbol{\alpha}^I)\,\mathbf{u}(\mathbf{a}_0,\boldsymbol{\alpha}^I,t)=\mathbf{f}(t) \tag{7}$$

where $\mathbf{M}$ is the mass matrix of the structure, $\mathbf{D}(\mathbf{a}_0,\boldsymbol{\alpha}^I)$ is the damping matrix which is in turn affected by the presence of uncertainty in the cracked elements and $\mathbf{K}(\mathbf{a}_0,\boldsymbol{\alpha}^I)$ is the stiffness matrix of the structure as defined in Eq. (6). The *interval* vector $\mathbf{u}(\mathbf{a}_0,\boldsymbol{\alpha}^I,t)$ collects the nodal displacements, while symbol dot over a variable denotes differentiation with respect to time *t*. In this paper, a proportional damping matrix (Rayleigh model) is considered and expressed as a linear combination of the mass and stiffness matrices as:

$$\mathbf{D}(\mathbf{a}_0,\boldsymbol{\alpha}^I)=d_0\,\mathbf{M}+d_1\mathbf{K}(\mathbf{a}_0,\boldsymbol{\alpha}^I) \tag{8}$$

where d_0 and d_1 are the Rayleigh damping constants to evaluate.

RESPONSE BOUNDS

The solution of the dynamical problem involving interval parameters as expressed in Eq. (7) requires the evaluation of the interval vector collecting the dynamical response set at each instant i.e.

$$\mathbf{u}(\mathbf{a}_0,\boldsymbol{\alpha}^I,t)=\left[\underline{\mathbf{u}}(\mathbf{a}_0,t),\,\bar{\mathbf{u}}(\mathbf{a}_0,t)\right] \tag{9}$$

In presence of uncertain-but-bounded parameters, the evaluation of the natural frequencies and associated mode shapes requires the solution of an interval eigenvalue problem expressed by:

$$\mathbf{K}(\mathbf{a}_0,\boldsymbol{\alpha}^I)\boldsymbol{\phi}_j(\mathbf{a}_0,\boldsymbol{\alpha}^I)=\lambda_j(\mathbf{a}_0,\boldsymbol{\alpha}^I)\mathbf{M}\boldsymbol{\phi}_j(\mathbf{a}_0,\boldsymbol{\alpha}^I);\quad (j=1,2,\ldots,n) \tag{10}$$

In Eq. (10), $\lambda_j(\mathbf{a}_0,\boldsymbol{\alpha}^I)=\omega_j^2(\mathbf{a}_0,\boldsymbol{\alpha}^I)$ represents the *j*-th interval eigenvalue and $\boldsymbol{\phi}_j(\mathbf{a}_0,\boldsymbol{\alpha}^I)$ is the associated interval eigenvector, which affected by the uncertainties, turns out to be bounded by interval vectors, namely $\boldsymbol{\phi}_j(\mathbf{a}_0)\in\boldsymbol{\phi}_j(\mathbf{a}_0,\boldsymbol{\alpha}^I)$. Among of all possible eigenvalues satisfying Eq. (10), the matrix $\mathbf{K}(\mathbf{a}_0,\boldsymbol{\alpha}^I)$ assumes all possible values inside the interval defined by

$$\mathbf{K}(\mathbf{a}_0,\boldsymbol{\alpha}^I)=\left[\underline{\mathbf{K}}(\mathbf{a}_0),\ \bar{\mathbf{K}}(\mathbf{a}_0)\right]=\left\{\mathbf{K}(\mathbf{a}_0,\boldsymbol{\alpha})\,\middle|\ \underline{k}_{ij}(\mathbf{a}_0)\le k_{ij}(\mathbf{a}_0)\le \bar{k}_{ij}(\mathbf{a}_0)\right\} \tag{11}$$

The aim is the evaluation of the narrowest interval enclosing all possible eigenvalues satisfying Eq. (10), i.e. [5-9]:

$$\lambda_j(\mathbf{a}_0,\boldsymbol{\alpha})=\left[\underline{\lambda}_j(\mathbf{a}_0),\bar{\lambda}_j(\mathbf{a}_0)\right] \tag{12}$$

where $\underline{\lambda}_j(\mathbf{a}_0)$ and $\bar{\lambda}_j(\mathbf{a}_0)$ with $(j=1,2,\ldots,n)$, represent the *LB* and *UB* of the *j*-th interval eigenvalue.

The problem can be handled by solving two deterministic eigenvalue problems reported in the following:

$$\begin{aligned}&\mathbf{K}\left(\mathbf{a}_0,\underline{\boldsymbol{\alpha}}\right)\boldsymbol{\phi}_j^{(\mathrm{LB})}(\mathbf{a}_0)=\underline{\lambda}_j(\mathbf{a}_0)\mathbf{M}\boldsymbol{\phi}_j^{(\mathrm{LB})}(\mathbf{a}_0);\quad \boldsymbol{\phi}_j^{(\mathrm{LB})}(\mathbf{a}_0)^T\mathbf{M}\boldsymbol{\phi}_k^{(\mathrm{LB})}(\mathbf{a}_0)=\Delta_{jk}\\&\mathbf{K}\left(\mathbf{a}_0,\bar{\boldsymbol{\alpha}}\right)\boldsymbol{\phi}_j^{(\mathrm{UB})}(\mathbf{a}_0)=\bar{\lambda}_j(\mathbf{a}_0)\mathbf{M}\boldsymbol{\phi}_j^{(\mathrm{UB})}(\mathbf{a}_0);\quad \boldsymbol{\phi}_j^{(\mathrm{UB})}(\mathbf{a}_0)^T\mathbf{M}\boldsymbol{\phi}_k^{(\mathrm{UB})}(\mathbf{a}_0)=\Delta_{jk},\ (j=1,2,\ldots n).\end{aligned} \tag{13a,b}$$

where Δ_{jk} is the Kronecker delta, $\boldsymbol{\phi}_j^{(\mathrm{LB})}(\mathbf{a}_0)$ and $\boldsymbol{\phi}_j^{(\mathrm{UB})}(\mathbf{a}_0)$ are the eigenvectors associated to the eigenproblem in which $\boldsymbol{\alpha}=\underline{\boldsymbol{\alpha}}$ and $\boldsymbol{\alpha}=\bar{\boldsymbol{\alpha}}$, respectively. The eigenvectors of both eigenproblems are real vectors, while the eigenvalues are real and positive quantities. Notice that the two stiffness matrices $\mathbf{K}\left(\mathbf{a}_0,\underline{\boldsymbol{\alpha}}\right)$ and $\mathbf{K}\left(\mathbf{a}_0,\bar{\boldsymbol{\alpha}}\right)$ as well as the mass matrix $\mathbf{M}$ are real, symmetric and positive definite matrices.

As it can be noted from Eq. (13a,b), the considered narrowest interval for each eigenvalue corresponds to the so-called *trivial* endpoint combinations for the interval parameters: lower bounds and upper bounds of the eigenvalues are provided by selecting simultaneously all the uncertain-but-bounded parameters to their lower bounds $\boldsymbol{\alpha}=\underline{\boldsymbol{\alpha}}$ and upper bounds $\boldsymbol{\alpha}=\bar{\boldsymbol{\alpha}}$, respectively.

To solve the double equations of motion (7), two coordinate transformations are introduced as follows:

$$\mathbf{u}(\underline{\boldsymbol{\alpha}},t)=\boldsymbol{\Phi}^{(\mathrm{LB})}(\mathbf{a}_0)\,\mathbf{q}(\underline{\boldsymbol{\alpha}},t)$$
$$\mathbf{u}(\bar{\boldsymbol{\alpha}},t)=\boldsymbol{\Phi}^{(\mathrm{UB})}(\mathbf{a}_0)\,\mathbf{q}(\bar{\boldsymbol{\alpha}},t) \tag{14}$$

where the interval vector of modal displacement $\mathbf{q}(\boldsymbol{\alpha}^I,t)$ and the matrices $\boldsymbol{\Phi}^{(\mathrm{LB})}(\mathbf{a}_0)$ and $\boldsymbol{\Phi}^{(\mathrm{UB})}(\mathbf{a}_0)$ whose $j-$th column is $\boldsymbol{\phi}_j^{(\mathrm{LB})}(\mathbf{a}_0)$ and $\boldsymbol{\phi}_j^{(\mathrm{UB})}(\mathbf{a}_0)$, respectively, are introduced. By applying these coordinate transformations, the equations of motion can be projected in the modal space:

$$\ddot{\mathbf{q}}(\underline{\boldsymbol{\alpha}},t)+\underline{\boldsymbol{\Xi}}(\mathbf{a}_0)\,\dot{\mathbf{q}}(\underline{\boldsymbol{\alpha}},t)+\underline{\boldsymbol{\Omega}}^2(\mathbf{a}_0)\,\mathbf{q}(\underline{\boldsymbol{\alpha}},t)=\boldsymbol{\Phi}^{(\mathrm{LB})}(\mathbf{a}_0)^T\,\mathbf{f}(t)$$
$$\ddot{\mathbf{q}}(\bar{\boldsymbol{\alpha}},t)+\bar{\boldsymbol{\Xi}}(\mathbf{a}_0)\,\dot{\mathbf{q}}(\bar{\boldsymbol{\alpha}},t)+\bar{\boldsymbol{\Omega}}^2(\mathbf{a}_0)\,\mathbf{q}(\bar{\boldsymbol{\alpha}},t)=\boldsymbol{\Phi}^{(\mathrm{UB})}(\mathbf{a}_0)^T\,\mathbf{f}(t) \tag{15}$$

where $\underline{\boldsymbol{\Omega}}^2(\mathbf{a}_0)$ and $\bar{\boldsymbol{\Omega}}^2(\mathbf{a}_0)$ are diagonal matrices whose $j-$th element is $\underline{\lambda}_j(\mathbf{a}_0)$ and $\bar{\lambda}_j(\mathbf{a}_0)$, respectively, and $\underline{\boldsymbol{\Xi}}(\mathbf{a}_0)$ and $\bar{\boldsymbol{\Xi}}(\mathbf{a}_0)$ are the generalized diagonal damping matrices, which according to the Rayleigh model, can be written as:

$$\underline{\boldsymbol{\Xi}}(\mathbf{a}_0)=d_0\,\mathbf{I}_m+d_1\,\underline{\boldsymbol{\Omega}}^2(\mathbf{a}_0)$$
$$\bar{\boldsymbol{\Xi}}(\mathbf{a}_0)=d_0\,\mathbf{I}_m+d_1\,\bar{\boldsymbol{\Omega}}^2(\mathbf{a}_0) \tag{16}$$

obtaining a set of decoupled differential equations.

Solution of Eq. (15) in terms of vectors $\mathbf{q}(\underline{\boldsymbol{\alpha}},t)$ and $\mathbf{q}(\bar{\boldsymbol{\alpha}},t)$ allows to provide the corresponding nodal responses by the coordinate transformation, Eq. (14).

Lower and upper bounds of the *k*-th component of the interval dynamic response $\mathbf{u}(\boldsymbol{\alpha}^I,t)$ can be calculated by the following relationships:

$$\underline{u}_k(t)=\min\left\{u_k(\underline{\boldsymbol{\alpha}},t),\,u_k(\bar{\boldsymbol{\alpha}},t)\right\};\quad \bar{u}_k(t)=\max\left\{u_k(\underline{\boldsymbol{\alpha}},t),\,u_k(\bar{\boldsymbol{\alpha}},t)\right\} \tag{17}$$

where the symbols $\min\{\cdot\}$ and $\max\{\cdot\}$ denote minimum (inferior) and maximum (superior) values, respectively.

NUMERICAL APPLICATION

In this section, the performance of the present procedure is illustrated through a numerical test. For comparison, the results of reference combinatorial method [10] are also included. The analysis is conducted on a steel 25-bar truss structure, as represented in Fig. 2, subjected to a UnitStep function applied at the node 9 in the x –direction, namely $f_{x,9}(t)=F_0U(t)$ with $F_0=15$ KN. All the bars are assumed to have cross-sectional area $A_i=A=0.01\ \mathrm{m}^2$ (prismatic section with *b* = *h* = 0.1 m) with $i=1,\ 2,\ldots,25$ and lengths L_i deducible from Fig.2 where $L=5.1$ m. Young's modulus and Poisson's ratio are $E=2.1\times10^8\ \mathrm{kN/m^2}$ and $\nu=0.3$, respectively. Furthermore, each node possess a lumped mass $M=500$ kg. All the vertical bars are supposed to be damaged (*r*=10) with crack depths modeled as interval parameters $a_i^I=a_{0,i}\left(1+\Delta\alpha_i e_i^I\right)$, $(i=1,2,...,10)$ with midpoint value $a_{0,i}=a_0=0.4h\ \forall i$ and deviation amplitudes $\Delta\alpha_i=\Delta\alpha=0.3$.

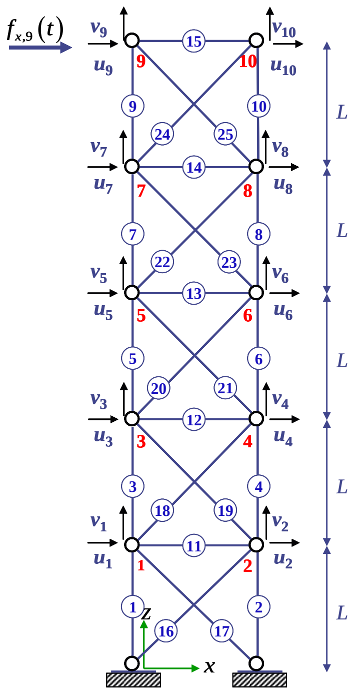

Figure 2: Truss structure: geometry and load condition.

Without loss of generality, the Rayleigh damping constants d_0 and d_1 were evaluated relating to the damaged mean configuration using as reference the mean stiffness matrix defined as $\mathbf{K}(\mathbf{a}_0,\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\mathbf{0}}$, see Eq. (6). As a consequence, in Eq. (8) the values $d_0 = 4.09071s^{-1}$ and $d_1 = 0.000347s$ are derived in such a way that the modal damping ratio for the first and second modes of the nominal structure is $\zeta_0 = 0.05$.

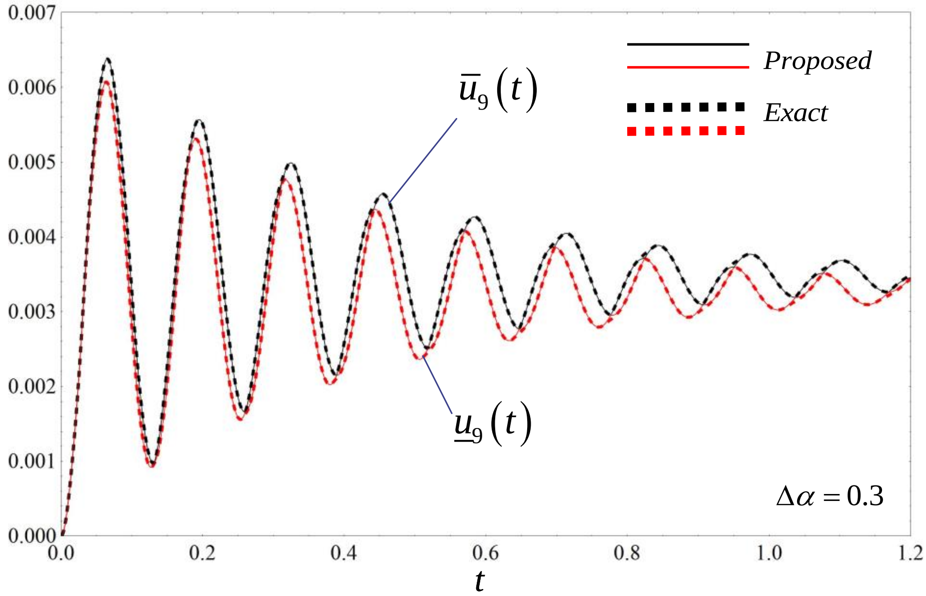

Figure 3: Time-varying lower and upper bounds of the 9-th node displacement in *x*-direction of the damaged truss structure with *r*=10 uncertain-but-bounded parameters:
proposed method (continuous lines) and exact solution (dotted lines).

Figure 3 shows the time-varying bounds of the 9-*th* node displacement in *x*-direction, obtained by applying the procedure explained in Section 3. The bounds provided by the proposed method (continuous lines) are compared with the results obtained by the combinatorial vertex method (dotted lines), which requires the evaluation of $2^{r=10}=1024$ problems and considered as the reference solution. It is worth to note the excellent agreement between the results of the two procedures.

CONCLUSIONS

In the present study a numerical approach that dealt with the linear dynamic analysis of truss structures with multiple cracked members was presented. The depth of the crack was modeled as an uncertain variable but bounded in a specific interval. The time-varying upper and lower bounds of the response were calculated for a 25-bar truss structure. Results obtained with the proposed procedure were validated by comparison with the response bounds derived via a combinatorial procedure, considered as the exact solution. The excellent agreement between the two approaches revealed the accuracy of the proposed approach in dealing with these type of problems.

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