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Published Version:
Convergence analysis of corner cutting algorithms refining nets of functions / Conti, Costanza; Dyn, Nira; Romani, Lucia. - In: MATHEMATICS AND COMPUTERS IN SIMULATION. - ISSN 0378-4754. - STAMPA. - 176:(2020), pp. 134-146. [10.1016/j.matcom.2020.01.012]

Availability:
This version is available at: https://hdl.handle.net/11585/726443 since: 2021-02-21

Published:
DOI: http://doi.org/10.1016/j.matcom.2020.01.012

Convergence analysis of corner cutting algorithms refining nets of functions

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Abstract

In this paper we propose a corner cutting algorithm for nets of functions and prove its convergence using some approximation ideas first applied to the case of corner cutting algorithms refining points with weights proposed by Gregory and Qu. In the net case convergence is proved for the above mentioned weights satisfying an additional condition. The condition requires a bound on the supremum of the relative sizes of the cuts.

Keywords: Corner cutting for polygonal lines; Coons transfinite interpolation; Corner cutting for nets of functions; Convergence; Lipschitz continuity

1. Introduction

This paper considers corner cutting algorithms refining nets of functions and proves the convergence of these algorithms by extending a new approach to the proof of convergence in the case of corner cutting algorithms refining points.

The first instance of a corner cutting algorithm for points was proposed by de Rham in [21, 22], where curves are obtained by repeatedly cutting off the corners of a given polygon. Precisely, at each iteration each edge of the current polygon is divided into three pieces in the ratio $w : (1-2w) : w$, where w is a given parameter. The de Rham process is convergent to a continuous curve if $w \in (0, \frac{1}{2})$ and to a differentiable curve if $w < \frac{1}{3}$ (see [20]). If $w = \frac{1}{4}$ the de Rham curve is a quadratic spline and the corresponding iterative algorithm is also known as the Chaikin algorithm introduced independently in [4]. A natural generalization of the de Rham algorithm is obtained by dividing each polygon edge into more than three pieces (see [11]) or by dividing each edge of the k-th iteration into three pieces in the level-dependent ratio $\alpha^k : (\beta^k - \alpha^k) : 1 - \beta^k$ where α^k and β^k are given parameters, or even by choosing α_i^k and β_i^k depending on the level and the location (see [1, 2, 14, 17, 19]). The latter case is the most relevant to our paper and its convergence was analyzed in [2].

Corner cutting algorithms are special instances of non-uniform subdivision schemes, and they can be analyzed by tools for subdivision (see, e.g., [5, 13]). Yet, the applications of these tools to the analysis of corner cutting for nets is not clear.

In this paper, we provide an alternative proof of convergence, based on approximation arguments, for the general class of corner cutting algorithms for points with weights as in [17]. The choice of this analysis method is motivated by the fact that the approximation arguments are naturally extendable to the case of nets. Indeed, the two main achievements of this paper are the design of a corner cutting algorithm for nets of functions (generalization of [8]) and the proof of its convergence for weights as in [17] but satisfying an additional condition which requires a bound on the supremum of the relative sizes of the cuts.

The key idea of our corner cutting algorithm for nets of functions is to construct, at each recursion step, a C^0-piecewise Coons interpolant [6] to the coarse net of functions, from which the new refined net is sampled. This

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approach extends to the net case the classical procedure used by corner cutting algorithms refining points where, at each recursion step, the refined points are sampled from a piecewise linear interpolant to the given points.

Besides the theoretical interest of the convergence result, corner cutting algorithms for nets of functions generate a variety of C^0 bivariate functions approximating the initial net, with the corner cutting weights acting as shape parameters. In a future work we plan to study the smoothness of the limits in the case of nets, and to derive conditions on the corner cutting weights which guarantee C^1 limit functions. This was investigated in the case of points in [3] and in [17].

The structure of the paper is as follows. In Section 2 we give our proof of the convergence of corner cutting algorithms refining points (polygonal lines) based on a nice approximation argument. In Section 3 we consider the case of bivariate nets of functions. First, in Subsection 3.1 we give preliminary results on Coons patches (see [6]) and their approximation properties since they are analogous to linear interpolants in the case of points. Then, in Subsection 3.2 we introduce the notion of bivariate nets of functions and present a corner cutting algorithm for them. The convergence theorem and its proof are given in Subsection 3.3. Conclusions are drawn in Section 4.

2. Corner cutting algorithms for points in $\mathbb{R}^n$

Corner cutting algorithms for points, as subdivision algorithms refining points (see [12]), are iterative methods that starting from a given sequence of points $\mathbf{p}^{[0]} = \{p_i^{[0]}, i \in \mathbb{Z}\}$ produce at each iteration denser and denser sequences of points $\mathbf{p}^{[k]}$, $k > 0$. Whenever convergent, they allow the user to define a continuous curve that approximates the shape described by the given polyline.

In this section we investigate the convergence of univariate corner cutting schemes under the assumption that the corner cutting weights are as in Gregory and Qu [17]. Convergence of corner cutting algorithms is equivalent to the uniform convergence of any sequence of continuous piecewise interpolants to the points generated through the iterative process. Our proof of convergence is based on the simple but crucial observation that the piecewise linear interpolant $\mathcal{L}(\mathbf{u}^{[k+1]}, \mathbf{p}^{[k+1]})$ to the points $\mathbf{p}^{[k+1]}$ at the corresponding parameter values $\mathbf{u}^{[k+1]}$ is a piecewise linear interpolant to $\mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})$ at $(\mathbf{u}^{[k+1]}, \mathbf{p}^{[k+1]})$, where the parameter values $\mathbf{u}^{[k]}$ are generated by the same corner cutting procedure as $\mathbf{p}^{[k]}$. Using an elementary error formula, we show that the sequence of piecewise linear interpolants $\{\mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})\}_{k\geq 0}$ is a Cauchy sequence.

Definition 2.1 (Corner cutting weights)**.** *Let $\ell(\mathbb{Z})$ be the set of scalar valued sequences indexed by $\mathbb{Z}$. We denote by $\mathcal{W}$ a subset of $\ell(\mathbb{Z}) \times \ell(\mathbb{Z})$ of the form*

$$\mathcal{W} := \left\{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \ell(\mathbb{Z}) \times \ell(\mathbb{Z}) \ : \ \inf_{i\in\mathbb{Z}} \{\alpha_i,\ 1-\beta_i,\ \beta_i - \alpha_i\} > 0\right\}. \tag{2.1}$$

Moreover, for $\boldsymbol{\gamma} := (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{W}$ we define

$$\mu(\boldsymbol{\gamma}) := \sup_{i\in\mathbb{Z}} \{\beta_i - \alpha_i,\ 1 - \beta_{i-1} + \alpha_i\}. \tag{2.2}$$

Now let $\ell^n(\mathbb{Z})$ denote the set of vector valued sequences indexed by $\mathbb{Z}$ and let $\mathbf{P} = \{P_i \in \mathbb{R}^n,\ i \in \mathbb{Z}\} \in \ell^n(\mathbb{Z})$. In the following we define the corner cutting operator for an arbitrary sequence $\mathbf{P}$ of points in $\mathbb{R}^n$.

Definition 2.2 (Corner cutting operator)**.** *The* corner cutting *operator with corner cutting weights $\boldsymbol{\gamma} := (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{W}$, denoted by $CC_{\boldsymbol{\gamma}}$, maps $\ell^n(\mathbb{Z})$ into $\ell^n(\mathbb{Z})$. For $\mathbf{P} \in \ell^n(\mathbb{Z})$*

$$\left(CC_{\boldsymbol{\gamma}}(\mathbf{P})\right)_{2i} = (1-\alpha_i)P_i + \alpha_i P_{i+1}, \qquad \left(CC_{\boldsymbol{\gamma}}(\mathbf{P})\right)_{2i+1} = (1-\beta_i)P_i + \beta_i P_{i+1}. \tag{2.3}$$

Remark 2.3. *The corner cutting operator given in Definition 2.2 is the same as the one studied in [17]. A more general corner cutting operator is considered in [2]. The condition required in* (2.1) *on the corner cutting weights is related to the observation that*

$$\alpha_i = \frac{\|Q_{2i} - P_i\|_2}{\|P_{i+1} - P_i\|_2}, \qquad 1 - \beta_i = \frac{\|P_{i+1} - Q_{2i+1}\|_2}{\|P_{i+1} - P_i\|_2}, \qquad \beta_i - \alpha_i = \frac{\|Q_{2i+1} - Q_{2i}\|_2}{\|P_{i+1} - P_i\|_2},$$

where $Q_{2i} = \left(CC_{\boldsymbol{\gamma}}(\mathbf{P})\right)_{2i}$, $Q_{2i+1} = \left(CC_{\boldsymbol{\gamma}}(\mathbf{P})\right)_{2i+1}$.

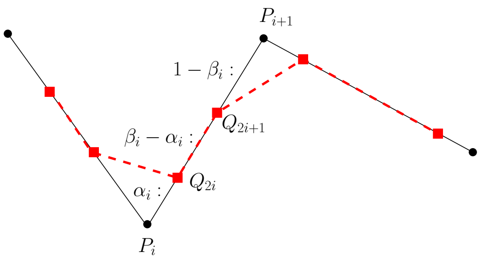

Figure 1: One application of the $CC_{\boldsymbol{\gamma}}$-operator on a sequence of points in $\ell^2(\mathbb{Z})$. Here $Q_{2i} = \left(CC_{\boldsymbol{\gamma}}(\mathbf{P})\right)_{2i}$ and $Q_{2i+1} = \left(CC_{\boldsymbol{\gamma}}(\mathbf{P})\right)_{2i+1}$.

Denoting by $\mathbf{P}^{[0]} \in \ell^n(\mathbb{Z})$ a sequence of points in $\mathbb{R}^n$ and assuming that, for each $k \geq 0$, a pair of scalar valued sequences $\boldsymbol{\gamma}^{[k]} := (\boldsymbol{\alpha}^{[k]}, \boldsymbol{\beta}^{[k]}) \in \mathcal{W}$ is assigned, we can formulate the *corner cutting algorithm*, for short the $CC_{\boldsymbol{\gamma}}$-algorithm, as follows.

Algorithm 2.4. *Corner cutting algorithm for points:*

Input: $\mathbf{P}^{[0]} \in \ell^n(\mathbb{Z})$

For $k = 0, 1, \dots,$

Input: $\boldsymbol{\gamma}^{[k]} \in \mathcal{W}$

Compute $\mathbf{P}^{[k+1]} = CC_{\boldsymbol{\gamma}^{[k]}}(\mathbf{P}^{[k]})$ *according to* (2.3)

In the remainder of this section we want to give a new simple proof of the fact that, for all choices of $\{\boldsymbol{\gamma}^{[k]} \in \mathcal{W},\ k \geq 0\}$ satisfying $\sup_{k\geq 0} \mu(\boldsymbol{\gamma}^{[k]}) < 1$, and for all sequences of points in $\mathbb{R}^n$ with bounded L_∞ distance between every two consecutive points, the corner cutting algorithm always converges. To this end we present two technical lemmas, where the first one is taken from [9, Lemma 18] and is here recalled for completeness.

Lemma 2.5. *Let f be a univariate function defined on $[a,b]$. If f is Lipschitz continuous with Lipschitz constant L, then the error in approximating f by the linear interpolating polynomial at the points a, b,*

$$\mathcal{L}(a,b;f(a),f(b))(x) = \frac{x-a}{b-a} f(b) + \frac{b-x}{b-a} f(a),$$

is bounded by

$$|f(x) - \mathcal{L}(a,b;f(a),f(b))(x)| \leq \frac{(b-a)L}{2}, \qquad x \in [a,b]\,.$$

Proof. It is well known that

$$f(x) - \mathcal{L}(a,b;f(a),f(b))(x) = (x-a)(x-b)[a,b,x]f \tag{2.4}$$

with $[a,b,x]f$ the divided difference of order 2 of f at the points a, b, x. By definition of divided differences we get

$$f(x) - \mathcal{L}(a,b;f(a),f(b))(x) = (x-a)(x-b)[a,b,x]f = \frac{(x-a)(x-b)}{b-a}\left(\frac{f(b)-f(x)}{b-x} - \frac{f(x)-f(a)}{x-a}\right). \tag{2.5}$$

Since $\frac{|(x-a)(x-b)|}{b-a} \leq \frac{b-a}{4}$, and f is Lipschitz continuous, then (2.5) yields

$$|f(x) - \mathcal{L}(a,b;f(a),f(b))(x)| \leq \frac{(b-a)}{4}\left(\frac{L|b-x|}{|b-x|} + \frac{L|x-a|}{|x-a|}\right) = \frac{(b-a)L}{2}.$$

□

The next lemma is about piecewise Lipschitz continuous functions. It is a well-known result but we provide the proof for convenience of the reader.

Lemma 2.6. *Let f be a continuous function, Lipschitz continuous on each interval of a partition $\cdots < x_i < x_{i+1} < \cdots$ of the real line $\mathbb{R} = \cup_{i\in\mathbb{Z}}[x_i, x_{i+1})$, with a bound L on the Lipschitz constants. Then f is Lipschitz continuous in $\mathbb{R}$ with Lipschitz constant L.*

Proof. Let $t_1,\ t_2 \in \mathbb{R}$, $t_1 < t_2$. If $t_1,\ t_2$ belong to the same interval of the partition, say $t_1,\ t_2 \in [x_i, x_{i+1})$, the inequality $|f(t_2) - f(t_1)| \le L\,|t_2 - t_1|$ holds by assumption. Otherwise, assuming $t_1 \in [x_i, x_{i+1})$, $t_2 \in [x_j, x_{j+1})$, $j \ge i+1$, using the continuity of f and writing

$$f(t_2) - f(t_1) = f(t_2) - f(x_j) + \sum_{l=i+1}^{j-1} (f(x_{l+1}) - f(x_l)) + f(x_{i+1}) - f(t_1) \tag{2.6}$$

we easily arrive at

$$|f(t_2) - f(t_1)| \le L\,|t_2 - x_j| + L \sum_{l=i+1}^{j-1} |x_{l+1} - x_l| + L\,|x_{i+1} - t_1| = L\,|t_2 - t_1|,$$

which concludes the proof. □

Theorem 2.7. *For $\{\boldsymbol{\gamma}^{[k]}\}_{k\ge0} \subset \mathcal{W}$ such that*

$$\sup_{k\ge0} \mu(\boldsymbol{\gamma}^{[k]}) < 1, \tag{2.7}$$

the corner cutting algorithm (Algorithm 2.4) converges uniformly for all initial sequences $\mathbf{P}^{[0]} = \{P_i^{[0]} \in \mathbb{R}^n,\ i \in \mathbb{Z}\} \in \ell^n(\mathbb{Z})$ satisfying

$$\|P_{i+1}^{[0]} - P_i^{[0]}\|_\infty < L, \quad \forall i \in \mathbb{Z},$$

with $L > 0$.

Proof. We prove convergence of the $CC\boldsymbol{\gamma}$-algorithm working component-wise. First we introduce a parametrization at each refinement level. Without loss of generality, we assume $\mathbf{u}^{[0]} = \mathbb{Z}$ and, for all $k \ge 0$, we denote by $\mathbf{u}^{[k]}$ the scalar sequence obtained from $\mathbf{u}^{[0]}$ by applying k steps of the $CC\boldsymbol{\gamma}$-algorithm (Algorithm 2.4). Precisely, from the $(k-1)$-th level parameters, the k-th level parameters are obtained by the rules

$$u_{2i}^{[k]} = (1 - \alpha_i^{[k-1]})u_i^{[k-1]} + \alpha_i^{[k-1]}u_{i+1}^{[k-1]}, \qquad u_{2i+1}^{[k]} = (1 - \beta_i^{[k-1]})u_i^{[k-1]} + \beta_i^{[k-1]}u_{i+1}^{[k-1]}.$$

Denoting by $p_i^{[k]}$ one component of $P_i^{[k]}$, we construct the piecewise linear interpolant to the data $(u_i^{[k]}, p_i^{[k]})$ and denote it by $\mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})$. In other words

$$\mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})(u) = \mathcal{L}(u_i^{[k]}, u_{i+1}^{[k]}; p_i^{[k]}, p_{i+1}^{[k]}), \quad u \in [u_i^{[k]}, u_{i+1}^{[k]}].$$

By the assumption on $\mathbf{P}^{[0]}$, we know that $|p_{i+1}^{[0]} - p_i^{[0]}| < L$ for all $i \in \mathbb{Z}$ and $\mathcal{L}(\mathbf{u}^{[0]}, \mathbf{p}^{[0]})$ is Lipschitz continuous with constant L on $[u_i^{[0]}, u_{i+1}^{[0]}] = [i, i+1]$. We show by induction that, for $k \ge 0$, $\mathcal{L}(\mathbf{u}^{[k+1]}, \mathbf{p}^{[k+1]})$ is Lipschitz continuous with constant L on $[u_i^{[k+1]}, u_{i+1}^{[k+1]}]$. Indeed, all points of $\mathbf{p}^{[k+1]}$ lie on $\mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})$ and therefore by the choice of $\mathbf{u}^{[k+1]}$ we know that $|p_{i+1}^{[k+1]} - p_i^{[k+1]}| \le L\,|u_{i+1}^{[k+1]} - u_i^{[k+1]}|$. Hence, by Lemma 2.6, we can conclude that $\mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})$ is Lipschitz continuous in $\mathbb{R}$ with constant L for all $k \ge 0$. Since $\mathcal{L}(\mathbf{u}^{[k+1]}, \mathbf{p}^{[k+1]})$ is, by construction, a piecewise linear interpolant to $\mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})$ at $(\mathbf{u}^{[k+1]}, \mathbf{p}^{[k+1]})$, we can regard $\mathcal{L}(\mathbf{u}^{[k+1]}, \mathbf{p}^{[k+1]})$ as an approximation of $\mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})$. In particular, for $u \in [u_{2i}^{[k+1]}, u_{2i+1}^{[k+1]}]$, we have $|\mathcal{L}(\mathbf{u}^{[k+1]}, \mathbf{p}^{[k+1]})(u) - \mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})(u)| = 0$ (see Figure 2). On the other hand, for $u \in [u_{2i-1}^{[k+1]}, u_{2i}^{[k+1]}]$, since $\mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})$ is Lipschitz continuous with constant L, we obtain by Lemma 2.5 that

$$|\mathcal{L}(\mathbf{u}^{[k+1]}, \mathbf{p}^{[k+1]})(u) - \mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})(u)| \le \frac{1}{2}\, L|u_{2i}^{[k+1]} - u_{2i-1}^{[k+1]}| \le \frac{1}{2}\, L\, d^{[k+1]}, \tag{2.8}$$

where $d^{[k]} = \sup_i |u_{i+1}^{[k]} - u_i^{[k]}|$. Now, we proceed by comparing $d^{[k+1]}$ with $d^{[k]}$. To this purpose we have to distinguish the following two cases (see Figure 2):

- Case 1: $u^{[k+1]}_{2i+1} - u^{[k+1]}_{2i} = (\alpha^{[k]}_i - \beta^{[k]}_i)u^{[k]}_i + (\beta^{[k]}_i - \alpha^{[k]}_i)u^{[k]}_{i+1} = (\beta^{[k]}_i - \alpha^{[k]}_i)(u^{[k]}_{i+1} - u^{[k]}_i)$;
- Case 2: $u^{[k+1]}_{2i} - u^{[k+1]}_{2i-1} = (1 - \beta^{[k]}_{i-1})(u^{[k]}_i - u^{[k]}_{i-1}) + \alpha^{[k]}_i(u^{[k]}_{i+1} - u^{[k]}_i)$.

Both cases yield that $d^{[k+1]} \le \mu^{[k]} d^{[k]}$ with $\mu^{[k]} := \mu(\gamma^{[k]})$. Thus, in view of (2.8), we get that $|\mathcal{L}(\mathbf{p}^{[k+1]}, \mathbf{u}^{[k+1]})(u) - \mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})(u)| \le \frac{1}{2} L d^{[k+1]} \le \frac{1}{2} L\, d^{[0]}\, (\prod_{h=0}^{k} \mu^{[h]})$. Taking into account also that $\prod_{h=0}^{k} \mu^{[h]} < \mu^{k+1}$ with $\mu := \sup_{k\ge 0} \mu^{[k]}$, for any arbitrary $m \in \mathbb{Z}_+$ we can write

$$\begin{aligned}|\mathcal{L}(\mathbf{u}^{[k+m]}, \mathbf{p}^{[k+m]})(u) - \mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})(u)| \le\ & \sum_{\ell=0}^{m-1} |\mathcal{L}(\mathbf{u}^{[k+\ell+1]}, \mathbf{p}^{[k+\ell+1]})(u) - \mathcal{L}(\mathbf{u}^{[k+\ell]}, \mathbf{p}^{[k+\ell]})(u)| \\ & \le \tfrac{1}{2} L\, d^{[0]} \mu^{k+1} \left(\sum_{\ell=0}^{m-1} \mu^{\ell} \right) \le \frac{L d^{[0]}}{2(1-\mu)}\, \mu^{k+1},\end{aligned}$$

from which we conclude that $\{\mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})\}_{k\ge 0}$ is a Cauchy sequence and therefore convergent. □

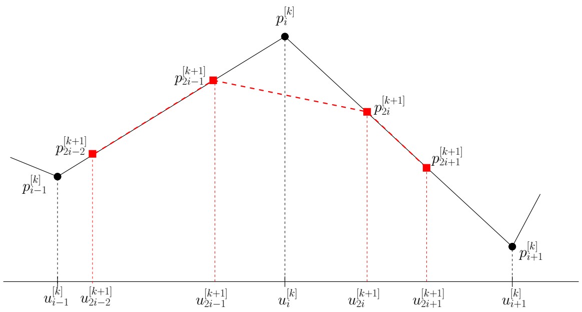

Figure 2: $\mathcal{L}(\mathbf{u}^{[k+1]}, \mathbf{p}^{[k+1]})$ (dashed red) versus $\mathcal{L}(\mathbf{u}^{[k]}, \mathbf{p}^{[k]})$ (solid black).

Remark 2.8. *Some important observations:*

(i) *The condition that the initial sequence of points $\mathbf{P}^{[0]} \in \ell^n(\mathbb{Z})$ is such that $\|P^{[0]}_{i+1} - P^{[0]}_i\|_\infty < L$ for all $i \in \mathbb{Z}$, is equivalent to requiring the piecewise linear interpolant to the data $(i, P^{[0]}_i)$, $i \in \mathbb{Z}$ to be Lipschitz continuous with Lipschitz constant L.*

(ii) *Convergence of the corner cutting algorithm can be obtained under weaker assumptions on $\alpha^{[k]}_i$ and $\beta^{[k]}_i$ than the ones required in Theorem 2.7, namely by requiring that $\lim_{k\to+\infty} \sum_{\ell=0}^{m-1} \prod_{h=0}^{k+\ell} \mu^{[h]} = 0$ for all $m \in \mathbb{Z}_+$.*

3. Corner cutting algorithms for nets of functions

The aim of this section is first to define a corner cutting algorithm refining nets of functions and then to show its convergence by suitably extending the results introduced in the previous section. For the first goal we need to recall the definition of Coons patch and to prove some of its properties.

3.1. Preliminary results on the Coons patch

Since our proof of convergence of corner cutting schemes refining nets of univariate functions (u-functions for short) is based on error estimates for Coons interpolation, we need to recall first the definition of bilinear patches and Coons patch (see [15], [16]). Then we point out some important properties of Coons patches that are relevant to our discussion.

Definition 3.1 (The bilinear patch)**.** *The bilinear patch interpolating the four points $\mathcal{P} = \{P_{ij},\ i, j \in \{0, 1\}\}$ is*

$$\mathcal{BL}(\mathcal{P}; \mathbf{h})(s,t) = (1 - \tfrac{s}{h_1})((1 - \tfrac{t}{h_2})P_{00} + \tfrac{t}{h_2}P_{01}) + \tfrac{s}{h_1}((1 - \tfrac{t}{h_2})P_{10} + \tfrac{t}{h_2}P_{11}),$$

where $\mathbf{h} = (h_1, h_2)$ and $(s,t) \in [0, h_1] \times [0, h_2]$.

It is easy to verify that

$$\mathcal{BL}(\mathcal{P};\mathbf{h})(ih_1, jh_2) = P_{ij}, \quad i,j \in \{0,1\}.$$

Definition 3.2 (The Coons patch)**.** *Let $\phi_0(s), \phi_1(s)$, $s \in [0,h_1]$ and $\psi_0(t), \psi_1(t)$, $t \in [0,h_2]$ be four continuous u-functions in $\mathbb{R}^3$ such that $P_{ji} = \phi_i(jh_1) = \psi_j(ih_2)$ for $i,j \in \{0,1\}$. The Coons patch interpolating the four u-functions $\phi_0, \phi_1, \psi_0, \psi_1$ is*

$$C(\phi_0,\phi_1,\psi_0,\psi_1;\mathbf{h})(s,t) = \quad (1-\tfrac{s}{h_1})\psi_0(t) + \tfrac{s}{h_1}\psi_1(t) + (1-\tfrac{t}{h_2})\phi_0(s) + \tfrac{t}{h_2}\phi_1(s) - \mathcal{BL}(\mathcal{P};\mathbf{h})(s,t), \tag{3.1}$$

where $\mathbf{h} = (h_1,h_2)$ and $(s,t) \in [0,h_1]\times[0,h_2]$.

In the following, to simplify the notation we write $C(\phi,\psi;\mathbf{h})$ in place of $C(\phi_0,\phi_1,\psi_0,\psi_1;\mathbf{h})$.

Remark 3.3. *It is easy to verify the transfinite interpolation properties of the Coons patch interpolant, i.e.*

$$C(\phi,\psi;\mathbf{h})(0,t) = \psi_0(t), \quad C(\phi,\psi;\mathbf{h})(h_1,t) = \psi_1(t),$$

$$C(\phi,\psi;\mathbf{h})(s,0) = \phi_0(s), \quad C(\phi,\psi;\mathbf{h})(s,h_2) = \phi_1(s).$$

Next, the notion of mixed second divided difference of a bivariate function F is introduced.

Definition 3.4. *The mixed second divided difference (MSDD) of a bivariate function F at the points $(\sigma_i,\tau_j) \in \mathbb{R}^2$, $i,j \in \{1,2\}$ is defined as*

$$[\sigma_1,\sigma_2;\tau_1,\tau_2]F = \frac{1}{(\sigma_1-\sigma_2)(\tau_1-\tau_2)}(F(\sigma_1,\tau_1)+F(\sigma_2,\tau_2)-F(\sigma_2,\tau_1)-F(\sigma_1,\tau_2)).$$

The following result expresses the error between a bivariate function F and the Coons patch interpolating its boundary u-functions.

Proposition 3.5. *Let F be a bivariate continuous function defined on a rectangular domain $R = [a,b]\times[c,d]$, and denote by $C(F_{|\partial R})$ the Coons patch interpolating $F_{|\partial R}$. Then*

$$F(s,t) - C(F_{|\partial R})(s,t) = \frac{(s-a)(s-b)(t-c)(t-d)}{(b-a)(d-c)}([b,s;d,t]F - [s,a;d,t]F + [s,a;t,c]F - [b,s;t,c]F), \quad (s,t) \in R.$$

Proof. Let $(\mathcal{L}_s(F))(s,t) = \frac{s-a}{b-a}F(b,t) + \frac{b-s}{b-a}F(a,t)$ and $(\mathcal{L}_t(F))(s,t) = \frac{t-c}{d-c}F(s,d) + \frac{d-t}{d-c}F(s,c)$. In view of (2.4) we get

$$((I-\mathcal{L}_s)(F))(s,t) = \frac{(s-a)(s-b)}{b-a}\left(\frac{F(b,t)-F(s,t)}{b-s} - \frac{F(s,t)-F(a,t)}{s-a}\right),$$

and

$$((I-\mathcal{L}_t)(F))(s,t) = \frac{(t-c)(t-d)}{d-c}\left(\frac{F(s,d)-F(s,t)}{d-t} - \frac{F(s,t)-F(s,c)}{t-c}\right).$$

Moreover, since $C(F_{|\partial R}) = \mathcal{L}_s(F) + \mathcal{L}_t(F) - \mathcal{L}_t(\mathcal{L}_s(F))$, we can also write

$$\begin{aligned} F(s,t) - C(F_{|\partial R})(s,t) &= ((I-\mathcal{L}_t)(I-\mathcal{L}_s)(F))(s,t) \\ &= (I-\mathcal{L}_t)\left(\tfrac{(s-a)(s-b)}{b-a}\left(\tfrac{F(b,t)-F(s,t)}{b-s} - \tfrac{F(s,t)-F(a,t)}{s-a}\right)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} F(s,t) - C(F_{|\partial R})(s,t) &= \tfrac{(t-c)(t-d)}{d-c}\tfrac{(s-a)(s-b)}{b-a}\left(\tfrac{[b,s]F(\cdot,d)-[s,a]F(\cdot,d)}{d-t} - \tfrac{[b,s]F(\cdot,t)-[s,a]F(\cdot,t)}{d-t}\right. \\ &\left. - \tfrac{[b,s]F(\cdot,t)-[s,a]F(\cdot,t)}{t-c} + \tfrac{[b,s]F(\cdot,c)-[s,a]F(\cdot,c)}{t-c}\right) \\ &= \tfrac{(t-c)(t-d)}{d-c}\tfrac{(s-a)(s-b)}{b-a}\left(\tfrac{[b,s](F(\cdot,d)-F(\cdot,t))}{d-t} - \tfrac{[s,a](F(\cdot,d)-F(\cdot,t))}{d-t}\right. \\ &\left. - \tfrac{[b,s](F(\cdot,t)-F(\cdot,c))}{t-c} + \tfrac{[s,a](F(\cdot,t)-F(\cdot,c))}{t-c}\right) \\ &= \tfrac{(t-c)(t-d)}{d-c}\tfrac{(s-a)(s-b)}{b-a}\left([b,s;d,t]F - [s,a;d,t]F + [s,a;t,c]F - [b,s;t,c]F\right). \end{aligned}$$

□

We introduce an important property of bivariate functions, which plays a key role in the convergence analysis of corner cutting schemes refining nets of functions.

Definition 3.6. *A bivariate function F defined on $\Omega \subset \mathbb{R}^2$ has the bounded MSDD property (BMSDD property) with constant L in Ω if for any $\sigma_1, \sigma_2; \tau_1, \tau_2 \in \mathbb{R}$ such that $(\sigma_i, \tau_j) \in \Omega$, $i, j \in \{1, 2\}$, satisfies*

$$|[\sigma_1, \sigma_2; \tau_1, \tau_2]F| \leq L.$$

Remark 3.7. *(i) Since the MSDD (mixed second divided difference) of a C^2 function is equal to its second mixed derivative at an intermediate point, as can be easily proved by Taylor expansion up to first order, a bivariate function with a bounded second mixed derivative has bounded MSDDs.*

(ii) A simple example of a class of bivariate functions that satisfy Definition 3.6 is $F(s, t) = f(s) + g(t)$. For such a bivariate function $L = 0$.

Combining Definition 3.6 with Proposition 3.5 we get

Corollary 3.8. *Let F be a bivariate continuous function defined on a rectangular domain $R = [a, b] \times [c, d]$, and denote by $C(F_{|\partial R})$ the Coons patch interpolating $F_{|\partial R}$. If F has the BMSDD property with constant L in R, then*

$$\|F(s, t) - C(F_{|\partial R})(s, t)\|_\infty \leq L\frac{(b - a)(d - c)}{4}, \quad (s, t) \in R.$$

Note that Corollary 3.8 is a generalization of Lemma 2.5 to bivariate transfinite interpolation. Note also that, in light of Remark 3.7, the set of functions to which Corollary 3.8 is applicable is rather wide.

3.2. Corner cutting of nets of u-functions

In this section we discuss a generalization of the Chaikin type corner-cutting algorithm for nets of u-functions, that was presented in [7] and [8]. To this purpose we start by introducing the notion of net of u-functions.

Definition 3.9 (Net of u-functions)**.** *A* net *N is a bivariate function defined on a* grid of lines

$$T = T((\mathbf{h}^{[s]}, \mathbf{h}^{[t]}), O) = \{s_i \times \mathbb{R},\ i \in \mathbb{Z}\} \cup \{\mathbb{R} \times t_j,\ j \in \mathbb{Z}\}, \tag{3.2}$$

with $\mathbf{h}^{[s]}, \mathbf{h}^{[t]}$ bi-infinite sequences of positive numbers, $O = (s_0, t_0)$, $s_{i+1} = s_i + h_i^{[s]}$, $i \in \mathbb{Z}$, and similarly for $\{t_j\}_{j \in \mathbb{Z}}$ with $h^{[t]}$ replacing $h^{[s]}$. In other words, N consists of the u-functions $N(s, t_j)$ and $N(s_j, t)$, $j \in \mathbb{Z}$ defined on $\mathbb{R}$. The point O is termed the origin *of T and the intervals $[s_j, s_{j+1}]$, $[t_j, t_{j+1}]$, $j \in \mathbb{Z}$ are termed* grid intervals.

To stress the relation between a net N of u-functions and the corresponding grid of lines we use the notation $N = N(T)$.

Definition 3.10 (C^0 net)**.** *A net N is termed a C^0 net if all the u-functions $\phi_j(s) = N(s, t_j)$, $\psi_j(t) = N(s_j, t)$, $j \in \mathbb{Z}$ are C^0.*

Definition 3.11 (Piecewise Coons patch)**.** *For a C^0 net N consisting of the u-functions ϕ_j, ψ_j, $j \in \mathbb{Z}$, we denote by $C(N)$ the* piecewise Coons patch *interpolating it, which is locally defined as*

$$C(N)(s, t) = C(\phi_i, \phi_{i+1}, \psi_j, \psi_{j+1}; h_{i,j})(s - s_i, t - t_j), \quad (s, t) \in [s_i, s_{i+1}] \times [t_j, t_{j+1}], \quad i, j \in \mathbb{Z}$$

with $h_{i,j} = (h_i^{[s]}, h_j^{[t]})$, $h_i^{[s]} = s_{i+1} - s_i$, $h_j^{[t]} = t_{j+1} - t_j$ $i, j \in \mathbb{Z}$.

We remark that for a C^0 net $N(T)$, the net obtained by evaluating the piecewise Coons patch $C(N)$ along the grid lines of any grid $\tilde{T}$, is also C^0, since $C(N)$ is continuous. Hence the following iterative procedure is well defined.

Algorithm 3.12. *Corner cutting algorithm for nets of functions:*

Input: a C^0 net $N^{[0]}(T^{[0]})$ with $T^{[0]} = \{s_i^{[0]} \times \mathbb{R},\ s_i^{[0]} = i \in \mathbb{Z}\} \cup \{\mathbb{R} \times t_j^{[0]},\ t_j^{[0]} = j \in \mathbb{Z}\}$

For $k = 0, 1, \ldots$

Input: $\boldsymbol{\gamma}^{[s],[k]} := (\boldsymbol{\alpha}^{[s],[k]}, \boldsymbol{\beta}^{[s],[k]}) \in \mathcal{W}$ and $\boldsymbol{\gamma}^{[t],[k]} := (\boldsymbol{\alpha}^{[t],[k]}, \boldsymbol{\beta}^{[t],[k]}) \in \mathcal{W}$

Compute $s_{2i}^{[k+1]} = (1 - \alpha_i^{[s],[k]})s_i^{[k]} + \alpha_i^{[s],[k]} s_{i+1}^{[k]}$ and $s_{2i+1}^{[k+1]} = (1 - \beta_i^{[s],[k]})s_i^{[k]} + \beta_i^{[s],[k]} s_{i+1}^{[k]}$, for $i \in \mathbb{Z}$

Compute $t_{2j}^{[k+1]} = (1 - \alpha_j^{[t],[k]})t_j^{[k]} + \alpha_j^{[t],[k]} t_{j+1}^{[k]}$ and $t_{2j+1}^{[k+1]} = (1 - \beta_j^{[t],[k]})t_j^{[k]} + \beta_j^{[t],[k]} t_{j+1}^{[k]}$, for $j \in \mathbb{Z}$

Define $T^{[k+1]} = \{s_i^{[k+1]} \times \mathbb{R},\ i \in \mathbb{Z}\} \cup \{\mathbb{R} \times t_j^{[k+1]},\ j \in \mathbb{Z}\}$

Compute $N^{[k+1]} = C(N^{[k]})|_{T^{[k+1]}}$

We denote the mapping from $N^{[k]}$ to $N^{[k+1]}$ in the above algorithm by $N^{[k+1]} = BC_{\boldsymbol{\gamma}^{[s],[k]},\boldsymbol{\gamma}^{[t],[k]}}(C(N^{[k]})$. In the next subsection we prove that Algorithm 3.12 is convergent under suitable assumptions on the initial net $N^{[0]}$ and the corner cutting weights. To state the assumption on the initial net, we introduce the notion of a BMSDD net of functions which is a direct analogue of Definition 3.6.

Definition 3.13. *A net of functions $N(T)$ has the BMSDD property with constant L, if*

$$|[\sigma_1, \sigma_2; \tau_1, \tau_2]N| \leq L, \quad \textit{for all} \quad (\sigma_i, \tau_j) \in T, \quad i, j \in \{1, 2\}, \tag{3.3}$$

where

$$[\sigma_1, \sigma_2; \tau_1, \tau_2]N = \frac{1}{(\sigma_1 - \sigma_2)(\tau_1 - \tau_2)}(N(\sigma_1, \tau_1) + N(\sigma_2, \tau_2) - N(\sigma_2, \tau_1) - N(\sigma_1, \tau_2)).$$

3.3. Convergence of the corner cutting algorithm for nets of functions

In this subsection we state and prove the main result of this paper that is convergence of Algorithm 3.12. We mention that we prove it by showing convergence of the sequence of continuous piecewise Coons interpolants to the generated nets.

Theorem 3.14. *Let $N^{[0]}$ be a C^0 net having the BMSDD property with constant L. Then the corner cutting algorithm for nets of functions (Algorithm 3.12) is convergent for all $\{\boldsymbol{\gamma}^{[s],[k]}, \boldsymbol{\gamma}^{[t],[k]}\}_{k\geq 0} \in \mathcal{W}$ such that*

$$\mu^* = \sup_{k\geq 0} \max\{\mu(\boldsymbol{\gamma}^{[s],[k]}),\ \mu(\boldsymbol{\gamma}^{[t],[k]})\} < \frac{\sqrt{3}}{3}. \tag{3.4}$$

To prove this theorem we need several intermediate results. The first is an important observation about the BMSDD property of nets of functions.

Lemma 3.15. *Let $N(T)$ satisfy the inequality in* (3.3) *for*

(a) $t_j \leq \tau_1, \tau_2 \leq t_{j+1}$; $\sigma_1 = s_i$, $\sigma_2 = s_{i+1}$, $i, j \in \mathbb{Z}$,

or

(b) $s_i \leq \sigma_1, \sigma_2 \leq s_{i+1}$; $\tau_1 = t_j$, $\tau_2 = t_{j+1}$, $i, j \in \mathbb{Z}$.

Then $N(T)$ has the BMSDD property with constant L.

Proof. Given $\sigma_1 < \sigma_2$, $\tau_1 < \tau_2$ such that $(\sigma_i, \tau_j) \in T$ for $i, j \in \{1, 2\}$, there are two possibilities:

(i) $\sigma_1 = s_i$, $\sigma_2 = s_{i+\ell}$, for some $i \in \mathbb{Z}$, $\ell \in \mathbb{N}$ and $\tau_1, \tau_2 \in \mathbb{R}$, or

(ii) $\tau_1 = t_j, \tau_2 = t_{j+\ell}$, for some $j \in \mathbb{Z}$, $\ell \in \mathbb{N}$ and $\sigma_1, \sigma_2 \in \mathbb{R}$.

We consider case (i); the proof in case (ii) is similar. We prove that the inequality in (3.3) holds for case (i) by induction. First we prove by induction on ℓ that the inequality in (3.3) holds in the case

$$\text{(iii)} \qquad \sigma_1 = s_i,\ \ \sigma_2 = s_{i+\ell},\ \ \text{for some } i \in \mathbb{Z},\ \ell \in \mathbb{N} \ \text{ and } \ t_j \le \tau_1, \tau_2 \le t_{j+1},\ \ \text{for some } j \in \mathbb{Z}.$$

The above claim holds for $\ell = 1$ by assumption (a). It remains to show that if the inequality in (3.3) holds for $\ell \le m$, it holds for $\ell = m+1$. Now,

$$\begin{aligned}
[s_i, s_{i+m+1}; \tau_1, \tau_2]N &= \tfrac{1}{(s_{i+m+1}-s_i)(\tau_2-\tau_1)} \left(N(s_{i+m+1}, \tau_2) + N(s_i, \tau_1) - N(s_{i+m+1}, \tau_1) - N(s_i, \tau_2)\right) \\
&= \tfrac{(s_{i+m}-s_i)}{(s_{i+m+1}-s_i)} \tfrac{1}{(s_{i+m}-s_i)(\tau_2-\tau_1)} \left(N(s_{i+m}, \tau_2) + N(s_i, \tau_1) - N(s_{i+m}, \tau_1) - N(s_i, \tau_2)\right) \\
&+ \tfrac{(s_{i+m+1}-s_{i+m})}{(s_{i+m+1}-s_i)} \tfrac{1}{(s_{i+m+1}-s_{i+m})(\tau_2-\tau_1)} \left(N(s_{i+m+1}, \tau_2) + N(s_{i+m}, \tau_1) - N(s_{i+m+1}, \tau_1) - N(s_{i+m}, \tau_2)\right).
\end{aligned} \tag{3.5}$$

Thus by the induction hypothesis and by (a) we get for case (iii)

$$|[s_i, s_{i+m+1}; \tau_1, \tau_2]N| \le \frac{(s_{i+m} - s_i)}{(s_{i+m+1} - s_i)} L + \frac{(s_{i+m+1} - s_{i+m})}{(s_{i+m+1} - s_i)} L = L,$$

and the inequality in (3.3) holds in case (iii). This concludes the first part of the proof.

Next we prove, again by induction, that the inequality in (3.3) holds in case (i). We assume that the inequality in (3.3) holds for $t_j \le \tau_1 \le t_{j+1}$ and $t_{j+m} \le \tau_2 \le t_{j+m+1}$ for some $m \in \mathbb{N}$, and show that the inequality in (3.3) holds for $t_j \le \tau_1 \le t_{j+1}$ and $t_{j+m+1} \le \tau_2 \le t_{j+m+2}$. This is sufficient since the case $m = 0$ corresponds to case (iii). Now, for $t_j \le \tau_1 \le t_{j+1}$ and $t_{j+m+1} \le \tau_2 \le t_{j+m+2}$

$$\begin{aligned}
[s_i, s_{i+\ell}; \tau_1, \tau_2]N &= \tfrac{1}{(s_{i+\ell}-s_i)(\tau_2-\tau_1)} \left(N(s_{i+\ell}, \tau_2) + N(s_i, \tau_1) - N(s_{i+\ell}, \tau_1) - N(s_i, \tau_2)\right) \\
&= \tfrac{(t_{j+1}-\tau_1)}{(\tau_2-\tau_1)} \tfrac{1}{(s_{i+\ell}-s_i)(t_{j+1}-\tau_1)} \Big(N(s_{i+\ell}, t_{j+1}) + N(s_i, \tau_1) - N(s_i, t_{j+1}) - N(s_{i+\ell}, \tau_1)\Big) \\
&+ \tfrac{(\tau_2-t_{j+1})}{(\tau_2-\tau_1)} \tfrac{1}{(s_{i+\ell}-s_i)(\tau_2-t_{j+1})} \Big(N(s_{i+\ell}, \tau_2) + N(s_i, t_{j+1}) - N(s_{i+\ell}, t_{j+1}) - N(s_i, \tau_2)\Big).
\end{aligned}$$

Thus,

$$[s_i, s_{i+\ell}; \tau_1, \tau_2]N = \frac{(t_{j+1} - \tau_1)}{(\tau_2 - \tau_1)} [s_i, s_{i+\ell}; t_{j+1}, \tau_1]N + \frac{(\tau_2 - t_{j+1})}{(\tau_2 - \tau_1)} [s_i, s_{i+\ell}; t_{j+1}, \tau_2]N.$$

The MSDD in the first term above corresponds to case (iii), since $\tau_1,\ t_{j+1} \in [t_j, t_{j+1}]$, and the MSDD in the second term above corresponds to case (i) with m, since $t_{j+1} \in [t_{j+1}, t_{j+2}]$ and $\tau_2 \in [t_{j+1+m}, t_{j+1+m+1}]$. By the first part of the proof we have $|[s_{i+\ell}, s_i; t_{j+1}, \tau_1]N| \le L$ and by the induction hypothesis we have $|[s_{i+\ell}, s_i; t_{j+1}, \tau_2]N| \le L$. Thus,

$$|[s_i, s_{i+\ell}; \tau_1, \tau_2]N| \le \frac{(t_{j+1} - \tau_1)}{(\tau_2 - \tau_1)} L + \frac{(\tau_2 - t_{j+1})}{(\tau_2 - \tau_1)} L = L, \quad \text{for} \quad t_j \le \tau_1 \le t_{j+1},\ \tau_{j+m+1} \le \tau_2 \le t_{j+m+2},$$

and the inequality in (3.3) holds in case (i) for $m+1$. □

A simple lemma follows from the linearity of the divided differences.

Lemma 3.16. *If $N_\ell(T)$, $\ell = 1, ..., m$ have the BMSDD property with constant L, then $\sum_{\ell=1}^m N_\ell(T)$ has the BMSDD property with constant mL.*

Using a similar induction to that in the second part of the proof of Lemma 3.15, we can prove

Lemma 3.17. *A bivariate function which has the BMSDD property with constant L on each rectangle of a grid T has the BMSDD property with constant L in $\mathbb{R}^2$.*

The next lemma considers bivariate functions which are piecewise linear in one variable.

Lemma 3.18. *Let F be a bivariate function of the form*

$$F(s,t) = \frac{s-s_i}{s_{i+1}-s_i}F(s_{i+1},t) + \frac{s_{i+1}-s}{s_{i+1}-s_i}F(s_i,t), \quad s\in[s_i,s_{i+1}],\ t\in\mathbb{R}, \quad i\in\mathbb{Z},$$

where $\{s_i\}_{i\in\mathbb{Z}}\subset\mathbb{R}$ is an increasing sequence. If F satisfies the inequality in (3.3) *for $\sigma_1,\sigma_2\in[s_i,s_{i+1}]$ for any $i\in\mathbb{Z}$ and $\tau_1,\tau_2\in\mathbb{R}$, then F has the BMSDD property with constant L in $\mathbb{R}^2$.*

Proof. First we show that F satisfies the inequality in (3.3) with constant L in each rectangle of a grid T defined by the parameters $\{s_i\}_{i\in\mathbb{Z}}$ and any increasing sequence $\{t_j\}_{j\in\mathbb{Z}}$. Let $\sigma_1,\sigma_2\in[s_i,s_{i+1}]$ and $\tau_1,\tau_2\in[t_j,t_{j+1}]$ for some $i,j\in\mathbb{Z}$. Since

$$[\sigma_1,\sigma_2;\tau_1,\tau_2]F = \frac{1}{(\tau_1-\tau_2)}\left([\sigma_1,\sigma_2]F(\cdot,\tau_1) - [\sigma_1,\sigma_2]F(\cdot,\tau_2)\right),$$

the linearity of F in s implies that

$$[\sigma_1,\sigma_2]F(\cdot,\tau_j) = \frac{1}{(s_{i+1}-s_i)}\Big(F(s_{i+1},\tau_j) - F(s_i,\tau_j)\Big), \quad j=1,2,$$

and we get

$$[\sigma_1,\sigma_2;\tau_1,\tau_2]F = \frac{1}{(\tau_1-\tau_2)}\frac{1}{(s_{i+1}-s_i)}\left(F(s_{i+1},\tau_1)+F(s_i,\tau_2)-F(s_{i+1},\tau_2)-F(s_i,\tau_1)\right).$$

From the assumption that F satisfies the inequality in (3.3) for $\sigma_1,\sigma_2\in[s_i,s_{i+1}]$ for any $i\in\mathbb{Z}$ and $\tau_1,\tau_2\in\mathbb{R}$, we conclude that F has the BMSDD property with constant L on each rectangle of T. Hence, by Lemma 3.17 F has the BMSDD property with constant L in $\mathbb{R}^2$. □

Remark 3.19. *It is obvious that the same result holds if F is linear in t in each rectangle of T.*

Another important observation is

Remark 3.20. *The restriction to a grid of a bivariate function which has the BMSDD property with constant L in $\mathbb{R}^2$ is a net which has the BMSDD property with constant L.*

The next Theorem is our first key result.

Theorem 3.21. *If $N(T)$ has the BMSDD property with constant L then $C(N)$ has the BMSDD property with constant $3L$.*

Proof. Define the bivariate functions related to N (similarly to the bivariate functions related to F in the proof of Proposition 3.5)

$$(\mathcal{L}_s(N))(s,t) = \tfrac{s-s_i}{s_{i+1}-s_i}N(s_{i+1},t) + \tfrac{s_{i+1}-s}{s_{i+1}-s_i}N(s_i,t),\ s\in[s_i,s_{i+1}],\ t\in\mathbb{R},\ i\in\mathbb{Z}$$

$$(\mathcal{L}_t(N))(s,t) = \tfrac{t-t_j}{t_{j+1}-t_j}N(s,t_{j+1}) + \tfrac{t_{j+1}-t}{t_{j+1}-t_j}N(s,t_j),\ s\in\mathbb{R},\ t\in[t_j,t_{j+1}],\ j\in\mathbb{Z},$$

$$\begin{aligned}(\mathcal{L}_s(\mathcal{L}_t(N))(s,t) = {}& \tfrac{s-s_i}{s_{i+1}-s_i}\left(\tfrac{t-t_j}{t_{j+1}-t_j}N(s_{i+1},t_{j+1}) + \tfrac{t_{j+1}-t}{t_{j+1}-t_j}N(s_{i+1},t_j)\right)\\ &+\tfrac{s_{i+1}-s}{s_{i+1}-s_i}\left(\tfrac{t-t_j}{t_{j+1}-t_j}N(s_i,t_{j+1}) + \tfrac{t_{j+1}-t}{t_{j+1}-t_j}N(s_i,t_j)\right), \quad s\in[s_i,s_{i+1}],\ t\in[t_j,t_{j+1}],\ i\in\mathbb{Z}, j\in\mathbb{Z}.\end{aligned}$$

Note that $\mathcal{L}_s(\mathcal{L}_t(N))$ is the piecewise bilinear function on the rectangles of T, interpolating the data $\{(s_i,t_j),N(s_i,t_j)\}_{i,j\in\mathbb{Z}}$. It follows from (3.1) and the definition of $C(N)$ that

$$C(N) = \mathcal{L}_s(N) + \mathcal{L}_t(N) - \mathcal{L}_s(\mathcal{L}_t(N)). \tag{3.6}$$

Next we show that the three functions in the right-hand side of the above equation have the BMSDD property with constant L in $\mathbb{R}^2$. By Lemma 3.18 and Remark 3.19 both $\mathcal{L}_s(N)$ and $\mathcal{L}_t(N)$ have the BMSDD property with constant L in each rectangle of T since by assumption $N(T)$ has the BMSDD property with constant L. Moreover, also $\mathcal{L}_s(\mathcal{L}_t(N))$ has the BMSDD property with constant L in each rectangle of T because for $\sigma_1, \sigma_2 \in [s_i, s_{i+1}]$ and $\tau_1, \tau_2 \in [t_j, t_{j+1}]$

$$[\sigma_1, \sigma_2; \tau_1, \tau_2]\mathcal{L}_s(\mathcal{L}_t(N)) = [s_i, s_{i+1}; t_j, t_{j+1}]N.$$

Now, by Lemma 3.17 the three functions have the BMSDD property with constant L in $\mathbb{R}^2$. Thus Lemma 3.16, in view of (3.6), implies that $C(N)$ has the BMSDD property with constant $3L$ in $\mathbb{R}^2$. $\square$

A direct consequence of Theorem 3.21 and Remark 3.20 is

Corollary 3.22. *If $N(T)$ has the BMSDD property with constant L, then $BC_{\boldsymbol{\gamma}^{[s]},\boldsymbol{\gamma}^{[t]}}(C(N))$ (defined after Algorithm 3.12) with $\boldsymbol{\gamma}^{[s]}, \boldsymbol{\gamma}^{[t]} \in \mathcal{W}$, has the BMSDD property with constant $3L$.*

Corollary 3.22 leads to our second key result.

Corollary 3.23. *Let $\{N^{[k]}\}_{k\in\mathbb{N}}$ be the nets generated by Algorithm 3.12 from $N^{[0]}$. If $N^{[0]}$ has the BMSDD property with constant L then $N^{[k]}$ has the BMSDD property with constant $3^k L$, for $k \geq 0$.*

We are now ready to prove the third key result.

Theorem 3.24. *In the notation of Algorithm 3.12, if $N^{[0]}$ has the BMSDD property with constant L then*

$$\|C(N^{[k+1]}) - C(N^{[k]})\|_\infty \leq 3^{k+1} L \frac{h_s^{[k+1]} h_t^{[k+1]}}{4} \tag{3.7}$$

where

$$h_s^{[k+1]} = sup_{i\in\mathbb{Z}}(s_{i+1}^{[k+1]} - s_i^{[k+1]}), \qquad h_t^{[k+1]} = sup_{i\in\mathbb{Z}}(t_{i+1}^{[k+1]} - t_i^{[k+1]}).$$

Proof. In view of Corollary 3.23 and Theorem 3.21, $C(N^{[k]})$ has the BMSDD property with constant $3^{k+1}L$, and therefore by Remark 3.20, also $N^{[k+1]} = C(N^{[k]})|_{T^{[k+1]}}$ has this property. Regarding $C(N^{[k+1]})$ as the piecewise Coons patch interpolating $C(N^{[k]})|_{T^{[k+1]}}$, we conclude (3.7) from Corollary 3.8. $\square$

We are now ready to prove Theorem 3.14.

Proof of Theorem 3.14. By the way $T^{[k+1]}$ is constructed from $T^{[k]}$ in steps 1-3 of Algorithm 3.12, we see that

$$h_s^{[k+1]} \leq \mu(\boldsymbol{\gamma}^{[s],[k]}) h_s^{[k]} \quad \text{and} \quad h_t^{[k+1]} \leq \mu(\boldsymbol{\gamma}^{[t],[k]}) h_t^{[k]},$$

with $\mu(\boldsymbol{\gamma}^{[s],[k]})$ and $\mu(\boldsymbol{\gamma}^{[t],[k]})$ defined as in (2.2). Defining $\mu^* = \sup_{k\geq 0} \max\{\mu(\boldsymbol{\gamma}^{[s],[k]}),\, \mu(\boldsymbol{\gamma}^{[t],[k]})\}$ we get from (3.7)

$$\|C(N^{[k+1]}) - C(N^{[k]})\|_\infty \leq \frac{3^{k+1} LH}{4} (\mu^*)^{2k} = \frac{3LH}{4} (3(\mu^*)^2)^k,$$

with $H = h_s^{[0]}\, h_t^{[0]}$. Thus, if $3(\mu^*)^2 < 1$, the sequence $\{C(N^{[k]}\}_{k\in\mathbb{N}}$ is a Cauchy sequence and therefore convergent. To conclude, the convergence of Algorithm 3.12 is guaranteed in case

$$\mu^* = \sup_{k\geq 0} \max\{\mu(\boldsymbol{\gamma}^{[s],[k]}),\, \mu(\boldsymbol{\gamma}^{[t],[k]})\} < \frac{\sqrt{3}}{3}.$$

$\square$

Remark 3.25. *The condition* (3.4) *in Theorem 3.14 can be relaxed to* $\sum_{k=0}^{\infty} 3^k \mu(\boldsymbol{\gamma}^{[s],[k]}),\, \mu(\boldsymbol{\gamma}^{[t],[k]}) < \infty.$

4. Conclusions

This short paper proposes a corner cutting algorithm for nets of functions and uses some simple but powerful approximation arguments to show its convergence. Even if we considered only convergence analysis, the proposed corner cutting generalization enriches the class of subdivision schemes for nets, currently consisting of few examples (see, e.g., [7, 8, 10, 18, 23]). In a future work we plan to study the smoothness of the limits in the case of nets, and to derive conditions on the corner cutting weights which guarantee C^1 limit functions as done in the univariate case in [3] and in [17].

Acknowledgements. The authors thank the reviewers for their useful comments. C. Conti and L. Romani are members of the INdAM Research group GNCS, which has partially supported this work.

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