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Stability and Thermodynamic Restrictions for a Dual-Phase-Lag Thermal Model

DOI 10.1515/jnet-2016-0039
Received May 2, 2016; revised November 4, 2016; accepted December 12, 2016

Abstract: In this paper, the seeming inconsistency highlighted by Fabrizio and Lazzari (Stability and second law of thermodynamics in dual-phase-lag heat conduction, *Int. J. Heat Mass Transfer* **74** (2014), 484–489) and Quintanilla and Racke (A note on stability of dual-phase-lag heat conduction, *Int. J. Heat Mass Transfer* **49** (2007), 1209–1213) for a thermoelastic material, between the thermodynamic restrictions and the stability conditions is studied. Actually, we show as these results are due to the use of different formulations of the thermodynamic principles, which are not always equivalent. So that, we prove by the model considered in the paper that these two formulations do not lead to the same restrictions on the constitutive equations. This analysis allowed us to restore the compatibility by an appropriate and wide representation of the Second Law.

Keywords: stability, thermodynamics, heat flux, fading memory

1 Introduction

The formulation of the Second Law of Thermodynamics has had over time various formulations, not all equivalent [1–3]. This is probably due to the complexity of the phenomenon to describe, but also to the broad class of materials to be studied [4].

In this paper, through a particular thermal model, we will show how it is necessary to represent the Second Law by a wide (or weak) formulation [5–7]. On the other hand, the notion of stability for a system can be conditioned by the topology used. Therefore, in this paper the suitable conditions that ensure the stability will be studied by a weak formulation of the Second Law.

For several materials it is sufficient to formulate the Second Law by restrictive representations, which allows equally to provide all the necessary restrictions on the constitutive equations.

The systems studied in Refs. [8–10] given by classical fading memory models (see Refs. [1, 2, 4]) and defined by the constitutive equation between the heat flow $\mathbf{q}$ and the gradient of the temperature $\nabla\theta$

$$\mathbf{q}(x,t) = -\frac{k}{\tau_q^2}\left[\tau_\theta^2\nabla\theta(x,t) + \int_0^\infty \kappa(s)\nabla\theta^t(x,s)\,\mathrm{d}s\right], \tag{1}$$

where $\theta^t(x,s) = \theta(x,t-s)$, $s \in (0,\infty)$ denotes the past history of the temperature, and

$$\kappa(s) = 2(\tau_q - \tau_\theta)e^{-s/\tau_q}\left[\frac{\tau_\theta}{\tau_q}\cos\frac{s}{\tau_q} + \sin\frac{s}{\tau_q}\right], \tag{2}$$

with suitable positive coefficients k, τ_q and τ_θ.

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So that, it has been shown in Ref. [8] that if the system (1) and (2) satisfies the Second Law on any closed cycles, then the coefficients τ_q and τ_θ of the constitutive eqs. (1) and (2) have to verify the restriction

$$(2-\sqrt{3})\tau_\theta < \tau_q < (2+\sqrt{3})\tau_\theta. \tag{3}$$

This result does not confirm the theory that there is a close relationship between the conditions which provide the stability and those which ensure the status of the Second Law [3]. Indeed, in Ref. [8] in agreement with Ref. [11], the stability of the problem is obtained for

$$\tau_\theta > (2-\sqrt{3})\tau_q. \tag{4}$$

This anomalous result suggests to better investigate the problem and in particular the formulation of the Second Law for this material system.

In the framework of the continuum thermomechanics, the formulation more used in the literature requires the existence of the entropy function combined with Clausius–Duhem inequality

$$\rho\dot{\eta}(t) \geq \frac{\mathcal{P}_m^i(t)}{\theta(t)} + \frac{1}{\theta^2(t)}\mathbf{q}(t)\cdot\nabla\theta(t), \tag{5}$$

where ρ is the density, η the entropy and $\mathcal{P}_m^i$ the internal mechanical power. This representation has the drawback of requiring the existence of the entropy function, which is not defined apriory by a constitutive equation.

To correct this view point, Coleman and Owen [2] proposed to formulate the Second Law on closed cycles, therefore the notions of state and process were introduced. This allows to define closed loops so, instead of eq. (5), the Second Law is formulated on closed cycles in the following form

$$\oint\left[\frac{\mathcal{P}_m^i(t)}{\theta(t)} + \frac{\mathbf{q}(t)\cdot\nabla\theta(t)}{\theta^2(t)}\right]dt \leq 0. \tag{6}$$

Now, it is worth to observe that the definition of a material system involves a precise definition of state and process sets that are compatible with the studied material. In the definition of the system considered in Ref. [8], the concept of state and process is related to a large set of histories, and accordingly of closed cycles. This means that inequality (6) requires the restrictions (3) on the coefficients τ_q and τ_θ.

In papers [8] and [12], the material system is defined by a more restrictive set of closed cycles. In fact in Ref. [8] it investigates the stability of the problem

$$\begin{aligned} &c\dot{\theta}(x,t) = \frac{k}{\tau_q^2}\nabla\cdot\left[\nabla\theta(x,t) + \int_0^t \kappa(s)\nabla\theta^t(x,s)\,ds\right] + f(x,t) \\ &\theta_{|\partial\Omega}(x,t) = 0\,, \qquad \theta(x,0) = 0 \end{aligned} \tag{7}$$

where the initial state is given by the zero history.

This restriction reduces the class of states and cycle processes. Therefore, the constraints deriving from eq. (3) will be weaker. Indeed for such a problem, we are able to show, also through numerical simulations, the inconsistence between the conditions that ensure the stability and the thermodynamic restrictions following by general formulation of Second Law. On the contrary, if we restrict the class of cyclic processes to those compatible with system (7), we show as the results of this comparison always ensure a consistency between these two view points. Therefore, we have reasons to believe that there is agreement between the conditions that ensure the stability and thermodynamic constraints, when we consider a consistent and suitable formulation of Second Law.

Finally, for this analysis it is important to note that the notion of material system must be defined not only by the constitutive equations but also by initial conditions and supplies.

The present paper is organized as follows. In the Chapter 2, we provide the classical basic elements of Thermodynamics, as the notion of state and process and the Second Law as proposed by Coleman and Owen [2]. In the Section 3, the differential system for a dual-phase-lag of a rigid heat conductor is formulated and the consequence restrictions of the Second Law on closed cycles are studied. The notion of minimal state is presented in the Section 4, where we prove as for this model, the state is represented by a vector of finite dimensions. Finally in the Section 5, a numerical analysis is shown for the study of the sign of the inequality (23). So that, we shown as the numerical simulations are in full agreement with thermodynamics.

2 Thermodynamics

In this section, following the framework of Coleman and Owen [2] and [13], we present the basic elements of Thermodynamics.

A material system is defined by the normed set Σ of the states σ and by the collection Π of the processes $P : [0, d_P) \to V \times \mathbb{R} \times \mathbb{R}^3$ of duration $d_P \in (0, \infty]$ and admissible with the system, defined by

$$P(t) = \left(\mathbf{L}(t), \dot{\theta}(t), \nabla\theta(t)\right),$$

where $\mathbf{L} \in V$ denotes the velocity gradient. So that, there exists a map $\hat{\rho} : \Sigma \times \Pi \to \Sigma$ that to the initial state $\sigma_i \in \Sigma$ and process $P \in \Pi$ provides the final state σ_f.

Following Ref. [2], we formulate

Second Law of Thermodynamics. There is at least one state $\sigma^0 \in \Sigma$ at which for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any process P_ε whereby $\left\|\hat{\rho}(\sigma^0, P_\varepsilon) - \sigma^0\right\| < \delta_\varepsilon$, we have

$$\int_0^{d_P} \left[\frac{\mathcal{P}^i_m(t)}{\theta(t)} + \frac{\mathbf{q}(t)\cdot\nabla\theta(t)}{\theta^2(t)}\right] dt < \varepsilon. \tag{8}$$

In this framework, the internal mechanical power is given by

$$\mathcal{P}^i_m(t) = \theta(t)\dot{e}(t) - \mathbf{T}(t)\cdot\mathbf{L}(t), \tag{9}$$

where e is the internal energy and $\mathbf{T}$ the stress tensor.

In this work, we study only thermal phenomena in the linear approximation, so that the process $P = (\dot{\theta}, \nabla\theta)$, while the state σ is reduced only to thermal phenomena. Then, the inequality (8) assumes the new form:

Second Law (only for thermal processes). There is at least one state $\sigma^0 \in \Sigma$, such that for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ so for any process P_ε for which $\left\|\hat{\rho}(\sigma^0, P_\varepsilon) - \sigma^0\right\| < \delta_\varepsilon$, we have

$$\int_0^{d_P} \mathbf{q}(t)\cdot\nabla\theta(t)dt < \varepsilon. \tag{10}$$

In this paper, the heat flux $\mathbf{q}$ is defined by eqs. (1) and (2) and the initial state σ^0 is given by the zero past history.

3 Statement of the problem

The differential problem for a rigid heat conductor in a smooth domain $\Omega \subset \mathbb{R}^3$, related with the energy balance, is given by

$$\rho \dot{e}(x,t) = -\nabla \cdot \mathbf{q}(x,t) + r(x,t) \tag{11}$$

where $e(x,t) = c\theta(x,t)$ $(c > 0)$ and r denotes the heat supply. In the following we suppose the density $\rho = 1$ and identify $\theta(x,t)$ with the relative temperature $(\theta(x,t) - \theta_r)$, where θ_r is the reference temperature of initial and boundary conditions. So, the differential problem related with the constitutive eqs. (1) and (2) is given by

$$c\dot{\theta}(x,t) = \frac{k}{\tau_q^2}\nabla \cdot \left[\tau_\theta^2 \nabla\theta(t) + \int_0^\infty \kappa(s)\nabla\theta^t(x,s)\,\mathrm{d}s\right] + r(x,t)\,, \tag{12}$$

with the initial and boundary conditions

$$\theta_{|\partial\Omega}(x,t) = 0\,,\ \theta(x,0) = 0,\ \ \theta^{t=0}(x,s) = \theta_0(x,s)\,, \quad s > 0 \tag{13}$$

In the following we suppose the supply $r(x,t) = e^{-t/\tau_q} r_0(x,t)$ with $r_0 \in L^\infty(0,\infty;L^2(\Omega))$, where

$$L^\infty(0,\infty;L^2(\Omega)) = \left\{\theta(x,t) : \theta(\bar{t},x) \in L^2(\Omega),\ \text{a.e. } \bar{t} \in (0,\infty);\ \theta(t,\bar{x}) \in L^\infty(0,\infty),\ \text{a.e. } \bar{x} \in \Omega\right\}.$$

Under these hypotheses, the eq. (12) assumes the new form

$$c\dot{\theta}(x,t) = \frac{k}{\tau_q^2}\nabla \cdot \left[\tau_\theta^2 \nabla\theta(t) + \int_0^t \kappa(s)\nabla\theta^t(x,s)\,\mathrm{d}s\right] + f(x,t) \tag{14}$$

where we suppose the initial history θ_0 such that the function

$$h(x,t) = \frac{k}{\tau_q^2}\nabla \cdot \int_t^\infty \kappa(s)\nabla\theta_0(x,s-t)\,\mathrm{d}s = \frac{k}{\tau_q^2}\nabla \cdot \int_{-\infty}^0 \kappa(t-\tau)\nabla\theta_0(x,\tau)\,\mathrm{d}\tau$$

is well defined and $f = h + r$.

The equivalence of the eq. (14) with the differential system without fading memory studied in Refs. [12, 14–16] was proved in Ref. [8].

Because κ is given by eq. (2), we have that

$$h(x,t) = e^{-t/\tau_q}\frac{k}{\tau_q^2}\nabla \cdot \int_{-\infty}^0 2(\tau_q - \tau_\theta)e^{\tau/\tau_q}\left[\frac{\tau_\theta}{\tau_q}\cos\frac{t-\tau}{\tau_q} + \sin\frac{t-\tau}{\tau_q}\right]\nabla\theta_0(x,\tau)\,\mathrm{d}\tau\,,$$

so that

$$h(x,t) = e^{-t/\tau_q} h_0(x,t) \tag{15}$$

where the initial history θ_0 is supposed such that $h_0 \in L^\infty(0,\infty;L^2(\Omega))$. So, we can prove that there exists two functions $A(x)$ and $B(x)$, whereby the supply $h(x,t)$ is given by

$$h(x,t) = e^{-t/\tau_q}\left[A(x)\sin\frac{t}{\tau_q} + B(x)\cos\frac{t}{\tau_q}\right]. \tag{16}$$

In the following, we work with quasi-cyclic processes that start from the state defined by the zero initial history and decay exponentially, but compatible within the system under consideration and the initial condition $\theta(x,0) = 0$. So that, by a suitable coefficient $\theta_C(x)$, we consider the functions

$$\theta(x,t) = \theta_C(x)e^{-t/\tau_q} \sin \frac{t}{\tau_q} \tag{17}$$

which satisfies the eq. (14) with the supply eq. (16).

Then from Second Law, we have that for any $\varepsilon > 0$, there exists a time T_ε such that

$$-\frac{k}{\tau_q^2}|\nabla\theta_C(x)|^2 \int_0^{T_\varepsilon} \left[\tau_\theta^2 e^{-t/\tau_q} \sin \frac{t}{\tau_q} \int_0^t \kappa(s)e^{-(t-s)/\tau_q} \sin \frac{t-s}{\tau_q} ds\right] \cdot e^{-t/\tau_q} \sin \frac{t}{\tau_q} dt < \varepsilon\,. \tag{18}$$

From eq. (18) and the definition (2) of κ, we obtain

$$\frac{k}{\tau_q^2}|\nabla\theta_C(x)|^2 \int_0^{T_\varepsilon} e^{-2t/\tau_q} \sin \frac{t}{\tau_q} \left\{\tau_\theta^2 \sin \frac{t}{\tau_q} + 2(\tau_q - \tau_\theta) \int_0^t \left[\frac{\tau_\theta}{\tau_q} \cos \frac{s}{\tau_q} + \sin \frac{s}{\tau_q}\right] \sin \frac{t-s}{\tau_q} dsdt\right\} > -\varepsilon \tag{19}$$

and the arbitrariness of ε yields

$$\int_0^\infty e^{-2t/\tau_q} \sin \frac{t}{\tau_q} \left\{\tau_\theta^2 \sin \frac{t}{\tau_q} + 2(\tau_q - \tau_\theta) \int_0^t \left[\frac{\tau_\theta}{\tau_q} \cos \frac{s}{\tau_q} + \sin \frac{s}{\tau_q}\right] \sin \frac{t-s}{\tau_q} dsdt\right\} \geq 0\,. \tag{20}$$

Now, we consider a more general supply of the type

$$\bar{f}(x,t) = e^{-t/\tau_q}\left[A(x) \sin \frac{t}{\tau_0} + B(x) \cos \frac{t}{\tau_0}\right] \tag{21}$$

with $\tau_0 \in R$. Then, we obtain solutions given by

$$\theta(x,t) = \theta_C(x)e^{-t/\tau_q} \sin \frac{t}{\tau_0} \tag{22}$$

In the Section 5, we study the value of τ_0 for which

$$\int_0^\infty e^{-2t/\tau_q} \sin \frac{t}{\tau_q} \left\{\tau_\theta^2 \sin \frac{t}{\tau_0} + 2(\tau_q - \tau_\theta) \int_0^t \left[\frac{\tau_\theta}{\tau_q} \cos \frac{s}{\tau_q} + \sin \frac{s}{\tau_q}\right] \sin \frac{t-s}{\tau_0} dsdt\right\} \geq 0 \tag{23}$$

so that, the Second Law is satisfied.

4 Minimal state

The kernel κ, defined in eq. (2) is solution of the linear ordinary differential equation

$$\ddot{\kappa}(t) + 2\tau_q\dot{\kappa}(t) + 2\tau_q^2\kappa(t) = 0\,.$$

By using a suitable linear combination of the function $\mathbf{q}$, defined in eq. (1) with its first and second derivative with respect to time, we obtained (see Refs. [8, 15, 17])

$$\frac{\tau_q^2}{2}\ddot{\mathbf{q}}(x,t) + \tau_q\dot{\mathbf{q}}(x,t) + \mathbf{q}(x,t) = -k\left[\nabla\theta(x,t) + \tau_\theta\nabla\dot{\theta}(x,t) + \frac{\tau_\theta^2}{2}\nabla\ddot{\theta}(x,t)\right]. \tag{24}$$

It should be noted that the constitutive eq. (24) as eq. (1) requires knowledge of the history of the temperature gradient. By the concept of minimal state, we prove that for this model the state is represented through a finite vector-valued variable.

Let us consider the vector space of the admissible histories of temperature gradient

$$\Gamma = \left\{ \nabla\theta^t : R^+ \to \mathcal{V}; \ \left| \int_0^{+\infty} \kappa(s+\tau)\nabla\theta^t(x,s)\,\mathrm{d}s \right| < +\infty\,, \quad \forall\tau \ge 0 \right\}, \tag{25}$$

it is possible to give the following equivalence relation.

Definition 4.1 *Two histories $\nabla\theta_j(t)$, $(j = 1, 2)$ are said to be equivalent, if*

$$\int_0^{+\infty} \kappa(s+\tau)\nabla\theta_1^t(x,s)\,ds = \int_0^{+\infty} \kappa(s+\tau)\nabla\theta_2^t(x,s)\,ds \qquad \forall\tau\,. \tag{26}$$

Therefore the function

$$\breve{\mathbf{I}}^t(x,\tau) = \int_0^{+\infty} \kappa(s+\tau)\nabla\theta^t(x,s)\,ds \tag{27}$$

characterizes the equivalence class $\Gamma_{\mathcal{R}}$ and defines the minimal state.

The definition 4.1 characterizes the set of equivalence histories, because provide the same heat flux. Let $\Gamma_{\mathcal{R}}$ be the quotient space with respect to the equivalence relation (26).

The evolution equation for $\breve{\mathbf{I}}$ is

$$\dot{\breve{\mathbf{I}}}^t(x,\tau) = \breve{\mathbf{I}}_\tau^t(x,\tau) + \kappa(\tau)\nabla\theta(x,t)\,, \tag{28}$$

where the index τ denotes the τ−derivative and the constitutive equation for the heat flux becomes

$$\mathbf{q}(t) = -\frac{k}{\tau_q^2}\left[\tau_\theta^2\nabla\theta(t) + \breve{\mathbf{I}}^t(x,0)\right]. \tag{29}$$

Later on the dependence on x will be omitted. Now we write explicitly the dependence of $\breve{\mathbf{I}}^t(\tau)$ on the variables t ant τ to prove the following result

Lemma 4.1 *For a material with memory, characterized by the kernel κ defined in eq. (2) the minimal state can be written as follow*

$$\breve{\mathbf{I}}^t(\tau) = e^{-\tau/\tau_q}\left[\mathbf{A}(t)\cos\frac{\tau}{\tau_q} + \mathbf{B}(t)\sin\frac{\tau}{\tau_q}\right], \tag{30}$$

where

$$\begin{aligned} \mathbf{A}(t) &= \breve{\mathbf{I}}^t(0)\,, \\ \mathbf{B}(t) &= \tau_q\dot{\breve{\mathbf{I}}}^t(0) + \breve{\mathbf{I}}^t(0) - 2(\tau_q - \tau_\theta)\tau_\theta\nabla\theta(t)\,. \end{aligned} \tag{31}$$

Proof. Introducing eq. (2) in eq. (27), we obtain

$$\begin{aligned} \breve{\mathbf{I}}^t(\tau) = & \frac{2(\tau_q - \tau_\theta)}{\tau_q} e^{-\tau/\tau_q} \times \\ & \times \left\{ \int_0^\infty e^{-s/\tau_q}\left[\tau_\theta\cos\frac{s}{\tau_q} + \tau_q\sin\frac{s}{\tau_q}\right]\nabla\theta^t(s)\,\mathrm{d}s\cos\frac{\tau}{\tau_q} \right. \\ & \left. + \int_0^\infty e^{-s/\tau_q}\left[\tau_q\cos\frac{s}{\tau_q} - \tau_\theta\sin\frac{s}{\tau_q}\right]\nabla\theta^t(s)\,\mathrm{d}s\sin\frac{\tau}{\tau_q} \right\}. \end{aligned} \tag{32}$$

then eq. $(31)_1$ holds.

To obtain eq. $(31)_2$, we observe that, by using an integration by parts, we get

$$\begin{aligned}\dot{\breve{\mathbf{I}}}^t(0) = & -\frac{2(\tau_q-\tau_\theta)}{\tau_q}\int_0^\infty e^{-s/\tau_q}\left[\tau_\theta\cos\frac{s}{\tau_q}+\tau_q\sin\frac{s}{\tau_q}\right]\frac{\mathrm{d}}{\mathrm{d}s}\nabla\theta^t(s)\,\mathrm{d}s\\ = & \frac{2(\tau_q-\tau_\theta)\tau_\theta}{\tau_q}\nabla\theta(t)\\ & -\frac{2(\tau_q-\tau_\theta)}{\tau_q^2}\int_0^\infty e^{-s/\tau_q}\left[\tau_\theta\cos\frac{s}{\tau_q}+\tau_q\sin\frac{s}{\tau_q}\right]\nabla\theta^t(s)\,\mathrm{d}s\\ & +\frac{2(\tau_q-\tau_\theta)}{\tau_q^2}\int_0^\infty e^{-s/\tau_q}\left[\tau_q\cos\frac{s}{\tau_q}-\tau_\theta\sin\frac{s}{\tau_q}\right]\nabla\theta^t(s)\,\mathrm{d}s\,,\end{aligned}\tag{33}$$

which gives eq. $(31)_2$. □

We also want to show that the evolution eq. (28) coincides with the differential eq. (24). In fact we have

$$\dot{\breve{\mathbf{I}}}^t(x,\tau) = e^{-\tau/\tau_q}\left[\dot{\mathbf{A}}(t)\cos\frac{\tau}{\tau_q}+\dot{\mathbf{B}}(t)\sin\frac{\tau}{\tau_q}\right]\tag{34}$$

while

$$\begin{aligned}\breve{\mathbf{I}}^t_\tau(x,\tau) = & -\frac{e^{-\tau/\tau_q}}{\tau_q}\left\{\left[\mathbf{A}(t)-\mathbf{B}(t)\right]\cos\frac{\tau}{\tau_q}+\left[\mathbf{A}(t)+\mathbf{B}(t)\right]\sin\frac{\tau}{\tau_q}\right\}\\ = & \frac{e^{-\tau/\tau_q}}{\tau_q}\left\{\left[\tau_q\dot{\breve{\mathbf{I}}}^t(0)-2(\tau_q-\tau_\theta)\tau_\theta\nabla\theta(t)\right]\cos\frac{\tau}{\tau_q}\right.\\ & \left.-\left[\tau_q\dot{\breve{\mathbf{I}}}^t(0)+2\breve{\mathbf{I}}^t(0)-2(\tau_q-\tau_\theta)\tau_\theta\nabla\theta(t)\right]\sin\frac{\tau}{\tau_q}\right\}\,.\end{aligned}\tag{35}$$

Replacing in eq. (28), we get

$$\begin{aligned}&\frac{1}{\tau_q}e^{-\tau/\tau_q}\cos\frac{\tau}{\tau_q}\left[\tau_q\dot{\mathbf{A}}(t)+\mathbf{A}(t)-\mathbf{B}(t)-2(\tau_q-\tau_\theta)\tau_\theta\nabla\theta(t)\right]\\ &\frac{1}{\tau_q}e^{-\tau/\tau_q}\sin\frac{\tau}{\tau_q}\left[\tau_q\dot{\mathbf{B}}(t)+\mathbf{A}(t)+\mathbf{B}(t)-2(\tau_q-\tau_\theta)\tau_q\nabla\theta(t)\right]=0\,.\end{aligned}\tag{36}$$

By using eq. (31), we obtain

$$\dot{\mathbf{A}}(t)=\dot{\breve{\mathbf{I}}}^t(x,0)\,,\quad \dot{\mathbf{B}}(t)=\tau_q\ddot{\breve{\mathbf{I}}}^t(0)+\dot{\breve{\mathbf{I}}}^t(0)-2(\tau_q-\tau_\theta)\tau_\theta\nabla\dot{\theta}(t)\,,$$

so that eq. (36) becomes

$$\frac{2}{\tau_q}e^{-\tau/\tau_q}\sin\frac{\tau}{\tau_q}\times\left[\frac{\tau_q^2}{2}\ddot{\breve{\mathbf{I}}}^t(0)+\tau_q\dot{\breve{\mathbf{I}}}^t(0)+\breve{\mathbf{I}}^t(0)-(\tau_q-\tau_\theta)\tau_\theta\tau_q\nabla\dot{\theta}(t)-(\tau_q^2-\tau_\theta^2)\nabla\theta(t)\right]=0\,,$$

for any $\tau \geq 0$.

Finally, by using eq. (29), we obtain

$$\begin{aligned}\breve{\mathbf{I}}^t(0) &= -\frac{\tau_q^2}{k}q(t)-\tau_\theta^2\nabla\theta(t)\,,\\ \dot{\breve{\mathbf{I}}}^t(0) &= -\frac{\tau_q^2}{k}\dot{q}(t)-\tau_\theta^2\nabla\dot{\theta}(t)\,,\\ \ddot{\breve{\mathbf{I}}}^t(0) &= -\frac{\tau_q^2}{k}\ddot{q}(t)-\tau_\theta^2\nabla\ddot{\theta}(t)\end{aligned}$$

and it is easy to show the equivalence between eqs. (24) and (28).

5 Numerical analysis for quasi-closed cycles

In this section we study the conditions for which the sign is not negative in the inequality (23).

The sign of the left-hand side of inequality (23) does not depend on the value of τ_q, in particular we can use τ_q as the unit of time, setting $\tau_q = 1$, so that eq. (23) becomes

$$\int_0^\infty e^{-2t} \sin \frac{t}{\tau_0} \left[\tau_\theta^2 \sin \frac{t}{\tau_0} + \int_0^t 2(1-\tau_\theta) \left[\tau_\theta \cos s + \sin s \right] \sin \frac{t-s}{\tau_0} ds \right] dt \geq 0$$

This integral can be calculated explicitly, and its value, setting $\omega = 1/\tau_0 > 0$, is

$$G(\omega, \tau_\theta) = \frac{\omega^2 \left[\left(\omega^4 - 1\right) \tau_\theta^2 + 8\left(\omega^2 + 1\right) \tau_\theta - 2\left(\omega^2 - 9\right) \right]}{4\left(\omega^2 + 1\right)\left(\omega^4 + 6\omega^2 + 25\right)}$$

Here we can see that the sign is determined only by the factor

$$P_\omega(\tau_\theta) = \left(\omega^4 - 1\right) \tau_\theta^2 + 8\left(\omega^2 + 1\right) \tau_\theta - 2\left(\omega^2 - 9\right)$$

If for $\omega \neq 1$ we consider this expression as a quadratic with respect to τ_θ, the discriminant is given by

$$\Delta = 8\left(\omega^2 + 1\right)\left[\left(\omega^2 - 1\right)^2 + 16\right] > 0$$

so for any value of $\omega \neq 1$ there are always two distinct values of τ_θ depending on ω, say $\tau_{\theta,1} < \tau_{\theta,2}$ for which $P_\omega(\tau_{\theta,i}) = 0$, with $i = 1, 2$.

We can study the given quadratic form in the domain $\tau_\theta > 0$ and in the following three intervals for $\omega : (0, 1]$, $(1, 3]$ and $(3, +\infty)$ ($\omega = 1$ and $\omega = 3$ are the positive values of ω in which the coefficients of the quadratic change sign).

- when $\omega \in (0, 1)$ the resulting functions of τ_θ are represented in Figure 1: the graphics of the function is represented by a downward parabola, starting at a positive value $P_\omega(0) = 2(9 - \omega^2)$, increasing until $\tau_\theta = 4/(1 - \omega^2)$, then decreasing and becoming negative at $\tau_{\theta,2} > \lim_{\omega \to 0} \tau_{\theta,2} = 4 + \sqrt{34} \approx 9.831$, the zero being increasing with ω. In particular, when $\omega = 1$ the function becomes the straight line $P_1(\tau_\theta) = 16(\tau_\theta + 1)$, so it is always positive in the considered domain.
- when $\omega \in (1, 3)$ the resulting functions of τ_θ are represented in Figure 2: the graphics of the function is represented by an upward parabola, starting at a positive value $P_\omega(0) = 2(9 - \omega^2)$, then increasing, so it is always positive in the considered domain. When $\omega = 3$ the starting point becomes 0, and so $\tau_{\theta,2} = 0$.

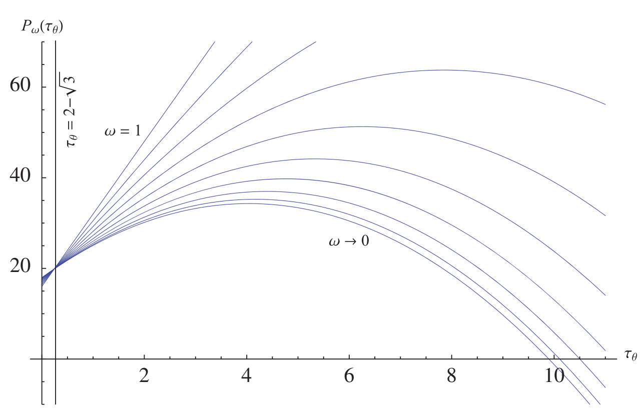

Figure 1: In this picture we study the function $P_\omega(\tau_\theta)$ depending on ω in the interval $0 < \omega < 1$.

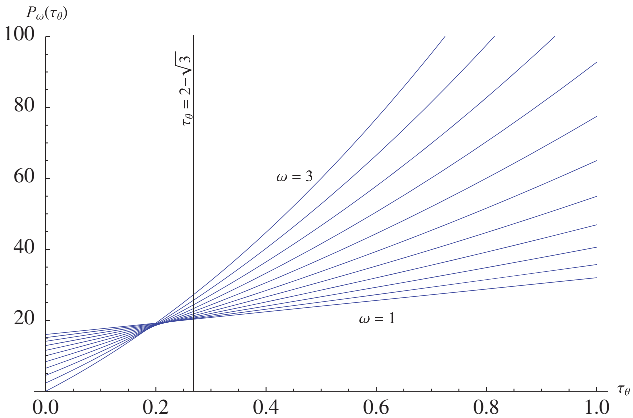

Figure 2: Here the function $P_\omega(\tau_\theta)$ is studied in the interval $1 < \omega < 3$.

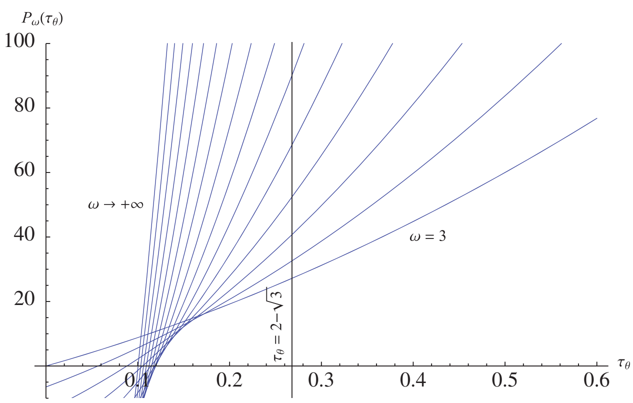

Figure 3: In this picture the function $P_\omega(\tau_\theta)$ is given in the interval $\omega > 3$.

- when $\omega \in (3, +\infty)$ the resulting functions of τ_θ are represented in Figure 3: the graphics of the function is still represented by an upward parabola, it starts at a negative value $P_\omega(0) = -2(\omega^2 - 9)$, then increases becoming positive at the zero $\tau_{\theta,2}$, then keeps increasing, so it is positive from this point on. It is interesting to analyze the behavior of the zero when ω ranges in $(3, +\infty)$, from which it is clear that the zero never reach the limiting value $2 - \sqrt{3} \approx 0.268$.

In all figures, the limit of the domain $\tau_\theta = 2 - \sqrt{3}$ is shown. With respect to this limit, we can see that when $\omega < 1$ there is the interval $(\tau_{\theta,2}, +\infty)$ where the function is negative. For $1 < \omega < 3$ the function is always positive, while for $\omega > 3$ the interval $(0, \tau_{\theta,2})$ where the function is negative falls in the zone where $\tau_\theta < 2-\sqrt{3}$, so it not of our interest.

From this analysis it results that restriction (4) is compatible with the thermodynamics on all quasi-closed cycles obtained through solution of type eq. (22) in the hypothesis $\omega \geq 1 \implies \tau_0 \leq \tau_q$. When $\omega < 1$, and so $\tau_0 > \tau_q$, there is compatibility only when $\tau_\theta < \tau_{\theta,2}$.

6 Conclusions

The examples presented in this paper show that for a complete characterization of a material system is not enough to give the constitutive equations. In fact, the thermodynamic properties of a material are defined, in addition to the constitutive equations, also by the set of initial conditions and sources of the differential system. In fact, this research shows that the conditions that ensure the compatibility of the system with thermodynamics can depend from the whole of definitions of initial data and sources.

References

[1] J. A. D. Appleby, M. Fabrizio, B. Lazzari and D. W. Reynolds, On exponential asymptotic stability in linear viscoelasticity, *Math. Models Methods Appl. Sci.* **16** (2006), no. 10, 1677–1694.
[2] B. D. Coleman and D. R. Owen, A mathematical foundation for thermodynamics, *Arch. Ration. Mech. Anal.* **54** (1974), 1–104.
[3] C. M. Dafermos, The entropy rate admissibility criterion in thermoelasticity, *Rend. Accad. Naz. Lincei, Ser. VIII* **57** (1974), 113–119.
[4] M. Fabrizio, B. Lazzari and R. Nibbi, Thermodynamics of non-local materials: extra fluxes and internal powers, *Continuum Mech. Thermodyn.* **23** (2011), 509–525.
[5] C. M. Dafermos, Second law of thermodynamics and stability, *Arch. Ration. Mech. Anal.* **70** (1979), 167–179.
[6] J. L. Ericksen, A thermo-kinetic view of elastic stability theory, *Int. J. Solids Struct.* **2** (1966), 573–580.
[7] M. E. Gurtin, Thermodynamics and stability, *Arch. Ration. Mech. Anal.* **59** (1975), 63–96.
[8] M. Fabrizio and B. Lazzari, Stability and second law of thermodynamics in dual-phase-lag heat conduction, *Int. J. Heat Mass Transfer* **74** (2014), 484–489.
[9] S. Chirita, M. Ciarletta and V. Tibullo, On the wave propagation in the time differential dual-phase-lag thermoelastic model, *Proc. R. Soc. A* **471** (2015), 20150400.
[10] M. Ciarletta, B. Straughan, V. Tibullo, Anisotropic effects on poroacoustic acceleration waves, *Mech. Res. Commun.* **37** (2010), 137–140.
[11] R. Quintanilla, Exponential stability in the dual-phase-lag heat conduction theory, *J. Non-Equilib. Thermodyn.* **27** (2002), 217–227.
[12] R. Quintanilla and R. Racke, A note on stability of dual-phase-lag heat conduction, *Int. J. Heat Mass Transfer* **49** (2007), 1209–1213.
[13] M. Fabrizio and A. Morro, Thermodynamics of electromagnetic isothermal systems with memory, *J. Non-Equilib. Thermodyn.* **22** (1997), 110–128.
[14] M. Fabrizio and F. Franchi, Delayed thermal models. Stability and thermodynamics, *J. Therm. Stress* **37** (2014), 160–173.
[15] D. Y. Tzou, A unified approach for heat conduction from macro to micro-scales, *ASME J. Heat Transfer* **117** (1995), 8–16.
[16] L. Wang, M. Xu and X. Zhou, Well-posedness and solution structure of dual-phase-lagging heat conduction, *Int. J. Heat Mass Transfer* **44** (2001), 1659–1669.
[17] D. Y. Tzou, The generalized lagging response in small-scale and high-rate heating, *Int. J. Heat Mass Transfer* **38** (1995), 3231–3240.