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Onset of thermal convection in a horizontal porous layer saturated by a power–law fluid

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Onset of thermal convection in a horizontal porous layer saturated by a power–law fluid

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Abstract. A horizontal porous layer saturated by a non–Newtonian fluid is taken into account. The fluid is a Ostwald–De Waele type of fluid. The layer is heated from below while the upper boundary is isothermal. The lower boundary is impermeable and the upper one is free. A fully developed basic throughflow is considered. The basic state is perturbed by employing small amplitude perturbations such that a linear stability analysis is performed. The disturbances are assumed to be normal modes and a system of ordinary differential equations governing the perturbation dynamics is obtained. These ODEs form an eigenvalue problem that is solved numerically: both the neutral stability curves and the threshold values for the onset of convection are presented.

1. Introduction

The thermally driven instabilities in fluid saturated porous media have been deeply investigated in the last century. The milestone studies of this research area are by Horton and Rogers [1] and Lapwood [2] who studied the so called Darcy–Bénard problem or, alternatively, Horton–Rogers and Lapwood problem: the analogous of the Rayleigh–Bénard problem when fluid saturated porous media are considered. The literature is mainly focused on the stability analysis of Newtonian fluids [3], while the stability analysis of non–Newtonian fluids saturating porous media is a topic that has been developing in the recent years. In this direction, some work has been done by Barletta and Nield [4] and by Celli et al. [5] by considering the classical Prats problem where a power–law fluid saturates the porous layer. The Prats problem is a variant of the Horton–Rogers and Lapwood problem with a horizontal throughflow. The system examined in this work is a horizontal porous layer saturated by a power–law fluid subject to a horizontal throughflow. The lower boundary is impermeable with a prescribed heat flux. The upper boundary is isothermal and isobaric. The novelty of the work is introduced by considering a permeable upper boundary. This hydrodynamic boundary condition may, in fact, influence drastically the threshold for the onset of thermal convection, as proved by Barletta [6]. The linear stability analysis of the basic state for the onset of thermal convection is performed. The momentum balance equations employed is the extended Darcy law for power–law fluids. The fluid is assumed to be incompressible and the energy balance equation employed is a convection–conduction equation with neither energy sources nor energy sinks.

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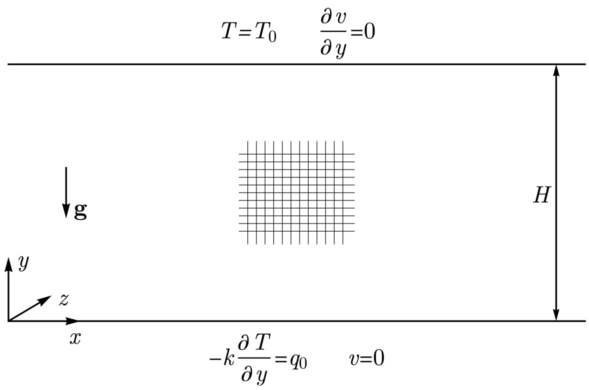

Figure 1. A sketch of the fluid saturated porous layer together with the boundary conditions

2. Mathematical model

An Ostwald–De Waele fluid saturating a horizontal porous layer is heated from below by prescribing a constant heat flux q_0. A basic horizontal throughflow is present. The lower boundary is impermeable while the upper boundary is open and isothermal at temperature T_0. The porous layer has thickness H and it is infinitely wide in both the horizontal directions. A sketch of the horizontal porous layer together with the boundary conditions is displayed in Fig. 1. A modified Darcy's law, together with the Oberbeck-Boussinesq approximation, is employed to model the momentum transport. The fluid is incompressible and no source/sink terms are considered in the local energy balance equation. The governing equations employed are

$$
\begin{gathered}
\overline{\boldsymbol{\nabla}} \cdot \overline{\mathbf{u}} = 0, \\
\frac{\eta}{K} |\overline{\mathbf{u}}|^{n-1} \overline{\mathbf{u}} = -\overline{\boldsymbol{\nabla}} \overline{p} - \rho_0 \overline{\mathbf{g}} \beta \left(\overline{T} - \overline{T_0} \right), \\
\sigma \frac{\partial \overline{T}}{\partial \overline{t}} + \overline{\mathbf{u}} \cdot \overline{\boldsymbol{\nabla}} \overline{T} = \chi \overline{\nabla}^2 \overline{T}, \\
\overline{y} = 0 : \quad \overline{v} = 0, \quad -k \frac{\partial \overline{T}}{\partial \overline{y}} = q_0, \\
\overline{y} = H : \quad \frac{\partial \overline{v}}{\partial \overline{y}} = 0, \quad \overline{T} = T_0,
\end{gathered}
\tag{1}
$$

where the line over the variables identifies dimensional quantities, $\overline{\mathbf{x}} = (\overline{x}, \overline{y}, \overline{z})$ is the Cartesian coordinates vector, $\overline{\mathbf{u}} = (\overline{u}, \overline{v}, \overline{w})$ is the filtration velocity vector, η is the consistency factor measured in $[Pa \cdot s^n]$, K is the generalised permeability of the porous medium measured in $[m^{n+1}]$, n is the power–law index, p is the total pressure head, ρ_0 is the fluid density evaluated at the reference temperature T_0, $\mathbf{g}$ is the gravitational acceleration vector, β is the thermal expansion coefficient of the fluid, σ is the ratio between the thermal capacity of the porous medium saturated by the fluid and the thermal capacity of the fluid, $\overline{t}$ it the time, χ is the actual thermal diffusivity coefficient of the porous medium measured in $[m^2 \cdot s^{-1}]$ and k is the effective thermal conductivity of the fluid saturated porous medium. The scaling

$$
\frac{\overline{\mathbf{x}}}{H} \to \mathbf{x}, \quad \frac{\chi}{\sigma H^2} \overline{t} \to t, \quad \frac{H}{\chi} \overline{\mathbf{u}} \to \mathbf{u}, \quad \frac{K}{\mu} \left(\frac{H}{\chi} \right)^n \overline{\boldsymbol{\nabla}} \overline{p} \to \boldsymbol{\nabla} p, \quad \frac{\overline{T} - T_0}{\Delta T} \to T,
\tag{2}
$$

where $\Delta T = q_0 H/k$, is employed to obtain the dimensionless governing equations, namely

$$\begin{aligned}
&\boldsymbol{\nabla}\cdot\mathbf{u} = 0,\\
&\boldsymbol{\nabla}\times\left(|\mathbf{u}|^{n-1}\mathbf{u}\right) = \mathrm{R}\,\boldsymbol{\nabla}\times(T\,\mathbf{e}_y),\\
&\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T,\\
&y=0: \qquad v=0, \quad \frac{\partial T}{\partial y} = -1,\\
&y=1: \qquad \frac{\partial v}{\partial y} = 0, \quad T=0,
\end{aligned} \tag{3}$$

where $\mathbf{e}_y$ is the unit vector in the y–direction. The curl operator is applied to the momentum balance equation to eliminate the contribution of the pressure. The modified Darcy–Rayleigh number R in Eq. (3) is defined as

$$\mathrm{R} = \frac{\rho\, g\, \beta\, K\, H^n\, \Delta T}{\chi^n\, \mu}, \tag{4}$$

where g is the modulus of gravitational acceleration.

3. Linear stability analysis

A fully developed basic flow is assumed such that $\bar{\mathbf{u}}_b = (U_0, 0, 0)$, where U_0 is the velocity of the throughflow and the subscript b indicates the basic state quantities. The dimensionless basic velocity is $\mathbf{u}_b = (\mathrm{Pe}, 0, 0)$, where $\mathrm{Pe} = U_0 H/\chi$ is the Péclet number.
By assuming that the temperature depends only on the vertical direction y, the Eq. (3) yields

$$T_b = 1 - y. \tag{5}$$

The basic state just defined is perturbed by superposing a small amplitude disturbance to it, namely

$$\mathbf{u} = \mathbf{u}_b + \varepsilon\mathbf{U}, \qquad T = T_b + \varepsilon\Theta, \tag{6}$$

where $\varepsilon \ll 1$. Equation (6) is substituted into Eq. (2) to obtain the following linearised governing equations for the disturbances:

$$\begin{aligned}
&\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0,\\
&\frac{\partial V}{\partial z} - \frac{\partial W}{\partial y} = \frac{\mathrm{R}}{\mathrm{Pe}^{n-1}}\frac{\partial \Theta}{\partial z},\\
&n\frac{\partial U}{\partial z} - \frac{\partial W}{\partial x} = 0,\\
&n\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} = -\frac{\mathrm{R}}{\mathrm{Pe}^{n-1}}\frac{\partial \Theta}{\partial x},\\
&\frac{\partial \Theta}{\partial t} + \mathrm{Pe}\frac{\partial \Theta}{\partial x} - V = \nabla^2\Theta,\\
&y=0: \qquad V=0, \quad \frac{\partial \Theta}{\partial y} = -1,\\
&y=1: \qquad \frac{\partial V}{\partial y} = 0, \quad \Theta = 0.
\end{aligned} \tag{7}$$

By manipulating Eq. (7), one may obtain a system of equations that contains only two dependent variables (V, Θ), namely

$$\begin{gathered}
\frac{\partial^2 V}{\partial x^2} + n\frac{\partial^2 V}{\partial y^2} + n\frac{\partial^2 V}{\partial z^2} = \frac{\mathrm{R}}{\mathrm{Pe}^{n-1}}\left(\frac{\partial^2 \Theta}{\partial x^2} + n\frac{\partial^2 \Theta}{\partial z^2}\right), \\
\frac{\partial \Theta}{\partial t} + \mathrm{Pe}\frac{\partial \Theta}{\partial x} - V = \nabla^2 \Theta, \\
y = 0: \qquad V = 0, \quad \frac{\partial \Theta}{\partial y} = -1, \\
y = 1: \qquad \frac{\partial V}{\partial y} = 0, \quad \Theta = 0.
\end{gathered} \tag{8}$$

The problem (8) can be solved by assuming that the perturbations have the form of plane waves

$$\begin{aligned}
V(x,y,z,t) &= f(y) e^{\eta t} e^{i(\alpha_x x + \alpha_z z + \omega t)}, \\
\Theta(x,y,z,t) &= h(y) e^{\eta t} e^{i(\alpha_x x + \alpha_z z + \omega t)},
\end{aligned} \tag{9}$$

where $\alpha = (\alpha_x^2 + \alpha_z^2)^{1/2}$ is the wavenumber, ω is the angular frequency and η is the growth/decay rate. The parameters $(\alpha_x, \alpha_z, \eta, \omega)$ are real numbers while the eigenfunctions (f, h) are complex valued. As function of the sign of η, the disturbances in Eq. (9) behave differently for $t \to \infty$:

- $\eta > 0$ yields growing disturbances (possibly unstable configuration)
- $\eta = 0$ yields neutral stability configuration
- $\eta < 0$ yields decaying disturbances (stable configuration)

Since this linear stability analysis is focused on the neutral stability configurations, in the following $\eta = 0$. By substituting definitions (9) into Eq. (8), by employing the rescaled angular frequency $\tilde{\omega} = \omega - \alpha_x \mathrm{Pe}$ and by employing the rescaled Darcy–Rayleigh number

$$\tilde{\mathrm{R}} = \frac{\mathrm{R}}{\mathrm{Pe}^{n-1}}, \tag{10}$$

one obtains the following set of ordinary differential equations

$$\begin{gathered}
f'' - \frac{\alpha^2 + (n-1)\alpha_z^2}{n}(f - \tilde{\mathrm{R}} h) = 0, \\
h'' - \left(\alpha^2 - i\,\tilde{\omega}\right) h + f = 0, \\
y = 0: \qquad f = 0, \quad h' = 0, \\
y = 1: \qquad f' = 0, \quad h = 0.
\end{gathered} \tag{11}$$

By indicating with ϕ the inclination angle of the wave vector $\boldsymbol{\alpha} = (\alpha_x, 0, \alpha_z)$ to the x–axis and by defining the rescaled power–law index

$$\tilde{n} = \frac{n}{1 + (n-1)\sin^2 \phi}, \tag{12}$$

one may simplify Eq. (11) to obtain

$$\begin{gathered}
f'' - \frac{\alpha^2}{\tilde{n}}(f - \tilde{\mathrm{R}} h) = 0, \\
h'' - h\left(\alpha^2 - i\,\tilde{\omega}\right) + f = 0, \\
y = 0: \qquad f = 0, \quad h' = 0, \\
y = 1: \qquad f' = 0, \quad h = 0.
\end{gathered} \tag{13}$$

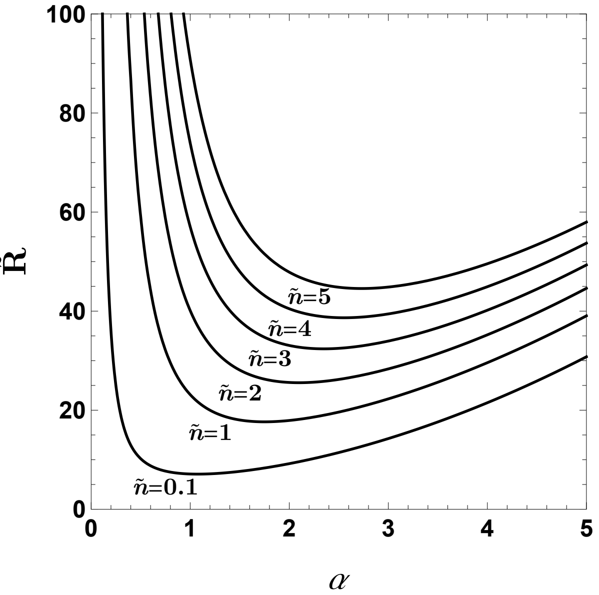

Figure 2. Neutral stability curves $\tilde{\mathrm{R}}(\alpha)$ for different values of $\tilde{n}$

4. Results

The solution of the eigenvalue problem Eq. (13) is carried out by coupling the Runge-Kutta method with the shooting method. The Runge–Kutta method solves the following initial value problem:

$$\begin{aligned} & f'' - \frac{\alpha^2}{\tilde{n}}(f - \tilde{\mathrm{R}}\, h) = 0, \\ & h'' - h\left(\alpha^2 - i\,\tilde{\omega}\right) + f = 0, \\ y = 0: \quad & f = 0, \quad f' = \xi_1 + i\,\xi_2, \quad h = 1, \quad h' = 0, \end{aligned} \tag{14}$$

where the conditions $f' = \xi_1 + i\,\xi_2$ and $h = 1$ are added to complete the initial conditions, the parameters ξ_1 and ξ_2 are unknowns and $h = 1$ is a scale fixing condition that can be imposed since the system (14) is homogeneous. The shooting method employs the solution obtained by the Runge–Kutta method to satisfy the target conditions at $y = 1$, namely

$$f' = 0, \quad h = 0. \tag{15}$$

The shooting method allows one to obtain the neutral stability curves $\tilde{\mathrm{R}}(\alpha)$: one curve for each value of $\tilde{n}$. The neutral stability curves are displayed in Fig. 2: the curves monotonically move upward for increasing values of $\tilde{n}$. The value of the modified angular frequency is found to be always zero, $\tilde{\omega} = 0$, for every solution obtained. One can thus induce that $\omega = \alpha_x \mathrm{Pe}$. Moreover, a zero modified angular frequency allows one to induce that the eigenvalue problem (13) is real valued.

The minima of the curves in Fig. 2 define the threshold parametric configurations for the onset of thermal convection. These values are conventionally named “critical values” and are indicated with the subscript *cr*. The curves $\tilde{\mathrm{R}}_{\mathrm{cr}}(\tilde{n})$ and $\tilde{\alpha}_{\mathrm{cr}}(\tilde{n})$ are reported in Table 1. It is worth noting that the case $\tilde{n} = 1$ yields $n = 1$. For $\tilde{n} = 1$ one, in fact, obtains the values reported in the literature for Newtonian fluids as presented in Tab 6.1 of Ref. [3].

Table 1. Critical values $\tilde{\mathrm{R}}_{\mathrm{cr}}$ and α_{cr} for $0.1 \leqslant \tilde{n} \leqslant 2$

$\tilde{n}$	$\tilde{\mathrm{R}}_{\mathrm{cr}}$	α_{cr}
0.1	7.0973158	1.0738772
0.2	8.9134045	1.2208534
0.3	10.359693	1.3256991
0.4	11.624207	1.4104158
0.5	12.775716	1.4827223
0.6	13.848240	1.5464415
0.7	14.861500	1.6037971
0.8	15.828121	1.6562145
0.9	16.756758	1.7046690
1.0	17.653653	1.7498612
1.1	18.523479	1.7923129
1.2	19.369847	1.8324247
1.3	20.195615	1.8705112
1.4	21.003092	1.9068248
1.5	21.794175	1.9415710
1.6	22.570446	1.9749199
1.7	23.333242	2.0070137
1.8	24.083704	2.0379731
1.9	24.822816	2.0679009
2.0	25.551432	2.0968860

5. Conclusions

The linear stability of a Ostwald–De Waele fluid saturating a porous layer is tested for the onset of thermal convection. The porous layer is heated from below and the upper boundary is open and isothermal. The eigenvalue problem obtained by employing the normal modes method depends on the modified Darcy–Rayleigh number $\tilde{\mathrm{R}}$ and the modified power–law index $\tilde{n}$. The critical values of the governing parameters are obtained numerically. The most relevant results are the following:

- The angular frequency of the perturbation is obtained analytically by inducing, from the numerical solutions, that the modified angular frequency is equal zero
- The modified power–law index $\tilde{n}$ has a monotonic influence on the critical values: increasing values of $\tilde{n}$ yield higher values of $\tilde{\mathrm{R}}_{\mathrm{cr}}$ and thus more stable configurations
- The case $\tilde{n} = 1$ coincides with the Newtonian fluid case and the critical values obtained agree with the values reported in the literature: $\tilde{\mathrm{R}}_{\mathrm{cr}} = 17.653653$ and $\alpha_{\mathrm{cr}} = 1.7498612$

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