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RESEARCH ARTICLE

Uniqueness and decay in local thermal non-equilibrium double porosity thermoelasticity

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Abstract

This paper studies a model for thermoelasticity where the body has a double porosity structure. There are the usual pores associated to a porous body, herein called macro pores. In addition, the solid skeleton contains cracks or fissures which give rise to a micro porosity. The fully anisotropic situation is analyzed. We firstly establish uniqueness of a solution to the boundary-initial value problem when the elastic coefficients are sign-indefinite and are required to satisfy only major symmetry. Furthermore, in the quasi-equilibrium case, where the solid acceleration is neglected, we demonstrate that a solution to the boundary-initial value problem with zero boundary conditions will decay to zero in a certain sense, under the assumption that there are no sources and external body force involved.

KEYWORDS:
double porosity, thermoelasticity, local-thermal non equilibrium, uniqueness, decay

1 | INTRODUCTION

Driven by the many real life applications, elastic materials with a double porosity structure have become the subject of intensive current research.

A double porosity elastic material is a solid which contains pores on a macro scale and pores on a much smaller scale. These are referred to as macro pores and micro pores, respectively. There are many application areas and these are detailed in Straughan [1], but to give an idea of the vastness we mention oil recovery, Olusola et al. [2]; hydraulic fracturing for gas, Kim and Moridis [3]; bone recovery and bone replacement, Svanadze and Scalia [4].

In this article we focus on an anisotropic theory for a double porosity elastic material under the effect of local thermal non-equilibrium, cf. Svanadze [5]. Local thermal non-equilibrium is where the solid and the fluid in a porous body may have different temperatures. In studies of fluid motion in a fixed porous body this concept was introduced by Nield [6] and striking differences have been observed by comparison with the single temperature theory, see e.g. Rees et al. [7], see also Straughan [8].

The basic double porosity theory we use is based on a single temperature model developed extensively by Svanadze [9,10], Ciarletta et al. [11], Svanadze and Scalia [4], Scarpetta and Svanadze [12], and within the local thermal non-equilibrium isotropic theory developed and used in Svanadze [5]. Thermal effects in porous elastic material are very important since cracking may occur due to thermally induced stress, see e.g. Siratovich et al. [13]. Thermally induced cracking in a porous elastic body is thus intimately connected to a double porosity theory.

In the first part of the paper we establish uniqueness of a solution without requiring any sign-definiteness of the elastic coefficients. We require only symmetry of elasticities. This is important since, for example, Xinchun and Lakes [14] and Ha et al. [15] have demonstrated that many materials have negative Poisson ratio. In fact, recent studies have abandoned positive-definiteness of the elastic coefficients in favour of the weaker condition of strong ellipticity, see e.g. Xinchun and Lakes [14], Chiriţă and Ciarletta [16], Chiriţă et al. [17], Chiriţă and Danescu [18], Chiriţă and Ghiba [19]. Thus, for an anisotropic body use of only the major symmetry is important.

In the quasi-equilibrium case where the solid acceleration is absent, see e.g. Scarpetta and Svanadze [12], the decay in time in a certain sense is established for a solution of the double porosity local thermal non-equilibrium problem, under the assumption that there are no sources and external body force involved.

2 | THE GOVERNING EQUATIONS

The porous elastic body consists of a solid elastic skeleton, macro pores, and micro pores in the skeleton. The temperature is allowed to be different in the skeleton, in the fluid in the macro pores and in the fluid in the micro pores. The relevant variables are the elastic displacement, u_i, the pressure in the macro pores, p, the pressure in the micro pores, q, the temperature in the solid skeleton, θ_1, the temperature of the fluid in the macro pores, θ_2, and the temperature of the fluid in the micro pores, θ_3. Each variable is a function of $\mathbf{x}$ and t and the elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^3$, which has boundary sufficiently smooth to allow application of the divergence theorem.

The basic equations for a double porosity thermoelastic body in local thermal non-equilibrium are derived in the isotropic case by Svanadze [5]. We here generalize this and present the fully anisotropic system of equations, namely

$$\begin{aligned}
\rho\ddot{u}_i &= \big(a_{ijhk}u_{h,k}\big)_{,j} - \big(\beta_{ij}p\big)_{,j} - \big(\gamma_{ij}q\big)_{,j} \\
&\quad - \Big(a^{(1)}_{ij}\theta_1\Big)_{,j} - \Big(a^{(2)}_{ij}\theta_2\Big)_{,j} - \Big(a^{(3)}_{ij}\theta_3\Big)_{,j} + \rho f_i\,, \\
\alpha\dot{p} &= \big(k_{ij}p_{,j}\big)_{,i} - \beta_{ij}\dot{u}_{i,j} - \gamma(p-q) + \rho s_1\,, \\
\beta\dot{q} &= \big(m_{ij}q_{,j}\big)_{,i} - \gamma_{ij}\dot{u}_{i,j} + \gamma(p-q) + \rho s_2\,, \\
a_1\dot{\theta}_1 &= \Big(\kappa_{ij}\theta_{1_{,j}}\Big)_{,i} - a^{(1)}_{ij}\dot{u}_{i,j} - d_1(\theta_1-\theta_2) - d_2(\theta_1-\theta_3) + \rho r_1\,, \\
a_2\dot{\theta}_2 &= \Big(\omega_{ij}\theta_{2_{,j}}\Big)_{,i} - a^{(2)}_{ij}\dot{u}_{i,j} + d_1(\theta_1-\theta_2) - d_3(\theta_2-\theta_3) + \rho r_2\,, \\
a_3\dot{\theta}_3 &= \Big(\pi_{ij}\theta_{3_{,j}}\Big)_{,i} - a^{(3)}_{ij}\dot{u}_{i,j} + d_3(\theta_2-\theta_3) + d_2(\theta_1-\theta_3) + \rho r_3\,,
\end{aligned} \tag{1}$$

where a superposed dot denotes $\partial/\partial t$, standard indicial notation is employed with $_{,i}$ denoting $\partial/\partial x_i$, and f_i, s_1, s_2, r_1, r_2, r_3 are the body force, sources for the pressure and externally supplied heat sources, respectively. The coefficients may all depend on $\mathbf{x}$, with $\rho > 0$ being the density of the solid elastic skeleton and also α, β, γ, d_1. d_2, d_3, a_1, a_2 and a_3 are positive. The tensors β_{ij}, γ_{ij}, $a^{(1)}_{ij}$, $a^{(2)}_{ij}$, $a^{(3)}_{ij}$ are all symmetric. Likewise the coefficients k_{ij}, m_{ij}, κ_{ij}, ω_{ij} and π_{ij} are all symmetric tensors.

Equations (1) hold on $\Omega \times (0,T]$ for some $T < \infty$. The boundary of Ω is denoted by Γ and on Γ the conditions are

$$\begin{aligned}
&u_i = h_i(\mathbf{x},t)\,, \quad p = p^B(\mathbf{x},t)\,, \quad q = q^B(\mathbf{x},t)\,, \\
&\theta_1 = \alpha_1(\mathbf{x},t)\,, \quad \theta_2 = \alpha_2(\mathbf{x},t)\,, \quad \theta_3 = \alpha_3(\mathbf{x},t)\,,
\end{aligned} \tag{2}$$

holding on $\Gamma \times [0,T]$, where h_1, p^B, q^B, α_1, α_2, α_3 are prescribed functions. In addition, the following initial conditions hold

$$\begin{aligned}
&u_i(\mathbf{x},0) = v_i(\mathbf{x})\,, \qquad \dot{u}_i(\mathbf{x},0) = w_i(\mathbf{x})\,, \\
&p(\mathbf{x},0) = P(\mathbf{x})\,, \qquad q(\mathbf{x},0) = Q(\mathbf{x})\,, \\
&\theta_1(\mathbf{x},0) = \Theta_1(\mathbf{x})\,, \quad \theta_2(\mathbf{x},0) = \Theta_2(\mathbf{x})\,, \quad \theta_3(\mathbf{x},0) = \Theta_3(\mathbf{x})\,,
\end{aligned} \tag{3}$$

with $\mathbf{x} \in \Omega$ and for given functions v_i, w_i, P, Q and Θ_α, $\alpha = 1\,,2\,,3$.

The boundary-initial value problem given by (1)–(3) is denoted by $\mathcal{P}$.

3 | UNIQUENESS

In this section we require the elastic coefficients to satisfy the symmetries

$$a_{ijhk} = a_{jihk} = a_{hkij}\,, \tag{4}$$

although we stress that no definiteness is required, and we require that k_{ij}, m_{ij}, κ_{ij}, ω_{ij} and π_{ij} are semi-positive-definite.

To establish uniqueness we let $(u_i^\kappa, p^\kappa, q^\kappa, \theta_1^\kappa, \theta_2^\kappa, \theta_3^\kappa)$, $\kappa = 1,2$ be two solutions to $\mathcal{P}$ which satisfy (2) and (3) for the same boundary and initial functions h_i, p^B, q^B, α_1, α_2, α_3, v_i, w_i, P, Q, Θ_1, Θ_2, Θ_3, and which satisfy (1) for the same body force f_i and for the same source terms s_1, s_2, r_1, r_2, r_3. Define the difference solution $\Lambda = (u_i, p, q, \theta_1, \theta_2, \theta_3)$ by

$$\begin{aligned} u_i = u_i^1 - u_i^2\,, \quad p = p^1 - p^2\,, \quad q = q^1 - q^2\,, \\ \theta_1 = \theta_1^1 - \theta_1^2\,, \quad \theta_2 = \theta_2^1 - \theta_2^2\,, \quad \theta_3 = \theta_3^1 - \theta_3^2\,. \end{aligned}$$

By subtraction one verifies that Λ satisfies the equations

$$\begin{aligned} \rho\ddot{u}_i &= \left(a_{ijhk}u_{h,k}\right)_{,j} - \left(\beta_{ij}p\right)_{,j} - \left(\gamma_{ij}q\right)_{,j} \\ &\quad - \left(a_{ij}^{(1)}\theta_1\right)_{,j} - \left(a_{ij}^{(2)}\theta_2\right)_{,j} - \left(a_{ij}^{(3)}\theta_3\right)_{,j}\,, \\ \alpha\dot{p} &= \left(k_{ij}p_{,j}\right)_{,i} - \beta_{ij}\dot{u}_{i,j} - \gamma(p-q)\,, \\ \beta\dot{q} &= \left(m_{ij}q_{,j}\right)_{,i} - \gamma_{ij}\dot{u}_{i,j} + \gamma(p-q)\,, \\ a_1\dot{\theta}_1 &= \left(\kappa_{ij}\theta_{1_{,j}}\right)_{,i} - a_{ij}^{(1)}\dot{u}_{i,j} - d_1(\theta_1-\theta_2) - d_2(\theta_1-\theta_3)\,, \\ a_2\dot{\theta}_2 &= \left(\omega_{ij}\theta_{2_{,j}}\right)_{,i} - a_{ij}^{(2)}\dot{u}_{i,j} + d_1(\theta_1-\theta_2) - d_3(\theta_2-\theta_3)\,, \\ a_3\dot{\theta}_3 &= \left(\pi_{ij}\theta_{3_{,j}}\right)_{,i} - a_{ij}^{(3)}\dot{u}_{i,j} + d_3(\theta_2-\theta_3) + d_2(\theta_1-\theta_3)\,, \end{aligned} \tag{5}$$

in $\Omega \times (0,T]$, together with the homogeneous boundary conditions

$$\begin{aligned} u_i = 0\,, \quad p = 0\,, \quad q = 0\,, \\ \theta_1 = 0\,, \quad \theta_2 = 0\,, \quad \theta_3 = 0\,, \end{aligned} \tag{6}$$

on $\Gamma \times [0,T]$, and the homogeneous initial conditions

$$\begin{aligned} &u_i(\mathbf{x},0) = 0\,, \quad \dot{u}_i(\mathbf{x},0) = 0\,, \\ &p(\mathbf{x},0) = 0\,, \quad q(\mathbf{x},0) = 0\,, \\ &\theta_1(\mathbf{x},0) = 0\,, \quad \theta_2(\mathbf{x},0) = 0\,, \quad \theta_3(\mathbf{x},0) = 0\,, \end{aligned} \tag{7}$$

for $\mathbf{x} \in \Omega$.

The uniqueness proof proceeds by a logarithmic convexity argument.

We firstly define the variables $\eta(\mathbf{x},t)$, $\zeta(\mathbf{x},t)$, $\phi(\mathbf{x},t)$, $\psi(\mathbf{x},t)$, $\xi(\mathbf{x},t)$ by

$$\begin{aligned} \eta = \int_0^t p(\mathbf{x},s)ds\,, \quad \zeta = \int_0^t q(\mathbf{x},s)ds\,, \\ \phi = \int_0^t \theta_1(\mathbf{x},s)ds\,, \quad \psi = \int_0^t \theta_2(\mathbf{x},s)ds\,, \quad \xi = \int_0^t \theta_3(\mathbf{x},s)ds\,. \end{aligned}$$

In terms of the variables η, ζ, ϕ, ψ and ξ we may integrate $(5)_{2-6}$ to derive the equations

$$\begin{aligned} \alpha p &= \left(k_{ij}\eta_{,j}\right)_{,i} - \beta_{ij}u_{i,j} - \gamma(\eta-\zeta)\,, \\ \beta q &= \left(m_{ij}\zeta_{,j}\right)_{,i} - \gamma_{ij}u_{i,j} + \gamma(\eta-\zeta)\,, \\ a_1\theta_1 &= \left(\kappa_{ij}\phi_{,j}\right)_{,i} - a_{ij}^{(1)}u_{i,j} - d_1(\phi-\psi) - d_2(\phi-\xi)\,, \\ a_2\theta_2 &= \left(\omega_{ij}\psi_{,j}\right)_{,i} - a_{ij}^{(2)}u_{i,j} + d_1(\phi-\psi) - d_3(\psi-\xi)\,, \\ a_3\theta_3 &= \left(\pi_{ij}\xi_{,j}\right)_{,i} - a_{ij}^{(3)}u_{i,j} + d_3(\psi-\xi) + d_2(\phi-\xi)\,. \end{aligned} \tag{8}$$

We denote by $< \cdot >$ integration over Ω, e.g.

$$< f > = \int_\Omega f\,dx\,.$$

Then define the functional F by

$$F(t) = < \rho u_i u_i > + \int_0^t < k_{ij}\eta_{,j}\eta_{,i} > ds + \int_0^t < m_{ij}\zeta_{,j}\zeta_{,i} > ds$$
$$+ \int_0^t < \gamma(\eta-\zeta)^2 > ds + \int_0^t < \kappa_{ij}\phi_{,j}\phi_{,i} > ds$$
$$+ \int_0^t < \omega_{ij}\psi_{,j}\psi_{,i} > ds + \int_0^t < \pi_{ij}\xi_{,j}\xi_{,i} > ds \tag{9}$$
$$+ \int_0^t < d_1(\phi-\psi)^2 > ds + \int_0^t < d_2(\phi-\xi)^2 > ds$$
$$+ \int_0^t < d_3(\psi-\xi)^2 > ds .$$

Differentiate F to obtain

$$F'(t) = 2 < \rho u_i \dot{u}_i > + 2\int_0^t < k_{ij}\eta_{,j}p_{,i} > ds + 2\int_0^t < m_{ij}\zeta_{,j}q_{,i} > ds$$
$$+ 2\int_0^t < \gamma(\eta-\zeta)(p-q) > ds + 2\int_0^t < \kappa_{ij}\phi_{,j}\theta_{1_{,i}} > ds$$
$$+ 2\int_0^t < \omega_{ij}\psi_{,j}\theta_{2_{,i}} > ds + 2\int_0^t < \pi_{ij}\xi_{,j}\theta_{3_{,i}} > ds \tag{10}$$
$$+ 2\int_0^t < d_1(\phi-\psi)(\theta_1-\theta_2) > ds + 2\int_0^t < d_2(\phi-\xi)(\theta_1-\theta_3) > ds + 2\int_0^t < d_3(\psi-\xi)(\theta_2-\theta_3) > ds .$$

After a further differentiation one finds

$$\begin{aligned} F''(t) =& 2 < \rho\dot{u}_i\dot{u}_i > + 2 < \rho u_i \ddot{u}_i > + 2 < k_{ij}\eta_{,j}p_{,i} > + 2 < m_{ij}\zeta_{,j}q_{,i} > \\ &+ 2 < \gamma(\eta-\zeta)(p-q) > + 2 < \kappa_{ij}\phi_{,j}\theta_{1_{,i}} > + 2 < \omega_{ij}\psi_{,j}\theta_{2_{,i}} > \\ &+ 2 < \pi_{ij}\xi_{,j}\theta_{3_{,i}} > + 2 < d_1(\phi-\psi)(\theta_1-\theta_2) > \\ &+ 2 < d_2(\phi-\xi)(\theta_1-\theta_3) > + 2 < d_3(\psi-\xi)(\theta_2-\theta_3) > . \end{aligned} \tag{11}$$

The next step is to multiply equation $(5)_1$ by u_i and integrate over Ω. Then multiply each of equations $(8)_1$ to $(8)_5$ by p, q, θ_1, θ_2 and θ_3, in turn, and integrate each over Ω with integration by parts and use of the homogeneous boundary conditions. The resulting six equations are added to yield the following equation

$$\begin{aligned} &< \rho u_i \ddot{u}_i > + < k_{ij}\eta_{,j}p_{,i} > + < m_{ij}\zeta_{,j}q_{,i} > + < \kappa_{ij}\phi_{,j}\theta_{1_{,i}} > \\ &< \omega_{ij}\psi_{,j}\theta_{2_{,i}} > + < \pi_{ij}\xi_{,j}\theta_{3_{,i}} > + < \gamma(\eta-\zeta)(p-q) > \\ &+ < d_1(\phi-\psi)(\theta_1-\theta_2) > + < d_3(\psi-\xi)(\theta_2-\theta_3) > \\ &+ < d_2(\phi-\xi)(\theta_1-\theta_3) > \\ =& - < a_{ijhk}u_{h,k}u_{i,j} > - < \alpha p^2 > - < \beta q^2 > \\ &- < a_1\theta_1^2 > - < a_2\theta_2^2 > - < a_3\theta_3^2 > . \end{aligned} \tag{12}$$

We now use equation (12) to substitute for the last ten terms in (11) to find

$$\begin{aligned} F''(t) =& 2 < \rho\dot{u}_i\dot{u}_i > - 2 < a_{ijhk}u_{h,k}u_{i,j} > - 2 < \alpha p^2 > - 2 < \beta q^2 > \\ &- 2 < a_1\theta_1^2 > - 2 < a_2\theta_2^2 > - 2 < a_3\theta_3^2 > . \end{aligned} \tag{13}$$

The next stage involves the energy equation for (5)–(7). Define the energy as

$$\begin{aligned} E(t) =&\frac{1}{2} < \rho \dot{u}_i \dot{u}_i > + \frac{1}{2} < a_{ijhk} u_{i,j} u_{h,k} > + \frac{1}{2} < \alpha p^2 > + \frac{1}{2} < \beta q^2 > \\ &+ \frac{1}{2} < a_1 \theta_1^2 > + \frac{1}{2} < a_2 \theta_2^2 > + \frac{1}{2} < a_3 \theta_3^2 > \ . \end{aligned} \tag{14}$$

To obtain the energy equation, one multiplies $(5)_1$ by $\dot{u}_i$, $(5)_{2-6}$ by p, q, θ_1, θ_2 and θ_3, respectively, and integrates each over Ω. Integration by parts, taking into account the homogeneous boundary conditions, and then integrating in time, readily yields

$$\begin{aligned} E(t) &+ \int_0^t < k_{ij} p_{,j} p_{,i} > ds + \int_0^t < m_{ij} q_{,j} q_{,i} > ds + \int_0^t < \gamma (p-q)^2 > ds \\ &+ \int_0^t < \kappa_{ij} \theta_{1_{,j}} \theta_{1_{,i}} > ds + \int_0^t < \omega_{ij} \theta_{2_{,j}} \theta_{2_{,i}} > ds + \int_0^t < \pi_{ij} \theta_{3_{,j}} \theta_{3_{,i}} > ds \\ &+ \int_0^t < d_1 (\theta_1 - \theta_2)^2 > ds + \int_0^t < d_2 (\theta_1 - \theta_3)^2 > ds + \int_0^t < d_3 (\theta_2 - \theta_3)^2 > ds = E(0) \, . \end{aligned} \tag{15}$$

From (7) it follows that $E(0) = 0$.

Substitute in (13) for the last six terms using (14) and (15) to see that

$$\begin{aligned} F''(t) =&4 < \rho \dot{u}_i \dot{u}_i > -4E(t) \\ =&4 < \rho \dot{u}_i \dot{u}_i > +4 \int_0^t < k_{ij} p_{,j} p_{,i} > ds + 4 \int_0^t < m_{ij} q_{,j} q_{,i} > ds \\ &+ 4 \int_0^t < \kappa_{ij} \theta_{1_{,j}} \theta_{1_{,i}} > ds \\ &+ 4 \int_0^t < \omega_{ij} \theta_{2_{,j}} \theta_{2_{,j}} > ds + 4 \int_0^t < \pi_{ij} \theta_{3_{,j}} \theta_{3_{,i}} > ds \\ &+ 4 \int_0^t < \gamma (p-q)^2 > ds + 4 \int_0^t < d_1 (\theta_1 - \theta_2)^2 > ds \\ &+ 4 \int_0^t < d_2 (\theta_1 - \theta_3)^2 > ds + 4 \int_0^t < d_3 (\theta_2 - \theta_3)^2 > ds \, . \end{aligned} \tag{16}$$

We now utilize (16), (9) and (10) to form the combination $FF'' - (F')^2$ to see that

$$FF'' - (F')^2 = 4S^2 \geq 0 \, , \tag{17}$$

where S^2, which is non-negative due to the Cauchy-Schwarz inequality, is given by

$$S^2 = AB - C^2 ,$$

where the terms A, B and C have form

$$A = < \rho u_i u_i > + \int_0^t < k_{ij}\eta_{,j}\eta_{,i} > ds + \int_0^t < m_{ij}\zeta_{,j}\zeta_{,i} > ds$$
$$+ \int_0^t < \kappa_{ij}\phi_{,j}\phi_{,i} > ds + \int_0^t < \omega_{ij}\psi_{,j}\psi_{,i} > ds + \int_0^t < \pi_{ij}\xi_{,j}\xi_{,i} > ds$$
$$+ \int_0^t < \gamma(\eta-\zeta)^2 > ds + \int_0^t < d_1(\phi-\psi)^2 > ds + \int_0^t < d_2(\psi-\xi)^2 > ds + \int_0^t < d_3(\phi-\xi)^2 > ds\,,$$

$$B = < \rho \dot{u}_i \dot{u}_i > + \int_0^t < k_{ij}p_{,j}p_{,i} > ds + \int_0^t < m_{ij}q_{,j}q_{,i} > ds$$
$$+ \int_0^t < \kappa_{ij}\theta_{1_{,j}}\theta_{1_{,i}} > ds + \int_0^t < \omega_{ij}\theta_{2_{,j}}\theta_{2_{,i}} > ds + \int_0^t < \pi_{ij}\theta_{3_{,j}}\theta_{3_{,i}} > ds$$
$$+ \int_0^t < \gamma(p-q)^2 > ds + \int_0^t < d_1(\theta_1-\theta_2)^2 > ds$$
$$+ \int_0^t < d_2(\theta_1-\theta_3)^2 > ds + \int_0^t < d_3(\theta_2-\theta_3)^2 > ds\,,$$

$$C = < \rho u_i \dot{u}_i > + \int_0^t < k_{ij}\eta_{,j}p_{,i} > ds + \int_0^t < m_{ij}\zeta_{,j}q_{,i} > ds$$
$$+ \int_0^t < \kappa_{ij}\phi_{,j}\theta_{1_{,i}} > ds + \int_0^t < \omega_{ij}\psi_{,j}\theta_{2_{,i}} > ds + \int_0^t < \pi_{ij}\xi_{,j}\theta_{3_{,i}} > ds$$
$$+ \int_0^t < \gamma(\eta-\zeta)(p-q) > ds + \int_0^t < d_1(\phi-\psi)(\theta_1-\theta_2) > ds$$
$$+ \int_0^t < d_2(\phi-\xi)(\theta_1-\theta_3) > ds + \int_0^t < d_3(\psi-\xi)(\theta_2-\theta_3) > ds\,.$$

From (17) one now deduces $F \equiv 0$ on $[0,T]$, see e.g. Straughan [1], chapter 1. Roughly, one divides (17) by F^2 to find $(\log F)'' \geq 0$, but care has to be taken since $F(0) = 0$. One has to use a contradiction argument involving $F(\epsilon)$, and then let $\epsilon \to 0$. Since $F \equiv 0$ it follows from (9) that $u_i \equiv 0$. Next, we appeal to the energy equation (15) using (14), to conclude also that

$$p \equiv 0\,, \quad q \equiv 0\,, \quad \theta_1 \equiv 0\,, \quad \theta_2 \equiv 0\,, \quad \theta_3 \equiv 0\,,$$

on $\Omega \times [0,T]$ and uniqueness follows, with only the symmetry conditions (4) on the elastic coefficients.

4 | DECAY

In double porosity elasticity theory several writers argue that for many applications one may dispense with the acceleration term in equation $(1)_1$, see e.g. Scarpetta and Svanadze [12] and the references therein. In this case the system is referred to as a quasi-equilibrium system, cf. Scarpetta and Svanadze [12], Svanadze [5].

In this section we analyze the quasi-equilibrium version of equations (1) with zero source terms, and so omit the $\rho \ddot{u}_i$ term. The boundary conditions adopted are those of (6), while the initial conditions are (3). The boundary-initial value problem thus defined is denoted by $\mathcal{M}$. We retain the symmetries of (4), but now we require a_{ijhk} to be positive definite, i.e.

$$a_{ijhk}\varepsilon_{hk}\varepsilon_{ij} \geq a_0\varepsilon_{ij}\varepsilon_{ij}\,, \tag{18}$$

for all ε_{ij}, where $a_0 > 0$ is a constant. In addition we impose the (realistic) bounds on the coefficients,

$$\beta_{ij}\beta_{ij}\,,\quad \gamma_{ij}\gamma_{ij}\,,\quad \gamma\,,\quad a_{ij}^{(1)}a_{ij}^{(1)}\,,\quad a_{ij}^{(2)}a_{ij}^{(2)}\,,\quad a_{ij}^{(3)}a_{ij}^{(3)}\quad \leq M\,, \tag{19}$$

for some constant M, where (19) hold $\forall\mathbf{x} \in \Omega$.

Furthermore, for positive constants α_0, β_0, $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, $\tilde{\alpha}_0$, $\tilde{\beta}_0$, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{a}_3$, k_1, m_1, κ_1, ω_1, π_1, we require

$$\alpha_0 \leq \alpha \leq \tilde{\alpha}_0\,,\quad \beta_0 \leq \beta \leq \tilde{\beta}_0\,,\quad \hat{a}_1 \leq a_1 \leq \tilde{a}_1\,,\quad \hat{a}_2 \leq a_2 \leq \tilde{a}_2\,,\quad \hat{a}_3 \leq a_3 \leq \tilde{a}_3\,, \tag{20}$$

and

$$\begin{aligned} &k_{ij}\varepsilon_j\varepsilon_i \geq k_1\varepsilon_i\varepsilon_i\,,\quad m_{ij}\varepsilon_j\varepsilon_i \geq m_1\varepsilon_i\varepsilon_i\,,\quad \kappa_{ij}\varepsilon_j\varepsilon_i \geq \kappa_1\varepsilon_i\varepsilon_i\,,\\ &\omega_{ij}\varepsilon_j\varepsilon_i \geq \omega_1\varepsilon_i\varepsilon_i\,,\quad \pi_{ij}\varepsilon_j\varepsilon_i \geq \pi_1\varepsilon_i\varepsilon_i\,, \end{aligned} \tag{21}$$

$\forall\mathbf{x} \in \Omega$ and for all ε_i.

We shall establish that a solution to $\mathcal{M}$ decays to zero in an appropriate manner. To do this we commence by multiplying the quasi-equilibrium version of $(5)_1$ by u_i and then integrate over Ω and use the homogeneous boundary conditions to derive

$$\begin{aligned} <a_{ijhk}u_{h,k}u_{i,j}>=\ & <\beta_{ij}pu_{i,j}> + <\gamma_{ij}qu_{i,j}>\\ & + <a_{ij}^{(1)}\theta_1 u_{i,j}> + <a_{ij}^{(2)}\theta_2 u_{i,j}> + <a_{ij}^{(3)}\theta_3 u_{i,j}>\ . \end{aligned} \tag{22}$$

We next employ the Cauchy-Schwarz and arithmetic-geometric mean inequalities on the terms on the right of (22). After, using the bounds (19) and (20) one may readily deduce the inequality

$$\frac{1}{2} <a_{ijhk}u_{h,k}u_{i,j}> \leq C^*\left[<\alpha p^2> + <\beta q^2> + <a_1\theta_1^2> + <a_2\theta_2^2> + <a_3\theta_3^2>\right]\,, \tag{23}$$

where

$$C^* = \frac{5MK}{2\hat{a}_3 a_0}$$

with

$$K = \frac{\hat{a}_3}{\alpha_0} + \frac{\hat{a}_3}{\beta_0} + \frac{\hat{a}_3}{\hat{a}_1} + \frac{\hat{a}_3}{\hat{a}_2} + 1\,.$$

The next step is to multiply the quasi-equilibrium version of $(5)_1$ by $\dot{u}_i$ and integrate over Ω, and then multiply $(5)_2$–$(5)_6$ by p, q, θ_1, θ_2 and θ_3, respectively, and integrate each over Ω. Denote by E_e and E_d the expressions

$$\begin{aligned} E_e =&\frac{1}{2} <a_{ijhk}u_{h,k}u_{i,j}>\,,\\ E_d =&\frac{1}{2} <\alpha p^2> + \frac{1}{2} <\beta q^2> + \frac{1}{2} <a_1\theta_1^2> + \frac{1}{2} <a_2\theta_2^2> + \frac{1}{2} <a_3\theta_3^2>\,, \end{aligned}$$

corresponding to the elastic energy and to the (thermo-bidispersive) dissipative term. By adopting the procedure above one may show

$$\frac{d}{dt}\left(E_e + E_d\right) = -f(t)\,, \tag{24}$$

where

$$\begin{aligned} f(t) =\ & <k_{ij}p_{,j}p_{,i}> + <m_{ij}q_{,j}q_{,i}> + <\kappa_{ij}\theta_{1_{,j}}\theta_{1_{,i}}>\\ & + <\omega_{ij}\theta_{2_{,j}}\theta_{2_{,i}}> + <\pi_{ij}\theta_{3_{,j}}\theta_{3_{,i}}> + <\gamma(p-q)^2>\\ & + <d_1(\theta_1-\theta_2)^2> + <d_2(\theta_1-\theta_3)^2> + <d_3(\theta_2-\theta_3)^2>\ . \end{aligned} \tag{25}$$

Upon integration, we see from (24) with $E = E_e + E_d$

$$0 \leq E(t) = E(0) - \int_0^t f(s)ds\,. \tag{26}$$

Next, using Poincaré's inequality together with (21) and the definition of E, from (26) we may deduce

$$
\begin{aligned}
E_d(t) \leq & E(t) = E(0) - \int_0^t f(s) ds \\
& \leq E(0) - \lambda_1 \int_0^t \left(k_1 < p^2(s) > + m_1 < q^2(s) > + \kappa_1 < \theta_1^2(s) > + < \omega_1 < \theta_2^2(s) > + < \pi_1 < \theta_3^2(s) > \right) ds \,,
\end{aligned}
\tag{27}
$$

λ_1 being the first eigenvalue in the membrane problem for Ω.

Therefore, using the upper bounds of (20), we may obtain

$$
\begin{aligned}
E_d(t) & \leq E(0) - \lambda_1 \int_0^t \left(\frac{k_1}{\tilde{\alpha}_0} < \alpha p^2(s) > + \frac{m_1}{\tilde{\beta}_0} < \beta q^2(s) > + \frac{\kappa_1}{\tilde{a}_1} < a_1 \theta_1^2(s) > + \frac{\omega_1}{\tilde{a}_2} < a_2 \theta_2^2(s) > + \frac{\pi_1}{\tilde{a}_3} < a_3 \theta_3^2(s) > \right) ds \\
& \leq E(0) - k \int_0^t E_d(s) ds \,,
\end{aligned}
\tag{28}
$$

where

$$
k = 2\lambda_1 \min \left\{ \frac{k_1}{\tilde{\alpha}_0}, \frac{m_1}{\tilde{\beta}_0}, \frac{\kappa_1}{\tilde{a}_1}, \frac{\omega_1}{\tilde{a}_2}, \frac{\pi_1}{\tilde{a}_3} \right\} .
$$

We integrate (27) with the aid of an integrating factor to find

$$
\int_0^t E_d(s) ds \leq \frac{E(0)}{k} \left(1 - e^{-kt} \right) , \quad \forall t > 0 \,,
$$

so

$$
\int_0^\infty E_d(s) ds \leq \frac{E(0)}{k} . \tag{29}
$$

However, from inequality (23) we see that

$$
\frac{E_e}{2C^*} \leq E_d \,.
$$

Hence, from (29) we may show

$$
\int_0^\infty E(s) ds \leq \frac{1 + 2C^*}{k} E(0) \,. \tag{30}
$$

Therefore, $E(t) \in L^1(0, \infty)$ and from (24) $E'(t) \leq 0$, $\forall t > 0$. Thus,

$$
\lim_{t \to \infty} E(t) = 0 \,. \tag{31}
$$

Relation (31) shows that the solution to $\mathcal{M}$ decays to zero in time in the measure $E(t)$.

5 | CONCLUSIONS

In this paper we have introduced a set of partial differential equations to model the evolutionary behaviour of an anisotropic elastic body which contains a double porosity structure under conditions of local thermal non-equilibrium. We have proved that a solution to the displacement boundary initial value problem is unique under the weak requirement of symmetry of the elastic coefficients. In addition, we have demonstrated that the solution to the equivalent problem for a quasi-equilibrium theory decays to zero in time in a precise sense, when there are no sources and external body force involved.

It is still an open problem to know the rate of decay, even in the quasi-equilibrium theory. It would also be very interesting to know properties of decay for the full theory along the lines of analogous results for classical linear thermoelasticity by Dafermos [20] and Lebeau and Zuazua [21]. Given that the decay rate even for thermoelasticity may be polynomial or exponential depending on the geometry, it is natural to expect something similar in the present more complicated model. In fact, in equation (1) there are two potential sources of damping. One is the analogous one from thermoelasticity involving the β_{ij}, γ_{ij}, $a_{ij}^{(s)}$,

$s = 1, 2, 3$, terms. However, a second source of damping could arise from the interaction terms between the macro and micro pressures, the terms involving γ, and between the temperatures, the d_α, $\alpha = 1, 2, 3$, terms. Since the γ terms involve the difference $p - q$, and d_α terms involve $\theta_\alpha - \theta_\beta$, this has some resemblance to the situation for a mixture of two elastic solids where the dissipation involves the difference of velocities, a problem resolved by Dafermos [22].

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