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The mathematical values of fraction signs in the Linear A script: A computational, statistical and typological approach

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A B S T R A C T

Minoan Linear A is still an undeciphered script mainly used for administrative purposes on Bronze Age Crete. One of its most enigmatic features is the precise mathematical values of its system of numerical fractions. The aim of this article is to address this issue through a multi-stranded methodology that comprises palaeographical examination and statistical, computational, and typological approaches. Taking on from previous analyses, which suggested hypothetical values for some fractions, we extended our probe into assessing values for some problematic ones. The results achieved, based, on the one hand, on a close palaeographical analysis and, on the other, on computational, statistical and typological strategies, show a remarkable convergence and point towards a systematic assignment of mathematical values for the Linear A fraction signs. This contribution sets the agenda for a combinatorial, novel, and unbiased approach that may help advance our comprehension of some standing issues related to ancient undeciphered writing systems.

1. Introduction

The aim of this paper is to shed light on the mathematical values of the Linear A writing system through a multi-stranded computational, statistical, typological, and palaeographical method. Linear A is a logo-syllabic script, still largely undeciphered, used on the island of Crete ca. 1700-1400 BC (Duhoux, 1989). Together with Cretan Hieroglyphic, it is one of two writing systems created by the Minoan civilisation, which thrived during the Bronze Age. Upon its template, the Mycenaeans later created the Linear B syllabary to register their dialect of ancient Greek. Today, the Linear A corpus comprises more than 7400 signs on 1527 inscriptions, 90% of which are clay documents of administrative nature, such as tablets, roundels, and nodules (Del Freo; Zurbach, 2011).

As for numerical notations, Linear A employs a decimal system, with signs representing four magnitudes: units are written with vertical strokes, tens with horizontal strokes or dots, hundreds with circles, and thousands with circles surrounded by strokes. The system is cumulative and additive, and numbers are written from left to right with the powers in descending order: thus, e.g. ‘6352’ would be written with six ‘1000’ signs, three ‘100’, five ‘10’ and two ‘1’. Linear A also includes a set of 17 signs that stand for fractions (Fig. 1). They are transcribed via capital letters: A, B, D, E, F, H, J, K, L, L2, L3, L4, L6, W, X, Y, and Ω (Godart and Olivier, 1985). Signs L2, L3, L4, and L6 have numbers because they consist of a basic semi-circular or triangular shape, called L, to which two, three, four, and six horizontal strokes (or dots) are added. However, the status of the basic L as a self-standing sign is uncertain, since it only appears joined to L2 (ZA 11a.4) and in other doubtful instances.[2] The shape was not considered as an independent sign by Bennett (1950, p. 206, Fig. 1; 1980), but it was distinguished in the standard Linear A corpus (Godart and Olivier, 1985) and later studies have treated it as a variable (Perna, 1990; Facchetti, 1994; Cash and Cash, 2011).

To shed light on the values of the fractions, we had to select a coherent epigraphical sample. It needs to be noted that the evidence at our disposal is not equally consistent for all sites and periods (Middle Minoan II, Middle Minoan III, and Late Minoan I), and thus it is not possible to conduct a regional and diachronic development study. We therefore focused on a specific set of Linear A material, which is more substantially represented, dated to the Late Minoan I period (ca. 1600-

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[2] These are:]Ḷ B B in KH 96 (Hallager et al., 1991), Ḷ in PH 7b.1, 2, Ḷ Ḷ in PH 12a (Godart and Olivier, 1976a), and]Ḟ Ḷ[in ZA 7b.8 (Godart and Olivier, 1976b). We excluded also two possible instances that could belong to either Cretan Hieroglyphic or Linear A (MA 4a.1 and CHIC #048.a, b) (Godart and Olivier, 1976a; Olivier and Godart, 1996). Finally, Younger (2000–2020) reads a combination L E on PH 7b.3. However, this is a damaged instance that cannot be read clearly, initially interpreted as *3̣0̣8̣ E (i.e. a logogram followed by fraction E) by Godart and Olivier (1976a) and then as J̣ E (Godart and Olivier, 1985).

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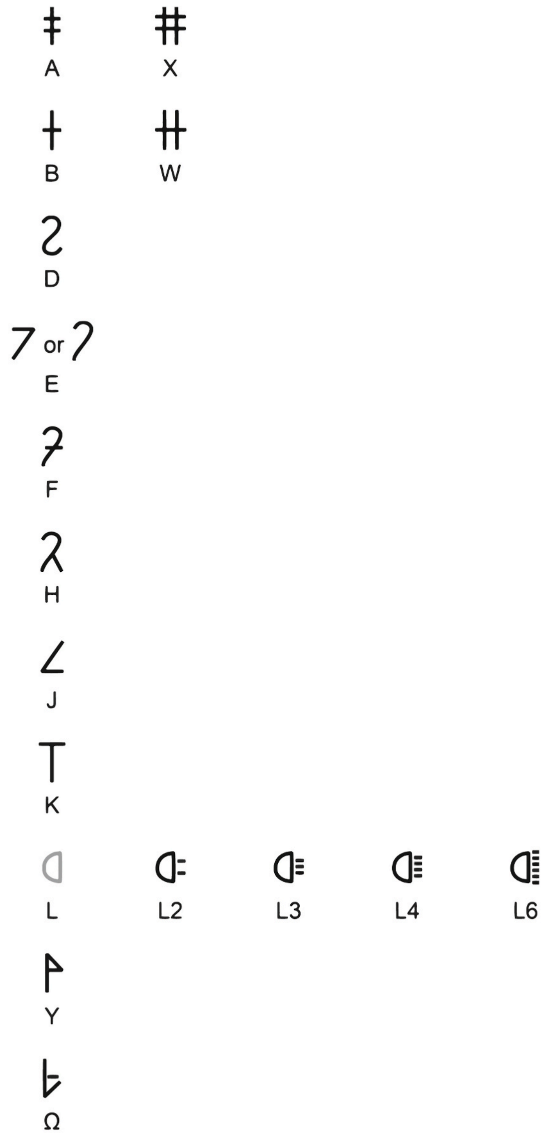


Fig. 1. Linear A fraction signs and their standard transcription.

Table 1
Number of attestations of Linear A fraction signs on their own, as part of combinations, and in ligatures with logograms, based on the standard corpus (Godart and Olivier, 1985) and inscriptions published later. Uncertain readings are excluded from this table, which includes attestations from doubtful Linear A inscriptions (*180+B and *180+SA + B from tablet MA 4, and HHH from CHIC #068.r.A) that were not used in Simulation 2 (see Section 3). Combination A A follows our reading of tablet KH 86.2 (Godart and Olivier, 1976b, p. 101, read A B B).

Single (also next to fracture) in numeral phrases		Combination in numeral phrases		Logogram + single		Logogram + combination	
J	101	J E	24	*303+D	20	GRA + K + L2	3
E	52	D D	14	*303+E	12	GRA + L3+L3	1
B	21	B B	5	GRA + F	4		
K	21	E F	5	*412VAS + E	3		
D	19	K L2	5	GRA + B	2		
H	15	E L2	3	*180+B	2		
A	13	J B	3	*412VAS + F	2		
F	9	J E L2	3	GRA + E	1		
L2	6	J L2	2	GRA + H	1		
W	4	A A	1	GRA + L2	1		
X	2	D D D D	1	*180+SA + B	1		
L3	2	E B	1	*303+K	1		
Y	1	E L4	1	*405+ Ω	1		
L	0	E L6	1	*414VAS + F	1		
L4	0	F K	1	*417VAS + L2	1		
L6	0	H H H	1	*418VAS + L2	1		
Ω	0	J A	1				
		J E B	1				
		J F	1				
		J H	1				
		J K	1				
		L L2	1				
		L2 L4	1				
		Y Y Y	1				

Table 2
Total number of attestations of Linear A fraction signs, based on the standard corpus (Godart and Olivier, 1985) and inscriptions published later.

Sign	J	E	D	B	K	L2	F	H	A
Total attestations	137	105	68	36	32	26	23	19	15
Sign	W	Y	L3	L4	L6	X	L	Ω	
Total attestations	4	4	3	2	2	2	1	1	

1450 BC), and more largely distributed in the west (Khania), central (Knossos, Tylissos, and Hagia Triada), and eastern portions of the island (Petras and Zakros).

Fraction signs are mainly attested on records of commodities (cereals, figs, wine, etc.), which are expressed using logograms that implicitly refer to units of measurement. Their amounts are represented by horizontal or vertical sequences of fraction signs, placed directly after the commodity if they were lower than one, and after signs for whole numbers if they were higher (e.g. OLIV 1 E in tablet HT 21.4). This shows that the notation of fractions was cumulative-additive and in descending order like the integers, with the larger values written to the left or above the smaller. At times, fractions appear also ligatured to commodity logograms, probably denoting fractions of the implicit unit of measurement (Tables 1 and 2).

Since Linear A seems not to have had distinct signs for units of measurement, except for weight AB 118, we cannot be sure if dry and liquid products were measured according to the same or different units (Schoep, 2002, p. 33). Nevertheless, on the ground of the comparison with other (ancient and modern) systems of measurement, the use of the unit of weight AB 118 for a limited range of products, and attestations of logograms ligatured with fractions, it seems more likely that different units of measurement were used depending on the nature of the products and the method and tools implied to measure them. In this case, the absolute quantities measured would have changed, but not the relative mathematical value of the fraction signs used to point them out. As no Linear A inscription containing totals of numeral phrases is without reading or calculation problems (Montecchi, 2009, 2013), it is difficult to deduce these mathematical values. This has led to diverging decipherment attempts, summarised in Table 3.

Bennett (1950, 1980; 1999) hypothesised that most Linear A fraction signs stood for fractions with numerator ‘1’ (1/2, 1/3, 1/4, etc.) via a comparison with the contemporary Egyptian script. He hypothesised their values through the frequency of signs and their combinations (Tables 1 and 2). With J as the most frequent sign both *per se* and as part of combinations, Bennett proposed its value to be 1/2. Since E is the second most common sign, and J is written above it or to its left, the notion that E = 1/4 still holds. Three Linear A texts support Bennett’s interpretation of J, E, F, as follows:

Clay tablet HT 104 (Fig. 2): Contains a sum and its total: 45 + J + 20 J̣ + 29 = 95. We deduce that J = 1/2, assuming the damaged J cannot be followed by any additional fractional sign (cf. Bennett, 1980, p. 16; Facchetti, 2012).

Clay tablet PE 1: Contains two entries, namely a sign-group followed by the logogram VIR/MUL (people) plus numerals, and the logogram

Table 3
Mathematical values previously proposed for Linear A fraction signs and the inferable results for combinations. Bennett (1950) proposed D = 2/3 and D D = 1/6 but suggested that these assignments might be inverted by later investigations.

Fraction	Bennett (1950)	Stoltenberg (1955)	Was (1971)	Facchetti(1994)	Younger (2000–2020)	Cash and Cash (2012)	Schrijver (2014)
J	1/2	1/2	1/4	1/4	1/2	1/2	1/2
E	1/4	1/4	1/2	1/2	1/4	1/4	1/4
B	1/12?	1/6	1/6	1/10	1/3?	1/5	1/6
K	<1/8	1/10			1/16	1/16?	1/16
D	2/3	1/5			1/5?	1/6	1/5
H	1/3	1/3			1/6?	3/10	1/3
A	<1/4	1/12			1/6?	1/20	1/12
F	1/8	1/8	3/8	3/8	1/8	1/8	1/8
L2	<1/4	1/9				3/20?	1/24?
W						= BB?	
X						9/20	
L3						1/10 or 3/40?	1/36?
Y							
L						>3/20 (17/40?)	
L4						3/40 or 1/40?	1/48?
L6						1/80?	1/64?
Ω							
J E	3/4	3/4	3/4	3/4	3/4	3/4	3/4
D D	1/6	2/5			2/5?	1/3	2/5
B B	<2/4		1/12	1/12	2/3?	2/5	2/6 = 1/3
E F	3/8	3/8	7/8	7/8	3/8	3/8	3/8
K L2	<2/4					17/80?	5/48
E L2	<2/4					2/5?	7/24?
J B	<3/4	2/3	5/12	7/20	5/6?	7/10	4/6
J E L2	<1					9/10?	19/24
J L2	<3/4	11/18				13/20?	13/24
A A	<2/4				2/6?	1/10	1/6
D D D D	2/6 = 1/3	4/5			4/5?	2/3	4/5
E B	<2/4		2/3	3/5	7/12?	9/20	5/12
E K	<2/4	9/40			3/16	3/16?	3/16
E L4	<2/4					13/40 or 11/40?	13/48?
E L6	<2/4					21/80?	17/64?
F K	<3/8	7/20			3/16	3/16?	3/16
H H H	1	1			1/2?	9/10	1
J A	<3/4	7/12			4/6?	11/20	7/12
J E B	<1	11/12	11/12	17/20	1 + 1/12?	19/20	11/12
J F	5/8	5/8	5/8	5/8	5/8	5/8	5/8
J H	5/6	5/6			4/6?	4/5	5/6
J K	<3/4	3/5			9/16	9/16?	9/16
L2 L4	<2/4					9/40 or 7/40?	3/48

GRA + PA (PA specifying the kind of cereal) plus numerals. The second entry registers 72 people and 36 GRA + PA, suggesting that each person is assigned or contributes 1/2 of grain. The same proportion appears in the first entry, where 50[(i.e. 50 plus something) people are registered along with 26 J of GRA + PA. In this case, 26 J corresponds to 26 + 1/2 and the damaged count of people is to be restored as double this amount, hence 53 (Tsipopoulou and Hallager, 1996). This supports the decipherment of J as 1/2.

Graffito on stucco HT Zd 156: Contains the inscription *319 1 *319 1 J *319 2 E *319 3 E F *ṭạ-jạ* K [. Sign *319 (I) is a possible separator and, if we consider only the numerical signs, we read 1, 1 J, 2 E, 3 E F. Next is the damaged syllabic sequence *ṭạ-jạ* (though the doubtful *ṭạ* may also be another instance of *319), fraction K, and, finally, a crack in the plaster. If we apply the value deduced for J (1/2) and those proposed for E and F (1/4 and 1/8), we obtain three steps of a geometric progression where each value is the previous one multiplied by 1.5: 1, 1+J = 1.5, 2+E = 2.25, 3+E + F = 3.375 (Pope, 1960, p. 204). The remainder of the inscription has been interpreted as another step in the progression, 5 + 1/16 (= 5.0625). In this way, K would be 1/16, but this relies on the questionable phonetic spelling of 'five' (*ṭạ-jạ*) (Olivier, 1992).

Given the difficulties in deciphering the mathematical values of the fractions uniquely through an analysis of the Linear A texts, we propose a new multi-stranded method for narrowing down their possible values. We combine logical constraints inferred from epigraphy (even if the Minoan scribes did not follow rules consistently), typological patterns garnered from numerical notations around the world, and computational approaches. Moreover, while previous studies have also included data from problematic inscriptions, we avoided uncertain readings as our starting point.

2. Methods

2.1. Points of departure for narrowing down the mathematical values of the signs

We can deduce a set of premises constraining the possible mathematical values of the Linear A fractions signs, as follows:

(1) *The values of J, E and F are 1/2, 1/4 and 1/8, respectively.*
(2) *Combinations of fractional signs are additive.*
(3) *The same fractional value cannot be represented by more than one sign, or by both a sign and a combination of signs.* Although the opposite is possible, this premise assumes that the Linear A system of fractions was economical. Conversely, we cannot exclude that more than one combination of signs expressed the same value. One implication is that Linear A would feature no special sign for 3/4, because this value is conveyed by J E (= 1/2 + 1/4).
(4) *Any fractional sign or sum of combined fractions* (Table 1) *in a numeral phrase is less than 1.* In numeral phrases containing signs for whole numbers (e.g. 9 J E B), any fractional signs equalling or surpassing a unit should be represented using signs for integers. This may have exceptions, as sometimes amounts may be notated cumulatively, without being totalled (e.g. J J possibly attested on tablets PH 9b and 22a, and E E possibly attested on PH 12.b and

Fig. 2. Linear A clay tablet HT 104 (Courtesy of the Heraklion Archaeological Museum and the Greek Ministry of Culture and Sport, Archaeological Resources Fund).

13.a.c, but these were excluded from our sample because they date to the Middle Minoan II period).

(5) *Any fraction is larger than the fraction placed to the right or below it.* Accordingly, unproblematic inscriptions show the following relationships: J > E > F > K > L2 > L4; J > A; J > H; E > B; L (?) > L2.[3]

(6) *The L series is divisive*, so that e.g. L2 = L/2 (whatever the value of "L" is). We follow the premise that signs in this series are mechanically related: i.e. strokes or dots are added as modifications of the same basic value "L" (Schrijver, 2014, p. 22). Yet, the relationship cannot be multiplicative or additive, as that would imply L4 > L2 and L2 > L, which violates Constraint (5). A subtractive mechanism can also be discounted because subtractions would lead to negative values. If we agree that the L series should be divisive, there would be various ways to design a divisive series, but the only solutions that appeared to show any reasonable sense are two: one with a divisive value to the horizontal lines (hl) = 10 (L2 = L/20, L3 = L/30, …) and the other with hl = 1 (L2 = L/2, L3 = L/3, …). One can certainly devise a series giving other divisive values to the horizontal lines (hl = 2, 3, 4, 5 …), but this method will produce L series with implausible values. During the development of our work, we considered these two possibilities, but for hl = 10 it would be very problematic to explain the absence of one single horizontal line (meaning L/10). Conversely, if hl = 1, it makes sense that we do not have L/1 (which would be equal to itself). For this reason, we chose the L-series with hl = 1.

2.2. A solution-driven approach through constraint programming

The literature on computational approaches to writing systems is substantial (e.g. Sproat, 2000; Rao et al., 2009; Winters and Morin, 2019; Miton and Morin, 2019). Their application has led, for instance, to the decipherment of ratios in the metrological systems of Mesopotamian proto-cuneiform (Nissen et al., 1993). In this light, and to a similar end, we applied constraint programming to the Linear A fractions. This is a paradigm that allows us to express constraints over variables and find solutions that satisfy them. Thus, we represented Constraints 1–6 for the values of Linear A fractions in mathematical form and let the solver find the possible assignments for the signs. Even if these techniques try to reduce the search space, the general problem remains difficult to solve computationally, with solutions found only for a limited number of variables and constraints. We expressed the problem by means of the MiniZinc constraint programming language (Nethercote et al., 2007) and the Gecode solver (Gecode Team, 2006).

2.3. Establishing the set of variables

Unfortunately, no constraints help narrowing down the values of W and X. Since their shapes would appear to double the shapes of B and A (Fig. 1), respectively, they may be identical to the combinations B B and A A (Godart and Olivier, 1985, p. xxi, n. 2) (Fig. 1). Therefore, we did not include them in our analysis. Y and Ω were also excluded because they are rare signs (Table 1), used only in very early inscriptions,[4] and our focus, as stated at the outset, is the bulk of the material dated to Late Minoan I.

2.4. Defining the possible values: a typology-based approach

Constraint programming requires us to select possible sign values, considering that larger sets lead to a higher number of solutions and more computation time. Thus, we limited the number of possibilities in three ways: (1) by defining a range 1/2 … 1/*n* for fractional values of numerator '1', where *n* represents the lowest reasonable denominator; (2) by excluding implausible values within the range 1/2 … 1/*n*; (3) by including a limited number of plausible values with numerator(s) other than '1'.

To achieve this, we resorted to typology. Linear A comprises fraction signs not based on the shapes used for whole numbers (henceforth 'special signs'), even though L2, L3, L4 and L6 have horizontal strokes or dots that match exactly the shape of the signs that stand for the tens, taken as possibly meaning 'part 20', 'part 30', 'part 40', 'part 60' (Schrijver, 2014, p. 22). This use of special signs makes Linear A comparable to ten other systems of fraction notation attested in the world (see Appendix A): they include four Egyptian systems (hieroglyphic, hieratic, 'Eye of Horus' and demotic), as well as Mesopotamian cuneiform (Old Akkadian/Old Babylonian phases), Greek alphabetical, Coptic alphabetical, North African Fez/Rumi, the 'vulgar' Arabic system, and Indian Grantha. Notably, Linear B is not included, since it does not comprise signs for fractions, but only units of measurement (see below section 5). As the lowest fraction represented by a special sign in the world is 1/320 (Grantha), we fixed the range of fractions at 1/2 … 1/320. We also included three typologically attested values with higher numerators, 5/6, 3/4, and 2/3. However, due to Constraint 1 we removed 1/2, 1/4 and 1/8 (represented by J, E, and F, respectively), as well as 3/4 (represented by J E). This range comprises 318 possible values, a figure that could make computation time-consuming and produce many implausible solutions.

Yet it would be restrictive to reduce the set of possibilities to the typologically attested values: 5/6, 2/3, 1/5, 1/6, 3/20, 1/10, 1/16, 1/

[3] Possible exceptions: tablets HT 123a.3–4 and ZA 8.4 contain doubtful but possible cases of E J, which stand in contradiction to 28 examples of J E (Table 1).

[4] Y is only used twice on fragmentary inscriptions (PH 9a and 26) dated to the end of Middle Minoan II (ca. 1700 BC), once in the combination E Y Y Y, thus possibly implying that Y < 1/4. Ω is only attested on a doubtful Cretan Hieroglyphic or Linear A inscription (MA 10b.1) dated to Middle Minoan III (ca. 1700/1650-1600 BC). In fact, Ω is identical in shape to the Cretan Hieroglyphic fraction sign 304 Λ (Godart and Olivier, 1985, p. xxi, n. 2).

Table 4
Correlation between attestations and common divisors in 10 random solutions from Simulation 1. Signs are considered to have an anomalous rank (marked by **) if after E they deviate substantially from the descending order of values in terms of number of common divisors.

Sign	Total attestations	Number of common divisors for each solution									
		Sol. 1	Sol. 2	Sol. 3	Sol. 4	Sol. 5	Sol. 6	Sol. 7	Sol. 8	Sol. 9	Sol. 10
J	138	11	11	10	10	10	11	11	10	10	10
E	105	6	6	6	8	6	8	8	8	7	6
D	68	6	*1*	*0*	6	*0*	*0*	3	*1*	5	3
B	36	6	6	6	2	5	4	2	*0*	2	2
K	32	5	5	5	2	3	3	2	2	2	*5*
L2	26	2	2	2	2	2	2	2	2	2	2
F	23	2	1	3	*5*	1	3	*5*	*5*	*3*	*5*
H	19	1	*0*	*0*	*0*	2	*0*	*0*	*0*	*0*	2
A	15	*0*	1	1	*5*	0	*0*	*5*	*0*	*0*	0
L3	3	1	1	1	1	1	1	1	1	1	1
L4	2	0	0	0	0	0	0	0	0	0	0
L6	2	0	0	0	0	0	0	0	0	0	0
L	1	*4*	*4*	*4*	*4*	*4*	*4*	*4*	*4*	*4*	*4*

20, 1/32, 1/40, 1/64, 1/80, 1/160, 1/320. Only two systems in our sample, the base-2 'Eye of Horus' and the base-20 Grantha, comprise values lower than 1/10, but all systems in our sample contain fewer than the 13 signs of Linear A. Hence the Linear A fractions, which certainly included values smaller than 1/10, may have had a different base. However, we can infer the range of plausible fractional bases and increase the set. All numerical notations in the world use a base of 10 or its multiples (base-20 and base-60 are attested) for representing natural numbers, and when they use sub-bases, these are divisors of the base (sub-bases 5 and 10 are attested) (Chrisomalis, 2010). Fraction systems do not necessarily use the same bases as the integers, but the only such case known to us, the base-2 'Eye of Horus' system, still uses a divisor of 10 (see Appendix A).

To cover all possibilities we included: all typological values, including the base-2 values of the 'Eye of Horus' and those of base-20 Grantha; all powers of 10, and all products of 10 and the sub-bases 2 and 5; all multiples of 10 up to 100; sexagesimal values as represented by the divisors of 240 (as 1/360 is excluded from our range) and the multiples of 12 (a possible sub-base of 60) up to 120; and 1/7 and 1/9, whose odd denominator (following Schrijver, 2014, p. 21) is a possible cause for the lack of combinatorial power of sign D (which is only ever combined with itself). Thus, our set comprises: 5/6, 2/3, 1/3, 1/5, 1/6, 3/20, 1/7, 1/9, 1/10, 1/12, 1/15, 1/16, 1/20, 1/24, 1/30, 1/32, 1/36, 1/40, 1/48, 1/50, 1/60, 1/64, 1/70, 1/72, 1/80, 1/84, 1/90, 1/96, 1/108, 1/100, 1/120, 1/160, 1/200, 1/240, 1/300, 1/320. We do not expect A, B, D, H, K to be very low fractions, because A, B, H, and K all combine with J, so the magnitude of their denominators should not be very disparate, and D is the third most attested sign. Moreover, B and K are higher than L2 (Constraint 5), which in turn must be equal or higher than 1/107, because L2 = L6*3 (cf. Constraint 6) and the lowest possible fraction in our computation is 1/320, hence $L2 \geq (1/320)*3 = 3/320 \approx 1/107$.

We were even less restrictive with the possibilities for the L series. The series must contain some of the lowest fractions, as Constraints 1 and 5 imply that L2, L3, L4, and L6 are smaller than 1/9 ($L4 < L2 < K < F = 1/8$). At the same time, according to Constraint 6, the series L, L2, L3, L4 and L6 is constructed in a specific divisive relationship: x, x/2, x/3, x/4, x/6. This means that L3 must equal L2*1.5 and typological values are not helpful: no two signs lower than 1/9 in the 'Eye of Horus' or the Grantha systems have values in such proportion, so by logic L, L2, L3, L4, L6 cannot mirror them exactly. Thus, we defined all values between 1/2 and 1/320 as possible for the L series.

3. The status of sign L

The scarce epigraphic evidence for L (Section 1) casts doubt on its status. Is it really an independent sign or is its only certain attestation (in a strange palaeographical combination LL2) something anomalous? To address this issue, before proceeding with our analysis, we performed two tests.

3.1. *Correlation between frequency and number of common divisors of fraction signs*

Bennett argued that the signs with the highest fractional values (lower denominator) in Linear A tend to be most frequent by themselves and in combinations. There is a mathematical basis to this. In a set of fractional signs that are combined to express certain values, like the Linear A one, the denominator of a given fraction is potentially a divisor of the denominators of other fractions represented in the set: for example, 1/2 equals 2*1/4, 3*1/6, 4*1/8, 5*1/10, … (and, as a fraction, 1/2 is a multiple of the other fractions). Therefore, it is not just because 1/2 has a low denominator that it is so frequent. Rather, it is because '2' is a divisor of several numbers—indeed, all even numbers. The more common divisors a given fraction has within a system, the more it will be able to render other values. As a result, any fraction higher than 1/2, that is, $1/2+n$, can generally be written with a sign standing for 1/2 plus only one or two more signs (depending on the value of n and the values represented in the system). This also means that even if the Minoans rarely used 1/2 by itself to measure goods, they could still employ a sign standing for it to express other fractions corresponding to $1/2+n$. Thus, in such a notation system we will observe a correlation between the number of common divisors of a fraction and the frequency of use of that same fraction.

This principle helps address the problem of the near-absence of L: we can estimate how frequent this sign should be, based on the number of common divisors for its value in any solution produced through constraint programming. Thus, after performing a first simulation (Simulation 1, with 2,172,836 solutions), we sorted the total attestations of each Linear A fraction sign (Table 1) listed in decreasing order. Given any solution, we

Table 5
Probability and significance values for the attestations of Linear A signs L and L6.

	Attestations of L	P-value (signif.[4])	Attestations of L6	P-value (signif.)
Real attestations	1	$4 * 10^{-12}$ (***)	2	0.029 (–)
Hypothetical attestations	2	$2 * 10^{-11}$ (***)	3	0.068 (–)
	5	$2 * 10^{-9}$ (***)	5	0.254 (–)
	10	$7 * 10^{-7}$ (***)		
	15	0.00004 (***)		
	20	0.00008 (***)		
	26	0.010 (*)		
	30	0.037 (–)		
	35	0.124 (–)		

4 (***) $p < 0.0001$; (**) $0.0001<p < 0.001$; (*) $0.001<p < 0.01$; (–) $p > 0.01$, no statistical significance.

computed for each Sign Assignment (e.g. sign J = 1/2) the number of other Sign Assignments (e.g. E = 1/4, F = 1/8, …) that can be considered a divisor of the latter, producing a list of divisors for each sign.

Table 4 illustrates the correlation in terms of attestations and number of common divisors of the signs for 10 solutions selected randomly from Simulation 1. Some signs (D, B, F, H, A) display anomalous ranks in terms of common divisors in certain solutions, but L is *always* very anomalous in this regard. This implies that if L were the highest member of a divisive L series, and regardless of its exact value, it would not be as rare a sign as it is.

For every solution, we used the Spearman's Rank Correlation Coefficient (*SRCC*) to measure the correlation between the two ranks—frequency (*f*) and number of common divisors (*ncd*). This measure, G_FreqVSDiv = *1-SRCC*(*f*,*ncd*), revealed that the lower (best) value for any solution in Simulation 1 was 0.14284. We performed a second simulation (Simulation 2) with the exclusion of L from the set. This simulation showed 3,794,740 solutions, of which 394,905 have values lower than 0.14284 (the better ones approach 0.00881, which is 16 times better), confirming that the correlation is more respected when L is not treated as an independent sign.

3.2. Implausible frequency of L

The second test focuses also on the number of common divisors of L in any given solution. Considering the divisive relation between the signs in the L series (Constraint 6), we observe the following: whatever the number of divisors of L2 is, say n_{L2}, L has a number of divisors at least equal to $n_{L2}+2$, because it is also divisible by L2 and by L3 (which in such a series cannot be a divisor of L2). Thus, we can use Pearson's $\chi 2$ goodness-of-fit statistical test to evaluate whether the difference between the expected and observed frequencies of L is statistically significant.

For our analysis of the frequency of L, we considered only the number of common divisors for signs within the subset L, L2, L3, L4, L6. Given Constraints 1–6, only other five signs (A, B, H, as well as problematic W, X) could conceal additional divisors of L and L2, and their existence would not dramatically change the ratio between the number of divisors of L and L2. It would be surprising if they were all additional divisors of the L series (e.g. L8 or L10) while not featuring the basic semicircle shape.

Considering the number of divisors of L and L2 within the L series, the probability of finding signs L and L2 when excluding all other signs is: P(L) = 2/3, P(L2) = 1/3. Table 5 shows the results of the test, including the probability values and the level of statistical significance (i.e. likelihood of the outcome not being random) for the real observed occurrences and for some hypothetical ones, if more instances of L and none of L2 were to be found. We use 0.01 as a threshold of statistical significance: p-values lower than 0.01 show a statistically significant discrepancy between the expected distribution and the observed frequencies.

This statistical test strongly supports our argument that the probability of having frequency L = 1 is so low that it cannot be due to chance. Applying the same analysis to L2 and L6 gives us a P(L6) = 1/4, with P-values for the significance test as also shown on Table 5. This shows that the occurrence of L6 is not so improbable as to be statistically significant, and that our line of reasoning applies equally to all L series signs.

In conclusion, in light of the frequencies of the signs, the semicircular shape is unlikely to correspond to a self-standing entity "L" if the L series is divisive ((L >) L2 > L3 > L4 > L6), for mathematical reasons as well as epigraphic ones, since its reading is uncertain. We therefore propose that the sign shape documented as "LL2" is anomalous and thus exclude L from our analysis.

4. Results

4.1. Rationalisation of the set of solutions

Using Constraints 1–6 and excluding W, X, Y, Ω and L, Simulation 2 still yielded 3,794,740 solutions. Clearly a constraint-based approach cannot decipher the values of Linear A fraction signs on its own, mainly because the system is severely underconstrained. While this high figure encompasses all valid assignments of values for our set of 12 signs (J, E, F, K, D, B, H, A, L2, L3, L4, L6), not every solution fits the existing evidence equally well. To narrow them down, we evaluated each solution according to four measures of goodness:

Measure 1. Correlation between frequency and number of common divisors (G_FreqVSDiv)

Introduced in Section 3.1. It measures each solution according to how suitable the frequency rank of its signs is when correlated to their number of common divisors within the same solution.

Measure 2. Attested ambiguity (G_AmbCombs)

No system of notation is operational if its signs are characterised by a high degree of ambiguity, so we do not expect the values of the fraction signs to produce a high number of redundant combinations. For any given solution, this measure counts the number of cases in which two *attested* combinations (Table 1) would yield the same fractional value. The higher the number of cases of attested ambiguity, the less likely the solution.

Measure 3. Typological probability of values (G_Typ)

Worldwide, systems of notation of fractions show certain tendencies in terms of the mathematical values they expressed using special signs (Appendix A). For example, no system represents 1/7 or 1/9 with a special sign. Linear A fractions certainly included some values unattested typologically, especially values lower than 1/10, if it used a fractional base different from the 'Eye of Horus' and Grantha systems. However, we do not expect a huge proportion of unattested values. For each solution we counted how many Sign Assignments are not typologically documented; the fewer the assignments, the lowest (and therefore better) value it receives in this metric. That all solutions in Simulation 2 have at least two unattested values confirms that this measure is not too severe.

Measure 4. Optimality (G_Optimal)

In a first instance, this measures the ability of a system to represent the fractional values corresponding to its own fractional base, by counting the number of values that cannot be expressed by any sign or combination of at most four signs,[5] within the range 1/2 through (D-1)/D. The fractional base or range is derived from the smallest value present in the system: e.g. for a solution corresponding to a system of notation where the lowest fraction represented by a sign is 1/80, the limit is defined by D (denominator) = 80 and N (numerator) = D - 1 = 80–1 = 79, hence 79/80.

However, simply counting the number of unrepresentable values in a solution creates a bias: the measure would penalise solutions containing signs with higher denominators. In other words, a system of fractions with a range 1/2–79/80 (where 1/80 is the lowest value of a sign) will have fewer unrepresented values than one with a range 1/2–119/120 (where 1/120 is the lowest value of a sign). Thus, to make our metric proportional to the range of each solution, we assigned a lower impact to

[5] We used at most four signs because this is the length of the longest combination in the Linear A corpus (D D D D, Table 1).

Table 6
Method for calculating measure G_Optimal for incompleteness (example: U=Unrepresented, R = Represented).

Denominators	Numerators				Incompleteness
	1	2	3	4	
2	R				0 * 1/1
3	R	U			1 * 1/2
4	R	R	R		0 * 1/3
5	R	R	R	U	1 * 1/4
Total	(0+1/2+0+1/4)/4=3/16				= 18.6 %

Table 7
Method for calculating measure G_Optimal for ambiguity (example: U=Unambiguous, A = Ambiguous).

Denominators	Numerators				Ambiguity
	1	2	3	4	
2	U				0 *1/1
3	U				0 *1/2
4	U	U	U		0 *1/3
5	U	A	A		2* 1/4
Total	(0+0+0+2/4)/4 = 1/8				= 12.5 %

fractions with a higher denominator: if the value 2/3 could not be notated, this was deemed more severe than if sign 59/60 could not be represented. For the calculation, we considered each possible denominator in a given solution and calculated the incompleteness metric with respect to each denominator through the formula 1/(D-1) *I (where I = number of unrepresented values with denominator D). For example, a system lacking 2/3 (3 being a denominator placed in a line of the table that has only two slots, 1/3 and 2/3) will have the metric 1/(3–1) * 1 = 1/2 for denominator 3. The metric value was then normalised by dividing the sum of incompleteness measures across all denominators by the total number of denominators in the system (Table 6).

G_Optimal also measures the ability of any system to represent its own fractional base while producing a minimum number of redundant values in the combinations of signs (regardless of whether the possible combination is attested in Linear A or not). The greater number of redundant additions a solution yields, the less efficient it is considered. We thus counted the number of fractional values from the range 1/2 through (D-1)/D that can, in principle, be represented by more than one sign or combination of up to four signs in the system. The procedure used for measuring the ambiguous values was the same: the number of ambiguous values for each denominator D was divided by D-1. Then, the sum of all the ambiguity measures for every denominator was divided by the total number of denominators in the system (Table 7). This makes the impact of ambiguous values inversely proportional to their denominator (thus redundancy in expressing 1/3 is more severe than redundancy in the representation of 1/30). Finally, to extract the G_Optimal value, the values of the incompleteness and the ambiguity components were summed.

The four measures described above were implemented at different stages, because G_Optimal could not be applied to the 3,794,740 solutions of Simulation 2: its complexity makes it computable in a reasonable timeframe only for a relatively small number of solutions. Thus, first we reduced the number of solutions by determining which 5000 had the best values of G_FreqVSDiv, G_Typ and G_AmbCombs. In practice, these represent the 5000 least problematic solutions in terms of correlation between frequency and number of common divisors for each fraction sign, presence of typologically unattested sign values, and ambiguity between attested combinations of signs. Thus, the individual results for these three measures were multiplied[6] to achieve a single metric (MULT) for each solution, and then we selected the 5000 solutions with the lowest MULT value. Afterwards, G_Optimal was applied.

4.2. Decimal-sexagesimal values for K and the L series

Once the 5000 solutions were sorted according to G_Optimal, two displayed the best value (0.67094). In these two solutions, seven signs have the same assignments: B = 1/5, D = 1/6, K = 1/10, L2 = 1/20, L3 = 1/30, L4 = 1/40, L6 = 1/60; differently, H and A take the values 1/32 and 1/36, and vice versa. Does this set of values represent the optimal solution? Since the results of our calculations depend on several variables, of epigraphical, typological, and mathematical nature, it is crucial to see consistency, namely how frequently they appear in the solutions displaying the lowest G_Optimal.

Assignments K = 1/10, L2 = 1/20, L3 = 1/30, L4 = 1/40, L6 = 1/60 appear in the 48 solutions with the best G_Optimal metric (values between 0.67094 and 0.75379) and in other 52 solutions, totalling 100 of the 5000 possible systems filtered in the previous step. They yield a decimal-sexagesimal subset of fractions, in which the frequent combination K L2 attains the mathematical value of 3/20, emerging from the sum of two fractions (1/10 + 1/20) whose denominators are both multiples of 10 and divisors of 60. The G_Optimal values of solutions containing these Sign Assignments strongly imply that they reflect the system best able to express values within its own fractional base (even discounting the values assigned to A, B, D, and H). In fact, it is clear that the L series is structurally important in the system, even if attested only 34 times (Table 2), because it combines not only with K, but also with the super-frequent J = 1/2 and E = 1/4 (Table 1). Thus, we filtered the 100 solutions containing K = 1/10, L2 = 1/20, L3 = 1/30, L4 = 1/40, L6 = 1/60.

4.3. The values of D and B

The majority (86) of the 100 solutions that comprise K = 1/10, L2 = 1/20, L3 = 1/30, L4 = 1/40, L6 = 1/60 also contain B = 1/5, D = 1/6, and this is the arrangement in the 28 solutions with the best G_Optimal metrics (0.67094–0.72676). These assignments rank far better and more consistently than the alternative B = 1/6, D = 1/5 (12 solutions, G_Optimal ≤ 0.71724), B = 1/5, D = 1/12 (1 solution, 0.79232) and B = 1/9, D = 1/6 (1 solution, 0.80510). This fact reflects the structural advantages of B = 1/5, D = 1/6 if the Linear A system had a decimal-sexagesimal base as determined by K = 1/10, L2 = 1/20 … L6 = 1/60. This is better illustrated by tabulating how the signs and combinations of signs attested in the Linear A corpus (Table 1) would have recorded the fractional values within the ranges 1/10–9/10, 1/20–19/20, and 1/60–59/60 (Appendices B and C):

If J = 1/2, E = 1/4, F = 1/8, K = 1/10, L2 = 1/20, L3 = 1/30, L4 = 1/40, L6 = 1/60 and B = 1/5, D = 1/6:

- Range 1/10–9/10: 1 value unattested.
- Range 1/20–19/20: 3 values unattested.
- Range 1/60–59/60: 35 values unattested, 1 uncertain (5/60 = 1/12 might be represented by A A in solutions where A = 1/24).

If J = 1/2, E = 1/4, F = 1/8, K = 1/10, L2 = 1/20, L3 = 1/30, L4 = 1/40, L6 = 1/60 and D = 1/5, B = 1/6:

- Range 1/10–9/10: 2 values unattested.
- Range 1/20–19/20: 6 values unattested.
- Range 1/60–59/60: 36 values unattested, 1 uncertain (5/60 = 1/12 might be represented by A A in solutions where A = 1/24).

Thus B = 1/5, D = 1/6 leads to fewer unattested values, especially in the decimal and vigesimal ranges. The latter would have been crucial to the Minoan scribes, judging from the greater frequency of K = 1/10 and L2 = 1/20 and comparatively scarce use of L3, L4 and L6. Moreover, since 5 is a divisor of 10 and 20, the sign representing 1/5 should combine often with other signs in such a system, and indeed sign B is relatively frequent in attested combinations (Table 1).

4.4. The values of H and A

The 28 solutions that contain B = 1/5, D = 1/6, K = 1/10, L2 = 1/20 … L6 = 1/60 and the lowest G_Optimal values have these possible assignments for H and A:

- H: 1/16, 1/24, 1/32, 1/36, 1/48, 1/64, 1/72, 1/84
- A: 1/24, 1/32, 1/36, 1/48, 1/64, 1/72, 1/84

The distribution of these assignments does not reveal any pattern, nor any arrangement repeating with a better G_Optimal metric. Those with the best value of optimality, H = 1/36, A = 1/32 and H = 1/32, A = 1/36 are not as consistent as the results achieved for B, D, K and the L series. Thus, specific values for H and A cannot be postulated through optimality.

The key for the value of H may lie partly in systemics, partly in palaeography: the sign seems modelled or modified from the same crooked shape that is shared by J = 1/2, E = 1/4 and F = 1/8 (Fig. 1). If J, E, F and H formed an interrelated subset, then assigning H the value of 1/16 would yield a binary series 1/2, 1/4, 1/8, 1/16. Notice that 1/16 is the only value possible for H (among the 28 most optimal solutions) that is not also possible for A.

If H = 1/16, there are thus structural advantages in fixing the value of A as 1/24. While the arrangement H = 1/16, A = 1/24 has a worse G_Optimal metric than H = 1/16 and A = 1/36 or 1/64 or 1/72, the implications change if we consider as an element of the system also the combination A A (=X?). If A = 1/24, then A A = 2/24 = 1/12, and this last value would represent 5/60 = 1/12 in a system that can fraction up to 1/60 (Appendix C). There are two further structural arguments in favour of H = 1/16, A = 1/24: (1) these signs should not have very high denominators (Section 2.4), and 1/16 and 1/24 are their lowest possible values in these solutions; (2) along with the denominators of 1/2, 1/4, 1/5, 1/6, 1/8, 1/10, 1/20, 1/30, 1/40 and 1/60, 16 and 24 are divisors of 240–and so is 12, the possible denominator of A A. In conclusion, 1/16, 1/24 and 1/12 fit well in a fractional base involving a sexagesimal component, as the values of K and the L series show.

5. Discussion

Our results can be expressed by a simple hypothesis: given Constraints 1–6, *if* the Linear A fractions had values that are typologically attested or part of a typologically attested base or sub-base of numerical notations; and *if* Linear A fractions formed an optimal system, in which the documented combinations of signs represented the maximum number of values within its own fractional base, while at the same time generating a minimum number of redundant additions; *then* the correct values of Linear A signs comprise J = 1/2, E = 1/4, F = 1/8, B = 1/5, D = 1/6, K = 1/10, L2 = 1/20, L3 = 1/30, L4 = 1/40, L6 = 1/60 (Tables 8 and 9). The assignments H = 1/16 and A = 1/24 are tentative because they do not violate typology, but their optimality is not as consistent.

Beyond the relatively consensual assignments for J, E, F, some values in our system had already been suggested (Table 3). Schrijver (2014, p. 22) considered identical assignments for L2, L3, L4, L6 because these

[6] We multiply the three metrics because the multiplication is similar to the logical conjunction and we require that all the metrics have a low value, favouring balanced solutions.

Table 8
Optimal system of mathematical values for Linear A fractions.

	Optimal value	Total attestations	Single attestations in numeral phrases	Common divisors
J	1/2	137	101	10
E	1/4	105	52	6
D	1/6	68	19	2
B	1/5	36	21	5
K	1/10	32	21	4
L2	1/20	26	6	2
F	1/8	23	9	3
H	1/16(?)	19	15	0
A	1/24(?)	15	13	0
L3	1/30	3	2	1
L4	1/40	2	0	0
L6	1/60	2	0	0

signs comprise the same basic shape plus strokes of Linear A numerals '20', '30', '40' and '60', but he assigned 1/24 to L2 from his reading of tablet HT 123b. This tablet, however, contains doubtful readings and scribal errors (Montecchi, 2009, pp. 37–38), thus we had to exclude it. Conversely, our approach, based on mathematical optimisation, has provided independent evidence for these values.

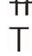
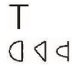
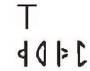
Cash and Cash (2012) and Montecchi (2013) suggested D = 1/6 and B = 1/5. The latter value is supported by the proportion B:4 = J:10 which may be inferred from tablet KH 7a. Previously, the value of D was proposed on two accounts that make its distribution very different: (1) it is never attested in combination with signs other than itself; (2) it occurs in double (D D) and quadruple (D D D D) combinations, though D D D is not attested. If D equals 1/5, then unattested D D D equals 3/5, which is not a problematic value; but if D equals 1/6, then D D D = 3/6 = 1/2, which can already be expressed by J. Thus, D as 1/6 might account for the absence of D D D, but this absence is not sufficient to exclude 1/5, as it might be accidental. Considering our optimal solutions, a potential disadvantage of D = 1/5 emerges from D D D D (attested once) = 4/5, which would express a value already reflected in J E L2 (attested three times) = 1/2 + 1/4 + 1/20 = 16/20 = 4/5. Yet, this cannot be proof that D = 1/5 is incorrect either because all possible solutions in Simulation 2 have at least two redundant combinations. Schrijver (2014, p. 21) argues that the high frequency of D D demonstrates that D had an odd denominator and was probably 1/5, but this depends on his other assignments (H = 1/3, K = 1/16), which contrast with our results.

The lack of combinatorial power of D cannot derive from the odd denominator of 1/5. The combinatorial ability of fraction signs is tied to their number of common divisors. In a system containing 1/5, 1/6, 1/10, 1/20, 1/30, 1/40, and 1/60, the value 1/5 shows at least 1/10, 1/20, 1/30 and 1/60 as common divisors, and 1/6 has common divisors in 1/30 and 1/60. Hence, signs standing for 1/5 and 1/6 might combine with K, L2, L3, L4, and L6 to express higher fractional values. Combinations with other fractions are equally possible. For example, J+1/5 could be used to express 7/10 and, indeed, our results show that this combination corresponds to the attested J B (Table 1). We can therefore discard D being odd as a cause. In conclusion, its peculiar behaviour may have been due more to contextual than mathematical reasons. For example, D = 1/6 and the combinations D D = 1/3 and D D D D = 2/3 may have been used as a special subset of signs by the Minoan scribes. This would be consistent with our typological survey (Appendix A), which suggests that any system that has a special way of notating 1/6 also has a specific notation for 1/3 and 2/3.

Table 9
Values of Linear A fraction combinations according to the proposed optimal system.

	Value	Value in denominator10	Value in denominator 20	Value in denominator 40	Value in denominator 60	Attestations
J E	**3/4**	–	15/20	30/40	45/60	24
D D	**1/3**	–	–	–	20/60	14
B B	**2/5**	4/10	8/20	16/40	24/60	5
E F	**3/8**	–	–	15/40	–	5
K L2	**3/20**	–	3/20	6/40	9/60	5
E L2	**3/10**	3/10	6/20	12/40	18/60	3
J B	**7/10**	7/10	14/20	28/40	42/60	3
J E L2	**4/5**	8/10	16/20	32/40	48/60	3
J L2	**11/20**	–	11/20	22/40	33/60	2
A A	**1/12(?)**	–	–	–	5/60	1
D D D D	**2/3**	–	–	–	40/60	1
E B	**9/20**	–	9/20	18/40	27/60	1
E L4	**11/40**	–	–	11/40	–	1
E L6	**4/15**	–	–	–	16/60	1
F K	**9/40**	–	–	9/40	–	1
J A	**13/24(?)**	–	–	–	–	1
J E B	**19/20**	–	19/20	38/40	57/60	1
J F	**5/8**	–	–	25/40	–	1
J H	**9/16(?)**	–	–	–	–	1
J K	**3/5**	6/10	12/20	24/40	36/60	1
L2 L4	**3/40**	–	–	3/40	–	1

Table 10
Comparison of Linear A fractions and Linear B units of measurement. The sign shape "L(n)" in the Linear A column represents the basic component in signs L2, L3, L4 and L6.

Linear A			Linear B		
Fraction	Sign shape	Value	Unit of measurement	Sign Shape	Value
D D		1/3	*117/M		1/3 of LANA
X = A A (?)		1/12	*116/N		1/12 of LANA
K		1/10	*112/T		1/10 of highest dry unit
L(n)		1/20–1/60	*111/V		1/60 of highest dry unit

There are also historical implications. Although Linear B was adapted from Linear A, the two scripts recorded quantities of commodities differently. Linear B employed no numerical fractions but used three special sets of logograms which stood for units of dry and liquid commodities and weights (Melena, 2014). Indeed, our optimal system of Linear A fraction values reveals possible relationships with the Linear B signs for measurements (Table 10):

- The shape of the Linear A combination D D = 1/3 is identical to Linear B sign *117/M, which represents 1/3 of the largest weight measure for wool (*145/LANA).
- The shape of Linear A K = 1/10 is identical to Linear B *112/T, which represents 1/10 of the largest dry capacity measure.
- The shape of Linear A X, most probably equal to A A = 1/12, is identical to Linear B *116/N, which represents 1/12 of the largest weight measure for wool (*145/LANA).
- The shape contained in the Linear A decimal-sexagesimal L series resembles Linear B *111/V, which represents 1/60 of the largest dry capacity measure.

This implies that some Linear B signs for weights and measures may have been adapted from the Linear A fraction signs, and even though the two systems registered quantities differently, the Linear B system took inspiration from Linear A (cf. already Younger, 2005). It is not hard to imagine how the adaptation may have taken place. Measures were implicit in Linear A commodity logograms, and some of them were ligatured with fractional signs (Section 1): for example, GRA + F most probably meant roughly ‘one-eighth of the grain measure’. Some fractional signs in time were reinterpreted to note a certain part of a measure. The frequent use of K L2 as a measure for cereal (GRA) gives some support to the notion that the sign shapes of K and L(n) became associated with dry capacity measures, and the creators of Linear B may have reused them as such. Thus, it is striking to see a notable historical continuum with the redeployment of Minoan fractions in the Mycenaean measuring system.

6. Conclusions

The mathematical values of the Minoan Linear A fraction signs are still an object of scientific debate. This contribution set it as its main goal to unravel this issue through a novel perspective. Taking on from the state-of-the-art on the topic, we set out to address the values of Linear A fractions through: (1) a thorough palaeographical and epigraphic assessment of fraction signs as they appear on the tablets; (2) a constraint-based computational approach narrowing down the possible spectrum of values for each sign; (3) multiple tests on frequencies of signs and the plausibility of a problematic instance; (4) optimality-driven searches evaluating ambiguity and efficiency levels. The results achieved by applying these methods converge in a coherent and systematic fashion. The application of computational strategies in close combination with the traditional epigraphic analysis of texts have thus opened the way to notable progress in the assignment of values to the Linear A fraction signs, adding to previous efforts directed to this goal, and contributing to shedding new, unbiased and objective evidence on an important yet unresolved problem in the study of this undeciphered script.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Typological survey of systems of fraction notation that use special signs

Feature	Old Akk./Old Bab. cuneiform	Egyptian hieroglyphic	Egyptian hieratic	‘Eye of Horus’	Demotic	Greek alphabetical	Coptic alphabetical	Fez/ Rumi	‘Vulgar’ Arabic	Grantha
5/6	x				x				x	
3/4		x							x	x
2/3	x	x	x		x	x	x	x	x	
1/2	x	x	x	x	x	x	x	x	x	x
1/3	x		x		x			x	x	
1/4			x	x	x			x	x	x
1/5										x
1/6					x				x	
3/20										x
1/8				x					x	x
1/10										x
1/16				x						
1/20										x
1/32				x						
1/40										x
1/64				x						
1/80										x
1/160										x
1/320										x
Signs based on integers	x	x	x		x	x	x	x	x	

References: Old Akkadian/Old Babylonian cuneiform: Friberg (2007, pp. 4–5, 11, 374, Fig. 0.4.3, A4.2); Egyptian hieroglyphic: Gardiner (1957, pp. 196–197); Austin and Guillemot (2017): Egyptian hieratic: Möller (1927, nos. 667–668); Gardiner (1957, pp. 196–197); Egyptian ‘Eye of Horus’: Gardiner (1957, pp. 197–198); Egyptian demotic: Griffith (1909, p. 418), Sethe (1916, Tab. III); Greek alphabetical: Chrisomalis (2003); Coptic alphabetical: Stern (1880, p. 471); Fez/Rumi: Lazrek (2006); ‘Vulgar’ Arabic: Perceval (1858, p. 116); Sethe (1916, pp. 66–67, Table III); Grantha: Grünendahl (2001, p. 58).
NB: The Romans employed a system of fractions whose base is 12 (a divisor of 60) and with values ranging from 1/12 to 12/12 = 1, but they represent units of the measure *uncia* ‘twelfth (of a pound)’ and the shapes of the signs reflect integers (Cagnat, 1898, p. 33; Chrisomalis, 2010).

Appendix B. Recording of fractional values in the ranges 1/10–9/10, 1/20–19/20 with attested Linear A signs and combinations of signs, if K = 10, L2 = 1/20, L3 = 1/30, L4 = 1/40, L6 = 1/60

10	*D=1/6; B=1/5*	*D=1/5; B=1/6*
1/10	**K**	**K**
2/10	**B**	**D**
3/10	**E L2**, B K	**E L2**, D K
4/10	**B B**	**D D**
5/10	**J**	**J**
6/10	**J K**	**J K**
7/10	**J B**	J D
8/10	**J E L2,** J B K	**J E L2,** J D K
9/10	J B B	J D D
1		

20	*D=1/6; B=1/5*	*D=1/5; B=1/6*
1/20	**L2**	**L2**
2/20	**K**	**K**
3/20	**K L2,** F L4	**K L2,** F L4
4/20	**B**	**D**
5/20	**E**	**E**
6/20	**E L2**, B K	**E L2**, D K
7/20	**E K**	**E K**
8/20	**B B**	**D D**
9/20	**E B**	E D
10/20	**J**	**J**
11/20	**J L2**	**J L2**
12/20	**J K**	**J K**
13/20	E B B, J F L4, J K L2	E D D, J F L4, J K L2
14/20	**J B**	J D
15/20	**J E**	**J E**
16/20	**J E L2**, J B K	**J E L2**, J D K
17/20	J E K	J E K
18/20	J B B	J D D
19/20	**J E B**	J E D
1		

Appendix C. Recording of fractional values in the range 1/60–59/60 with attested Linear A signs and combinations of signs, if K = 10, L2 = 1/20, L3 = 1/30, L4 = 1/40, L6 = 1/60. The aim of this table is twofold: (1) demonstrate that B = 1/5, D = 1/6 forms a more optimal base-60 system with K, L2-L6 than D = 1/5, B = 1/6; (2) that A would also take part in such system if its value was 1/24

60	*D=1/6; B=1/5*	*D=1/5; B=1/6*
1/60	**L6**	**L6**
2/60	**L3**	**L3**
3/60	**L2**	**L2**
4/60	**L3 L3**, L2 L6, 1/24 L4	**L3 L3**, L2 L6, 1/24 L4
5/60	1/12, 1/24 1/24, L2 L3	1/12, 1/24 1/24, L2 L3
6/60	**K**	**K**
7/60	K L6	K L6
8/60	K L3	K L3
9/60	**K L2,** F L4	**K L2,** F L4
10/60	**D**	**B**
11/60	D L6	**Ḅ L6**
12/60	**B**	**D**
13/60	**Ḅ L6**, D L2	D L6, B L2
14/60	B L3	D L3
15/60	**E**	**E**
16/60	**E L6,** D K	**E L6,** B K
17/60	E L3	E L3
18/60	**E L2**, B K	**E L2**, D K
19/60	E L2 L6, E 1/24 L4, E L3 L3, B K L6, D K L2, D F L4	E L2 L6, E 1/24 L4, E L3 L3, D K L6, B K L2, B F L4
20/60	**D D**	**B B**
21/60	**E K**	**E K**
22/60	B D	D B
23/60	B D L6, D D L2, E K L3	D **B L6**, B B L2, E K L3
24/60	**B B**	**D D**
25/60	E D	**E B**
26/60	B B L3, D D K, E D L6	D D L3, B B K, E B L6
27/60	**E B**	E D
28/60	E B L6, E D L2, B D K	E D L6, E B L2, D B K

29/60	E B L3	E D L3
30/60	**J**	**J**
31/60	J L6	J L6
32/60	J L3	J L3
33/60	**J L2**	**J L2**
34/60	B B D, J 1/24 L4, J L2 L6, J L3 L3	D D B, J 1/24 L4, J L2 L6, J L3 L3
35/60	E D D, J 1/24 1/24, J L2 L3	E B B, J 1/24 1/24, J L2 L3
36/60	**J K**	**J K**
37/60	J K L6, E B D	J K L6, E D B
38/60	J K L3	J K L3
39/60	J K L2, J F L4, E B B	J K L2, J F L4, E D D
40/60	**D D D D**, J D	B B B B, **J B**
41/60	J D L6	J B L6
42/60	**J B**	J D
43/60	J B L6, J D L2	J D L6, J B L2
44/60	J B L3	J D L3
45/60	**J E**	**J E**
46/60	J E L6, J D K	J E L6, J B K
47/60	J E L3	J E L3
48/60	**J E L2**, J B K	**J E L2**, J D K, **D D D D**
49/60	J E L3 L3, J E L2 L6, J E 1/24 L4, J D K L2, J D F L4, J B K L6, E B B D	J E L3 L3, J E L2 L6, J E 1/24 L4, J B K L2, J B F L4, J D K L6, E D D B
50/60	J D D	J B B
51/60	J E K	J E K
52/60	J B D	J D B
53/60	J E K L3, J D D L2, J B D L6	J E K L3, J B B L2, J D B L6
54/60	J B B	J D D
55/60	J E D	**J E B**
56/60	J E D L6, J D D K, J B B L3	J E B L6, J B B K, J D D L3
57/60	**J E B**	J E D
58/60	J E B L6, J E D L2, J B D K	J E D L6, J E B L2, J D B K
59/60	J E B L3	J E D L3
1		

. (*continued*).

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