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A Fourier-based Picard-iteration approach for a class of McKean-Vlasov SDEs with Lévy jumps

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Abstract

We consider a prototype class of Lévy-driven stochastic differential equations (SDEs) with McKean-Vlasov (MK-V) interaction in the drift coefficient. It is assumed that the drift coefficient is affine in the state variable, and only measurable in the law of the solution. We study the equivalent functional fixed-point equation for the unknown time-dependent coefficients of the associated linear Markovian SDE. By proving a contraction property for the functional map in a suitable normed space, we infer existence and uniqueness results for the MK-V SDE, and derive a discretized Picard iteration scheme that approximates the law of the solution through its characteristic function. Numerical illustrations show the effectiveness of our method, which appears to be appropriate to handle the multi-dimensional setting.
Keywords: Nonlinear stochastic differential equations, Lévy processes, McKean-Vlasov model, Picard iteration, Fourier transform
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1 Introduction

We study a prototype class of McKean-Vlasov (MK-V) stochastic differential equations (SDEs) where the coefficients are functions of both the state variable and the law of the solution. Introduced by McKean in the 60's, these equations have received increasing attention in the last few decades due to their wide range of applications in several fields, which include physics, neurosciences, economics and finance among others. In particular, the link with mean-field interacting particle systems, and the advent of mean-field games, have boosted the research on MK-V SDEs.

From the numerical perspective, the study of the solutions to MK-V SDEs has been mainly conducted by exploiting the so-called "propagation of chaos" results. It is usually shown that, in the limit $N \to \infty$, the empirical law of a Markovian system with N interacting particles, converges to the law of the solution to the MK-V SDE under study, which can be then approximated via time-discretization and simulation. Following the work by Sznitman in [21], where the first propagation of chaos result was proved, many authors have contributed to this stream of literature, see, for example, [2], [23], [11], and [20], by supposing different forms and regularity assumptions on the MK-V SDE coefficients. Although a very powerful approximating tool which is applicable in many settings, the simulation of large-particle systems can be computationally very expensive. For this reason, several authors investigated alternative approaches to the resolution of MK-V SDEs. Szpruch et al. [22] provided an alternative iterative particle representation that can be combined with multilevel Monte Carlo techniques in order to simulate the solutions. Sun et al. [20] developed Itô-Taylor schemes of Euler- and Milstein-type for numerically estimating the solution of MK-V SDEs with Lipschitz regular coefficients and square-integrable initial law. Gobet and Pagliarani [8] recently developed analytical

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approximations of the transition density of the solutions by extending a perturbation technique that was previously developed for standard SDEs. In [4], Chaudru De Raynal and Garcia Trillos developed a cubature method to obtain estimates for the solution of forward-backward SDEs of MK-V type. Furthermore, Belomestny and Schoenmakers [3] developed a novel projection-based particle method for a class of MK-V SDEs with affine structure. They used generalised Fourier transform to demonstrate the convergence of their numerical method for the case where SDE coefficients are allowed to have upto a linear growth in state variable, in addition to other regularity conditions.

In this work, we propose a Picard-iteration scheme for a class of MK-V SDEs driven by a Lévy process. Precisely, we assume a linear mean-field interaction which arises via expectation of the state variable, with a drift coefficient that is affine in the state variable. We reformulate the problem as an equivalent fixed-point equation for the unknown time-dependent coefficients of the related Markovian SDE. The special structure of our class of MK-V SDEs, allows us to exploit a priori estimates on the characteristic function of the solution, and to show that the related functional map is a contraction. This implies existence and uniqueness of the solution to the MK-V SDE. We then discretize the functional map in order to obtain a fully implementable Picard-iteration scheme to accurately approximate the functions that determine the law of the solution. We also provide the rate of convergence of our scheme with respect to both the time-discretization step and number of Picard iterations, and show that it is independent of the dimension.

We point out that this approach of characterizing the (unknown) time-dependent coefficients, which determine the solution to a MK-V SDE as the fixed-point of a functional contraction map, has also been recently employed by Hambly et al. [12]. They proved the contraction property for a diffusive MK-V SDE with endogenous jumps that are induced, at a microscopic level, by the single particle hitting a certain boundary (see also Delarue et al. in [6]).

In our setting, we consider a general underlying Lévy process and a general initial law, but with jumps which are not dependent on the law of the MK-V SDE. To deal with such a Lévy measure, we show the contraction result by proving suitable estimates in the Fourier space. For this reason, we need a weak assumption on integrability of the initial moment and the Lévy measure, namely finiteness of the first moment. Also, our initial assumptions require the drift coefficients to be in L^1. However, we are able to overcome this limitation by using the so-called *damping method* (see, for example, [7]). By a suitable modification of the functions appearing in the drift coefficients using penalization functions, and exploiting the properties of Fourier transform derivatives, we are able to prove the contraction property and all the consequent results in the case of non-integrable (though bounded) coefficients. However straightforward and effective, this strategy to circumvent the L^1 assumption on the coefficients does not seem completely satisfactory. In fact, on the one hand, this generalization requires additional hypotheses, namely higher order integrability of the underlying Lévy process and the initial law. On the other hand, while it is possible to drop the integrability assumption on the coefficients, we cannot avoid requiring the Fourier transform of the *damped* (penalized) coefficients being in L^p with $p > 1$, and suitably close to 1, save the the case $d = 1$ for which the latter assumption can be also dropped. It is important to point out that all these restrictions do not have a clear probabilistic interpretation, but they rather seem to be related to our choice of carrying out the analysis in the Fourier space (Belomestny and Schoenmakers [3] handled more general MK-V SDE coefficients compared to our case by using generalised Fourier transform, however, in the absence of Lévy jumps). Although this approach offers a great deal of advantage in that it allows us to deal with a generic Lévy measure, it may not be the optimal choice in order to prove the contraction property under minimal assumptions, at least in some specific cases. In order to substantiate this claim, in Section 5.4 we demonstrate numerically that the discretized Picard iteration method converges with the expected rate even when choosing an initial law with infinite first moment. We aim to come back to this point in a future work, where we plan to extend the approach to more general coefficients (not necessarily linear in the state variable) by performing the analysis in the original space as opposed to the Fourier one.

Although the class of MK-V SDEs approximated in this paper is relatively small, we point out that the idea of translating the mean-field SDE into a fixed-point equation on the coefficients space turns out to be effective, and leads to a Picard iteration method that is numerically efficient. Moreover, it allows for generic Lévy jumps in the dynamics. To the best of our knowledge, most numerical methods available in the literature for MK-V equations with Lévy jumps make use of propagation of chaos results and simulation of the related large-particle system, albeit with different integrability assumptions and more involved dependence structure for the Lévy process (see, for example, [1], [13], [9], [10]).

The numerical approach developed here, based on the Fourier transform and Picard iteration for a prototype class of equations, paves the way to handle more general instances of MK-V SDEs. A straightforward extension, which would require minimal modifications of the proofs in our work, would consist in dropping the boundedness assumption on the coefficients w.r.t. the law. This would allow us to consider models with coefficients possessing polynomial growth. We discuss this case in Remark 3.12 below. However, such a relaxation would require additional moment conditions on the initial datum and the Lévy measure. A more general dependence (nonlinear) of the coefficients with respect to the state variable could also be considered by making use of suitable estimates on the density kernels in both pure- and jump-diffusion settings. We sketch these possible generalisations in Remark 2.2 and 2.3 below. Another interesting extension consists in dropping the ellipticity hypothesis on the diffusion coefficient to consider degenerate MK-V diffusions. This would involve a study of time-dependent Hörmander-type conditions. Along with the above extensions, in future works, we also aim to consider the case of common noise, such as common Brownian motion and/or Lévy jumps, for which propagation of chaos results have also been obtained. See, for example, [1] [5], [15], [16].

The rest of the paper is organized as follows: In Section 2, we introduce the MK-V SDE under study. In Section 2.1 we provide some preliminary results on linear SDEs with jumps. In Section 3, we prove the contraction property of the functional mapping which provides the existence and uniqueness result of the solution. We prove the contraction property for L^1 coefficients in Section 3.1. We introduce the damping method and prove the contraction property for bounded functions in Section 3.2. The discretized Picard iteration scheme to obtain solution estimates and its rate of convergence is presented in Section 4. Numerical experiments to validate our theoretical results are illustrated in Section 5. The proofs of intermediate lemmas and a priori estimates are relegated to Appendix A and B respectively.

2 Linear MK-V SDEs with jumps

Let us consider the following MK-V SDE on $\mathbb{R}^d$

$$\begin{cases} \mathrm{d}X_t = \mathbb{E}[a(X_t)x + b(X_t)]|_{x=X_t}\mathrm{d}t + \mathrm{d}L_t, \quad t > 0, \\ X_0 = Y, \end{cases} \tag{1}$$

where Y is a $\mathbb{R}^d$-valued random variable, $a : \mathbb{R}^d \to \mathcal{M}^{d\times d}$, $b : \mathbb{R}^d \to \mathbb{R}^d$, and L is a d-dimensional Lévy process with characteristic triplet $\big(0, \theta, \nu(\mathrm{d}y)\big)$, meaning that

$$L_t = \sigma W_t + \int_0^t \int_{|y|\geq 1} yN(\mathrm{d}s, \mathrm{d}y) + \int_0^t \int_{|y|<1} y\big(N(\mathrm{d}s, \mathrm{d}y) - \nu(\mathrm{d}y)\mathrm{d}s\big). \tag{2}$$

In the above, $N(\mathrm{d}s, \mathrm{d}y)$ is a d-dimensional Poisson measure with compensator $\nu(\mathrm{d}y)\mathrm{d}s$, W is a q-dimensional Brownian motion, and σ is a $d \times q$ matrix such that $\sigma\sigma^\top = \theta$ is positive definite.

We observe that solving (1) up to time $T > 0$ is equivalent to finding measurable functions $\alpha : [0, T] \to \mathcal{M}^{d\times d}$ in the space of square matrices with dimension d, and $\beta : [0, T] \to \mathbb{R}^d$ that solve the following MK-V fixed-point equation

$$(\alpha_t, \beta_t) = \Big(\mathbb{E}\big[a\big(X_t^{(\alpha,\beta)}\big)\big], \mathbb{E}\big[b\big(X_t^{(\alpha,\beta)}\big)\big]\Big) =: \Psi_t(\alpha, \beta), \quad t \in [0, T], \tag{3}$$

where $X^{(\alpha,\beta)}$ denotes the solution to

$$\begin{cases} \mathrm{d}X_t^{(\alpha,\beta)} = \big(\alpha_t X_t^{(\alpha,\beta)} + \beta_t\big)\mathrm{d}t + \mathrm{d}L_t, \quad t \in]0, T], \\ X_0^{(\alpha,\beta)} = Y. \end{cases} \tag{4}$$

Clearly, (α, β) solves (3) if and only if $X^{(\alpha,\beta)}$ solves MK-V SDE (1). In the above, we work with the concept of strong solution of an SDE. Since the distribution of $X_t^{(\alpha,\beta)}$ can be analytically characterized through an explicit characteristic function if $(\alpha, \beta) \in L^\infty([0, T])$, we will look for solutions in this class. Thus, we will look at $\Psi = (\Psi_1, \Psi_2)$ defined in (3) as an operator from[1] $L^\infty([0, T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$ onto itself which is equipped with the family of norms

$$\|\gamma\|_{T,\lambda} := \max\{\|\alpha\|_{T,\lambda}, \|\beta\|_{T,\lambda}\},\ \lambda > 0, \quad \gamma = (\alpha, \beta) \in L^\infty([0, T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d),$$

where

$$\|\alpha\|_{T,\lambda} := \operatorname*{ess\,sup}_{t\in[0,T]} \mathrm{e}^{-\lambda t}|\alpha_t|, \quad \alpha \in L^\infty([0, T] : \mathcal{M}^{d\times d}),$$
$$\|\beta\|_{T,\lambda} := \operatorname*{ess\,sup}_{t\in[0,T]} \mathrm{e}^{-\lambda t}|\beta_t|, \quad \beta \in L^\infty([0, T] : \mathbb{R}^d),$$

Above, and throughout the paper, $|\cdot|$ denotes the spectral norm when it is applied to a matrix in $\mathcal{M}^{d\times d}$, and the Euclidean norm when applied to a vector of $\mathbb{R}^d$. Hereafter, we denote the marginal law of the solution to (4) by $\mu_{X_t^{(\alpha,\beta)}}$.

Remark 2.1. We consider a weighted norm on $L^\infty([0, T] : \mathcal{M}^{d\times d})$ and $L^\infty([0, T] : \mathbb{R}^d)$ to prove the contraction property of the map Ψ for an arbitrarily large terminal time $T > 0$. Indeed, the proofs of our contraction theorems (for example, Theorem 3.7) rely heavily on the upper bounds in Lemma 2.5 and Lemma 2.6, which are unbounded as functions of T. It then seems difficult to obtain a contraction result with respect to a norm that is independent of T, such as the plain L^∞ norm. Instead, due to exponential decay of the weight function $\mathrm{e}^{-\lambda t}$, we can prove that Ψ is a contraction on $L^\infty([0, T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{T,\lambda}$ for a suitably large $\lambda = \lambda(T)$. Note that this kind of exponential weighting is also widely employed in the literature concerning backward SDEs, to prove existence and uniqueness results, as well as estimates of the solutions.

In the next remarks we sketch some possible extensions of the above framework, which could be covered by suitable generalisations of our results.

Remark 2.2. The equivalence between (1) and (3) allows us to characterize the solution to (1) by means of two univariate functions. This approach does not rely on the affine structure of the drift as a function of the state variable x. Indeed, one could consider a more general dynamics

$$\begin{cases} \mathrm{d}X_t = \mathbb{E}\Big[\sum_{i=1}^n a_i(X_t) f_i(x)\Big]\Big|_{x=X_t} \mathrm{d}t + \mathrm{d}L_t, \quad t > 0, \\ X_0 = Y, \end{cases} \tag{5}$$

with $f_i : \mathbb{R}^d \to \mathbb{R}^d$. Again, we have that (5) is satisfied if and only if

$$\begin{aligned}(\alpha_{1,t}, \cdots, \alpha_{n,t}) &= \Big(\mathbb{E}\big[a_1\big(X_t^{(\alpha_1,\cdots,\alpha_n)}\big)\big], \cdots, \mathbb{E}\big[a_n\big(X_t^{(\alpha_1,\cdots,\alpha_n)}\big)\big]\Big) \\ &=: \Psi_t(\alpha_1, \cdots, \alpha_n)\end{aligned}$$

holds, with $X^{(\alpha_1,\cdots,\alpha_n)}$ denoting the solution to

$$\begin{cases} \mathrm{d}X_t^{(\alpha_1,\cdots,\alpha_n)} = \Big(\sum_{i=1}^n \alpha_{i,t} f_i\big(X_t^{(\alpha_1,\cdots,\alpha_n)}\big)\Big)\mathrm{d}t + \mathrm{d}L_t, \quad t \in]0, T], \\ X_0^{(\alpha_1,\cdots,\alpha_n)} = Y. \end{cases} \tag{6}$$

[1] $L^\infty([0, T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$ denotes the space of $L^\infty([0, T])$ functions with values on $\mathcal{M}^{d\times d} \times \mathbb{R}^d$.

This framework includes, for instance, the MK-V equation associated to the stochastic Kuramoto model, that is,

$$\begin{cases} \mathrm{d}X_t = \mathbb{E}\big[\sin X_t \cos x - \cos X_t \sin x\big]\big|_{x=X_t}\mathrm{d}t + \mathrm{d}L_t, \quad t > 0, \\ X_0 = Y, \end{cases}$$

However, the distribution of the solution to the Markovian SDE (6) cannot not be analytically characterized in general. Therefore, one would have to resort to theoretical upper bounds on the transition density in order to prove the contraction property of the map Ψ, and to some approximations of such density, or of its Fourier transform, in order to implement the Picard iteration algorithm. For instance, the analytical approximations in [17], [18] could serve this purpose.

Another extension would consist in adding MK-V interactions to the Lévy part of (5). In this case, one could stick to the affine dynamics in order to analytically characterize the characteristic function of the related Markovian systems, or resort to suitable estimates and approximations in the general case.

Remark 2.3. An even further generalization would be to consider a general drift coefficient, that is,

$$\begin{cases} \mathrm{d}X_t = \mathbb{E}[a(X_t, x)]|_{x=X_t}\mathrm{d}t + \mathrm{d}L_t, \quad t > 0, \\ X_0 = Y, \end{cases} \tag{7}$$

In this case, one could still re-write the MK-V equation as a fixed-point equation in the space of coefficient functions. However, this time, the unknown coefficient would be a function defined on $[0, T] \times \mathbb{R}^d$. One could still aim to prove a contraction property, under appropriate assumptions, by means of suitable upper/lower bounds fo the transition density of the related Markovian systems. However, a numerical method based on the Picard iteration would not be effective in this case. Instead, one could first approximate (7) with a system in the form of (5), and then proceed to determine the n time-dependent functions that determine its solution.

2.1 Preliminaries on linear SDEs with jumps

For any probability measure μ on $\mathcal{B}(\mathbb{R}^d)$, we define its Fourier transform as

$$\hat{\mu}(\eta) := \int_{\mathbb{R}^d} \mathrm{e}^{i\langle \eta, x\rangle}\mu(\mathrm{d}x), \quad \eta \in \mathbb{R}^d.$$

Analogously, for any function $f \in L^1(\mathbb{R}^d)$ we define its Fourier transform as

$$\hat{f}(\eta) := \int_{\mathbb{R}^d} \mathrm{e}^{i\langle \eta, x\rangle} f(x)\mathrm{d}x, \quad \eta \in \mathbb{R}^d,$$

and, the inverse Fourier transform as

$$f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-i\langle \eta, x\rangle}\hat{f}(\eta)\mathrm{d}\eta, \quad x \in \mathbb{R}^d.$$

The inverse Fourier transform makes sense if $\hat{f} \in L^1(\mathbb{R}^d)$. In general, we will resort to Plancherel's theorem which states that if $f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then $\hat{f}, \hat{g} \in L^2(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{f}(\eta)\hat{g}(-\eta)\mathrm{d}\eta.$$

In Lemma 2.4 below, we have the first preliminary result regarding the Fourier transform of $\mu_{X_t^{(\alpha,\beta)}}$. It is a standard result, but due to the lack of a precise statement in the literature, we provide its proof in Appendix A.

Lemma 2.4. *For any $T>0$ and $(\alpha,\beta)\in L^\infty([0,T]:\mathcal{M}^{d\times d}\times\mathbb{R}^d)$, the Fourier transform of the law $\mu_{X_t^{(\alpha,\beta)}}$ is given as*

$$\hat{\mu}_{X_t^{(\alpha,\beta)}}(\eta)=\exp\Big(-\frac{1}{2}\langle\eta,C_t^\alpha\eta\rangle+i\,\langle\eta,m_t^{\alpha,\beta}\rangle+n_t^\alpha(\eta)\Big)\hat{\mu}_Y\big((\Phi_{0,t}^\alpha)^\top\eta\big), \tag{8}$$

where

$$m_t^{\alpha,\beta}=\int_0^t\Phi_{s,t}^\alpha\beta_s\mathrm{d}s,\qquad C_t^\alpha=\int_0^t\Phi_{s,t}^\alpha\,\sigma\sigma^\top(\Phi_{s,t}^\alpha)^\top\,\mathrm{d}s, \tag{9}$$

$$n_t^\alpha(\eta)=\int_0^t\int_{\mathbb{R}^d}\Big(\mathrm{e}^{i\langle\eta,\Phi_{s,t}^\alpha y\rangle}-1-\mathbf{1}_{\{|y|<1\}}i\big\langle\eta,\Phi_{s,t}^\alpha y\big\rangle\Big)\nu(\mathrm{d}y)\mathrm{d}s. \tag{10}$$

In the above, $\Phi_{s,\cdot}^\alpha:[s,T]\to\mathcal{M}^{d\times d}$ is the unique solution of

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\Phi_{s,t}^\alpha=\alpha_t\Phi_{s,t}^\alpha, & s<t\le T,\\ \Phi_{s,s}^\alpha=I_d, & \end{cases} \tag{11}$$

which is an absolutely continuous function such that $\Phi_{s,t}^\alpha=I_d+\int_s^t\alpha_u\Phi_{s,u}^\alpha\mathrm{d}u$, for $0\le t\le T$ and μ_Y is the law of the initial datum $Y=X_0^{(\alpha,\beta)}$.

We also have the following estimates on the quantities arising in $\hat{\mu}_{X_t^{(\alpha,\beta)}}$, which will be used repeatedly throughout the paper. The first estimate (12) for Φ can be obtained immediately from Grönwall's lemma and the definition of Φ in (11).The other results can be deduced similarly and the details are postponed until Appendix A.

Lemma 2.5. *For any $T>0, 0<\delta<1$ and $(\alpha,\beta)\in L^\infty([0,T]:\mathcal{M}^{d\times d}\times\mathbb{R}^d)$, we have*

$$|\Phi_{s,t}^\alpha|\le\mathrm{e}^{T\|\alpha\|_{T,0}}, \tag{12}$$

$$|C_t^\alpha|\le t|\theta|\mathrm{e}^{2T\|\alpha\|_{T,0}}, \tag{13}$$

$$|m_t^{\alpha,\beta}|\le t\mathrm{e}^{T\|\alpha\|_{T,0}}\|\beta\|_{T,0}, \tag{14}$$

$$|n_t^\alpha(\eta)|\le 2t\mathrm{e}^{2T\|\alpha\|_{T,0}}\bigg(|\eta|^2\int_{|y|<\delta}|y|^2\nu(\mathrm{d}y)+(|\eta|+1)\int_{|y|\ge\delta}\nu(\mathrm{d}y)\bigg), \tag{15}$$

$$|\langle\eta,C_t^\alpha\eta\rangle|\ge\lambda_{min}(\theta)\mathrm{e}^{-2T\|\alpha\|_{T,0}}t|\eta|^2, \tag{16}$$

for any $0\le s\le t\le T$ and $\eta\in\mathbb{R}^d$. In the above, $\lambda_{min}(\theta)$ denotes the minimum eigenvalue of θ.

Lemma 2.6. *For any $T>0$ and $(\alpha,\beta)\in L^\infty([0,T]:\mathcal{M}^{d\times d}\times\mathbb{R}^d)$, we have that*

$$|\Phi_{s,t'}^\alpha-\Phi_{s,t}^\alpha|\le(t'-t)\|\alpha\|_{T,0}\mathrm{e}^{T\|\alpha\|_{T,0}}, \tag{17}$$

$$|C_{t'}^\alpha-C_t^\alpha|\le(t'-t)|\theta|\mathrm{e}^{4T\|\alpha\|_{T,0}}, \tag{18}$$

$$|m_{t'}^{\alpha,\beta}-m_t^{\alpha,\beta}|\le(t'-t)\|\beta\|_{T,0}\mathrm{e}^{2T\|\alpha\|_{T,0}}, \tag{19}$$

(and in addition, if Assumption 3.5 holds)

$$|n_{t'}^\alpha(\eta)-n_t^\alpha(\eta)|\le 2(t'-t)\mathrm{e}^{3T\|\alpha\|_{T,0}}\big(|\eta|^2+|\eta|+1\big)\int_{\mathbb{R}^d}|y|(1\wedge|y|)\nu(\mathrm{d}y), \tag{20}$$

for any $0\le s\le t\le t'\le T$ and $\eta\in\mathbb{R}^d$.

3 Contraction property in the Fourier space

In this section we continue to suppose that $a,b\in L^\infty(\mathbb{R}^d)$. In Section 3.1, we additionally assume that $a,b\in L^1(\mathbb{R}^d)$ whereas in Section 3.2, we handle the case when $a,b\notin L^1(\mathbb{R}^d)$ using the 'damping' method.

3.1 The case $a, b \in L^1(\mathbb{R}^d)$

In this section, we perform the analysis of the fixed-point equation (3) under the simplified assumption that the coefficients a, b are integrable and prove that the function Ψ_t defined in equation (3) is a contraction map.

Assumption 3.1. The coefficients $a, b \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

Assumption 3.2. The matrix $\theta = \sigma\sigma^\top$ is positive definite, that is, its smallest eigenvalue $\lambda_{\min}(\theta)$ is positive.

Proposition 3.3. *[Fourier representation] Under Assumptions 3.1 and 3.2, for any $T > 0$ and $(\alpha, \beta) \in L^\infty([0,T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$, we have that*

$$\Psi_{1,t}(\alpha,\beta) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{a}(\eta)\exp\left(-\frac{1}{2}\langle \eta, C_t^\alpha \eta\rangle - i\,\langle \eta, m_t^{\alpha,\beta}\rangle + n_t^\alpha(-\eta)\right)\hat{\mu}_Y\left(-(\Phi_{0,t}^\alpha)^\top\eta\right)\mathrm{d}\eta, \quad (21)$$

$$\Psi_{2,t}(\alpha,\beta) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{b}(\eta)\exp\left(-\frac{1}{2}\langle \eta, C_t^\alpha \eta\rangle - i\,\langle \eta, m_t^{\alpha,\beta}\rangle + n_t^\alpha(-\eta)\right)\hat{\mu}_Y\left(-(\Phi_{0,t}^\alpha)^\top\eta\right)\mathrm{d}\eta, \quad (22)$$

for any $t \in [0,T]$, where $C^\alpha, m^{\alpha,\beta}, n^\alpha, \Phi^\alpha$ are as defined in (9)-(10)-(11).

Proof. By (8), together with (16) and (15) with δ suitably small, we have that $\hat{\mu}_{X_t^{(\alpha,\beta)}} \in L^p(\mathbb{R}^d)$ for any $p \in \mathbb{N}$. In particular $\hat{\mu}_{X_t^{(\alpha,\beta)}} \in L^1(\mathbb{R}^d)$, and thus, $\mu_{X_t^{(\alpha,\beta)}}$ has a density $f_{X_t^{(\alpha,\beta)}}$ that is given by

$$f_{X_t^{(\alpha,\beta)}}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-i\langle x,\eta\rangle}\hat{\mu}_{X_t^{(\alpha,\beta)}}(\eta)\mathrm{d}\eta, \quad x \in \mathbb{R}^d. \quad (23)$$

This yields

$$\Psi_{1,t}(\alpha,\beta) = \int_{\mathbb{R}^d} a(x)\mu_{X_t^{(\alpha,\beta)}}(\mathrm{d}x) = \int_{\mathbb{R}^d} a(x) f_{X_t^{(\alpha,\beta)}}(x)\mathrm{d}x. \quad (24)$$

Moreover, since $f_{X_t^{(\alpha,\beta)}}(x) = \hat{\hat{\mu}}_{X_t^{(\alpha,\beta)}}(-x)$ and $\hat{\mu}_{X_t^{(\alpha,\beta)}} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we also have that $f_{X_t^{(\alpha,\beta)}} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Also, by Assumption 3.1, $a \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Therefore, by employing Plancherel's theorem in (24), we get that

$$\Psi_{1,t}(\alpha,\beta) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{a}(\eta)\hat{\mu}_{X_t^{(\alpha,\beta)}}(-\eta)\mathrm{d}\eta.$$

Finally, applying the result in Lemma 2.4 concludes the proof of (21). The proof of (22) is identical. □

Remark 3.4. Under Assumption 3.1 and 3.2, the function $t \mapsto \Psi_t(\alpha,\beta)$ is continuous on $]0,T]$ for any $(\alpha,\beta) \in L^\infty([0,T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$. This can be easily verified by using Proposition 3.3 together with Lebesgue dominated convergence theorem.

Assumption 3.5. The Lévy measure ν is such that

$$\bar{n} := \int_{\mathbb{R}^d} |y|(1 \wedge |y|)\nu(\mathrm{d}y) < \infty, \quad (25)$$

and the initial datum Y is such that

$$\mathbb{E}[|Y|] < \infty.$$

Assumption 3.6. The Fourier transforms $\hat{a}, \hat{b} \in L^{(d+1)/d}(\mathbb{R}^d)$.

Assumptions 3.5 and 3.6 are necessary in order to prove the contraction property for the map Ψ in the Fourier space. We note that Assumption 3.6 is always satisfied for $d = 1$ in light of Plancherel's theorem ($a, b \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ imply $a, b \in L^2(\mathbb{R}^d)$). In general, a sufficient condition for Assumption 3.6 to hold is $\hat{a}, \hat{b} \in L^1(\mathbb{R}^d)$ ($\hat{a}, \hat{b}$ are also bounded). The latter is satisfied, for instance, if

$a, b \in C^2(\mathbb{R}^d)$ with derivatives in $L^1(\mathbb{R}^d)$, or if a, b are in the first-order Sobolev space $H^1(\mathbb{R}^d)$. However, we claim that Assumption 3.6 is satisfied by many natural instances of $a, b \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ that do not satisfy the sufficient conditions mentioned above. We also point out that neither one of Assumptions 3.5 or 3.6 are necessary to prove Proposition 3.3, which can be used for computational purposes as long as $a, b \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

Theorem 3.7. *[Contraction property] Suppose that Assumption 3.1, 3.2, 3.5 and 3.6 hold. Then, for any $T, c > 0$, there exists $\lambda > 0$, only dependent on c, T, θ, $\|a\|_{L^\infty}$, $\|b\|_{L^\infty}$, $\frac{1}{(2\pi)^d}\|\hat{a}\|_{L^{(d+1)/d}}$, $\frac{1}{(2\pi)^d}\|\hat{b}\|_{L^{(d+1)/d}}$, $\nu(\mathrm{d}y)$ and $\mathbb{E}[|Y|]$, such that*

$$\|\Psi(\alpha, \beta) - \Psi(\alpha', \beta')\|_{T,\lambda} \leq c\|(\alpha - \alpha', \beta - \beta')\|_{T,\lambda}, \tag{26}$$

for any $(\alpha, \beta), (\alpha', \beta') \in L^\infty([0, T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$ with $\|(\alpha, \beta)\|_{T,0}, \|(\alpha', \beta')\|_{T,0} \leq \|a\|_{L^\infty} + \|b\|_{L^\infty}$.

The proof of Theorem 3.7 follows below after the proof of Corollary 3.8.

Corollary 3.8. *Under the assumptions of Theorem 3.7, for any $T > 0$, we have that Ψ is a contraction map from the set*

$$B = \left\{(\alpha, \beta) \in L^\infty([0, T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d) : \|(\alpha, \beta)\|_{T,0} \leq \|a\|_{L^\infty} + \|b\|_{L^\infty}\right\}$$

onto itself, with respect to the norm $\|\cdot\|_{T,\lambda}$, with λ as in Theorem 3.7. In particular, there exists a unique solution $(\bar{\alpha}, \bar{\beta})$ in $L^\infty([0, T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$ to the MK-V fixed-point equation (3), and it is continuous on $]0, T]$.

Proof. The fact that $a, b \in L^\infty(\mathbb{R}^d)$, and that $X_t^{(\alpha,\beta)}$ has a density for any $t \in]0, T]$ and $(\alpha, \beta) \in L^\infty([0, T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$, imply that $\Psi(\alpha, \beta) \in B$. Therefore, as a result of Theorem 3.7, Ψ is a contraction from B onto itself, and (3) has a unique solution in $L^\infty([0, T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$, which belongs to B. The continuity of the solution is immediate from Remark 3.4. $\square$

Remark 3.9. It is worth noticing that under the assumptions of Theorem 3.7, which include $a, b \in L^\infty(\mathbb{R}^d)$, it is not excluded that the MK-V fixed-point equation (3) has a solution that does not belong to $L^\infty([0, T])$. This can happen provided that the solution to (1) does not have a density. Indeed, if $X_t^{(\alpha,\beta)}$ has no density, then it is not possible to establish a priori that $\Psi(\alpha, \beta)$ belongs to B. However, by reinforcing Assumption 3.1 which requires a, b to be pointwise bounded instead of simply in $L^\infty(\mathbb{R}^d)$, the continuous solution $(\bar{\alpha}, \bar{\beta})$ in Corollary 3.8 is the only possible solution to (3). In fact, the pointwise boundedness of a, b implies that $\Psi(\alpha, \beta) \in B$ for any choice of $(\alpha, \beta) : [0, T] \to \mathcal{M}^{d\times d} \times \mathbb{R}^d$ such that (4) has a solution. Thus, a solution to (3) necessarily belongs to B and (by Remark 3.4) is continuous on $]0, T]$. Of course, in many natural instances of (1), which fall outside the hypotheses of Theorem 3.7, we could have maps $t \mapsto \mathbb{E}[a(X_t)]$ and/or $t \mapsto \mathbb{E}[b(X_t)]$ failing to be in $L^\infty([0, T])$ without this having any relation to there not being a density of X_t. For instance, the expectation could explode in finite time due to a or b not being in $L^\infty(\mathbb{R}^d)$. Such an example is given by $\mathrm{d}X_t = \mathbb{E}[X_t]X_t\mathrm{d}t + \mathrm{d}B_t$, whose solution up to explostion time $c_0 = 1/\mathbb{E}[X_0]$ is $X_t = X_0 \mathrm{e}^{\int_0^t (c_0 - s)^{-1}\mathrm{d}s} \mathrm{e}^{B_t - t/2}$. In this case we have $\mathbb{E}[X_t] = 1/(c_0 - t)$.

The proof of Theorem 3.7 is preceded by the following a priori estimates, whose proofs are postponed to Appendix B.

Lemma 3.10. *Suppose that Assumption 3.1 and 3.5 hold. For any $T > 0$, and for any $(\alpha, \beta), (\alpha', \beta') \in$*

$L^\infty([0,T]:\mathcal{M}^{d\times d}\times\mathbb{R}^d)$ with $\|\alpha\|_{T,0},\|\alpha'\|_{T,0}\le\|a\|_{L^\infty(\mathbb{R}^d)}$ and $\|\beta\|_{T,0}\le\|b\|_{L^\infty(\mathbb{R}^d)}$, it holds that

$$\big|\langle\eta,C_t^\alpha\eta\rangle-\langle\eta,C_t^{\alpha'}\eta\rangle\big|\le\kappa\,t|\eta|^2\int_0^t|\alpha_u-\alpha'_u|\mathrm{d}u, \tag{27}$$

$$\big|\langle\eta,m_t^{\alpha,\beta}\rangle-\langle\eta,m_t^{\alpha',\beta}\rangle\big|\le\kappa\,t|\eta|\int_0^t|\alpha_u-\alpha'_u|\mathrm{d}u, \tag{28}$$

$$\big|\langle\eta,m_t^{\alpha,\beta}\rangle-\langle\eta,m_t^{\alpha,\beta'}\rangle\big|\le\kappa\,|\eta|\int_0^t|\beta_u-\beta'_u|\mathrm{d}u, \tag{29}$$

$$\big|\big((\Phi_{s,t}^\alpha)^\top-(\Phi_{s,t}^{\alpha'})^\top\big)\eta\big|\le\kappa\,|\eta|\int_0^t|\alpha_u-\alpha'_u|\mathrm{d}u, \tag{30}$$

$$\big|n_t^\alpha(\eta)-n_t^{\alpha'}(\eta)\big|\le\kappa\,t|\eta|(1+|\eta|)\int_0^t|\alpha_u-\alpha'_u|\mathrm{d}u, \tag{31}$$

for any $0\le s\le t\le T,\eta\in\mathbb{R}^d$, where κ is a positive constant that depends only on $T,\theta,\|a\|_{L^\infty}$, $\|b\|_{L^\infty}$, and $\nu(\mathrm{d}y)$.

We are now ready to prove Theorem 3.7.

Proof of Theorem 3.7. Throughout this proof, we denote by κ any positive constant that depends at most on T, θ, $\|a\|_{L^\infty}$, $\|b\|_{L^\infty}$, $\frac{1}{(2\pi)^d}\|\hat{a}\|_{L^{(d+1)/d}}$, $\frac{1}{(2\pi)^d}\|\hat{b}\|_{L^{(d+1)/d}}$, $\nu(\mathrm{d}y)$ and $\mathbb{E}[|Y|]$.
Step 1: For any $0\le t\le T$, we first prove that

$$\begin{aligned}&|\Psi_{1,t}(\alpha,\beta)-\Psi_{1,t}(\alpha',\beta')|+|\Psi_{2,t}(\alpha,\beta)-\Psi_{2,t}(\alpha',\beta')|\\&\le\kappa\bigg(\int_0^t|\alpha_s-\alpha'_s|\mathrm{d}s+\int_0^t|\beta_s-\beta'_s|\mathrm{d}s\bigg)\int_{\mathbb{R}^d}\big(|\hat{a}(\eta)|+|\hat{b}(\eta)|\big)\Big(t|\eta|^2+|\eta|\big(t+1\big)\Big)\mathrm{e}^{-\frac{t|\eta|^2}{2\kappa}}\mathrm{d}\eta.\end{aligned} \tag{32}$$

By Proposition 3.3 and by triangular inequality, we obtain the following

$$|\Psi_{1,t}(\alpha,\beta)-\Psi_{1,t}(\alpha',\beta')|\le\frac{1}{(2\pi)^d}\sum_{i=1}^5 I_i,$$

where

$$\begin{aligned}I_1&=\int_{\mathbb{R}^d}|\hat{a}(\eta)|\left|\mathrm{e}^{-\frac12\langle\eta,C_t^\alpha\eta\rangle}-\mathrm{e}^{-\frac12\langle\eta,C_t^{\alpha'}\eta\rangle}\right|\left|\mathrm{e}^{-i\langle\eta,m_t^{\alpha,\beta}\rangle+n_t^\alpha(-\eta)}\right|\left|\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big)\right|\mathrm{d}\eta,\\I_2&=\int_{\mathbb{R}^d}|\hat{a}(\eta)|\left|\mathrm{e}^{-\frac12\langle\eta,C_t^{\alpha'}\eta\rangle+n_t^\alpha(-\eta)}\right|\left|\mathrm{e}^{-i\langle\eta,m_t^{\alpha,\beta}\rangle}-\mathrm{e}^{-i\langle\eta,m_t^{\alpha',\beta}\rangle}\right|\left|\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big)\right|\mathrm{d}\eta,\\I_3&=\int_{\mathbb{R}^d}|\hat{a}(\eta)|\left|\mathrm{e}^{-\frac12\langle\eta,C_t^{\alpha'}\eta\rangle+n_t^\alpha(-\eta)}\right|\left|\mathrm{e}^{-i\langle\eta,m_t^{\alpha',\beta}\rangle}-\mathrm{e}^{-i\langle\eta,m_t^{\alpha',\beta'}\rangle}\right|\left|\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big)\right|\mathrm{d}\eta,\\I_4&=\int_{\mathbb{R}^d}|\hat{a}(\eta)|\left|\mathrm{e}^{-\frac12\langle\eta,C_t^{\alpha'}\eta\rangle-i\langle\eta,m_t^{\alpha',\beta'}\rangle}\right|\left|\mathrm{e}^{n_t^\alpha(-\eta)}-\mathrm{e}^{n_t^{\alpha'}(-\eta)}\right|\left|\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big)\right|\mathrm{d}\eta,\\I_5&=\int_{\mathbb{R}^d}|\hat{a}(\eta)|\left|\mathrm{e}^{-\frac12\langle\eta,C_t^{\alpha'}\eta\rangle-i\langle\eta,m_t^{\alpha',\beta'}\rangle+n_t^{\alpha'}(-\eta)}\right|\left|\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big)-\hat{\mu}_Y\big(-(\Phi_{0,t}^{\alpha'})^\top\eta\big)\right|\mathrm{d}\eta.\end{aligned}$$

By (27), combined with (16), we obtain that

$$\left|\mathrm{e}^{-\frac12\langle\eta,C_t^\alpha\eta\rangle}-\mathrm{e}^{-\frac12\langle\eta,C_t^{\alpha'}\eta\rangle}\right|\le\kappa\,t|\eta|^2\mathrm{e}^{-\frac{t|\eta|^2}{2\kappa}}\int_0^t|\alpha_s-\alpha'_s|\mathrm{d}s,$$

while by (15), for any $\varepsilon>0$, we have that

$$\left|\mathrm{e}^{n_t^\alpha(-\eta)}\right|\le\kappa\,\mathrm{e}^{t(\varepsilon|\eta|^2+\kappa|\eta|)}\le\kappa\,\mathrm{e}^{2\varepsilon|\eta|^2}. \tag{33}$$

Taking ε suitably small in (33), yields that

$$I_1 \le \kappa \int_{\mathbb{R}^d} |\hat{a}(\eta)|\, t|\eta|^2 \mathrm{e}^{-\frac{t|\eta|^2}{2\kappa}} \mathrm{d}\eta \int_0^t |\alpha_s - \alpha'_s| \mathrm{d}s.$$

Similarly, from (28)–(31) and again from (15)–(16), and the fact that

$$\big|\hat{\mu}_Y\big(-(\Phi^{\alpha}_{0,t})^\top \eta\big) - \hat{\mu}_Y\big(-(\Phi^{\alpha'}_{0,t})^\top \eta\big)\big| \le \big|\big((\Phi^{\alpha}_{0,t})^\top - (\Phi^{\alpha'}_{0,t})^\top\big)\eta\big|\, \mathbb{E}[|Y|],$$

we obtain that

$$\begin{aligned}
I_2 &\le \kappa \int_{\mathbb{R}^d} |\hat{a}(\eta)|\, t|\eta| \mathrm{e}^{-\frac{t|\eta|^2}{2\kappa}} \mathrm{d}\eta \int_0^t |\alpha_s - \alpha'_s| \mathrm{d}s,\\
I_3 &\le \kappa \int_{\mathbb{R}^d} |\hat{a}(\eta)|\, |\eta| \mathrm{e}^{-\frac{t|\eta|^2}{2\kappa}} \mathrm{d}\eta \int_0^t |\beta_s - \beta'_s| \mathrm{d}s,\\
I_4 &\le \kappa \int_{\mathbb{R}^d} |\hat{a}(\eta)|\, t|\eta|(1+|\eta|) \mathrm{e}^{-\frac{t|\eta|^2}{2\kappa}} \mathrm{d}\eta \int_0^t |\alpha_s - \alpha'_s| \mathrm{d}s,\\
I_5 &\le \kappa \int_{\mathbb{R}^d} |\hat{a}(\eta)|\, |\eta| \mathrm{e}^{-\frac{t|\eta|^2}{2\kappa}} \mathrm{d}\eta \int_0^t |\alpha_s - \alpha'_s| \mathrm{d}s.
\end{aligned}$$

These, together with analogous estimates for $|\Psi_{2,t}(\alpha,\beta) - \Psi_{2,t}(\alpha',\beta')|$, prove (32).
Step 2: Combining (32) with the standard Gaussian estimate (see Lemma A.1. in [8])

$$|\eta|^\gamma \mathrm{e}^{-\frac{\rho|\eta|^2}{2}} \le \Big(\frac{2\gamma}{\mathrm{e}}\Big)^{\frac{\gamma}{2}} \rho^{-\frac{\gamma}{2}} \mathrm{e}^{-\frac{\rho|\eta|^2}{4}}, \quad \eta \in \mathbb{R}^d,\ \rho > 0,\ \gamma \ge 0, \tag{34}$$

we obtain that

$$\begin{aligned}
&|\Psi_{1,t}(\alpha,\beta) - \Psi_{1,t}(\alpha',\beta')| + |\Psi_{2,t}(\alpha,\beta) - \Psi_{2,t}(\alpha',\beta')|\\
&\le \frac{\kappa}{\sqrt{t}} \bigg(\int_0^t |\alpha_s - \alpha'_s| \mathrm{d}s + \int_0^t |\beta_s - \beta'_s| \mathrm{d}s\bigg) \bigg(\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (|\hat{a}(\eta)| + |\hat{b}(\eta)|) \mathrm{e}^{-\frac{t|\eta|^2}{4\kappa}} \mathrm{d}\eta\bigg).
\end{aligned} \tag{35}$$

Next we note that, by Hölder's inequality,

$$\int_{|\eta|\le 1} (|\hat{a}(\eta)| + |\hat{b}(\eta)|) \mathrm{e}^{-\frac{t|\eta|^2}{4\kappa}} \mathrm{d}\eta \le \kappa\big(\|\hat{a}\|_{L^{(d+1)/d}} + \|\hat{b}\|_{L^{(d+1)/d}}\big). \tag{36}$$

Moreover, by setting $\gamma = 1 - \frac{1}{2(d+1)}$ and $\rho = \frac{t}{2\kappa}$ in (34), we obtain

$$\begin{aligned}
\int_{|\eta|>1} (|\hat{a}(\eta)| + |\hat{b}(\eta)|) \mathrm{e}^{-\frac{t|\eta|^2}{4\kappa}} \mathrm{d}\eta &= \int_{|\eta|>1} \frac{|\hat{a}(\eta)| + |\hat{b}(\eta)|}{|\eta|^{1-\frac{1}{2(d+1)}}} |\eta|^{1-\frac{1}{2(d+1)}} \mathrm{e}^{-\frac{t|\eta|^2}{4\kappa}} \mathrm{d}\eta\\
&\le \kappa\, t^{-\frac{1}{2}+\frac{1}{4+4d}} \int_{|\eta|>1} \frac{|\hat{a}(\eta)| + |\hat{b}(\eta)|}{|\eta|^{1-\frac{1}{2(d+1)}}} \mathrm{e}^{-\frac{t|\eta|^2}{8\kappa}} \mathrm{d}\eta\\
&\le \kappa\, t^{-\frac{1}{2}+\frac{1}{4+4d}} \big(\|\hat{a}\|_{L^{(d+1)/d}} + \|\hat{b}\|_{L^{(d+1)/d}}\big) \bigg(\int_{|\eta|>1} |\eta|^{-(d+\frac{1}{2})} \mathrm{d}\eta\bigg)^{\frac{1}{d+1}}.
\end{aligned} \tag{37}$$

In the above, the final inequality follows by applying Hölder's inequality.

Combining now (35) with (36)-(37) we obtain, for any $t \in [0,T]$,

$$\begin{aligned}
\mathrm{e}^{-\lambda t} |\Psi_{1,t}(\alpha,\beta) - \Psi_{1,t}(\alpha',\beta')| &\le \kappa\, t^{-1+\frac{1}{4+4d}} \int_0^t \mathrm{e}^{-\lambda(t-s)} \mathrm{e}^{-\lambda s} \big(|\alpha_s - \alpha'_s| + |\beta_s - \beta'_s|\big) \mathrm{d}s\\
&\le \big(\|\alpha - \alpha'\|_{T,\lambda} + \|\beta - \beta'\|_{T,\lambda}\big) \frac{\kappa}{\lambda^{\frac{1}{4+4d}}} \frac{(1-\mathrm{e}^{-\lambda t})}{(\lambda t)^{1-\frac{1}{4+4d}}}\\
&\le \big(\|\alpha - \alpha'\|_{T,\lambda} + \|\beta - \beta'\|_{T,\lambda}\big) \kappa\, \lambda^{-\frac{1}{4+4d}}.
\end{aligned}$$

The same estimate holds for $|\Psi_{2,t}(\alpha,\beta) - \Psi_{2,t}(\alpha',\beta')|$ and thus, taking λ suitably large yields the result. $\square$

3.2 The case $a, b \notin L^1(\mathbb{R}^d)$

In order to extend the previous proposition to the case where a, b are not necessarily integrable, we make use of the so-called *damping method*. The intuitive idea behind it is rather simple and can be summarized, loosely, as follows. Assume that we wish to compute the expectation $\mathbb{E}[g(Z)]$ for a given function g that has no Fourier transform (say, $g \in L^\infty$), and for random variable Z whose density f_Z is fast decreasing in the tails. Then, we seek a *damping function* φ such that both $\overline{g}(x) := \varphi(x)g(x)$ and f_Z/φ admit a Fourier transform. The inversion formula then yields

$$\int_{\mathbb{R}} g(x) f_Z(x)\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\overline{g}}(\eta)\,\mathcal{F}(f_Z/\varphi)(-\eta)\mathrm{d}\eta,$$

which is useful as long as we have an explicit expression for $\mathcal{F}(f_Z/\varphi)(-\eta)$. In our case, we choose a damping function for a and b of the type $\varphi(x) = (1+\sum_{j=1}^d x_j^q)^{-1}$, for a suitable even $q \in \mathbb{N}$ such that $a\varphi, b\varphi \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. This choice allows us to take advantage of the Fourier transform properties and compute their Fourier transform as

$$\mathcal{F}(f_Z/\varphi)(-\eta) = \Big(1 + i^q\sum_{j=1}^d \partial_{\eta_j}^q\Big)\hat{f}_Z(-\eta).$$

Hereafter, we set $q = 2\lceil (d+1)/2\rceil$ (the smallest even integer greater or equal than $d+1$) and define:

$$\overline{a}(x) := \frac{a(x)}{1+\sum_{j=1}^d x_j^q}, \qquad \overline{b}(x) := \frac{b(x)}{1+\sum_{j=1}^d x_j^q}, \qquad x \in \mathbb{R}^d. \tag{38}$$

We note that, for this choice of q, the functions $\overline{a}, \overline{b} \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ under the assumption that $a, b \in L^\infty(\mathbb{R}^d)$. We are then able to weaken Assumption 3.1 as following:

Assumption 3.11. The coefficients $a, b \in L^\infty(\mathbb{R}^d)$.

Remark 3.12. Assumption 3.11 could be weakened by requiring $a, b \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ with polynomial growth, provided that the order q of the damping polynomial in (38) is sufficiently high. This would certainly extend the class of models under consideration. However, the contraction property could be proved only for small times, and not up to an arbitrary horizon $T > 0$. This is coherent with the fact that explosions in finite time can occur when the coefficients a, b are unbounded (see for instance the example at the end of Remark 3.9).

To replace Assumption 3.1 with Assumption 3.11, we pay the following cost which is an additional condition on the Lévy measure $\nu(\mathrm{d}y)$ and distribution of the initial datum Y, in order to ensure that the function $x \mapsto \big(1+\sum_{j=1}^d x_j^q\big) f_{X_t^{(\alpha,\beta)}}(x)$ belongs to $L^2(\mathbb{R}^d)$.

Assumption 3.13. For $q = 2\lceil (d+1)/2\rceil$, the Lévy measure ν is such that

$$\bar{n}_q := \int_{|y|\geq 1} |y|^{q+1}\nu(\mathrm{d}y) < \infty, \tag{39}$$

and the initial datum Y is such that

$$\mathbb{E}[|Y|^{q+1}] < \infty. \tag{40}$$

Proposition 3.14. *[Fourier representation] Under Assumption 3.2, 3.11 and 3.13, for any $T > 0$ and $(\alpha, \beta) \in L^\infty([0,T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$, $t \in [0,T]$, we have that*

$$\Psi_{1,t}(\alpha,\beta) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{\bar{a}}(\eta)\,\mathcal{L}\Big(\exp\Big(-\frac{1}{2}\langle\eta, C_t^\alpha\eta\rangle - i\,\langle\eta, m_t^{\alpha,\beta}\rangle + n_t^\alpha(-\eta)\Big)\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big)\Big)\mathrm{d}\eta, \tag{41}$$

$$\Psi_{2,t}(\alpha,\beta) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{\bar{b}}(\eta)\,\mathcal{L}\Big(\exp\Big(-\frac{1}{2}\langle\eta, C_t^\alpha\eta\rangle - i\,\langle\eta, m_t^{\alpha,\beta}\rangle + n_t^\alpha(-\eta)\Big)\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big)\Big)\mathrm{d}\eta, \tag{42}$$

for any $t \in [0,T]$, where $C^\alpha, m^{\alpha,\beta}, n^\alpha, \Phi^\alpha$ are defined as in (9)–(11)*, and $\mathcal{L}$ is the operator defined as*

$$\mathcal{L} := \Big(1 + i^q\sum_{j=1}^d \partial_{\eta_j}^q\Big), \quad q := 2\lceil (d+1)/2\rceil. \tag{43}$$

Remark 3.15. Note that the damping functions in (38) are not the only possible choices. For instance, choosing

$$\bar{a}(x) := \frac{a(x)}{\prod_{j=1}^d(1+x_j^2)}, \qquad \bar{b}(x) := \frac{b(x)}{\prod_{j=1}^d(1+x_j^2)}, \qquad x \in \mathbb{R}^d,$$

and slightly reinforcing Assumption 3.13, Proposition 3.14 would still hold true with $\mathcal{L} = \prod_{j=1}^d(1 + \partial_{\eta_j}^2)$. This choice could be more suitable, sometimes, in order to explicitly compute $\hat{\bar{a}}(\eta)$ and $\hat{\bar{b}}(\eta)$.

Proof. For any $j = 1, \cdots, d$, by employing (12), (39), and the dominated convergence theorem, we obtain that

$$\partial_{\eta_j} n_t^\alpha(\eta) = i\int_0^t \Big(\int_{|y|<1}\big(\Phi_{s,t}^\alpha y\big)_j\Big(\mathrm{e}^{i\langle\eta,\Phi_{s,t}^\alpha y\rangle} - 1\Big)\nu(\mathrm{d}y) + \int_{|y|\geq 1}\big(\Phi_{s,t}^\alpha y\big)_j \mathrm{e}^{i\langle\eta,\Phi_{s,t}^\alpha y\rangle}\nu(\mathrm{d}y)\Big)\mathrm{d}s, \tag{44}$$

$$\partial_{\eta_j}^m n_t^\alpha(\eta) = i^m\int_0^t\int_{\mathbb{R}^d}\big(\Phi_{s,t}^\alpha y\big)_j^m \mathrm{e}^{i\langle\eta,\Phi_{s,t}^\alpha y\rangle}\nu(\mathrm{d}y)\mathrm{d}s, \qquad m = 2,\cdots,q,$$

and

$$|\partial_{\eta_j} n_t^\alpha(\eta)| \leq t\Big(|\eta|\mathrm{e}^{2T\|\alpha\|_{T,0}}\int_{|y|<1}|y|^2\nu(\mathrm{d}y) + \mathrm{e}^{T\|\alpha\|_{T,0}}\int_{|y|\geq 1}|y|\nu(\mathrm{d}y)\Big) \tag{45}$$

$$|\partial_{\eta_j}^m n_t^\alpha(\eta)| \leq t\mathrm{e}^{mT\|\alpha\|_{T,0}}\int_{\mathbb{R}^d}|y|^m\nu(\mathrm{d}y), \qquad m = 2,\cdots,q. \tag{46}$$

Moreover, for any $j = 1, \cdots, d$ and $m = 1, \cdots, q$, by (40) it is straightforward to see that

$$\partial_{\eta_j}^m \hat{\mu}_Y\big((\Phi_{0,t}^\alpha)^\top\eta\big) = i^m\int_{\mathbb{R}^d}\big(\Phi_{0,t}^\alpha y\big)_j^m \mathrm{e}^{i\langle\eta,\Phi_{0,t}^\alpha y\rangle}\mu_Y(\mathrm{d}y),$$

and

$$\big|\partial_{\eta_j}^m \hat{\mu}_Y\big((\Phi_{0,t}^\alpha)^\top\eta\big)\big| \leq \mathrm{e}^{mT\|\alpha\|_{T,0}}E\big[|Y|^m\big]. \tag{47}$$

Combining these estimates with (8), (15) and (16), with δ suitably small, we conclude that

$$\partial_{\eta_j}^m \hat{\mu}_{X_t^{(\alpha,\beta)}} \in L^p(\mathbb{R}^d), \quad j = 1,\cdots,d, \quad m = 1,\cdots,q, \quad p \in \mathbb{N}.$$

In particular, recalling that $\mu_{X_t^{(\alpha,\beta)}}$ has a density $f_{X_t^{(\alpha,\beta)}}$ given by (23), basic properties of the Fourier transform on $L^2(\mathbb{R}^d)$ yield that

$$\mathcal{F}^{-1}\big(i^m\partial_{\eta_j}^m\hat{\mu}_{X_t^{(\alpha,\beta)}}\big)(x) = x_j^m f_{X_t^{(\alpha,\beta)}}(x) \in L^2(\mathbb{R}^d).$$

Assumption 3.11 and (38) also yield that $\bar{a}, \bar{b} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Summing up, we can apply Plancherel's theorem and obtain the following

$$\Psi_{1,t}(\alpha,\beta) = \int_{\mathbb{R}^d} a(x) f_{X_t^{(\alpha,\beta)}}(x)\mathrm{d}x = \int_{\mathbb{R}^d} \bar{a}(x)\Big(1+\sum_{j=1}^d x_j^q\Big) f_{X_t^{(\alpha,\beta)}}(x)\mathrm{d}x$$
$$= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{\bar{a}}(\eta)\, \mathcal{L}\hat{\mu}_{X_t^{(\alpha,\beta)}}(-\eta)\mathrm{d}\eta,$$

which, combined with (8), proves (41). The proof of (42) is identical. □

Assumption 3.16. The Fourier transforms $\hat{\bar{a}}, \hat{\bar{b}} \in L^{(d+1)/d}(\mathbb{R}^d)$.

Remark 3.17. In order to prove the contraction property of the map Ψ in the Fourier space, we were able to relax the assumption that $a, b \in L^1(\mathbb{R}^d)$. However, the assumption that $\hat{\bar{a}}, \hat{\bar{b}} \in L^{(d+1)/d}(\mathbb{R}^d)$ cannot be relaxed. It is necessary in order to handle general Lévy jumps in the dynamics. For only diffusive dynamics, i.e. $L_t = \sigma W_t$ in (2), the contraction property, and thus, the existence and uniqueness for the fixed-point equation (3), can be proved by working in the original space under the sole assumption that $a, b \in L^\infty(\mathbb{R}^d)$. We point out that Assumption 3.16 is always satisfied if $d = 1$ and $a, b \in L^\infty(\mathbb{R}^d)$ (by Plancherel's theorem). Also note that Proposition 3.14 does not rely on $\hat{\bar{a}}, \hat{\bar{b}} \in L^{(d+1)/d}(\mathbb{R}^d)$, and thus the Fourier representation (41)-(42) can be used for computational purposes as long as $a, b \in L^\infty(\mathbb{R}^d)$.

Theorem 3.18. *[Contraction property] Suppose that Assumption 3.2, 3.11, 3.13 and 3.16 hold. Then, for any $T, c > 0$, there exists $\lambda > 0$, only dependent on c, T, θ, $\|a\|_{L^\infty}$, $\|b\|_{L^\infty}$,$\|\hat{\bar{a}}\|_{L^{(d+1)/d}}$, $\|\hat{\bar{b}}\|_{L^{(d+1)/d}}$, $\nu(\mathrm{d}y)$, $\mathbb{E}[|Y|^q]$ (with q as in (43)), and dimension d, such that*

$$\|\Psi(\alpha,\beta) - \Psi(\alpha',\beta')\|_{T,\lambda} \le c\|(\alpha-\alpha', \beta-\beta')\|_{T,\lambda},$$

for any $(\alpha,\beta), (\alpha',\beta') \in L^\infty([0,T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$ with $\|(\alpha,\beta)\|_{T,0}, \|(\alpha',\beta')\|_{T,0} \le \|a\|_{L^\infty} + \|b\|_{L^\infty}$.

Corollary 3.19. *Under the assumptions of Theorem 3.18, the same conclusion as of Corollary 3.8 holds.*

The proof Corollary 3.19 is identical to the proof of its counterpart Corollary 3.8. The proof of Theorem 3.18 is preceded by the following lemma on a priori estimates whose proof is relegated to Appendix B.

Lemma 3.20. *Suppose that Assumption 3.11 and 3.13 hold. For any $T > 0$, and $\alpha, \alpha' \in L^\infty([0,T] : \mathcal{M}^{d\times d})$ with $\|\alpha\|_{T,0}, \|\alpha'\|_{T,0} \le \|a\|_{L^\infty(\mathbb{R}^d)}$, it holds that*

$$\big|\partial^m_{\eta_j} n^\alpha_t(\eta) - \partial^m_{\eta_j} n^{\alpha'}_t(\eta)\big| \le \kappa\, t(|\eta|+1)\int_0^t |\alpha_u - \alpha'_u|\mathrm{d}u, \qquad j = 1,\cdots,d, \quad m = 1,\cdots,q, \tag{48}$$

for any $\eta \in \mathbb{R}^d, 0 \le t \le T$, where κ is a positive constant that depends only on T, $\|a\|_{L^\infty}$, and $\nu(\mathrm{d}y)$.

Poof of Theorem 3.18. Throughout this proof, we denote by κ any positive constant that depends at most on T, θ, $\|a\|_{L^\infty}$, $\|b\|_{L^\infty}$,$\|\hat{\bar{a}}\|_{L^{(d+1)/d}}$, $\|\hat{\bar{b}}\|_{L^{(d+1)/d}}$, $\nu(\mathrm{d}y)$, $\mathbb{E}[|Y|^q]$, and dimension d. It is enough to prove that

$$|\Psi_{1,t}(\alpha,\beta) - \Psi_{1,t}(\alpha',\beta')| + |\Psi_{2,t}(\alpha,\beta) - \Psi_{2,t}(\alpha',\beta')|$$
$$\le \frac{\kappa}{\sqrt{t}}\left(\int_0^t |\alpha_s - \alpha'_s|\mathrm{d}s + \int_0^t |\beta_s - \beta'_s|\mathrm{d}s\right)\left(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\big(|\hat{\bar{a}}(\eta)| + |\hat{\bar{b}}(\eta)|\big)\mathrm{d}\eta\right), \tag{49}$$

for any $0 \le t \le T$. With (49) at hand, the proof can be concluded exactly like the proof of Theorem 3.7.

By Proposition 3.14 together with triangular inequality, we obtain that

$$|\Psi_{1,t}(\alpha,\beta) - \Psi_{1,t}(\alpha',\beta')| \leq \frac{1}{(2\pi)^d}\Big(\sum_{i=1}^{5} I_i + \sum_{i=1}^{5} J_i\Big),$$

where

$$J_1 = \int_{\mathbb{R}^d} |\hat{\hat{a}}(\eta)| \Big|\partial_\eta^q\Big(\Big(\mathrm{e}^{-\frac{1}{2}\langle\eta, C_t^{\alpha}\eta\rangle} - \mathrm{e}^{-\frac{1}{2}\langle\eta, C_t^{\alpha'}\eta\rangle}\Big)\mathrm{e}^{-i\,\langle\eta, m_t^{\alpha,\beta}\rangle + n_t^{\alpha}(-\eta)}\hat{\mu}_Y\big(-(\Phi_{0,t}^{\alpha})^\top\eta\big)\Big)\Big|\,\mathrm{d}\eta,$$
$$J_2 = \int_{\mathbb{R}^d} |\hat{\hat{a}}(\eta)| \Big|\partial_\eta^q\Big(\mathrm{e}^{-\frac{1}{2}\langle\eta, C_t^{\alpha'}\eta\rangle + n_t^{\alpha}(-\eta)}\Big(\mathrm{e}^{-i\,\langle\eta, m_t^{\alpha,\beta}\rangle} - \mathrm{e}^{-i\,\langle\eta, m_t^{\alpha',\beta}\rangle}\Big)\hat{\mu}_Y\big(-(\Phi_{0,t}^{\alpha})^\top\eta\big)\Big)\Big|\,\mathrm{d}\eta,$$
$$J_3 = \int_{\mathbb{R}^d} |\hat{\hat{a}}(\eta)| \Big|\partial_\eta^q\Big(\mathrm{e}^{-\frac{1}{2}\langle\eta, C_t^{\alpha'}\eta\rangle + n_t^{\alpha}(-\eta)}\Big(\mathrm{e}^{-i\,\langle\eta, m_t^{\alpha',\beta}\rangle} - \mathrm{e}^{-i\,\langle\eta, m_t^{\alpha',\beta'}\rangle}\Big)\hat{\mu}_Y\big(-(\Phi_{0,t}^{\alpha})^\top\eta\big)\Big)\Big|\,\mathrm{d}\eta,$$
$$J_4 = \int_{\mathbb{R}^d} |\hat{\hat{a}}(\eta)| \Big|\partial_\eta^q\Big(\mathrm{e}^{-\frac{1}{2}\langle\eta, C_t^{\alpha'}\eta\rangle - i\,\langle\eta, m_t^{\alpha',\beta'}\rangle}\Big(\mathrm{e}^{n_t^{\alpha}(-\eta)} - \mathrm{e}^{n_t^{\alpha'}(-\eta)}\Big)\hat{\mu}_Y\big(-(\Phi_{0,t}^{\alpha})^\top\eta\big)\Big)\Big|\,\mathrm{d}\eta,$$
$$J_5 = \int_{\mathbb{R}^d} |\hat{\hat{a}}(\eta)| \Big|\partial_\eta^q\Big(\mathrm{e}^{-\frac{1}{2}\langle\eta, C_t^{\alpha'}\eta\rangle - i\,\langle\eta, m_t^{\alpha',\beta'}\rangle + n_t^{\alpha'}(-\eta)}\Big(\hat{\mu}_Y\big(-(\Phi_{0,t}^{\alpha})^\top\eta\big) - \hat{\mu}_Y\big(-(\Phi_{0,t}^{\alpha'})^\top\eta\big)\Big)\Big)\Big|\,\mathrm{d}\eta.$$

The terms I_i are like those in the proof of Theorem 3.7 and are obtained by replacing $\hat{a}$ with $\hat{\hat{a}}$, and can be bounded in the same way. Analogous bounds for J_i can be obtained by repeatedly applying the estimates of Lemma 2.5, 3.10 and 3.20, and the estimates (45)-(46)-(47). Eventually, applying (34) yields (49). We omit the details to avoid repeating the arguments from the proof of Theorem 3.7. □

4 Discretized Picard iteration scheme

In principle, we can compute the unique solution $(\bar{\alpha},\bar{\beta})$ to the MK-V fixed-point equation (3) via a Picard iteration scheme. The result of Corollary 3.8 (or Corollary 3.19) together with the Fourier representation (21)-(22) (or (41)-(42)) provide us with a convergent scheme. However, for a given initial point $\gamma^0 = (\alpha^0,\beta^0) \in L^\infty([0,T] : \mathcal{M}^{d\times d}\times\mathbb{R}^d)$, the approximating sequence

$$\gamma^m = (\alpha^m,\beta^m) := \Psi(\gamma^{m-1}) = \Psi(\alpha^{m-1},\beta^{m-1}), \quad m\in\mathbb{N}, \tag{50}$$

cannot be computed explicitly at each step. Although, we do not consider here the effect of the error introduced by numerically approximating the space integral in (21)-(22) (or (41)-(42)), we do analyse the impact of time-discretization of functions $\Phi_{s,t}^{\alpha,\beta}$, C_t^{α}, $m_t^{\alpha,\beta}$ on the convergence rate of the numerical scheme.

For any $\gamma = (\alpha,\beta) \in L^\infty([0,T] : \mathcal{M}^{d\times d}\times\mathbb{R}^d)$ and $n\in\mathbb{N}$, we define the piece-wise constant function $\gamma^{(n)}$ as

$$\gamma_t^{(n)} := \sum_{i=1}^{n} \gamma_{t_i}\mathbf{1}_{[t_{i-1},t_i[}(t), \quad t\in[0,T], \quad \text{where } t_i := \frac{T}{n}i.$$

We also define operator $\Psi^{(n)}$ from $L^\infty([0,T] : \mathcal{M}^{d\times d}\times\mathbb{R}^d)$ onto itself acting as

$$\Psi^{(n)}(\gamma) := \big(\Psi(\gamma)\big)^{(n)}, \quad \gamma\in L^\infty([0,T] : \mathcal{M}^{d\times d}\times\mathbb{R}^d).$$

The map $\Psi^{(n)}$ has to be interpreted as a step-wise approximation of Ψ, and is the map we compute in our Picard iteration scheme. The idea is to repeatedly apply operator $\Psi^{(n)}$ instead of Ψ, in order to take advantage of the fact that $\Phi_{s,t}^{\alpha,\beta}$, C_t^{α}, $m_t^{\alpha,\beta}$ are explicitly computable if α and β are step-functions. Precisely, for a given initial step-function $\gamma^{0,n}\in L^\infty([0,T] : \mathcal{M}^{d\times d}\times\mathbb{R}^d)$, the approximating sequence

$$\gamma^{m,n} = (\alpha^{m,n},\beta^{m,n}) := \Psi^{(n)}(\gamma^{m-1,n}) = \Psi^{(n)}(\alpha^{m-1,n},\beta^{m-1,n}), \quad m\in\mathbb{N}, \tag{51}$$

can be computed explicitly at each step, up to computing a space integral on $\mathbb{R}^d$. This is due to the fact that the solution Φ^α to ODE (11) can be computed explicitly in terms of matrix exponentials whenever α is a piece-wise constant. Hereafter, we assume that the continuous and discretized Picard iterations, defined by (50) and (51) respectively, are both initialized by the same constant function, i.e.,

$$\gamma_t^0 = \gamma_t^{0,n} \equiv \gamma_0 = (\alpha_0, \beta_0) \in \mathcal{M}^{d\times d} \times \mathbb{R}^d, \quad t \in [0,T]. \tag{52}$$

Note that $\gamma^{m,n} \neq (\gamma^m)^{(n)}$ which means that $\gamma^{m,n}$ is not the discretized version of γ^m.

In order to be able to control the error introduced by the time-discretization, we must be able to study the regularity of the function $t \mapsto \Psi_t(\gamma)$ on $[0,T]$. For this purpose, we need to introduce some further assumptions on the coefficients a, b and/or on distribution of the initial datum Y, which are needed to ensure Lipschitz continuity of $\Psi_t(\gamma)$ near $t = 0$.

Assumption 4.1. $\hat{a}, \hat{b}$ and $\hat{\mu}_Y$ satisfy the following conditions

$$\int_{\mathbb{R}^d} \big(|\hat{a}(\eta)| + |\hat{b}(\eta)|\big)|\hat{\mu}_Y(\eta)||\eta|^2 \mathrm{d}\eta < \infty, \qquad \int_{\mathbb{R}^d} \big(|\hat{a}(\eta)| + |\hat{b}(\eta)|\big)|\eta| \mathrm{d}\eta < \infty. \tag{53}$$

Remark 4.2. Note that Assumption 4.1 is related to the regularity of functions a, b and of the distribution of the initial datum Y. For instance, the second condition in (53) is equivalent to requiring that $\hat{a}(\eta)|\eta|$ and $\hat{b}(\eta)|\eta|$ belong to $L^1(\mathbb{R}^d)$, which implies that a, b are continuously differentiable. Also, the first condition in (53) is satisfied if either the functions $\hat{a}(\eta)|\eta|^2$, $\hat{b}(\eta)|\eta|^2$, or the function $\hat{\mu}_Y(\eta)|\eta|^2$ belong to $L^1(\mathbb{R}^d)$, which in turn is satisfied if the coefficients a, b, or density of the initial datum Y are $d+3$ times continuously differentiable with derivatives in $L^1(\mathbb{R}^d)$. Alternatively, the first condition in (53) is also ensured if $\hat{a}(\eta)|\eta|$, $\hat{b}(\eta)|\eta|$ and $\hat{\mu}_Y(\eta)|\eta|$ all belong to $L^2(\mathbb{R}^d)$, which in turn is ensured by requiring a, b and the density of Y belong to the first-order Sobolev space $H^1(\mathbb{R}^d)$. All these conditions seem rather strong, but we claim that they are not necessary in many particular cases. Once again we emphasise that this is the cost we incur for carrying out the analysis in the Fourier space, which enables us to deal with general Lévy measures.

When working under the assumptions of Section 3.2 (a, b only in $L^\infty(\mathbb{R}^d)$), we will need instead the following additional assumptions.

Assumption 4.3. $\hat{\bar{a}}, \hat{\bar{b}}$ and $\hat{\mu}_Y$ satisfy the following conditions

$$\int_{\mathbb{R}^d} \big(|\hat{\bar{a}}(\eta)| + |\hat{\bar{b}}(\eta)|\big)|\hat{\mu}_Y(\eta)||\eta|^2 \mathrm{d}\eta < \infty, \qquad \int_{\mathbb{R}^d} \big(|\hat{\bar{a}}(\eta)| + |\hat{\bar{b}}(\eta)|\big)|\eta| \mathrm{d}\eta < \infty,$$

where $\bar{a}$ and $\bar{b}$ are as defined in (38).

Remark 4.4. Note that Assumption 4.1 and 4.3 imply Assumption 3.6 and 3.16, respectively.

We now state the two main results of this section.

Theorem 4.5. *Let $(\gamma^{m,n})_{m,n\in\mathbb{N}}$ be the sequence as defined by (51)-(52). Suppose that Assumption 3.1, 3.2, 3.5 and 4.1 also hold. For any $T > 0, \gamma_0 = \big(\alpha_0, \beta_0\big) \in \mathcal{M}^{d\times d} \times \mathbb{R}^d$ with $\max\{|\alpha_0|, |\beta_0|\} \leq \|a\|_{L^\infty} + \|b\|_{L^\infty}$, there exist $\lambda, \kappa > 0$, only dependent on T, θ, $\nu(\mathrm{d}y)$, Y and the coefficients a, b, such that*

$$\|\overline{\gamma} - \gamma^{m,n}\|_{T,\lambda} \leq \kappa\Big(\frac{1}{2^m} + \frac{1}{n}\Big), \qquad n, m \in \mathbb{N},$$

where $\overline{\gamma} = (\overline{\alpha}, \overline{\beta})$ is the unique solution in $L^\infty([0,T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$ to McKean-Vlasov fixed-point equation (3).

In analogy with the results of Section 3.2, for coefficients a, b that are not in $L^1(\mathbb{R}^d)$ we have the following extension.

Theorem 4.6. *Under Assumption 3.11, 3.2, 3.13 and 4.3, the same result as of Theorem 4.5 holds.*

The remaining part of the section is devoted to prove Theorem 4.5. The proof of Theorem 4.6 is analogous and thus, is omitted. **From now on, through the rest of this section, we will assume that the hypotheses of Theorem 4.5 are satisfied. In particular, we fix an arbitrary T, and a suitable $\lambda > 0$ such that (26) holds with $c = 1/2$. Also, we will denote by κ any positive constant that depends at most on T, θ, $\nu(\mathrm{d}y)$, Y and coefficients a, b. Finally, we initialize the sequences $\gamma^{m,n}$ and γ^m as in (52) with $\gamma_0 = (\alpha_0, \beta_0)$ satisfying $\max\{|\alpha_0|, |\beta_0|\} \leq \|a\|_{L^\infty} + \|b\|_{L^\infty}$.**

Lemma 4.7. *For any $\gamma \in L^\infty([0,T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$ with $\|\gamma\|_{T,0} \leq \|a\|_{L^\infty} + \|b\|_{L^\infty}$, we have the following*

$$|\Psi_{j,t}(\gamma) - \Psi_{j,t'}(\gamma)| \leq \kappa|t-t'|, \qquad t, t' \in [0,T], \quad j = 1, 2, \tag{54}$$

Proof. By Proposition 3.3 and triangular inequality we obtain that

$$|\Psi_{1,t}(\alpha,\beta) - \Psi_{1,t'}(\alpha,\beta)| \leq \frac{1}{(2\pi)^d}\sum_{i=1}^{4} I_i,$$

where

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^d} |\hat{a}(\eta)| \left|\mathrm{e}^{-\frac{1}{2}\langle \eta, C_t^\alpha \eta\rangle} - \mathrm{e}^{-\frac{1}{2}\langle \eta, C_{t'}^\alpha \eta\rangle}\right| \left|\mathrm{e}^{-i\,\langle \eta, m_t^{\alpha,\beta}\rangle + n_t^\alpha(-\eta)}\right| \left|\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big)\right| \mathrm{d}\eta,\\
I_2 &= \int_{\mathbb{R}^d} |\hat{a}(\eta)| \left|\mathrm{e}^{-\frac{1}{2}\langle \eta, C_{t'}^\alpha \eta\rangle + n_t^\alpha(-\eta)}\right| \left|\mathrm{e}^{-i\,\langle \eta, m_t^{\alpha,\beta}\rangle} - \mathrm{e}^{-i\,\langle \eta, m_{t'}^{\alpha,\beta}\rangle}\right| \left|\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big)\right| \mathrm{d}\eta,\\
I_3 &= \int_{\mathbb{R}^d} |\hat{a}(\eta)| \left|\mathrm{e}^{-\frac{1}{2}\langle \eta, C_{t'}^\alpha \eta\rangle - i\,\langle \eta, m_{t'}^{\alpha,\beta}\rangle}\right| \left|\mathrm{e}^{n_t^\alpha(-\eta)} - \mathrm{e}^{n_{t'}^\alpha(-\eta)}\right| \left|\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big)\right| \mathrm{d}\eta,\\
I_4 &= \int_{\mathbb{R}^d} |\hat{a}(\eta)| \left|\mathrm{e}^{-\frac{1}{2}\langle \eta, C_{t'}^\alpha \eta\rangle - i\,\langle \eta, m_{t'}^{\alpha,\beta}\rangle + n_{t'}^\alpha(-\eta)}\right| \left|\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big) - \hat{\mu}_Y\big(-(\Phi_{0,t'}^\alpha)^\top\eta\big)\right| \mathrm{d}\eta.
\end{aligned}$$

We note that

$$\left|\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big)\right| \leq |\hat{\mu}_Y(\eta)| + \left|\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big) - \hat{\mu}_Y(\eta)\right| \leq |\hat{\mu}_Y(\eta)| + \left|\big((\Phi_{0,t}^\alpha)^\top - (\Phi_{0,0}^\alpha)^\top\big)\eta\right| \mathbb{E}[|Y|]$$

(by (17))

$$\leq |\hat{\mu}_Y(\eta)| + \kappa t|\eta|. \tag{55}$$

By (18) and (16), we obtain that

$$\left|\mathrm{e}^{-\frac{1}{2}\langle \eta, C_t^\alpha \eta\rangle} - \mathrm{e}^{-\frac{1}{2}\langle \eta, C_{t'}^\alpha \eta\rangle}\right| \leq \kappa\,|t-t'||\eta|^2 \mathrm{e}^{-\frac{t|\eta|^2}{2\kappa}},$$

which, combined to (33) with ε suitably small, yields that

$$I_1 \leq \frac{\kappa|t-t'|}{(2\pi)^d}\int_{\mathbb{R}^d} |\hat{a}(\eta)|\mathrm{e}^{-\frac{t|\eta|^2}{2\kappa}}|\eta|^2 \left|\hat{\mu}_Y\big(-(\Phi_{0,t}^\alpha)^\top\eta\big)\right|\mathrm{d}\eta$$

(by (55) along with (34))

$$\leq \frac{\kappa|t-t'|}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{t|\eta|^2}{2\kappa}}|\hat{a}(\eta)|\big(|\eta|^2\left|\hat{\mu}_Y(\eta)\right| + |\eta|\big)\mathrm{d}\eta \leq \kappa|t-t'|.$$

In the last inequality above, we employed Assumption 4.1. Similarly, we find the same bound for I_2, I_3, I_4 by applying (19), (20) and (15)-(16) again. This proves (54) for $j = 1$. The proof for $j = 2$ is identical. □

Lemma 4.8. *For any $\gamma, \gamma' \in L^\infty([0,T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$ with $\|\gamma\|_{T,0} \leq \|a\|_{L^\infty} + \|b\|_{L^\infty}$, we have that*

$$\|\Psi^{(n)}(\gamma') - \Psi(\gamma)\|_{T,\lambda} \leq \frac{\kappa}{n} + \frac{1}{2}\|\gamma' - \gamma\|_{T,\lambda}, \quad n \in \mathbb{N},$$

Proof. By triangular inequality, we obtain that

$$\|\Psi^{(n)}(\gamma') - \Psi(\gamma)\|_{T,\lambda} \le \|\Psi^{(n)}(\gamma') - \Psi(\gamma')\|_{T,\lambda} + \|\Psi(\gamma') - \Psi(\gamma)\|_{T,\lambda}.$$

For the second term above, we use the contraction property of Theorem 3.7 with $c = 1/2$. To see the bound for the first term, we write the following

$$\big|\big(\Psi^{(n)}_{j,t}(\gamma')\big) - \big(\Psi_{j,t}(\gamma')\big)\big| \le \sum_{i=1}^{n} \big|\big(\Psi_{j,t_i}(\gamma')\big) - \big(\Psi_{j,t}(\gamma')\big)\big|\mathbf{1}_{[t_{i-1},t_i[}, \quad t \in [0,T], \quad j = 1,2,$$

which yields that

$$\big\|\Psi^{(n)}_{j}(\gamma') - \Psi_j(\gamma')\big\|_{T,\lambda} \le \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} \sum_{i=1}^{n} \big|\big(\Psi_{j,t_i}(\gamma')\big) - \big(\Psi_{j,t}(\gamma')\big)\big|\mathbf{1}_{[t_{i-1},t_i[}(t)$$

(by applying Lemma 4.7)

$$\le \kappa \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} \sum_{i=1}^{n} |t - t_i|\mathbf{1}_{[t_{i-1},t_i[}(t) \le \frac{\kappa}{n}, \quad t \in [0,T], \quad j = 1,2.$$

This concludes the proof. □

Remark 4.9. For any $\gamma \in L^\infty([0,T] : \mathcal{M}^{d\times d} \times \mathbb{R}^d)$, it is easy to observe from the definition of $\Psi^{(n)}$ that

$$\|\Psi^{(n)}(\gamma)\|_{T,0} \le \|\Psi(\gamma)\|_{T,0}.$$

Lemma 4.10. *For any $m, n \in \mathbb{N}$, we have that*

$$\|\gamma^{m,n} - \gamma^m\|_{T,\lambda} \le 2\Big(1 - \frac{1}{2^m}\Big)\frac{\kappa}{n} \le \frac{\kappa}{n}. \tag{56}$$

Proof. First note that, by the assumptions on $\gamma^{0,n}, \gamma^n$ and Remark 4.9, we have that

$$\|\gamma^{m,n}\|_{T,0}, \|\gamma^m\|_{T,0} \le \|a\|_{L^\infty} + \|b\|_{L^\infty}, \quad m, n \in \mathbb{N} \cup \{0\}.$$

We now prove the result by induction. For $m = 1$, by Lemma 4.8 and the fact that $\gamma^{0,n} = \gamma^n$, we obtain that

$$\|\gamma^{1,n} - \gamma^1\|_{T,\lambda} = \|\Psi^{(n)}(\gamma^{0,n}) - \Psi(\gamma^0)\|_{T,\lambda} \le \frac{\kappa}{n}.$$

Now suppose that (56) holds for $m-1$, and we prove it true for m. Again, applying Lemma 4.8 yields that

$$\|\gamma^{m,n} - \gamma^m\|_{T,\lambda} = \|\Psi^{(n)}(\gamma^{m-1,n}) - \Psi(\gamma^{m-1})\|_{T,\lambda} \le \frac{\kappa}{n} + \frac{1}{2}\|\gamma^{m-1,n} - \gamma^{m-1}\|_{T,\lambda}$$

(by inductive hypothesis)

$$\le \frac{\kappa}{n} + \Big(1 - \frac{1}{2^{m-1}}\Big)\frac{\kappa}{n} = 2\Big(1 - \frac{1}{2^m}\Big)\frac{\kappa}{n}.$$

□

We are now in the position to prove Theorem 4.5.

Proof of Theorem 4.5. First note that, by the assumptions on γ^0 and by Corollary 3.8, we have that

$$\|\gamma^0 - \overline{\gamma}\|_{T,\lambda} \le \|\gamma^0\|_{T,0} + \|\overline{\gamma}\|_{T,0} \le 2(\|a\|_{L^\infty} + \|b\|_{L^\infty}).$$

Therefore, by applying the contraction property in Theorem 3.7 with $c = 1/2$ it is straightforward to see that

$$\|\gamma^m - \overline{\gamma}\|_{T,\lambda} \le 2^{-(m-1)}(\|a\|_{L^\infty} + \|b\|_{L^\infty}). \tag{57}$$

Now, by the triangular inequality, we obtain that

$$\|\gamma^{m,n} - \overline{\gamma}\|_{T,\lambda} \le \|\gamma^{m,n} - \gamma^m\|_{T,\lambda} + \|\gamma^m - \overline{\gamma}\|_{T,\lambda}.$$

The result follows from applying Lemma 4.10 to the first term, and (57) to second. □

5 Numerical results

In this section, we demonstrate the applicability of our theoretical results by testing them on examples for which semi-explicit solutions are available. We verify the convergence of the discretized Picard iteration scheme and the rate of error convergence as discussed in Section 4. We performed all the numerical computations on a computing device with 2,4 GHz Intel i5 processor and 16 GB RAM.

5.1 Gaussian benchmark

We first consider the following SDE in one dimension

$$\mathrm{d}X_t = \big(aX_t + \mathbb{E}[\cos(X_t)]\big)\mathrm{d}t + \sigma dW_t,\ t > 0, \quad X_0 = Y, \tag{58}$$

with $a \in \mathbb{R}$ and $\sigma > 0$, and where Y has Laplace distribution $\mu_Y(\mathrm{d}y) = \frac{1}{2}\mathrm{e}^{-|y|}\mathrm{d}y$. Comparing with our setting of (1), it gives us that

$$a(x) \equiv a, \quad b(x) = \cos(x),$$

and that $\nu(\mathrm{d}y) \equiv 0$, i.e., there are no jumps. Here, a and b are bounded functions but they do not belong to $L^1(\mathbb{R})$. However, considering the damped coefficients $\overline{a}(x) = \frac{a}{(1+x^2)}$ and $\overline{b}(x) = \frac{\cos x}{(1+x^2)}$, it is immediate to check that the Fourier transforms $\hat{\overline{a}}$, $\hat{\overline{b}}$ satisfy Assumption 3.16 and 4.3. Moreover, Assumption 3.2 and 3.13 are also satisfied, and thus the results of Sections 3.2 and 4 do apply to (58). In particular: we have the existence and uniqueness for the solution of the fixed-point equation (3) from Corollary 3.19, and that the Picard sequence in (51)-(52) converges to the solution $(\bar{\alpha}, \bar{\beta})$ from Theorem 4.6.

For this specification of a, b it is possible to obtain semi-explicit solutions $(\bar{\alpha}, \bar{\beta})$ for (3), which we use as a benchmark to test the rate of convergence of our approximating scheme. In fact, by Lemma 2.4 we have that $(\bar{\alpha}, \bar{\beta})$ satisfies

$$\begin{aligned}
\bar{\alpha}_t &= a,\\
\bar{\beta}_t &= \mathbb{E}[\cos(X_t^{(\bar{\alpha},\bar{\beta})})] = \frac{\hat{\mu}_{X_t^{(\bar{\alpha},\bar{\beta})}}(1) + \hat{\mu}_{X_t^{(\bar{\alpha},\bar{\beta})}}(-1)}{2} = \frac{\mathrm{e}^{-\frac{1}{2}C_t}}{1+\mathrm{e}^{2at}}\cos(m_t),
\end{aligned}$$

where we set

$$m_t := m_t^{\bar{\alpha},\bar{\beta}} = \mathrm{e}^{at}\int_0^t \mathrm{e}^{-as}\bar{\beta}_s\mathrm{d}s, \qquad C_t := C_t^{\bar{\alpha}} = \frac{\sigma^2}{2a}(\mathrm{e}^{2at}-1), \tag{59}$$

respectively the mean and variance of the solution to (58). We then obtain that $\bar{\beta}_t = m_t' - am_t$, where m_t solves the following equation

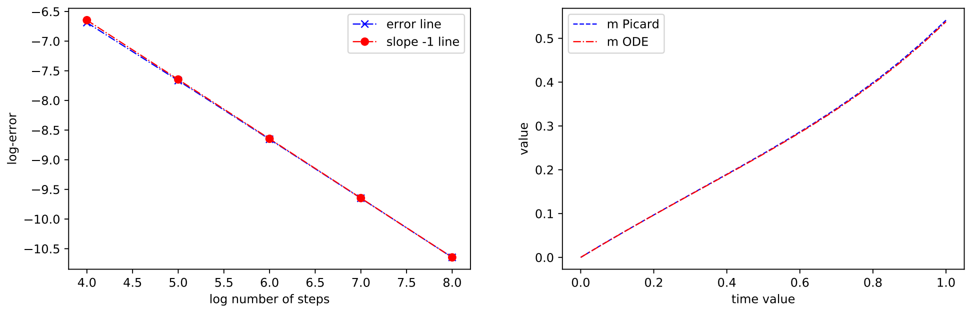
$$m_t' = a\,m_t + \frac{\mathrm{e}^{-\frac{1}{2}C_t}}{1+\mathrm{e}^{2at}}\cos(m_t). \tag{60}$$

In the absence of a closed-form expression for $\bar{\beta}_t$, we treat the numerical solution from ODE (60) as a proxy for the true value. We also point that, in this specific case, no numerical integration is required to compute $\gamma^{m,n}$, neither in the Fourier space nor in the original one.

In order to verify the convergence rate of our method as derived in Theorem 4.5, we set the number of Picard iteration steps to be $m = \log_2(n)$ where n is the number of time discretization steps. In Figure 1(a), for parameter values $a = 1.5$, $\sigma = 0.8$, $T = 1.0$, we vary $n = 2^k, 4 \le k \le 8$, and observe that the slope of log-error, i.e.,

$$\log\Big(\max_{k=0,\cdots,n}|\beta_{t_k}^{m,n} - \bar{\beta}_{t_k}|\Big) \quad (\text{with } t_k = \frac{kT}{n}), \tag{61}$$

indeed matches the result in Theorem 4.5. In Figure 1(b), we compare the Picard scheme approximation $m_t^{\alpha^{m,n},\beta^{m,n}}$ of $m_t^{\bar{\alpha},\bar{\beta}}$ with $n = 2^4$ against the numerical solution obtained by solving ODE (60). This comparison shows that we obtain an accurate approximation even for small values of n and m in the discretized Picard iteration scheme.



(a) (b)

Figure 1: Error convergence rate and comparison of approximations for one-dimensional model.

5.2 Jumps in one dimension

In this section, we generalize the previous example by adding Lévy jumps to the McKean-Vlasov SDE in (58). In particular, we consider a compound Poisson process with jump-intensity λ, where the distribution of jumps is defined in terms of an asymmetric double exponential density, i.e.,

$$\chi(\mathrm{d}y) = \big(p\lambda_1 \mathrm{e}^{-\lambda_1 y}\mathbf{1}_{\{y>0\}} + (1-p)\lambda_2 \mathrm{e}^{-\lambda_2 y}\mathbf{1}_{\{y<0\}}\big)\mathrm{d}y,$$

where $\lambda_1, \lambda_2 > 0$ and $p \in [0,1]$ represents the probability of upward jumps. The Lévy measure of this process, which appears in some financial applications (see Kou model [14]), is $\nu = \lambda\chi$, and thus satisfies Assumption 3.13. Therefore, the theoretical results of Sections 3.2 and 4 still apply to this case. We also note that the density of $\mu_{X_t^{(\alpha,\beta)}}$ is not known in closed form and thus, our method based on Fourier transform has even more significance.

Since compound Poisson processes have finite activity, $\nu(\mathrm{d}y)$ is a finite measure on $\mathbb{R}\setminus\{0\}$. Thus it is convenient to simplify the Lévy-Ito representation in (2) by writing a pure-jump (non-compensated) stochastic integral on $\mathbb{R}$. This choice also simplifies the integral part in the characteristic exponent of $X_t^{(\alpha,\beta)}$. Denoting once again by $(\bar\alpha, \bar\beta)$ the unique solution to (3), we obtain the following (see Pascucci [19, Page 465])

$$\int_{\mathbb{R}} (\mathrm{e}^{i\xi y} - 1)\nu(\mathrm{d}y) = i\lambda\xi\left(\frac{p}{\lambda_1 - i\eta} - \frac{1-p}{\lambda_2 + i\eta}\right),$$

which in turn gives us

$$n_t(\eta) = n_t^{\bar\alpha}(\eta) = \frac{p\lambda}{a}\log\left(\frac{i\eta - \lambda_1}{i\eta \mathrm{e}^{at} - \lambda_1}\right) + \frac{(1-p)\lambda}{a}\log\left(\frac{i\eta + \lambda_2}{i\eta\mathrm{e}^{at} + \lambda_2}\right).$$

Due to the above result we can write that

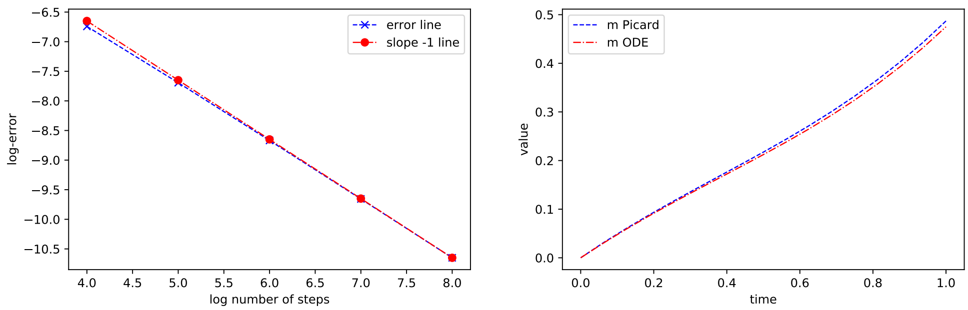
$$\begin{aligned}\bar\beta_t &= \mathbb{E}[\cos(X_t^{(\bar\alpha,\bar\beta)})] = \frac{\hat\mu_{X_t^{(\bar\alpha,\bar\beta)}}(1) + \hat\mu_{X_t^{(\bar\alpha,\bar\beta)}}(-1)}{2} \\ &= \frac{\mathrm{e}^{-\frac{1}{2}C_t}}{1+\mathrm{e}^{2at}}\left(\frac{1+\lambda_1^2}{\mathrm{e}^{2at}+\lambda_1^2}\right)^{\frac{p\lambda}{2a}}\left(\frac{1+\lambda_2^2}{\mathrm{e}^{2at}+\lambda_2^2}\right)^{\frac{(1-p)\lambda}{2a}}\cos\left(m_t + \frac{p\lambda}{a}(\theta_1 - \theta_1') + \frac{(1-p)\lambda}{a}(\theta_2 - \theta_2')\right),\end{aligned}$$

with m_t and C_t as in (59) and where

$$\theta_1 = \arctan\Big(-\frac{1}{\lambda_1}\Big), \qquad \theta_2 = \arctan\Big(\frac{1}{\lambda_2}\Big), \qquad \theta_1' = \arctan\Big(-\frac{\mathrm{e}^{at}}{\lambda_1}\Big), \qquad \theta_2' = \arctan\Big(\frac{\mathrm{e}^{at}}{\lambda_2}\Big).$$

Proceeding now like in Example 5.1, we obtain a suitable modification of ODE (60) for m_t, which can be solved numerically to obtain a reference benchmark for m_t and $\bar{\beta}_t$.

In Figure 2(a), like in Example 5.1, we plot again the quantity in (61), and observe that for parameter values $a = 0.25$, $\sigma = 1.0$, $T = 1.0$, $\lambda = 0.8$, $\lambda_1 = 0.5$, $\lambda_2 = 0.6$, $p = 0.35$, by varying $n = 2^k, 4 \le k \le 8$, we confirm the result in Theorem 4.5. In Figure 2(b), results of the numerical solution from the ODE are compared with the discretized Picard iteration scheme approximation $m_t^{\alpha^{m,n},\beta^{m,n}}$ of $m_t^{\bar{\alpha},\bar{\beta}}$ with $n = 2^4$ which once again illustrates the accuracy of our method even for small values of n and m.



(a)

(b)

Figure 2: Error convergence rate and comparison of approximations for the model with jumps.

5.3 Convergence rate in multiple dimensions

In this example, we demonstrate that the error convergence rate of the Picard iteration scheme remains independent of the dimension modulo the error of the numerical approximation of the space integral needed to compute $E[b(X_t^{(\alpha,\beta)})]$. We generalize the MK-V SDE (58) in a way that the computation of $E[b(X_t^{(\alpha,\beta)})]$ still does not require numerical integration, and a semi-explicit benchmark (up to solving an ODE) for the solution is still available. We suppose that Y is an $\mathbb{R}^d$-valued random variable with a Laplace type distribution given as $\mu_Y(\mathrm{d}y) = \frac{1}{2^d}\mathrm{e}^{-\sum_{i=1}^d |y_i|}\mathrm{d}y$ and consider the following MK-V SDE

$$\mathrm{d}X_t = \Big(aX_t + \mathbb{E}\big[\cos\big(\textstyle\sum_{i=1}^d X_t^i\big)\big]\mathbf{1}\Big)\mathrm{d}t + \sigma dW_t,\ t > 0, \qquad X_0 = Y, \tag{62}$$

where $a \in \mathbb{R}$, $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^d$, W is a q-dimensional Brownian motion, and $\sigma \in \mathcal{M}^{d\times q}$ such that $\theta = \sigma\sigma^\top$ is positive definite. Note that (62) can be put in the form (1)-(2) by setting

$$a(x) \equiv aI_d, \quad b(x) = \cos(\textstyle\sum_{i=1}^d x_i)\mathbf{1}, \quad \nu(\mathrm{d}y) \equiv 0.$$

The solution $(\bar{\alpha}, \bar{\beta})$ to (3) then satisfies the following

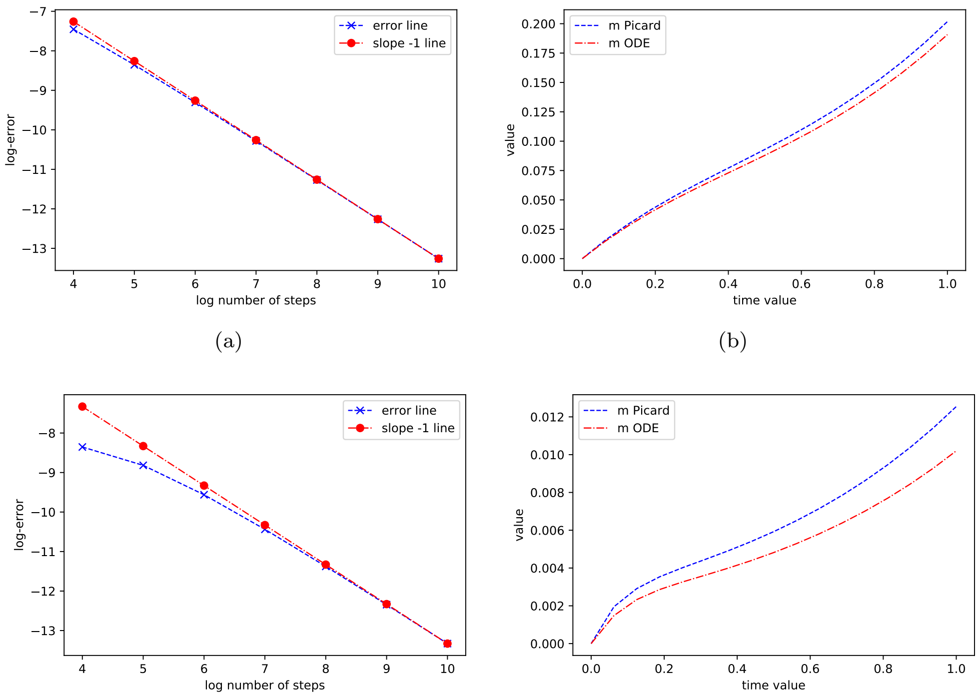
$$\bar{\alpha}_t \equiv aI_d, \quad \bar{\beta} = \tilde{\beta}_t\mathbf{1},$$

where

$$\tilde{\beta}_t := \mathbb{E}\big[\cos\big(\textstyle\sum_{i=1}^d X_t^{(\bar{\alpha},\bar{\beta}),i}\big)\big] = \frac{\hat{\mu}_{X_t^{(\bar{\alpha},\bar{\beta})}}(\mathbf{1}) + \hat{\mu}_{X_t^{(\bar{\alpha},\bar{\beta})}}(-\mathbf{1})}{2}$$

(by Lemma 2.4)

$$= \frac{\mathrm{e}^{-\frac{1}{2}\mathbf{1}^\top C_t \mathbf{1}}}{(1 + \mathrm{e}^{2a_0 t})^d}\cos(d\,\tilde{m}_t),$$



(a) (b)

(c) (d)

Figure 3: Error convergence rate (a) $d = 2$ (c) $d = 5$ and comparison of approximations (b) $d = 2$ (d) $d = 5$.

with

$$C_t = C_t^{\bar{\alpha}} = \frac{1}{2a}(\mathrm{e}^{2at} - 1)\theta, \qquad \tilde{m}_t = \mathrm{e}^{at}\int_0^t \mathrm{e}^{-as}\tilde{\beta}_s \mathrm{d}s.$$

In the above, C_t is the covariance matrix and $\tilde{m}_t$ is the mean of all the components of $X_t^{(\bar{\alpha},\bar{\beta})}$. Proceeding once again like in Example 5.1, we obtain a suitable modification of the ODE (60) for $\tilde{m}_t$, which can be solved numerically to obtain a reference benchmark for $\tilde{m}_t$ and $\tilde{\beta}_t$. In Figure 3, we plot quantities that are analogous to those in Figures 1 and 2. We demonstrate that, for $a = 0.25$ and randomly generated σ matrix, the convergence rate is independent of dimension as proven in Theorem 4.5, and also provide approximations for $\tilde{m}_t$ using the discretized Picard iteration scheme for $n = 2^4$.

5.4 Convergence rate for non-integrable initial datum

In the final example, we consider the case of a non-integrable initial law given by a multivariate γ-stable distribution with independent components. The characteristic function of Y with $\gamma = 1$ is then given as

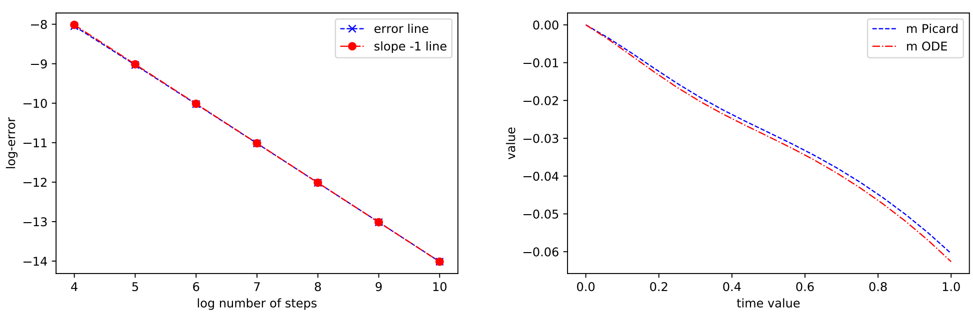
$$\hat{\mu}_Y(\eta) = \mathbb{E}\big[\exp(i\langle \eta, Y\rangle)\big] = \exp\big(i\langle \eta, \mathbf{1}\rangle - \textstyle\sum_{k=1}^d |\eta_k|\big).$$

In the above formula, the shift parameter is represented by a unit vector $\mathbf{1} \in \mathbb{R}^d$, and for each component, the skewness parameter is set to zero and the scale parameter is set to one. The above distribution does not admit a first moment. With this example, we wish to test the applicability of our method by relaxing the moment conditions on the initial datum. As expected, it turns out that the finite-moment assumptions on the initial distributions are not essential for the Picard iteration method that we propose (at least in some cases), but they are rather related to the fact that we proved the contraction properties in the Fourier space.

We work with the MK-V SDE setting of Section 5.3. Applying Lemma 2.4 it is easy to show that

$$\tilde{\beta}_t := \mathbb{E}\big[\cos\big(\textstyle\sum_{i=1}^d X_t^{(\bar{\alpha},\bar{\beta}),i}\big)\big] = \frac{\hat{\mu}_{X_t^{(\bar{\alpha},\bar{\beta})}}(\mathbf{1}) + \hat{\mu}_{X_t^{(\bar{\alpha},\bar{\beta})}}(-\mathbf{1})}{2} = \mathrm{e}^{-\frac{1}{2}\mathbf{1}^\top C_t \mathbf{1} - d\mathrm{e}^{at}}\cos\big(d(\mathrm{e}^{at} + \tilde{m}_t)\big),$$

and a benchmark for $\tilde{\beta}_t$ can be obtained by solving a suitable modification of the ODE (60). In Figure 4, we demonstrate that even though the initial datum has an undefined mean, our Picard iteration scheme converges and the rate is independent of dimension as proven in Theorem 4.5. We also provide approximation for $\tilde{m}_t$ using the discretized Picard iteration scheme for $n = 2^4$.



(a) (b)

Figure 4: Error convergence rate and comparison of approximations for the model with initial datum with an undefined mean in $d = 2$.

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A Proofs of Lemma 2.4, 2.5 and 2.6

Proof of Lemma 2.4. To shorten notation, throughout the proof, we set $X_t = X_t^{(\alpha,\beta)}$ and $\Phi_{s,t} = \Phi^{\alpha}_{s,t}$.

Let us set $\tilde{X}_t = \Phi_{0,t}^{-1} X_t$, with Φ as in (11). By Itô's formula we obtain that

$$\tilde{X}_t = Y + \int_0^t \Phi_{0,s}^{-1}\beta_s \mathrm{d}s + \int_0^t \Phi_{0,s}^{-1}\mathrm{d}L_s,$$

and thus, we have that

$$\hat{\mu}_{X_t}(\eta) = \mathbb{E}\big[\exp\big(i\big\langle \eta, \Phi_{0,t}\tilde{X}_t\big\rangle\big)\big] = \mathbb{E}\big[\exp\big(i\big\langle \Phi_{0,t}^{\top}\eta, \tilde{X}_t\big\rangle\big)\big] = \mathbb{E}\big[\exp\big(i\big\langle \Phi_{0,t}^{\top}\eta, \tilde{X}_t - Y\big\rangle\big)\big]\hat{\mu}_Y\big(\Phi_{0,t}^{\top}\eta\big). \tag{63}$$

Set $Z_t := \mathrm{e}^{i\langle \xi, \tilde{X}_t - Y\rangle}$. By Itô's formula, we obtain that

$$\begin{aligned} Z_t = 1 &+ \int_0^t Z_s \int_{\mathbb{R}^d}\Big(\mathrm{e}^{i\langle \xi, \Phi_{0,s}^{-1}y\rangle} - 1 - \mathbf{1}_{\{|y|<1\}} i\big\langle \xi, \Phi_{0,s}^{-1}y\big\rangle\Big)\nu(\mathrm{d}y)\mathrm{d}s \\ &+ \int_0^t Z_s\Big(i\big\langle \xi, \Phi_{0,s}^{-1}\beta_s\big\rangle - \frac{1}{2}\big\langle \xi, \Phi_{0,s}^{-1}\sigma\big(\Phi_{0,s}^{-1}\sigma\big)^{\top}\xi\big\rangle\Big)\mathrm{d}s \\ &+ i\int_0^t Z_s \xi^{\top}\Phi_{0,s}^{-1}\sigma \mathrm{d}W_s + \int_0^t Z_{s-}\int_{\mathbb{R}^d}\Big(\mathrm{e}^{i\langle \xi, \Phi_{0,s}^{-1}y\rangle} - 1\Big)\big(N(\mathrm{d}s,\mathrm{d}y) - \nu(\mathrm{d}y)\mathrm{d}s\big). \end{aligned}$$

Now set $\phi_t(\xi) := \mathbb{E}[Z_t]$. By the martingale property of the Itô and the jump integrals, combined with Fubini's theorem, we obtain that

$$\phi_t(\xi) = 1 + \int_0^t \phi_s(\xi)\psi_s(\xi)\mathrm{d}s,$$

where

$$\psi_s(\xi) = \int_{\mathbb{R}^d}\Big(\mathrm{e}^{i\langle \xi, \Phi_{0,s}^{-1}y\rangle} - 1 - \mathbf{1}_{\{|y|<1\}} i\big\langle \xi, \Phi_{0,s}^{-1}y\big\rangle\Big)\nu(\mathrm{d}y) + i\big\langle \xi, \Phi_{0,s}^{-1}\beta_s\big\rangle - \frac{1}{2}\big\langle \xi, \Phi_{0,s}^{-1}\sigma\big(\Phi_{0,s}^{-1}\sigma\big)^{\top}\xi\big\rangle.$$

By differentiating both terms, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(\xi) = \phi_t(\xi)\psi_t(\xi), \quad t > 0, \qquad \phi_0(\xi) = 1,$$

which yields that

$$\mathbb{E}\big[\exp\big(i\big\langle \xi, \tilde{X}_t - Y\big\rangle\big)\big] = \phi_t(\xi) = \mathrm{e}^{\int_0^t \psi_s(\xi)\mathrm{d}s},$$

which in turn, combined with (63), yields (8) and concludes the proof. □

Proof of Lemma 2.5. To ease the notation, we set $\Phi_{s,t} = \Phi^{\alpha}_{s,t}$, $C_t = C^{\alpha}_t$, $m_t = m_t^{\alpha,\beta}$ and $n_t(\eta) = n_t^{\alpha}(\eta)$.

The inequality (12) is a straightforward consequence of Grönwall's Lemma. We now prove (13). By (11) it holds that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} C_t = \theta + 2\alpha_t C_t, \quad s < t \le T, \\ \Phi_0 = 0, \end{cases} \tag{64}$$

which means that C is absolutely continuous and $C_t = t\theta + 2\int_0^t \alpha_u C_u \mathrm{d}u$. Grönwall's lemma again yields (13). The proof of (14) is completely analogous.

To prove (15), set

$$f(x) := \int_{\mathbb{R}^d} \left(\mathrm{e}^{i\langle x,y\rangle} - 1 - \mathbf{1}_{\{|y|<1\}} i\langle x,y\rangle \right) \nu(\mathrm{d}y), \quad x \in \mathbb{R}^d. \tag{65}$$

Now, for any $0 \le \delta < 1$, Taylor's theorem yields that

$$|f(x)| \le 2\bigg(|x|^2 \int_{|y|<\delta} |y|^2 \nu(\mathrm{d}y) + (|x|+1) \int_{|y|\ge\delta} \nu(\mathrm{d}y) \bigg), \quad x \in \mathbb{R}^d,$$

which, combined with (12) gives that

$$\left| f\left(\Phi_{s,t}^\top \eta\right) \right| \le 2\mathrm{e}^{2T\|\alpha\|_{T,0}} \left(|\eta|^2 \int_{|y|<\delta} |y|^2 \nu(\mathrm{d}y) + (|\eta|+1) \int_{|y|\ge\delta} \nu(\mathrm{d}y) \right). \tag{66}$$

This, together with the fact that

$$n_t(\eta) = \int_0^t f\left(\Phi_{s,t}^\top \eta\right) \mathrm{d}s, \tag{67}$$

proves (15).

We finally prove (16). We have that

$$|\langle \eta, C_t \eta\rangle| \ge \lambda_{\min}(\theta) \int_0^t \eta^\top \Phi_{s,t} \Phi_{s,t}^\top \eta \, \mathrm{d}s, \tag{68}$$

and

$$\lambda_{\min}\left(\Phi_{s,t}\Phi_{s,t}^\top\right) = \frac{1}{\lambda_{\max}\left(\left(\Phi_{s,t}\Phi_{s,t}^\top\right)^{-1}\right)} = \frac{1}{\left|\left(\Phi_{s,t}\right)^{-1}\right|^2}.$$

Noting that $\Phi_{s,t}^{-1} = I_d - \int_0^t \Phi_{s,u}^{-1} \alpha_u \mathrm{d}u$, and applying Grönwall's Lemma yields that

$$\left|\Phi_{s,t}^{-1}\right| \le \mathrm{e}^{T\|\alpha\|_{T,0}}. \tag{69}$$

Eventually, (16) results from (68)–(69). $\square$

Proof of Lemma 2.6. To ease the notation, we set $\Phi_{s,t} = \Phi_{s,t}^\alpha$, $C_t = C_t^\alpha$, $m_t = m_t^{\alpha,\beta}$ and $n_t(\eta) = n_t^\alpha(\eta)$.

By definition of Φ, we obtain that

$$|\Phi_{s,t} - \Phi_{s,t'}| = \left| \int_t^{t'} \Phi_{s,u}\alpha_u \mathrm{d}u \right| \le \int_t^{t'} |\Phi_{s,u}||\alpha_u| \mathrm{d}u,$$

and (17) follows from (12).

We now prove (18). By definition of C and by the triangular inequality, we obtain that

$$|C_t - C_{t'}| \le \int_t^{t'} \left|\Phi_{s,t'}\sigma\sigma^\top \Phi_{s,t'}^\top\right| \mathrm{d}s + \int_0^t \left| \left(\Phi_{s,t'}\sigma\sigma^\top \Phi_{s,t'}^\top\right) - \left(\Phi_{s,t}\sigma\sigma^\top\Phi_{s,t}^\top\right) \right| \mathrm{d}s. \tag{70}$$

Applying (12) and (17) yields that

$$\left|\Phi_{s,t'}\sigma\sigma^\top\Phi_{s,t'}^\top\right| \le \mathrm{e}^{2T\|\alpha\|_{T,0}}|\theta|,$$

and

$$\begin{aligned}\left|\left(\Phi_{s,t'}\sigma\sigma^\top\Phi_{s,t'}^\top\right)-\left(\Phi_{s,t}\sigma\sigma^\top\Phi_{s,t}^\top\right)\right| &\le \left|\Phi_{s,t'}\sigma\sigma^\top(\Phi_{s,t'}-\Phi_{s,t})^\top\right|+\left|(\Phi_{s,t'}-\Phi_{s,t})\sigma\sigma^\top\Phi_{s,t}^\top\right|\\ &\le 2\|\alpha\|_{T,0}\mathrm{e}^{2T\|\alpha\|_{T,0}}|\theta|(t'-t).\end{aligned}$$

Plugging these into (70), we have that

$$|C_t - C_{t'}| \le (1+2T\|\alpha\|_{T,0})\mathrm{e}^{2T\|\alpha\|_{T,0}}|\theta|(t'-t),$$

which proves (18). The proof of (19) is analogous and thus is omitted.

We finally prove (20). Consider the function f as defined in (65). For any $M>0$, an application of Lagrange's mean-value theorem yields the following

$$|f(x)-f(x')| \le 2(M+1)\,\bar{n}\,|x-x'|, \quad |x|,|x'|<M, \tag{71}$$

with $\bar{n}$ as in (25). This, together with (12), yields that

$$\left|f\left(\Phi_{s,t'}^\top\eta\right)-f\left(\Phi_{s,t}^\top\eta\right)\right| \le 2\left(|\eta|\mathrm{e}^{T\|\alpha\|_{T,\lambda}}+1\right)\bar{n}\left|(\Phi_{s,t'}^\top-\Phi_{s,t}^\top)\eta\right|$$

(by (17))

$$\le (t'-t)2\|\alpha\|_{T,0}\mathrm{e}^{2T\|\alpha\|_{T,0}}\left(|\eta|^2+|\eta|\right)\bar{n}, \tag{72}$$

while (66) gives the following

$$\left|f\left(\Phi_{s,t'}^\top\eta\right)\right| \le 2\mathrm{e}^{2T\|\alpha\|_{T,0}}\left(|\eta|^2+|\eta|+1\right)\bar{n}. \tag{73}$$

Now, by (67) and by the triangular inequality we obtain that

$$|n_t(\eta)-n_{t'}(\eta)| \le \int_t^{t'}\left|f\left(\Phi_{s,t'}^\top\eta\right)\right|\mathrm{d}s + \int_0^t\left|f\left(\Phi_{s,t'}^\top\eta\right)-f\left(\Phi_{s,t}^\top\eta\right)\right|\mathrm{d}s$$

(by (72)-(73))

$$\le 2(1+T\|\alpha\|_{T,0})\mathrm{e}^{2T\|\alpha\|_{T,0}}\left(|\eta|^2+|\eta|+1\right)\bar{n}(t'-t),$$

which proves (20) and concludes the proof. □

B Proofs of Lemma 3.10 and Lemma 3.20

Proof of Lemma 3.10. We first show (27). Set $w := C^\alpha - C^{\alpha'}$. By (64), we obtain that w is the solution to the following equation

$$\begin{cases} w_t' = 2\left(\alpha_t C_t^\alpha - \alpha_t' C_t^{\alpha'}\right), & t>0\\ w_0 = 0,\end{cases}$$

which means that w is absolutely continuous and is given as

$$w_t = 2\int_0^t \left(\alpha_s C_s^\alpha - \alpha_s' C_s^{\alpha'}\right)\mathrm{d}s = 2\int_0^t \left(\alpha_s w_s + \left(\alpha_s-\alpha_s'\right)C_s^{\alpha'}\right)\mathrm{d}s,$$

and thus satisfies the following

$$|w_t| \le 2\int_0^t \Big(|\alpha_s||w_s| + |\alpha_s - \alpha'_s|\big|C_s^{\alpha'}\big|\Big)\mathrm{d}s.$$

Now, by Grönwall's Lemma we obtain that

$$|w_t| \le 2\,\mathrm{e}^{2\int_0^t |\alpha_s|\mathrm{d}s}\int_0^t |\alpha_s - \alpha'_s|\big|C_s^{\alpha'}\big|\mathrm{d}s,$$

which, together with (13), yields that

$$|w_t| \le 2t|\theta|\,\mathrm{e}^{2T(\|\alpha\|_{T,0}+\|\alpha'\|_{T,0})}\int_0^t |\alpha_s - \alpha'_s|\mathrm{d}s \le 2t|\theta|\,\mathrm{e}^{4T\|a\|_{L^\infty}}\int_0^t |\alpha_s - \alpha'_s|\mathrm{d}s. \tag{74}$$

Now, combining the following

$$|\langle \eta, C_t^{\alpha}\eta\rangle - \langle \eta, C_t^{\alpha'}\eta\rangle| \le |\eta|\big|(C_t^{\alpha} - C_t^{\alpha'})\eta\big| \le |\eta|^2\big|C_t^{\alpha} - C_t^{\alpha'}\big| = |\eta|^2|w_t|$$

with (74) proves (27).

The proof of (30) is analogous, and so are the proofs of (28) and (29) after observing that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\big(m_t^{\alpha,\beta} - m_t^{\alpha',\beta}\big) = \alpha_t m_t^{\alpha,\beta} - \alpha'_t m_t^{\alpha',\beta}, \quad t > 0, \\ m_0^{\alpha,\beta} - m_0^{\alpha',\beta} = 0, \end{cases}$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\big(m_t^{\alpha,\beta} - m_t^{\alpha,\beta'}\big) = \alpha_t\big(m_t^{\alpha,\beta} - m_t^{\alpha,\beta'}\big) + (\beta_t - \beta'_t), \quad t > 0, \\ m_0^{\alpha,\beta} - m_0^{\alpha,\beta'} = 0. \end{cases}$$

We next show (31). By (71) and (12) we obtain that

$$\big|f\big((\Phi_{s,t}^{\alpha})^\top\eta\big) - f\big((\Phi_{s,t}^{\alpha'})^\top\eta\big)\big| \le 2\big(|\eta|\mathrm{e}^{T\|a\|_{L^\infty}} + 1\big)\bar{n}\big|(\Phi_{s,t}^{\alpha})^\top\eta - (\Phi_{s,t}^{\alpha'})^\top\eta\big|$$

(by (30))

$$\le \kappa|\eta|(|\eta|+1)\int_0^t |\alpha_s - \alpha'_s|\mathrm{d}s,$$

which in turn, combined with the following

$$n_t^{\alpha}(\eta) - n_t^{\alpha'}(\eta) = \int_0^t \Big(f\big((\Phi_{s,t}^{\alpha})^\top\eta\big) - f\big((\Phi_{s,t}^{\alpha'})^\top\eta\big)\Big)\mathrm{d}s,$$

yields (31) and concludes the proof. □

Proof of Lemma 3.20. By (44) we have that

$$\begin{aligned} \partial_{\eta_j}\big(n_t^{\alpha} - n_t^{\alpha'}\big)(\eta) = i\int_0^t\int_{\mathbb{R}^d} \big(\Phi_{s,t}^{\alpha}y - \Phi_{s,t}^{\alpha'}y\big)_j\Big(\mathrm{e}^{i\langle\eta,\Phi_{s,t}^{\alpha}y\rangle} - \mathbf{1}_{\{|y|<1\}}\Big)\nu(\mathrm{d}y)\mathrm{d}s, \\ + i\int_0^t\int_{\mathbb{R}^d} \big(\Phi_{s,t}^{\alpha'}y\big)_j\Big(\mathrm{e}^{i\langle\eta,\Phi_{s,t}^{\alpha}y\rangle} - \mathrm{e}^{i\langle\eta,\Phi_{s,t}^{\alpha'}y\rangle}\Big)\nu(\mathrm{d}y)\mathrm{d}s, \end{aligned}$$

for any $j = 1, \dots, d$, from which we obtain that

$$\begin{aligned} |\partial_{\eta_j}\big(n_t^{\alpha} - n_t^{\alpha'}\big)(\eta)| \le \int_0^t\int_{\mathbb{R}^d} \big|\Phi_{s,t}^{\alpha} - \Phi_{s,t}^{\alpha'}\big||y|\Big(\big|\Phi_{s,t}^{\alpha}\big||y||\eta| + \mathbf{1}_{\{|y|\ge 1\}}\Big)\nu(\mathrm{d}y)\mathrm{d}s, \\ + \int_0^t\int_{\mathbb{R}^d} \big|\Phi_{s,t}^{\alpha'}\big||y|^2|\eta|\big|\Phi_{s,t}^{\alpha} - \Phi_{s,t}^{\alpha'}\big|\nu(\mathrm{d}y)\mathrm{d}s \end{aligned}$$

by (12)

$$\leq 2\big(\mathrm{e}^{T\|\alpha\|_{T,0}} + \mathrm{e}^{T\|\alpha'\|_{T,0}}\big)(|\eta|+1)\int_{\mathbb{R}^d}|y|^2\nu(\mathrm{d}y)\int_0^t \big|\Phi^{\alpha}_{s,t} - \Phi^{\alpha'}_{s,t}\big|\mathrm{d}s.$$

The estimate (48) for $m = 1$ then follows from (31). The proof for $m > 1$ is completely analogous and thus it is omitted. ☐

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