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Generalized Samuel Multiplicities of Monomial Ideals and Volumes

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Generalized Samuel Multiplicities of Monomial Ideals and Volumes

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ABSTRACT

We describe conjecturally the generalized Samuel multiplicities c_0, \dots, c_{d-1} of a monomial ideal $I \subset K[x_1, \dots, x_d]$ in terms of its Newton polyhedron $\text{NP}(I)$. More precisely, we conjecture that c_i equals the sum of the normalized $(d-i)$ -volumes of pyramids over the projections of the $(d-i-1)$ -dimensional compact faces of $\text{NP}(I)$ along the infinite-directions of i -unbounded facets in which they are contained. For c_0 proofs are known (Guibert, Jeffries and Montaña) and for c_{d-1} a proof is given.

KEYWORDS

monomial ideal; generalized Samuel multiplicity; Newton polyhedron

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1. Introduction

In this paper, based on computations with the free softwares Germenes [14] and REDUCE [11], we give a conjecture that in the case of monomial ideals links the generalized multiplicities defined algebraically in [3] with volumes derived from the Newton polyhedra of the ideals, thus extending a result of Teissier [17].

In 1988, Teissier [17, p. 131] proved that for an \mathfrak{m} -primary monomial ideal I of a local ring A the Samuel multiplicity is equal to the normalized volume of the complement of the Newton polyhedron of the ideal I . In 1999, Guibert [9] generalized Teissier's result. Precisely, Guibert defines the local Segre class of an ideal generated by a set of germs of holomorphic functions and, under a non-degeneracy condition, he describes such a class by Minkowski mixed volumes of polytopes. As a special case he obtains that for a certain class of monomial ideals the local Segre class is a normalized volume of the simplex generated by the origin and the vertices of the Newton polyhedron, see [9, 4.2]. By [4], the local Segre class is the so called j -multiplicity of the ideal. In 2013, Jeffries and Montaña [13] gave a different proof that the j -multiplicity of a monomial ideal is the normalized volume of the pyramid of the ideal.

The j -multiplicity of an ideal is different from zero if and only if its analytic spread is maximal, that is, equal to the Krull-dimension d of A . A result of Bivià-Ausina [6] states that the analytic spread diminished by one is the maximum of the dimensions of compact faces of the Newton polyhedron of I .

According to [3] the j -multiplicity is the first coordinate of the generalized Samuel multiplicity vector $c(I) = (c_0(I), \dots, c_d(I))$. Here we present and illustrate a conjecture which expresses the other components of $c(I)$ in terms of

the Newton polyhedron of I . Our conjecture holds for $c_0(I)$ by the known result of Guibert and of Jeffries and Montaña, and we shall prove it here for $c_{d-1}(I)$.

2. Generalized Samuel multiplicities

This section is a quick review of a generalization of Samuel's multiplicity by a sequence of numbers, the so-called generalized Samuel multiplicity, which we introduced in [3].

Let A be a d -dimensional Noetherian local ring (A, \mathfrak{m}) with unique maximal ideal \mathfrak{m} or a standard graded algebra $A = \bigoplus_{i \geq 0} A_i$ such that A_0 is a field and $\mathfrak{m} = (A_1)A$ is the unique homogeneous maximal ideal of A . Let $I \subset A$ be an arbitrary ideal (not necessarily \mathfrak{m} -primary).

In order to define the generalized Samuel multiplicity $c(I)$, consider $G_I(A) := \bigoplus_{j \geq 0} I^j / I^{j+1}$, the associated graded ring of A with respect to I and the bigraded ring

$$T = \bigoplus_{i, j \geq 0} T_{ij} = G_{\mathfrak{m}}(G_I(A)) = \bigoplus_{i, j \geq 0} \frac{\mathfrak{m}^i I^j + I^{j+1}}{\mathfrak{m}^{i+1} I^j + I^{j+1}},$$

where $T_{00} = A/\mathfrak{m} = K$ is a field.

Let $H^{(0,0)}(i, j) := \dim_K T_{ij}$ be the Hilbert function of the bigraded ring T and let

$$H^{(1,1)}(i, j) := \sum_{q=0}^j \sum_{p=0}^i H^{(0,0)}(p, q)$$

be its twofold sum transform. For both $i, j \gg 1$ this function becomes a polynomial in (i, j) , which can be written in the form

$$\sum_{k+l \leq d} a_{k,l}^{(1,1)} \binom{i+k}{k} \binom{j+l}{l}.$$

Following [3] define the *generalized Samuel multiplicity* to be the vector

$$\begin{aligned} (a_{0,d}^{(1,1)}, a_{1,d-1}^{(1,1)}, \dots, a_{d,0}^{(1,1)}) &=: (c_0(T), c_1(T), \dots, c_d(T)) =: c(T) \\ &=: (c_0(I), c_1(I), \dots, c_d(I)) =: c(I). \end{aligned}$$

The first coefficient $c_0(I)$ plays an important role as an intersection number and was introduced in [2]. It is called the j -multiplicity $j(I) := c_0(I)$. The last coefficient $c_d(I)$ is zero if $\dim(A/I) < d = \dim(A)$, see [3, Proposition 2.3]. In particular, if $I \neq 0$ is a monomial ideal in a polynomial ring, then $c_d(I) = 0$.

The generalized Samuel multiplicities depend only on the highest dimensional components of T , see [18] or [3, Proposition 1.2]:

Proposition 1. *Let Q_i , $i = 1, \dots, q$, be the highest dimensional primary ideals in a minimal primary decomposition of the zero ideal of the bigraded ring $T = G_m(G_I(R))$ and P_i their associated prime ideals. Then there is an equality of vectors*

$$c(I) = c(T) = \sum_{i=1}^q \text{length}(T_{P_i}) \cdot c(T/P_i) = \sum_{i=1}^q c(T/Q_i).$$

By analogy with the application of $c(I)$ to intersection theory, we shall call $c_k(T/Q_i) \neq 0$ a *movable contribution* to $c_k(I)$ if there is an integer $\ell > k$ such that $c_\ell(T/Q_i) \neq 0$.

3. A conjecture and some results

Let I be an ideal in $R = K[x_1, \dots, x_d] = K[\mathbf{x}]$ (K a field) minimally generated by the monomials

$$\mathbf{x}^{v_1} := x_1^{v_1(1)} \dots x_d^{v_1(d)}, \dots, \mathbf{x}^{v_r} := x_1^{v_r(1)} \dots x_d^{v_r(d)},$$

that is, $v_1 = (v_1(1), \dots, v_1(d)), \dots, v_r = (v_r(1), \dots, v_r(d))$ are the points of $\mathbb{Z}_{\geq 0}^d$ corresponding to the exponents of the generators of I .

The *Newton polyhedron* $\text{NP}(I)$ of I is defined as the convex hull of $\{v \in \mathbb{Z}_{\geq 0}^d \mid \mathbf{x}^v \in I\}$ in \mathbb{R}^d , that is,

$$\begin{aligned} \text{NP}(I) &:= \text{conv}(\{v \in \mathbb{Z}_{\geq 0}^d \mid x_1^{v(1)} \dots x_d^{v(d)} \in I\}) \\ &= \text{conv}(\{v_1, \dots, v_r\}) + \mathbb{R}_{\geq 0}^d, \end{aligned}$$

where $+$ denotes the Minkowski sum (for the equality see [15, Lemma 4.3]). It is well-known (see, for example, [12, Proposition 1.4.6]) that the set of integer lattice points of $\text{NP}(I)$ equals the exponent set of the integral closure of I , which is again a monomial ideal. Our conjectures involve both $\text{NP}(I)$ and the generalized Samuel multiplicities $c(I)$, which are also known to be invariant up to the integral closure of I , see [7, Proposition 2.7].

A hyperplane

$$H = \{v \in \mathbb{R}^d \mid \langle v, a \rangle = b\} \quad (\text{with } a \in \mathbb{R}_{\geq 0}^d, b \in \mathbb{R})$$

is called a *supporting hyperplane* of the Newton polyhedron $\text{NP}(I)$ if

$$\text{NP}(I) \subset H^+ = \{v \in \mathbb{R}^d \mid \langle v, a \rangle \geq b\} \text{ and } \text{NP}(I) \cap H \neq \emptyset.$$

A subset $F \subset \text{NP}(I)$ is called a *proper face* of $\text{NP}(I)$ if there exists a supporting hyperplane H of $\text{NP}(I)$ such that $F = \text{NP}(I) \cap H$. The boundary of $\text{NP}(I)$ is a set of faces of

dimension $d - 1$, called *facets* of $\text{NP}(I)$, some of them may be compact.

The zero-dimensional faces are called *vertices* or *extreme points* of $\text{NP}(I)$. We shall denote the set of vertices by $\text{vert}(I)$. Note that the monomials corresponding to the points in $\text{vert}(I)$ are part of the set of minimal generators of I , so by renumbering we will assume that

$$\text{vert}(I) = \{v_1, \dots, v_s\} \text{ with some } s \leq r,$$

hence

$$\text{NP}(I) = \text{conv}(\{v_1, \dots, v_r\}) + \mathbb{R}_{\geq 0}^d = \text{conv}(\{v_1, \dots, v_s\}) + \mathbb{R}_{\geq 0}^d.$$

The monomials corresponding to the points in $\text{vert}(I)$ generate the unique minimal monomial reduction ideal of I , see [16, Proposition 2.1].

Any face F can be described using its vertices and infinite-directions. Let e_j denote the unit vector with non-zero j th component, let H be a supporting hyperplane such that $F = \text{NP}(I) \cap H$ and let a be a normal vector to H . We call the coordinate direction e_j an *infinite-direction* of F if the j th component $a(j)$ of a is zero. If v_1, \dots, v_s are the vertices of F , then

$$F = \text{conv}(\{v_1, \dots, v_s\}) + \sum_{j:a(j)=0} \mathbb{R}_{\geq 0} e_j.$$

Often we shall write simply $v_1 \dots v_s$ instead of $\text{conv}(\{v_1, \dots, v_s\})$. Of course, the compact or bounded faces are precisely those that do not have infinite-directions e_j .

By the Minkowski-Weyl Theorem for convex polyhedra, there are uniquely determined finitely many closed half spaces

$$\begin{aligned} H_i^+ &= \{v \in \mathbb{R}^d \mid \langle v, a_i \rangle \geq b_i\} \text{ (with } a_i \in \mathbb{Z}_{\geq 0}^d, b_i \in \mathbb{Z}_{\geq 0}), \\ &i = 1, \dots, t, \end{aligned}$$

such that

$$\text{NP}(I) = H_1^+ \cap \dots \cap H_t^+.$$

Then $F_i := H_i \cap \text{NP}(I)$, $i = 1, \dots, t$, are the facets of $\text{NP}(I)$. We will assume that H_1, \dots, H_u are the hyperplanes corresponding to the unbounded facets and that H_{u+1}, \dots, H_t are those corresponding to the compact facets.

To each bounded facet $F = \text{conv}(\{v_{i_1}, \dots, v_{i_s}\})$ of $\text{NP}(I)$ we associate the polytope (or pyramid)

$$\hat{F} := \text{conv}(0, F) = \text{conv}(\{0, v_{i_1}, \dots, v_{i_s}\})$$

and denote by $\text{vol}_d(\hat{F})$ its d -dimensional volume and by

$$\text{Vol}_d(\hat{F}) := d! \text{vol}_d(\hat{F})$$

its *normalized volume*.

The normalized volumes of pyramids over projections of bounded faces of $\text{NP}(I)$ play a crucial role in our conjectures. Since it was known that $c_0(I)$ equals the sum of the normalized d -dimensional volumes of the pyramids over the bounded facets of $\text{NP}(I)$, our guess was that $c_{d-(k+1)}(I)$ should be a sum of $(k+1)$ -dimensional volumes coming from the bounded faces F^k of dimension k . We succeeded in proving this for $k=0$ by projecting the vertices of $\text{NP}(I)$ on the

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coordinate axes, see Theorem 2. Thus, in order to obtain $c_{d-(k+1)}(I)$, we tried to sum up the normalized $(k+1)$ -dimensional volumes of the pyramids over the projections of the bounded faces F^k of $\text{NP}(I)$ on coordinate $(k+1)$ -planes. This did not work well, and by [5, Section 2.2] we realized that one should consider only projections along the infinite-directions of facets F^{d-1} that contain F^k . Then we refined our guess by computing many examples. To formulate our conjectures, we proceed as follows.

We call a facet $F \subset \text{NP}(I)$ an *h-unbounded facet* if the normal vector a to its supporting hyperplane has at least $h > 0$ coordinates $a(j)$ which are zero, that is, if the facet has at least h infinite-directions e_j .

Let $\mathcal{F}(k)$ be the set of all $(d-(k+1))$ -unbounded facets containing at least one k -dimensional compact face F^k , $0 \leq k \leq d-2$. We define $\mathcal{F}(d-1)$ to be the set of all compact or bounded facets of $\text{NP}(I)$.

If $F^{d-1} \in \mathcal{F}(k)$ and F^k is a k -dimensional compact face contained in F^{d-1} , then we associate to the pair (F^k, F^{d-1}) a $(k+1)$ -dimensional normalized volume $\text{Vol}(F^k, F^{d-1})$ as follows. Let $L \subset \{1, \dots, d\}$ be such that $\{e_\ell : \ell \in L\}$ is the set of all infinite-directions of F^{d-1} and let M_1, \dots, M_q be all subsets of L consisting of exactly $d-(k+1)$ infinite-directions. For $i = 1, \dots, q$, let

$$\pi_i : \mathbb{R}^d \rightarrow \sum_{1 \leq j \leq d, j \notin M_i} \mathbb{R}e_j \cong \mathbb{R}^{k+1}$$

be the orthogonal projection which removes from $v \in \mathbb{R}^d$ the coordinates $v(m)$ with $m \in M_i$. Then $\pi_i(F^k)$ is a polytope of dimension at most k . By renumbering, assume that $\dim(\pi_i(F^k)) = k$ for $1 \leq i \leq p$ and $\dim(\pi_i(F^k)) < k$ for $p+1 \leq i \leq q$.



The volume associated to the pair (F^k, F^{d-1}) is

$$\text{Vol}(F^k, F^{d-1}) := \min_{1 \leq i \leq p} \text{Vol}_{k+1}(\text{conv}(\{0, \pi_i(F^k)\})). \quad (3-1)$$

Conjecture 1. For each $k = 0, \dots, d-1$, the generalized Samuel multiplicity of a monomial ideal I is

$$c_{d-(k+1)}(I) = \sum_{F^{d-1} \in \mathcal{F}(k)} \min_{F^k} \{\text{Vol}(F^k, F^{d-1})\}, \quad (3-2)$$

where the minimum is taken over all compact faces F^k of $\text{NP}(I)$ that are contained in the facet F^{d-1} .

In the extremal cases $k = d-1$ and $k = 0$ the formula (3.2) can be simplified.

The case $k = d-1$. Since $\mathcal{F}(d-1)$ is the set of compact facets of $\text{NP}(I)$, $F^{d-1} \in \mathcal{F}(d-1)$ does not have infinite-directions. Hence L above is empty, $q = 1$, $M_1 = \emptyset$ and π_1 is the identity map on \mathbb{R}^d . The only $(d-1)$ -dimensional compact face contained in F^{d-1} is F^{d-1} itself and

$$\begin{aligned} \text{Vol}(F^{d-1}, F^{d-1}) &:= \text{Vol}_d(\text{conv}(\{0, \pi_1(F^{d-1})\})) \\ &= \text{Vol}_d(\text{conv}(\{0, F^{d-1}\})). \end{aligned}$$

Then the formula (3-2) reads

$$c_0(I) = \sum_{F^{d-1} \in \mathcal{F}(d-1)} \text{Vol}_d(\text{conv}(\{0, F^{d-1}\})). \quad (3-3)$$

The case $k = 0$. At first we describe $\mathcal{F}(0)$.

Proposition 2. Let $I \subset K[x_1, \dots, x_d] = K[\mathbf{x}]$ be an ideal generated by the monomials $\mathbf{x}^{v_1}, \dots, \mathbf{x}^{v_r}$. For $j = 1, \dots, d$, set $m_j := \min\{v_1(j), \dots, v_r(j)\}$ and

$$F_j := \text{conv}(\{v \in \text{vert}(I) \mid v(j) = m_j\}) + \sum_{1 \leq i \leq d, i \neq j} \mathbb{R}_{\geq 0} e_i.$$

Then $\mathcal{F}(0) = \{F_1, \dots, F_d\}$.

Proof. Since each $v \in \text{NP}(I)$ is the sum of a convex combination of the vertices v_1, \dots, v_s of $\text{NP}(I)$ and of some $w \in \mathbb{R}_{\geq 0}^d$, we have

$$v(j) \geq \min\{v_1(j), \dots, v_s(j)\} + w(j) \geq \min\{v_1(j), \dots, v_s(j)\},$$

hence

$$m_j := \min\{v_1(j), \dots, v_r(j)\} = \min\{v_1(j), \dots, v_s(j)\} = \min_{v \in \text{NP}(I)} \{v(j)\}.$$

It follows that F_1, \dots, F_d are precisely the $(d-1)$ -unbounded facets of $\text{NP}(I)$, that is, $\mathcal{F}(0) = \{F_1, \dots, F_d\}$. \square

By the preceding proposition, for $k = 0$ the formula (3-2) reads

$$c_{d-1}(I) = \sum_{j=1}^d \min_v \{\text{Vol}(v, F_j)\}, \quad (3-4)$$

where the minimum is taken over all vertices v of $\text{NP}(I)$ that are contained in the facet F_j . In order to compute $\text{Vol}(v, F_j)$, note that each F_j has $d-1$ infinite-directions, more precisely, $L = \{1, \dots, \hat{j}, \dots, d\}$, hence $q = 1$, $L = M_1$, and $\pi_1 : \mathbb{R}^d \rightarrow \mathbb{R}e_j$ sends each $v \in F_j$ to its j th coordinate $v(j) = m_j$. It follows that $\text{Vol}(v, F_j) = \text{Vol}_1(\text{conv}(\{0, \pi_1(v)\})) = v(j) = m_j$ for each vertex v of $\text{NP}(I)$ that is contained in F_j , and no minimum has to be taken in (3-4). Thus the formula of Conjecture 1 becomes

$$c_{d-1}(I) = m_1 + \dots + m_d. \quad (3-5)$$

Conjecture 2. With the notation of Proposition 1, for each $k = 0, \dots, d-1$, there is a one-to-one correspondence between the non-zero $c_{d-(k+1)}(T/Q_i)$ and the non-zero summands $\min_{F^k} \{\text{Vol}(F^k, F^{d-1})\}$ in the formula of Conjecture 1 such that the corresponding numbers are equal.

In particular, the number of compact facets of $\text{NP}(I)$ is equal to the number of d -dimensional associated prime ideals of T that contain $\mathfrak{m} = (x_1, \dots, x_d)\mathbb{R}$.

Moreover, if $\text{Vol}(F^k, F^{d-1}) \neq 0$ contributes to $c_{d-(k+1)}(I)$ and can be obtained by more than one projection \mathfrak{p}_α , then it is a movable contribution.

Note that in general, if K is algebraically closed and

$$T = G_{\mathfrak{m}}(G_I(\mathbb{R})) \cong K[x_1, \dots, x_d, y_1, \dots, y_r]/\mathfrak{n},$$

then the bigraded ideal \mathfrak{n} is a binomial but not a monomial ideal, see [8, Corollary 1.9].

Our conjectures are confirmed by many examples, but so far we do not have a proof except for Conjecture 1 in the extremal cases $k = 0$ (formula (3-3)) and $k = d-1$ (formula (3-5)), as it is stated in the following two theorems.

Theorem 1. (Jeffries and Montaña [13, Theorem 3.2]). If $I \subset K[x_1, \dots, x_d]$ is a monomial ideal and F_{r+1}, \dots, F_t

are the compact facets of the Newton polyhedron $\text{NP}(I)$, then

$$c_0(I) = \sum_{i=r+1}^t d! \text{vol}(\hat{F}_i) = \sum_{i=r+1}^t \text{Vol}(\hat{F}_i).$$

Theorem 2. Let I be a monomial ideal in $R = K[x_1, \dots, x_d] = K[\mathbf{x}]$ generated by $\mathbf{x}^{v_1}, \dots, \mathbf{x}^{v_r}$ and $m_j = \min\{v_1(j), \dots, v_r(j)\}, j = 1, \dots, d$. Then

$$c_{d-1}(I) = m_1 + \dots + m_d.$$

Proof. By [3, Proposition 2.3], $c_{d-1}(I) \neq 0$ if and only if $\dim R/I = d-1$. If $\dim R/I < d-1$, then none of the variables x_j appears in all monomials generating I , hence $m_j = 0$ for all j , and the result is true. If $\dim R/I = d-1$, then again by [3, loc. cit.],

$$c_{d-1}(I) = \sum_P e(IR_P) \cdot e(R/P),$$

where P runs through all $(d-1)$ -dimensional associated prime ideals of R/I , that is, prime ideals of the form (x_j) for some j . Therefore $IR_P = (x_j^{m_j})R_P$ and $e(IR_P) = m_j$. The $(d-1)$ -dimensional part of the primary decomposition of I is $(x_1^{m_1}) \cap (x_2^{m_2}) \cap \dots \cap (x_d^{m_d})$, which is of degree $m_1 + \dots + m_d$. \square

Corollary 3. Let $I \subset K[x_1, x_2]$ be a monomial ideal generated by $\mathbf{x}^{v_1}, \dots, \mathbf{x}^{v_r}$. We assume that v_1, \dots, v_s ($1 \leq s \leq r$) are the vertices of the Newton polyhedron $\text{NP}(I)$ numbered such that $v_1(1) > \dots > v_s(1)$, hence $v_1(2) < \dots < v_s(2)$. Then the set of unbounded facets of $\text{NP}(I)$ is

$$\mathcal{F}(0) = \{F_1 = v_s + \mathbb{R}_{\geq 0} e_2, F_2 = v_1 + \mathbb{R}_{\geq 0} e_1\},$$

the set of bounded facets of $\text{NP}(I)$ is

$$\mathcal{F}(1) = \{v_1 v_2, v_2 v_3, \dots, v_{s-1} v_s\} (= \emptyset \text{ if } s = 1),$$

and the generalized Samuel multiplicities are

$$c_0(I) = \det \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \dots + \det \begin{pmatrix} v_{s-1} \\ v_s \end{pmatrix},$$

$$c_1(I) = v_1(2) + v_s(1), c_2 = 0.$$

Proof. The corollary follows immediately from Proposition 2, Theorem 1, Theorem 2 and [3, Proposition 2.3 (i)]. \square

4. Examples

We illustrate the theorems and the conjecture by examples of monomial ideals $I \subset R = K[x_1, \dots, x_d]$, K an arbitrary field. We set $\mathfrak{m} := (x_1, \dots, x_d)R$ and $T := G_{\mathfrak{m}}(G_I(R))$. All the examples will show a close relation between the summands in the formula of Conjecture 1 and the highest dimensional primary components of T and confirm Conjecture 2.

In our first two examples we consider monomial ideals in polynomial rings of dimension two. Note that Conjecture 1

holds in this case (formulas (3-3) and (3-5)), see the Theorems 1 and 2 and Corollary 3.

Example 1 (Figure 1). We begin with the simplest case of a monomial ideal generated by one monomial in two variables in order to illustrate Corollary 3 if the Newton polyhedron has only one vertex.

Let $I = (x^3 y^2) \subset R = K[x, y]$. We have

$$c(I) = (c_0(I), c_1(I), c_2(I)) = (0, 5, 0) = 2 \cdot (0, 1, 0) + 3 \cdot (0, 1, 0),$$

where the summands are the contributions of the components of the bigraded ring $G_{\mathfrak{m}}(G_I(R))$, see Proposition 1.

The Newton polyhedron $\text{NP}(I)$ has only one vertex $v = (3, 2)$ and two (unbounded) facets $F_1 = v + \mathbb{R}_{\geq 0} e_2$ and $F_2 = v + \mathbb{R}_{\geq 0} e_1$ (see Figure 1), hence $\mathcal{F}(1) = \emptyset$ and $c_0(I) = 0$. We have $\mathcal{F}(0) = \{F_1, F_2\}$ and $\text{Vol}(v, F_1) = 3$ and $\text{Vol}(v, F_2) = 2$, hence $c_1(I) = 5$.

Example 2 (Figure 2). This example is to illustrate Corollary 3 in the case of two and more vertices. The vertices are numbered as in the corollary.

Let $I = (x^6 y, x^4 y^2, x^2 y^5, x^3 y^4) \subset R = K[x, y]$. We have

$$c(I) = (c_0(I), c_1(I), c_2(I)) = (24, 3, 0) = 16 \cdot (1, 0, 0) + 8 \cdot (1, 0, 0) + 2 \cdot (0, 1, 0) + (0, 1, 0),$$

where the summands are the contributions of the components of the bigraded ring $G_{\mathfrak{m}}(G_I(R))$, see Proposition 1.

The Newton polyhedron $\text{NP}(I)$ has three vertices $v_1 = (6, 1), v_2 = (4, 2), v_3 = (2, 5)$, two unbounded facets $F_1 = v_3 + \mathbb{R}_{\geq 0} e_2, F_2 = v_1 + \mathbb{R}_{\geq 0} e_1$ and two bounded facets: the line segments $F_3 = \text{conv}(v_1, v_2), F_4 = \text{conv}(v_2, v_3)$ (see Figure 2), hence $\mathcal{F}(1) = \{F_3, F_4\}$ and

$$c_0(I) = \text{Vol}(\text{conv}(0, F_3)) + \text{Vol}(\text{conv}(0, F_4)) = \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} = 8 + 16.$$

We have $\mathcal{F}(0) = \{F_1, F_2\}$ and

$$c_1(I) = \text{Vol}(v_3, F_1) + \text{Vol}(v_1, F_2) = 1 + 2 = 3.$$

Example 3 (Figures 3-5). The purpose of this example is twofold. It shows that there can be compact faces of $\text{NP}(I)$ that do not contribute to the generalized Samuel multiplicity $c(I)$. Furthermore it aims to discuss a movable contribution (to $c_1(I)$). In this example, because of the movable contribution, the number of the highest dimensional components of T is one less than the number of summands in the conjectured formula (3-2).

Let $I = (x^2 y, x^2 z, x y^2, x z^2) \subset K[x, y, z]$. By a computer computation (using [1]) we have

$$c(I) = (c_0(I), c_1(I), c_2(I), c_3(I)) = (9, 3, 1, 0) = 3 \cdot (3, 0, 0, 0) + (0, 1, 0, 0) + (0, 2, 1, 0),$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T , see Proposition 1. The contribution 2 in the last vector is a

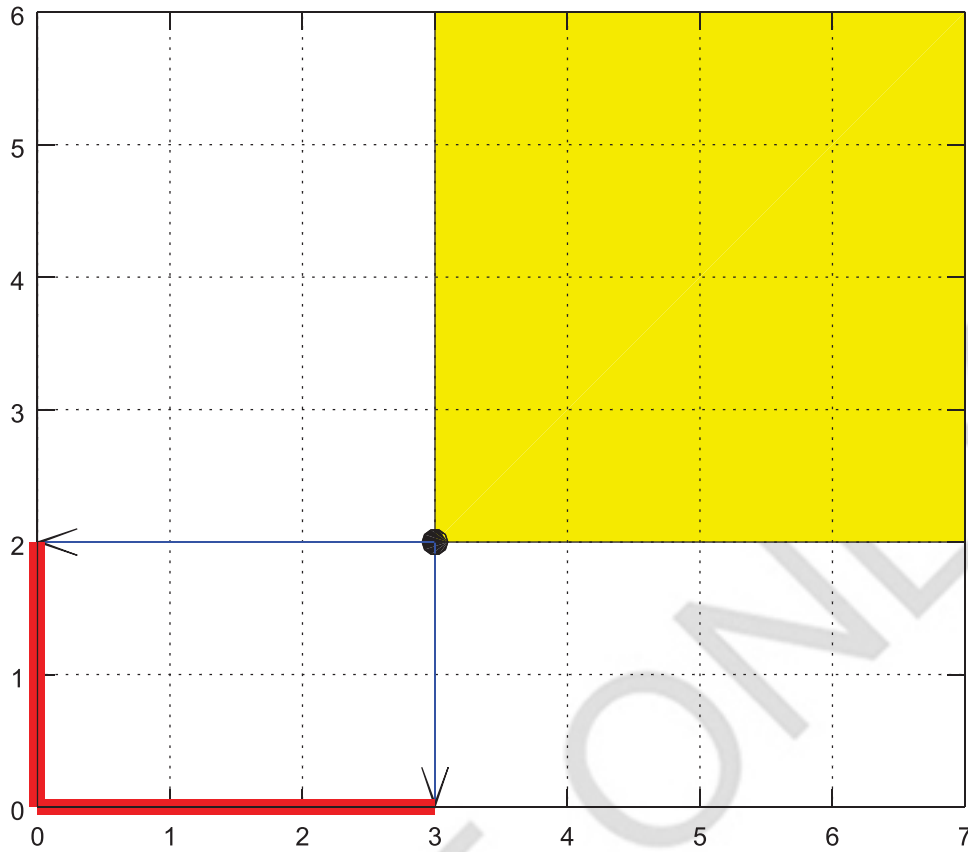


Figure 1. Projection along the infinite-directions of the facets gives $c_1(I)$, which is the red distance.

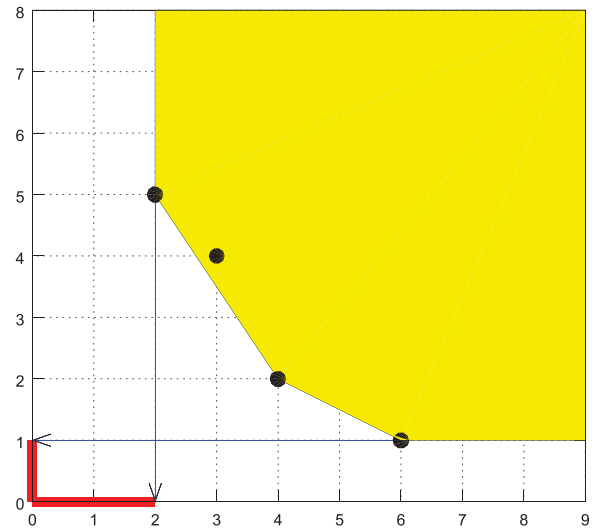
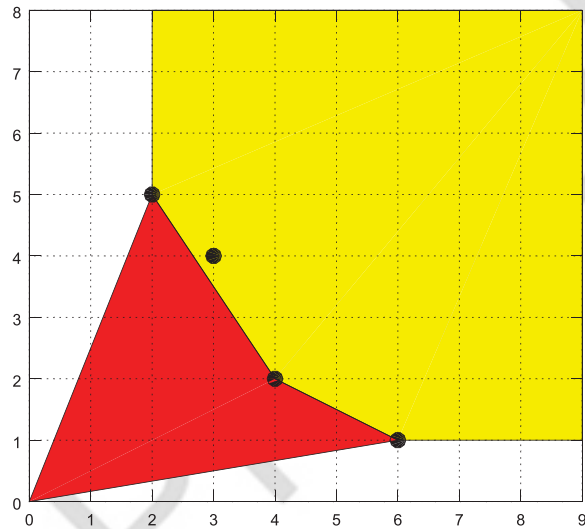


Figure 2. The red area is $c_0(I)/2$, the red distance $c_1(I)$.

movable contribution to $c_1(I)$. This can be read off also from the Newton polyhedron $NP(I)$, see Figure 5.

According to the program Germenes [14], the compact faces of $NP(I)$ are the vertices $v_1 = (2, 1, 0)$, $v_2 = (2, 0, 1)$, $v_3 = (1, 2, 0)$, $v_4 = (1, 0, 2)$, the line segments $v_1v_2, v_1v_3, v_2v_4, v_3v_4$ and the quadrilateral facet $v_1v_2v_4v_3$. The unbounded facets are

$$F_1 = v_3v_4 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3, \quad F_2 = v_2v_4 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_3,$$

$$F_3 = v_1v_3 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2, \quad F_4 = v_1v_2 + \mathbb{R}_{\geq 0} e_1.$$

We observe that the set of bounded facets is $\mathcal{F}(2) = \{v_1v_2v_4v_3\}$ and

$$c_0(I) = \text{Vol}(\text{conv}(0, v_1, v_2, v_4, v_3)) = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 3 + 6 = 9,$$

see Figure 4.

The set of 1-unbounded facets that contain a compact one-dimensional face is $\mathcal{F}(1) = \{F_1, F_2, F_3, F_4\}$, and we have

$$\text{Vol}(v_1v_2, F_4) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

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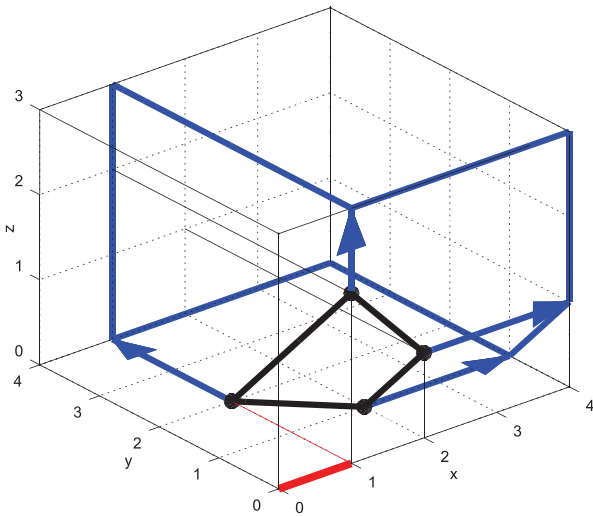
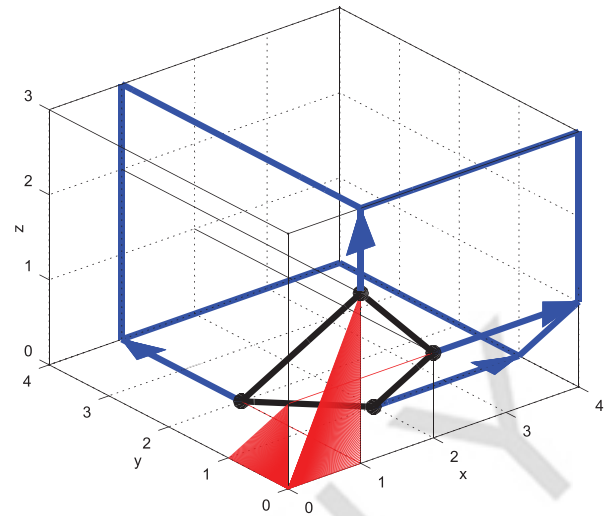


Figure 3. Infinite-directions (blue arrows) of the unbounded facets, $c_2(I)$ (red distance) and $c_1(I)/2$ (red area).



$$c_2(I) = \min\{\text{Vol}(v_3, F_1), \text{Vol}(v_4, F_1)\} + \text{Vol}(v_4, F_2) + \min\{\text{Vol}(v_1, F_3), \text{Vol}(v_3, F_3)\} = 1 + 0 + 0 = 1.$$

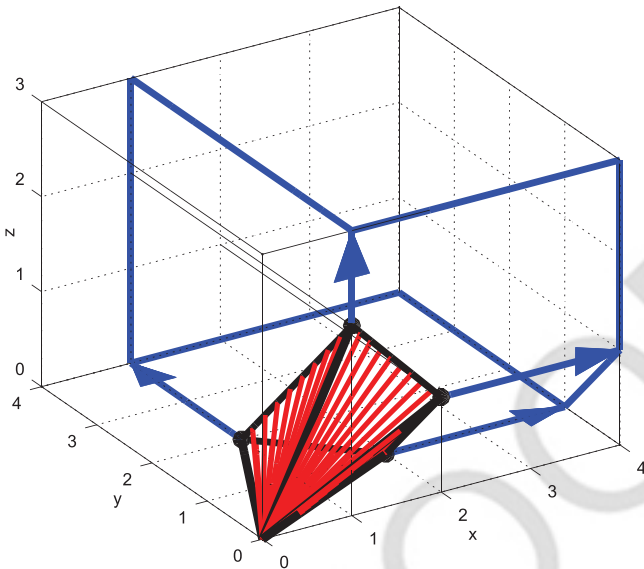


Figure 4. The volume of the red pyramid is $c_0(I)/6$.

$$\text{Vol}(v_1 v_3, F_3) = \min \left\{ \left| \begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 \end{array} \right| \right\} = 0,$$

$$\text{Vol}(v_2 v_4, F_2) = \min \left\{ \left| \begin{array}{cc|cc} 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \end{array} \right| \right\} = 0,$$

$$\text{Vol}(v_3 v_4, F_1) = \min \left\{ \left| \begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 \end{array} \right| \right\} = 2$$

(the last minimum is given by two different projections and is a movable contribution, see Figure 5), hence

$$\begin{aligned} c_1(I) &= \text{Vol}(v_1 v_2, F_4) + \text{Vol}(v_1 v_3, F_3) + \text{Vol}(v_2 v_4, F_2) \\ &\quad + \text{Vol}(v_3 v_4, F_1) \\ &= 1 + 0 + 0 + 2 = 3. \end{aligned}$$

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$. We have $\text{Vol}(v_3, F_1) = 1, \text{Vol}(v_4, F_1) = 1, \text{Vol}(v_4, F_2) = 0, \text{Vol}(v_1, F_3) = 0, \text{Vol}(v_3, F_3) = 0$, hence

Example 4 (Figure 6). Here we give a monomial ideal I such that its Newton polyhedron has compact edges that do not lie on any 1-unbounded facet and therefore, according to Conjecture 1, have not to be taken into account in order to compute $c_1(I)$.

Let $I = (xy^4z^5, x^2y^5z^2, xy^5z^3, x^5yz^2, x^2yz^5, x^5y^2z) \subset K[x, y, z]$. By a computer computation we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I)) = (168, 26, 3, 0) = \\ &= 19 \cdot (1, 0, 0, 0) + 103 \cdot (1, 0, 0, 0) + 22 \cdot (1, 0, 0, 0) + \\ &\quad + 24 \cdot (1, 0, 0, 0) + 7 \cdot (0, 1, 0, 0) + (0, 3, 1, 0) + \\ &\quad + 8 \cdot (0, 1, 0, 0) + (0, 1, 0, 0) + 4 \cdot (0, 1, 0, 0) + \\ &\quad + 3 \cdot (0, 1, 0, 0) + (0, 0, 1, 0) + (0, 0, 1, 0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T , see Proposition 1. The contribution 3 in the sixth vector is a movable contribution to $c_1(I)$.

The program Germenes [14] gives the following description of the Newton polyhedron $\text{NP}(I)$. The compact faces of $\text{NP}(I)$ are the 6 vertices $v_1 = (1, 4, 5), v_2 = (2, 5, 2), v_3 = (1, 5, 3), v_4 = (5, 1, 2), v_5 = (2, 1, 5), v_6 = (5, 2, 1)$, the 9 line segments $v_4 v_6, v_2 v_6, v_5 v_6, v_4 v_5, v_3 v_6, v_3 v_2, v_3 v_5, v_1 v_5, v_1 v_3$ and the 4 triangles (bounded facets) $v_4 v_5 v_6, v_2 v_3 v_6, v_3 v_5 v_6, v_1 v_3 v_5$. There are 7 unbounded facets:

$$\begin{aligned} F_1 &= v_1 v_3 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3, \quad F_2 = v_4 v_5 + \mathbb{R}_{\geq 0} e_1 \\ &\quad + \mathbb{R}_{\geq 0} e_3, \\ F_3 &= v_6 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2, \quad F_4 = v_1 v_5 + \mathbb{R}_{\geq 0} e_3, \\ F_5 &= v_2 v_3 + \mathbb{R}_{\geq 0} e_2, \quad F_6 = v_2 v_6 + \mathbb{R}_{\geq 0} e_2, \\ F_7 &= v_4 v_6 + \mathbb{R}_{\geq 0} e_1. \end{aligned}$$

From the set of bounded facets $\mathcal{F}(2) = \{v_4 v_5 v_6, v_2 v_3 v_6, v_3 v_5 v_6, v_1 v_3 v_5\}$ we get

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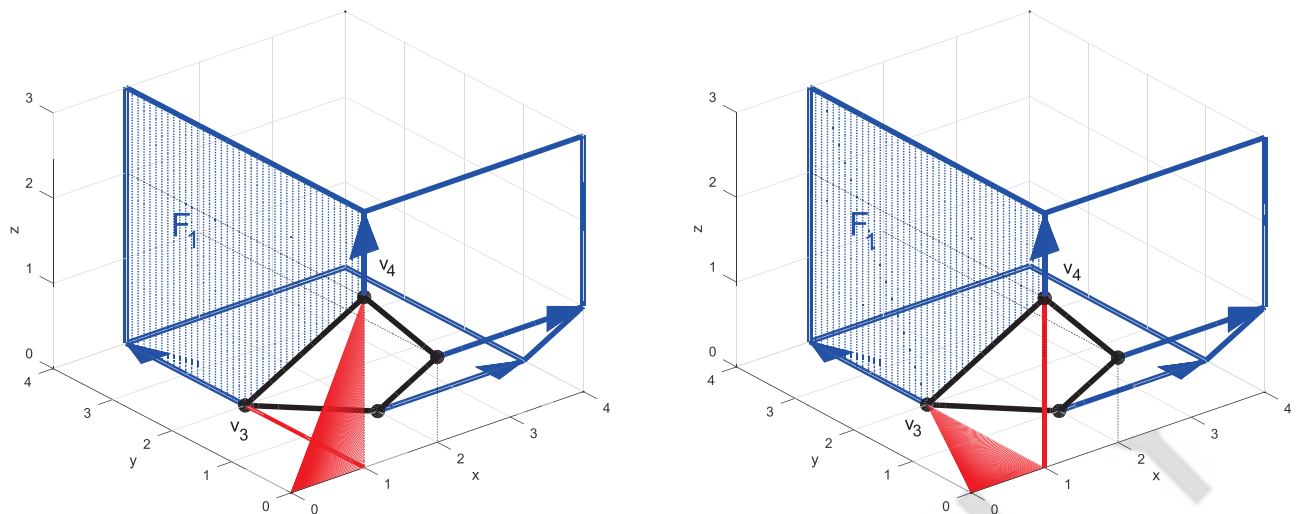


Figure 5. A movable contribution (to $c_1(I)/2$, red area) can be realized by at least two projections. Here the volume associated to the pair (v_3v_4, F_1) is obtained both by the projection of v_3v_4 along the y -axis and the z -axis, that is, along the two infinite-directions of F_1 .

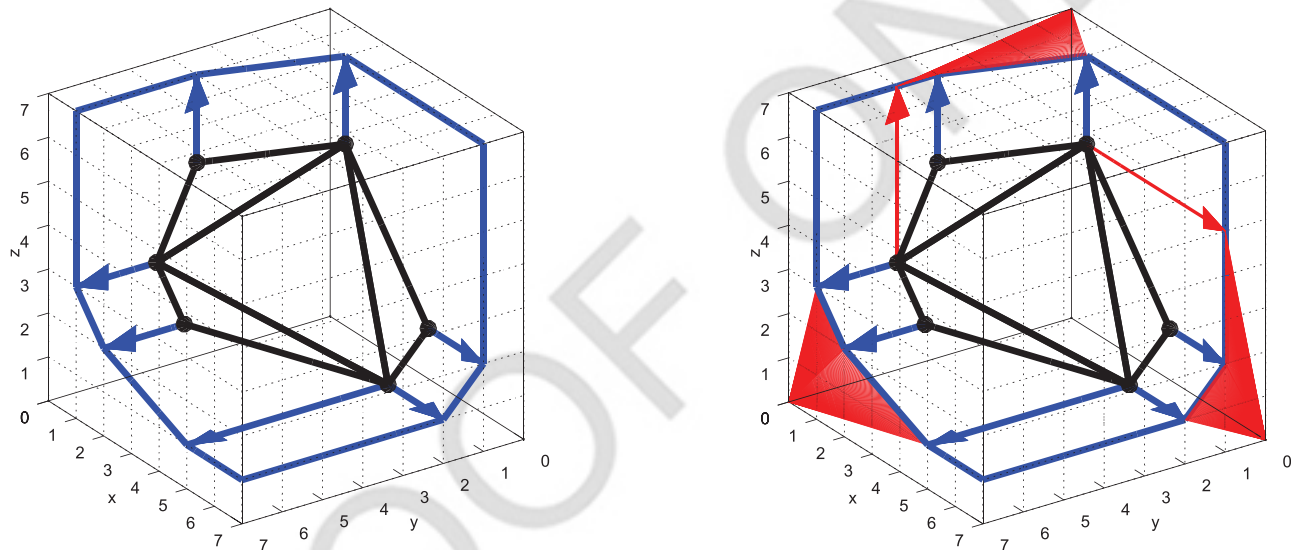


Figure 6. $NP(I)$ with compact (black) and unbounded (blue) facets; projections of compact edges along infinite-directions (blue arrows) give $c_1(I)/2$ (red area).

$$\begin{aligned}
 c_0(I) &= \text{Vol}(\text{conv}(0, v_4, v_5, v_6)) + \text{Vol}(\text{conv}(0, v_2, v_3, v_6)) + \\
 &\quad + \text{Vol}(\text{conv}(0, v_3, v_5, v_6)) + \text{Vol}(\text{conv}(0, v_1, v_3, v_5)) = \\
 &= \begin{vmatrix} 5 & 1 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 5 & 2 \\ 1 & 5 & 3 \\ 5 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 3 \\ 2 & 1 & 5 \\ 5 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 & 5 \\ 2 & 1 & 5 \\ 1 & 5 & 3 \end{vmatrix} = \\
 &= 24 + 22 + 103 + 19 = 168.
 \end{aligned}$$

We have $\mathcal{F}(1) = \{F_1, F_2, F_4, F_5, F_6, F_7\}$ and

$$\begin{aligned}
 \text{Vol}(v_1v_3, F_1) &= \min\left\{\begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix}\right\} = \min\{1, 2\} = 1, \\
 \text{Vol}(v_4v_5, F_2) &= \min\left\{\begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix}\right\} = \min\{3, 3\} = 3
 \end{aligned}$$

(here the minimum is attained twice, that is, by two different projections which indicates a movable contribution),

$$\text{Vol}(v_1v_5, F_4) = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7, \quad \text{Vol}(v_2v_3, F_5) = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4,$$

$$\text{Vol}(v_2v_6, F_6) = \begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix} = 8, \quad \text{Vol}(v_4v_6, F_7) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3,$$

hence

$$\begin{aligned}
 c_1(I) &= \text{Vol}(v_1v_3, F_1) + \text{Vol}(v_4v_5, F_2) + \text{Vol}(v_1v_5, F_4) + \\
 &\quad + \text{Vol}(v_2v_3, F_5) + \text{Vol}(v_2v_6, F_6) + \text{Vol}(v_4v_6, F_7) = \\
 &= 1 + 3 + 7 + 4 + 8 + 3 = 26.
 \end{aligned}$$

We observe that the compact 1-dimensional faces v_5v_6 , v_3v_6 , v_3v_5 , that is, the edges of the big triangle $v_3v_5v_6$, do not contribute to $c_1(I)$ since they lie on no 1-unbounded facet. Moreover, as in the previous example, there is a movable contribution, namely $\text{Vol}(v_4v_5, F_2) = 3$.

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$, and we have $\text{Vol}(v_1, F_1) = 1$, $\text{Vol}(v_3, F_1) = 1$, $\text{Vol}(v_4, F_2) = 1$, $\text{Vol}(v_5, F_2) = 1$, $\text{Vol}(v_6, F_3) = 1$, hence

$$\begin{aligned}
 c_2(I) &= \min\{\text{Vol}(v_1, F_1), \text{Vol}(v_3, F_1)\} \\
 &\quad + \min\{\text{Vol}(v_4, F_2), \text{Vol}(v_5, F_2)\} + \\
 &\quad + \text{Vol}(v_6, F_3) = 1 + 1 + 1 = 3.
 \end{aligned}$$

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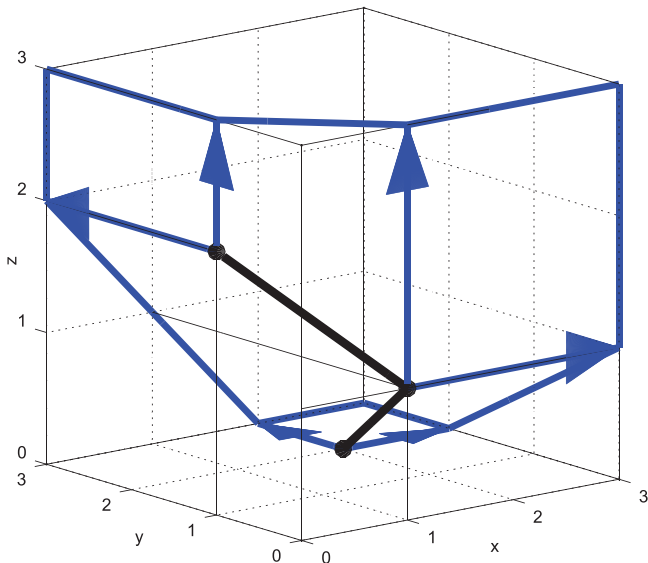


Figure 7. The triangle defined by 3 affinely independent vertices is not a compact facet.

Example 5 (Figure 7). The scope of this example is to give a monomial ideal in dimension three such that its Newton polyhedron has three affinely independent vertices v_1, v_2, v_3 , but its local Segre class is zero and thus not equal to the normalized volume of the simplex generated by the origin and v_1, v_2, v_3 as erroneously stated in [9, 4.2].

Let $I = (xz, x^2y^2, yz^2) \subset K[x, y, z]$. By a computer computation we have

$$c(I) = (c_0(I), c_1(I), c_2(I), c_3(I)) = (0, 7, 0, 0) = 2 \cdot (0, 1, 0, 0) + 2 \cdot (0, 1, 0, 0) + 2 \cdot (0, 1, 0, 0) + (0, 1, 0, 0),$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T , see Proposition 1.

A computation with the program Germenes [14] shows that the compact faces of $\text{NP}(I)$ are the vertices $v_1 = (1, 0, 1), v_2 = (2, 2, 0), v_3 = (0, 1, 2)$ and the line segments v_1v_2, v_1v_3 . There are no compact facets, but 6 unbounded facets:

$$\begin{aligned} F_1 &= v_3 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3, & F_2 &= v_1 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_3, \\ F_3 &= v_2 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2, & F_4 &= v_1v_2 + \mathbb{R}_{\geq 0} e_1, \\ F_5 &= v_1v_3 + \mathbb{R}_{\geq 0} e_3, & F_6 &= v_1v_2v_3 + \mathbb{R}_{\geq 0} e_2. \end{aligned}$$

We observe that $\mathcal{F}(2) = \emptyset$, hence

$$c_0(I) = 0 \neq \text{Vol}_3(\text{conv}(\{0, v_1, v_2, v_3\})) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 6.$$

This means that v_1, v_2, v_3 are affinely independent, but the local Segre class is zero and not equal to the normalized volume of the simplex generated by the origin and v_1, v_2, v_3 as claimed in [9, 4.2]. The reason is that the triangle $v_1v_2v_3$ is not a compact facet of $\text{NP}(I)$.

The set of 1-unbounded facets which contain a compact 1-dimensional face is $\mathcal{F}(1) = \{F_4, F_5, F_6\}$, and we have

$$\begin{aligned} \text{Vol}(v_1v_2, F_4) &= \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2, & \text{Vol}(v_1v_2, F_6) &= \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2, \\ \text{Vol}(v_1v_3, F_5) &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, & \text{Vol}(v_1v_3, F_6) &= \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2, \end{aligned}$$

hence

$$\begin{aligned} c_1(I) &= \text{Vol}(v_1v_2, F_4) + \text{Vol}(v_1v_2, F_6) + \text{Vol}(v_1v_3, F_5) \\ &\quad + \text{Vol}(v_1v_3, F_6) \\ &= 2 + 2 + 1 + 2 = 7. \end{aligned}$$

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$ and we have

$$\text{Vol}(v_3, F_1) = 0, \quad \text{Vol}(v_1, F_2) = 0, \quad \text{Vol}(v_2, F_3) = 0,$$

hence $c_2(I) = 0$.

Example 6. With the notation of the paragraph before Conjecture 1, this example is to have pairs (F^k, F^{d-1}) such that $\dim(\pi_i(F^k)) < k$ for some of the projections π_i . In our example we have $k = 1, d = 4$, and the pairs (v_1v_2, F_2) and (v_1v_2, F_3) have the desired property.

Let $d = 4, I = (x_1^3x_2x_3x_4, x_1x_2x_3x_4^2)$. By a computer computation we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I), c_4(I)) = (0, 0, 7, 4, 0) = \\ &= 5 \cdot (0, 0, 1, 0, 0) + (0, 0, 1, 0, 0) + (0, 0, 1, 0, 0) + \\ &\quad + (0, 0, 0, 1, 0) + (0, 0, 0, 1, 0) + (0, 0, 0, 1, 0) + \\ &\quad + (0, 0, 0, 1, 0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T , see Proposition 1.

The compact faces of the Newton polyhedron $\text{NP}(I)$ are the vertices $v_1 = (3, 1, 1, 1), v_2 = (1, 1, 1, 2)$ and the line segment v_1v_2 . There are no compact facets, but 5 unbounded facets:

$$\begin{aligned} F_1 &= v_2 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_4, \\ F_2 &= v_1v_2 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_4, \\ F_3 &= v_1v_2 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_4, \\ F_4 &= v_1 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3, \\ F_5 &= v_1v_2 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3. \end{aligned}$$

Obviously $\mathcal{F}(3) = \mathcal{F}(2) = \emptyset$, hence $c_0(I) = c_1(I) = 0$. We have $\mathcal{F}(1) = \{F_2, F_3, F_5\}$ and

$$\begin{aligned} \text{Vol}(v_1v_2, F_2) &= \min \left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right\} = \min\{2, 1\} = 1, \\ \text{Vol}(v_1v_2, F_3) &= \min \left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right\} = \min\{2, 1\} = 1, \\ \text{Vol}(v_1v_2, F_5) &= \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5. \end{aligned}$$

We observe that in the computations of $\text{Vol}(v_1v_2, F_2)$ and $\text{Vol}(v_1v_2, F_3)$ the projection of the line segment v_1v_2 on the $\{x_2, x_3\}$ -plane gives the point $(1, 1)$ and must not be considered. We obtain

$$\begin{aligned} c_2(I) &= \text{Vol}(v_1v_2, F_2) + \text{Vol}(v_1v_2, F_3) + \text{Vol}(v_1v_2, F_5) \\ &= 1 + 1 + 5 = 7. \end{aligned}$$

The set of 3-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3, F_4\}$, and we have

$$\begin{aligned} \text{Vol}(v_2, F_1) &= 1, & \text{Vol}(v_1, F_2) &= 1, & \text{Vol}(v_2, F_2) &= 1, \\ \text{Vol}(v_1, F_3) &= 1, & \text{Vol}(v_2, F_3) &= 1, & \text{Vol}(v_1, F_4) &= 1, \end{aligned}$$

hence

$$\begin{aligned} c_3(I) &= \text{Vol}(v_2, F_1) + \min\{\text{Vol}(v_1, F_2), \text{Vol}(v_2, F_2)\} + \\ &\quad + \min\{\text{Vol}(v_1, F_3), \text{Vol}(v_2, F_3)\} + \text{Vol}(v_1, F_4) = \\ &= 1 + 1 + 1 + 1 = 4. \end{aligned}$$

Example 7. This example is to show that in the formula of Conjecture 1 with $k > 0$ one needs to take the minimum over all compact faces F^k of $\text{NP}(I)$ that are contained in the facet F^{d-1} . In our example we have $d = 5$, $k = 1$ and three compact 1-dimensional faces lie on the same 3-unbounded facet F_3 or F_4 .

Let $d = 5$, $I = (x_1^3 x_2 x_3 x_4 x_5, x_1 x_2^2 x_3 x_4 x_5, x_1 x_2 x_3 x_4 x_5^5)$. By a computer computation we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I), c_4(I), c_5(I)) = (0, 0, 26, 6, 5, 0) = \\ &= 22 \cdot (0, 0, 1, 0, 0, 0) + 2 \cdot (0, 0, 1, 0, 0, 0) + 2 \cdot (0, 0, 1, 0, 0, 0) + \\ &\quad + 2 \cdot (0, 0, 0, 1, 0, 0) + (0, 0, 0, 1, 0, 0) + (0, 0, 0, 1, 0, 0) + \\ &\quad + (0, 0, 0, 1, 0, 0) + (0, 0, 0, 1, 0, 0) + (0, 0, 0, 0, 1, 0) + \\ &\quad + (0, 0, 0, 0, 1, 0) + (0, 0, 0, 0, 1, 0) + (0, 0, 0, 0, 1, 0) + \\ &\quad + (0, 0, 0, 0, 1, 0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T , see Proposition 1.

The program Germenes [14] shows that the compact faces of $\text{NP}(I)$ are the vertices $v_1 = (3, 1, 1, 1, 1)$, $v_2 = (1, 2, 1, 1, 1)$, $v_3 = (1, 1, 1, 1, 5)$, the line segments $v_1 v_2, v_1 v_3, v_2 v_3$ and the triangle $v_1 v_2 v_3$. The facets, all of them unbounded, are:

$$\begin{aligned} F_1 &= v_2 v_3 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_4 + \mathbb{R}_{\geq 0} e_5, \\ F_2 &= v_1 v_3 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_4 + \mathbb{R}_{\geq 0} e_5, \\ F_3 &= v_1 v_2 v_3 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_4 + \mathbb{R}_{\geq 0} e_5, \\ F_4 &= v_1 v_2 v_3 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_5, \\ F_5 &= v_1 v_2 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_4, \\ F_6 &= v_1 v_2 v_3 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_4. \end{aligned}$$

Obviously $\mathcal{F}(4) = \mathcal{F}(3) = \emptyset$, hence $c_0(I) = c_1(I) = 0$. We have $\mathcal{F}(2) = \{F_3, F_4, F_6\}$ and

$$\begin{aligned} \text{Vol}(v_1 v_2 v_3, F_3) &= \text{Vol}(v_1 v_2 v_3, F_4) \\ &= \min \left\{ \begin{vmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 5 \end{vmatrix} \right\} = \\ &= \min\{2, 4\} = 2, \\ \text{Vol}(v_1 v_2 v_3, F_6) &= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 5 \end{vmatrix} = 22. \end{aligned}$$

Observe that in the computations of $\text{Vol}(v_1 v_2 v_3, F_3)$ and $\text{Vol}(v_1 v_2 v_3, F_4)$ four projections of the compact triangle $v_1 v_2 v_3$ give a line segment and must not be considered. Summing up we get

$$\begin{aligned} c_2(I) &= \text{Vol}(v_1 v_2 v_3, F_3) + \text{Vol}(v_1 v_2 v_3, F_4) + \text{Vol}(v_1 v_2 v_3, F_6) \\ &= 2 + 2 + 22 = 26. \end{aligned}$$

The set of 3-unbounded facets containing 1-dimensional compact faces is $\mathcal{F}(1) = \{F_1, F_2, F_3, F_4, F_5\}$, and we have

$$\begin{aligned} \text{Vol}(v_2 v_3, F_1) &= \min \left\{ \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} = \min\{1, 4\} = 1, \\ \text{Vol}(v_1 v_3, F_2) &= \min \left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} = \min\{2, 4\} = 2, \\ \text{Vol}(v_1 v_2, F_3) &= \text{Vol}(v_1 v_2, F_4) = \min \left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \right\} \\ &= \min\{2, 1\} = 1, \\ \text{Vol}(v_1 v_3, F_3) &= \text{Vol}(v_1 v_3, F_4) = \min \left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} \\ &= \min\{2, 4\} = 2, \\ \text{Vol}(v_2 v_3, F_3) &= \text{Vol}(v_2 v_3, F_4) = \min \left\{ \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} \\ &= \min\{1, 4\} = 1, \\ \text{Vol}(v_1 v_2, F_5) &= \min \left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \right\} = \min\{2, 1\} = 1, \end{aligned}$$

hence

$$\begin{aligned} c_3(I) &= \text{Vol}(v_2 v_3, F_1) + \text{Vol}(v_1 v_3, F_2) + \\ &\quad + \min\{\text{Vol}(v_1 v_2, F_3), \text{Vol}(v_1 v_3, F_3), \text{Vol}(v_2 v_3, F_3)\} + \\ &\quad + \min\{\text{Vol}(v_1 v_2, F_4), \text{Vol}(v_1 v_3, F_4), \text{Vol}(v_2 v_3, F_4)\} + \\ &\quad + \text{Vol}(v_1 v_2, F_5) = 1 + 2 + 1 + 1 + 1 = 6. \end{aligned}$$

From the list of the facets we see that there are five 4-unbounded facets, precisely $\mathcal{F}(0) = \{F_1, F_2, F_3, F_4, F_5\}$ and we have

$$\begin{aligned} \text{Vol}(v_2, F_1) &= 1, & \text{Vol}(v_3, F_1) &= 1, & \text{Vol}(v_1, F_2) &= 1, \\ \text{Vol}(v_3, F_2) &= 1, & & & & \\ \text{Vol}(v_1, F_3) &= 1, & \text{Vol}(v_2, F_3) &= 1, & \text{Vol}(v_3, F_3) &= 1, \\ \text{Vol}(v_1, F_4) &= 1, & & & & \\ \text{Vol}(v_2, F_4) &= 1, & \text{Vol}(v_3, F_4) &= 1, & \text{Vol}(v_1, F_5) &= 1, \\ \text{Vol}(v_2, F_5) &= 1, & & & & \end{aligned}$$

$$\begin{aligned} c_4(I) &= \min\{\text{Vol}(v_2, F_1), \text{Vol}(v_3, F_1)\} + \min\{\text{Vol}(v_1, F_2), \\ &\quad \text{Vol}(v_3, F_2)\} + \\ &\quad + \min\{\text{Vol}(v_1, F_3), \text{Vol}(v_2, F_3), \text{Vol}(v_3, F_3)\} + \\ &\quad + \min\{\text{Vol}(v_1, F_4), \text{Vol}(v_2, F_4), \text{Vol}(v_3, F_4)\} + \\ &\quad + \min\{\text{Vol}(v_1, F_5), \text{Vol}(v_2, F_5)\} = 1 + 1 + 1 + 1 + 1 = 5. \end{aligned}$$

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