THE FRACTIONAL MEAN CURVATURE FLOW
IL FLUSSO PER CURVATURA MEDIA FRAZIONARIA

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Abstract. In this note, we present some recent results in the study of the fractional mean curvature flow, that is a geometric evolution of the boundary of a set whose speed is given by the fractional mean curvature. The flow under consideration is of nonlocal type and presents several interesting difference with respect to the classical mean curvature flow. We will describe the main contributions in this field, with particular emphasis on some tipically nonlocal behaviors which are in contrast with the classical local case.

Sunto. In questa nota, presentiamo alcuni risultati recenti riguardanti lo studio del moto per curvatura media frazionaria, che descrive l'evoluzione del bordo di un insieme la cui velocita è data dalla curvatura media frazionaria. Tale flusso ha natura nonlocale e presenta alcune interessanti differenze rispetto al flusso per curvatura media classica. Descriviamo i principali contributi in questo ambito, con particolare enfasi ai comportamente tipicamente nonlocali che sono in contrasto col caso classico.

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1. Introduction

This contribution concerns the study of the, so called, *fractional mean curvature flow* (FMCF, for short), that is a geometric flow driven by the fractional mean curvature. Let us start by stating our problem.

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Let E_0 be a subset of $\mathbb{R}^n$, and let $\mathcal{M}_0 = \partial E_0$. For a fixed $s \in (0,1)$, we consider the family of immersions $F : \mathcal{M}_0 \times [0,T) \to \mathbb{R}^n$ which satisfies

$$
\begin{cases} \partial_t F(p,t) = -H_s(p,t)\,\nu(p,t), & p \in \mathcal{M}_0, t \geq 0 \\ F(p,0) = p & p \in \mathcal{M}_0, \end{cases} \tag{1}
$$

where $H_s(p,t)$ and $\nu(p,t)$ denote, respectively, the fractional (or nonlocal) mean curvature of order s (see formula (3) below) and the outward unit normal to the hypersurface $\mathcal{M}_t := F(\mathcal{M}_0, t)$ at the point $F(p,t)$.

The notion of fractional mean curvature arises naturally when one computes the first variation of the fractional perimeter, a nonlocal notion of perimeter which was introduced by Caffarelli, Roquejoffre and Savin in [12]. Given E a bounded subset of $\mathbb{R}^n$, the fractional perimeter of E is given by

$$
\mathrm{Per}_s(E) = c_s \int_E \int_{\mathbb{R}^n \setminus E} \frac{1}{|x-y|^{n+s}}\,dx\,dy = c_s[\chi_E]_{W^{s,1}(\mathbb{R}^n)}, \tag{2}
$$

where χ_E denotes the characteristic function of the set E, $[\cdot]_{W^{s,1}(\mathbb{R}^n)}$ denotes the seminorm in the fractional Sobolev space $W^{s,1Ev}$, and c_s is a constant depending on s which behaves like $(1-s)$ as $s \uparrow 1$.

One can see the analogy with the notion of classical perimeter in the sense of De Giorgi, defined as

$$
\mathrm{Per}(E) = [\chi_E]_{BV(\mathbb{R}^n)},
$$

where $[\cdot]_{BV(\mathbb{R}^n)}$ denotes the seminorm in the space BV. In (2) we are considering a fractional order derivative of the characteristic function of a set and the two notions are consistent in the sense that $\mathrm{Per_s} \to \mathrm{Per}$ as $s \uparrow 1$ (see e.g. [1, 14, 28]).

Roughly speaking, the s-perimeter captures the interactions between a set E and its complement, these interactions take place in the whole $\mathbb{R}^n$ and are weighted by a kernel with polynomial decay. Due to its nonlocal character the s-perimeter has several application, for example in image reconstruction and nonlocal capillarity models, see e.g. [7, 43].

A set E which is a minimizer for the fractional perimeter is called a *fractional (or nonlocal) minimizing minimal set*, and its boundary is referred as a *nonlocal minimizing* *minimal surface*. In [12] the Euler-Lagrange equation for this functional has been derived: similarly to the classical case, a nonlocal minimizing minimal set E must have vanishing fractional mean curvature H_s (in the viscosity sense), where H_s is given by the following expression

$$H_s(x) = c_s \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{\chi_{\mathbb{R}^n\setminus E}(y) - \chi_E(y)}{|x-y|^{n+s}} dy, \tag{3}$$

where c_s denotes again a constant depending on s which behaves like $(1-s)$ as $s \uparrow 1$. To use the same terminology as for the classical local case, we call a surface with vanishing fractional mean curvature a *nonlocal minimal surface* (even if it is a stationary and not a minimizing surface).

The study of nonlocal minimizing minimal surfaces, with particular interest to their regularity properties, has attracted much interest in the last years. They are known to be C^∞ surfaces outside of a singular set of Hausdorff dimension at most $n-3$ for any $s \in (0,1)$ (see [12, 6, 47]). Moreover, when s is sufficiently close to 1 (but not in a quantifiable way) they are known to recover the same regularity properties of classical area-minimizing surfaces, that is, their singular set has Hausdorff dimension at most $n-8$ (see [14]). The biggest open problem in this field, which would give the optimal Hausdorff dimension for the singular set for any $s \in (0,1)$, is the classification of nonlocal minimal cones. The only available result which holds for any $s \in (0,1)$ was established in [47], where flatness of nonlocal minimal cones in $\mathbb{R}^2$ has been proven. This two-dimensional result has then been extended to the more general notion of *stable* sets and to more general nonlocal perimeters in [22]. The only other available results in this setting hold only for s close to 1; more precisely in [14] flatness for nonlocal minimal cones has been proven up to dimension $n = 8$, by a compactness argument, which therefore does not give quantitative value of such s; recently in [8] flatness for nonlocal minimal cones in dimension $n = 3$ for s close to 1 has been established, but now in a quantifiable way (this last result holds actually for the more general class of stable nonlocal minimal cones). Some computations in [30] suggest that when s is close to 0 there could be singular nonlocal minimal cones in dimension $n = 7$, differently from the classical local case. Another interesting difference with respect to the local case is related to the fact that fractional minimal surfaces can stick at the boundary of (even smooth and convex) domains (see [31]).

As already mentioned above, a necessary condition for being a nonlocal minimizing minimal set, consists in having vanishing fractional mean curvature. Some examples of surfaces with vanishing nonlocal mean curvature (apart from hyperplanes) are helicoids and a nonlocal version of catenoids, see [21, 30]. Other interesting related results concern the study of sets with constant nonlocal mean curvature. The first intuitive example of a surface having constant (positive) nonlocal mean curvature is the sphere. Other examples are Delaunay-type surfaces, which have been studied in [11, 29, 9, 10]. Moreover a nonlocal analogue of the Alexandrov Theorem has been established in [11, 25]: in these papers the authors established that any bounded open set, with sufficiently regular boundary, having constant fractional mean curvature, is necessarily a sphere. In [25] a quantitative version of this result was also established. We emphasize an interesting difference with respect to the local case: here the set is not required to be connected since, basically, the nonlocal character of the fractional mean curvature excludes by itself the possibility to have several connected components (the union of disconnected balls does not have constant fractional mean curvature!). This nonlocal Alexandorv Theorem will be useful in our last Section 4.

Let us now pass to describe some well known results in the study of the *classical* mean curvature flow, that is a geometric flow in which the speed is given by the usual mean curvature H, i.e.

$$\begin{cases} \partial_t F(p,t) = -H(p,t)\,\nu(p,t), & p \in \mathcal{M}_0, t \geq 0 \\ F(p,0) = p & p \in \mathcal{M}_0. \end{cases} \tag{4}$$

The classical mean curvature flow arises in the description of the evolution of interfaces in several physical models (see for example [49]), in particular it is a natural evolution model in problems in which the relevant energy is of interface type. Indeed mean curvature flow can be seen as the gradient flow of the perimeter functional. On the other hand, one can show, by a direct computation, that $-H(p,t)\,\nu(p,t) = \Delta_{g(t)}F(p,t)$, where $\Delta_{g(t)}$ denotes the Laplace-Beltrami operator on the surface $\mathcal{M}$ associated to the metric $g(t)$, induced by the immersion $F(t)$. Hence this flow can be interpreted also as a sort of heat flow, but differently from the heat equation, it is nonlinear since the Laplace-Beltrami operator changes with the evolving surface. Nevertheless, the flow share some "good" properties with the classical heat equation: it is of parabolic type, it has a unique solution for small time, it satisfies a comparison principle and has a smoothing effect. An important feature of this flow is that, even if smooth solutions exists for small time, they can become singular in finite time (when the curvature blows-up). For this reason, different notions of weak solutions have been introduced, allowing to continue the evolution after the formation of singularities, see e.g. [20, 34].

There is a huge literature on the mean curvature flow. In the next sections we will recall some of the main well known results and will focuse on the analogy or differences with respect to the nonlocal case. For the moment, let us just mention a celebrated convergence result due to Huisken [39], which is a crucial ingredient in the analysis of singularities. In [39], Huisken proved that *convex hypersurfaces remain smooth up to a finite maximal time at which they shrink to a point, and they converge to a round sphere after rescaling.* An analogous result (convergence to a sphere) was proven to hold also in the volume preserving case again by Huisken in [40]. Other interesting feature of this flow have been studied, such as fattening phenomena, examples of self-shrinkers, evolution of graphs, discrete approximation for the flow, classification of singularities. We will mention some of them, later on in this note. Let us finally recall that variants of the mean curvature flow have been also used to give different proofs of some gemetric inequalities, such has the isoperimetric or Minkowski inequalities.

On the other hand the study of nonlocal mean curvature flows has been started only in the last years and very few results are known. Concerning existence of solutions, the existence and uniqueness of viscosity solutions, together with some basic properties, such has the validity of comparison principles, have been established in [41, 13, 16]. More precisely, in [13] Caffarelli and Souganidis proved a discrete approximation result, which is the fractional analogue of the so called MBO approximation (see Section 2 for details). This type of discrete approximation has been generalized to the case of an anisotropic version of the fractional mean curvature flow and to the precence of an external force, by Chambolle, Novaga and Ruffini in [17]. In the same work [17], they also proved that these more general flows preserves convexity, a property which will be useful later on in Section 4. Some geometric properties of smooth solutions and the evolution equations of geometric quantities in the smooth setting, have been studied by Saez and Valdinoci in [46], the formation of fattening phenomena have been addressed in [26] by Cesaroni, Dipierro, Novaga and Valdinoci, examples of self-shrinkers were considered in [27] by Cesaroni and Novaga, while examples of surfaces which develope neckpinch singularities have been provided in [23]. Regarding the study of regularity of solutions, we mention the two following very recent contributions: in [42], Julin and La Manna established the short-time existence of a unique classical solution starting from a $C^{1,1}$ initial datum; in [15], Cameron proved a quite interesting regularizing effect of the fractional mean curvature flow, which is false for the classical flow: any set that is at finite distance away from a Lipschitz subgraph will become a Lipschitz subgraph in finite time. Finally, concerning asymptotic convergence result á la Huisken, in [24], convergence to a sphere for the volume preserving fractional mean curvature flow has been established.

This is, to our knowledge, a complete list of references on the fractional mean curvature flow. We will describe some of these results in the next Sections. More precisely:

- In Section 2, we recall the classical MBO approximation scheme for the mean curvature flow and describe its analogue in the nonlocal setting considered in [13];
- In Section 3, we describe and give an idea of the proof of the formation of neckpinch singularities established in [23], focusing on the difference with respect to the classical setting;
- Finally, in Section 4, we describe the convergence to a sphere in the volume preserving case, proven in [24].

2. Discrete approximation results

An important question from the point of view of applications is to find efficient computational schemes for mean curvature (or more general geometric) flows. The first contribution in this direction, for what regards the classical mean curvature flow, is due to Merriman, Bence and Osher, who introduced a time-discretization, nowadays called the

MBO scheme, to generate motion by mean curvature [45]. The idea, that we make precise later in this Section, is to start with a bounded and open set E_0 in $\mathbb{R}^n$ (the initial datum), consider the function $\chi_{E_0} - \chi_{(E_0)^c}$ (which takes value 1 in E_0 and -1 in E_0^c) and make it evolve by the heat equation for a short time $h > 0$. Heat equation produces a smooth solution w (which therefore takes continuous values and not just $-1, 1$). One then define a new initial datum E_1 to be the super-level set of w $\{w > 0\}$, and repeat the procedure. In this way, the scheme produces a discrete sequences of surfaces $\mathcal{M}^h_{nh}$ which are the boundaries of the sets E_n. Few years later that this scheme was proposed, Evans [33] and, independently, Barles and Georgelin [5], gave rigourous proofs of the convergence of the MBO approximation to mean curvature flow. Their proofs is based on the, so-called, *level-set approach* for the motion by mean curvature, which allows to introduce a notion of viscosity solutions, for which a comparison principle holds. The analogue, in the nonlocal setting, of this type of discrete approximation, was established by Caffarelli and Souganidis in [13]: the main idea relies in following the MBO scheme, but where the heat equation is replaced by the fractional heat equation. This was the actual motivation for the definition of the fractional mean curvature.

2.1. **The classical setting.** To better understand these results, let us start by considering the classical setting and introducing the notion of viscosity solution for the classical mean curvature flow (4). As already mentioned above, this is done by using the level-set approach. The idea is the following: given an initial surface $\mathcal{M}_0 = \partial E_0$, we choose any continuous function $u_0 : \mathbb{R}^n \to \mathbb{R}$ such that

$$E_0 = \{x \in \mathbb{R}^n \,:\, u_0 > 0\} \quad \text{and} \quad \mathcal{M}_0 = \{x \in \mathbb{R}^n \,:\, u_0(x) = 0\}. \tag{5}$$

The geometric equation satisfied by the evolution $\mathcal{M}_t$ of $\mathcal{M}_0$ can then be translated into an equation satisfied by a function $u(x,t)$, where $u(x,0) = u_0(x)$ and at each time

$$E_t = \{x \in \mathbb{R}^n \,:\, u(\cdot,t) > 0\} \quad \text{and} \quad \mathcal{M}_t = \{x \in \mathbb{R}^n \,:\, u(\cdot,t) = 0\}. \tag{6}$$

If $\mathcal{M}_t$ evolves by mean curvature flow, then one can see that the equation satisfied by u is

$$\partial_t u + H[x, u(\cdot,t)]|Du(x,t)| = 0 \quad \text{in } \mathbb{R}^n \times (0,+\infty), \tag{7}$$

where u satisfies the initial condition

$$u(x,t) = u_0(x) \quad \text{in } \mathbb{R}^n.$$

Since $\nabla u/|\nabla u|$ is a unit normal to a level set of u (here ∇u denotes the spatial gradient of u), and hence $\operatorname{div}(\nabla u/|\nabla u|)$ represents its mean curvature (unless $|\nabla u| = 0$), we can write the above equation as

$$\partial_t u = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{8}$$

The fact that this definition of $\mathcal{M}_t$ is well posed, that is that equation (7) has a unique solution and definition (5) does not depend on the choice of the function u_0, was established in [20, 34].

We can now give the notion of viscosity solution considered in [34, 20].

Definition 2.1. i) *An upper semicontinuous function* $u : [0,T] \times \mathbb{R}^n$ *is a viscosity subsolution of* (8) *if, for every smooth test function* ϕ *such that* $u - \phi$ *admits a global maximum at* (t,x), *we have*

$$\partial_t \phi \leq |\nabla \phi| \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) \tag{9}$$

if $D\phi(x,t) \neq 0$, *and* $\partial_t \phi(x,t) \leq 0$ *if not.*

ii) *A lower semicontinuous function* $u : [0,T] \times \mathbb{R}^n$ *is a viscosity supersolution of* (8) *if for every smooth test function* ϕ *such that* $u - \phi$ *admits a global minimum at* (t,x), *we have*

$$\partial_t \phi \geq |\nabla \phi| \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) \tag{10}$$

if $D\phi(x,t) \neq 0$, *and* $\partial_t \phi(x,t) \geq 0$ *if not.*

iii) *A locally bounded function* u *is a viscosity solution of* (8) *if its upper semicontinuous envelope is a subsolution and its lower semicontinuous envelope is a supersolution of* (8).

As established in [20, 34], a comparison principle holds for viscosity solutions, meaning that of u and v are, respectively, a subsolution and a supersolution of (8) starting from initial data u_0 and v_0 satisfying $u_0 \leq v_0$, then for all later times it holds $u \leq v$.

Let us describe now more in detail the MBO scheme, in order to give a rigorous statement of the convergence result.

Let E_0 be an open and bounded set in $\mathbb{R}^n$ and let χ_{E_0} denote its characteristic function. We start by considering the heat flow with initail datum $\chi_{E_0} - \chi_{E_0^c}$:

$$(11) \qquad \begin{cases} \partial_t w = \Delta w & \text{in } \mathbb{R}^n \times (0, +\infty) \\ w(\cdot, 0) = \chi_{E_0} - \chi_{E_0^c} & \text{on } \mathbb{R}^n. \end{cases}$$

The (unique bounded) solution w at time $h > 0$ is given by convolution with the heat kernel:

$$w(x,h) = \frac{1}{(4\pi h)^{n/2}} \int_{E_0} e^{-\frac{|x-y|^2}{4h}}\, dy.$$

After this first time-step h we stop the flow, and we define a new set E_1^h in the following way:

$$E_1^h := \{w(\cdot, h) > 0\}.$$

We then repeat the procedure, solving the heat equation in a time interval again of step h and with initial datum $w^h(x,0) = \chi_{E_1^h} - \chi_{(E_1^h)^c}$. Proceeding in this way, we define a sequence of sets E_{nh}^h, of surfaces $\mathcal{M}_{nh}^h = \partial E_{nh}^h$, and of functions $w^h(\cdot, nh)$, where

$$E_{nh}^h = \{x \in \mathbb{R}^n \, : \, G_h * w(\cdot, (n-1)h)(x) > 0\},$$

being G_h the heat kernel at time h:

$$G_h = \frac{1}{(4\pi h)^{n/2}} e^{-\frac{|z|^2}{4h}}.$$

We can now state the convergence result proven in [5, 33].

Theorem 2.1 (Theorem 1.2 in [5]). *Let E_t and $\mathcal{M}_t$ be defined as in* (6)*, that is $E_t = \{u > 0\}$ and $\mathcal{M}_t = \{u = 0\}$, where u is the viscosity solution of* (7)*.*

Then, for all $t \geq 0$

$$\liminf_{y \to x \; nh \to t} w^h(y, nh) = 1 \quad \text{in } E_t,$$

$$\limsup_{y \to x \; nh \to t} w^h(y, nh) = -1 \quad \text{in } (E_t \cup \mathcal{M}_t)^c.$$

The above theorem tells us that the MBO scheme is an approximation for mean curvature flow if we put 1 in the region inside the front $\{u > 0\}$ and -1 in the region outside the front $\{u < 0\}$. However, whether the regions where w_h converges to -1 and 1 are exactly the region inside and outside the front, depends on the fact that $\mathcal{M}_t$ could become "fat", that is it could develop an interior. Adding an additional condition which excludes this possibility, we have the following more precise result:

Theorem 2.2 (Corollary 1.3 in [5]). *Let* $\mathcal{F} := \bigcup_{t>0} \mathcal{M}_t \times \{t\}$ *and* $\mathcal{F}^h := \bigcup_n \mathcal{M}^h_{nh} \times \{nh\}$. *If* $\mathcal{F} = \partial\{(x,t) : u(x,t) > 0\} = \partial\{(x,t) : u(x,t) < 0\}$, *then* $\mathcal{F}^h$ *converges to* $\mathcal{F}$ *in the sense of the Hausdorff distance.*

The condition in the above theorem basically says that $\mathcal{M}_t$ does not become fat.

Let us now pass to describe the analogue of the MBO approximation scheme in the fractional setting.

2.2. **The nonlocal setting.** As already mentioned in the Introduction, the MBO approximation for the fractional mean curvature flow was proved by Caffarelli and Souganidis in [13].

We start by recalling the definition of viscosity solution, which was already introduced in [41] for the FMCF, using again the level set approach. In this situation, equation (7) is replaced by

$$\partial_t u + C_s H^s[x, u(\cdot,t)]|Du(x,t)| = 0 \quad \text{in } \mathbb{R}^n \times (0,+\infty), \tag{12}$$

where C_s is a constant, whose precise value can be found in [13], and $H_s[x, u(\cdot,t)]$ denotes the fractional mean curvature of the superlevel set of $u(\cdot,t)$ at the point x, i.e.

$$H^s[x, u(\cdot,t)] = H^s_{\{y\in\mathbb{R}^n : u(y,t)>u(x,t)\}}(x).$$

We recall now the notion of viscosity solution from [41].

Let $\mathcal{M} = \{x \in \mathbb{R}^n : u(x) = 0\} = \partial\{x \in \mathbb{R}^n : u(x) > 0\}$. If $u \in C^{1,1}$ and $Du \neq 0$, we can define the following quantities

$$\begin{aligned} k^*[x,\mathcal{M}] = k^*[x,u] &= \int_{\mathbb{R}^n} \frac{\chi_{\{u(x+z)\geq u(x)\}}(z) - \chi_{\{u(x+z)<u(x)\}}(z)}{|z|^{n+s}}\,dz, \\ k_*[x,\mathcal{M}] = k_*[x,u] &= \int_{\mathbb{R}^n} \frac{\chi_{\{u(x+z)>u(x)\}}(z) - \chi_{\{u(x+z)\leq u(x)\}}(z)}{|z|^{n+s}}\,dz. \end{aligned} \tag{13}$$

It is easy to see that if $u \in C^{1,1}$ and its gradient Du does not vanish on $\{z \in \mathbb{R}^n : u(z) = u(x)\}$, then k^* are finite and

$$k^*[x,u] = k_*[x,u] = -H_s[x,u].$$

We can now give the definition of *viscosity solution* for (12) (see [41], Sec. 3).

Definition 2.2. i) *An upper semicontinuous function $u : [0,T] \times \mathbb{R}^n$ is a viscosity subsolution of* (12) *if for every smooth test function ϕ such that $u - \phi$ admits a global maximum at (t,x), we have*

$$\partial_t\phi \leq C_s k^*[x,\phi(\cdot,t)]|D\phi|(x,t) \tag{14}$$

if $D\phi(x,t) \neq 0$, and $\partial_t\phi(x,t) \leq 0$ if not.

ii) *A lower semicontinuous function $u : [0,T] \times \mathbb{R}^n$ is a viscosity supersolution of* (12) *if for every smooth test function ϕ such that $u - \phi$ admits a global minimum at (t,x), we have*

$$\partial_t\phi \geq C_s k_*[x,\phi(\cdot,t)]|D\phi|(x,t) \tag{15}$$

if $D\phi(x,t) \neq 0$, and $\partial_t\phi(x,t) \geq 0$ if not.

iii) *A locally bounded function u is a viscosity solution of* (12) *if its upper semicontinuous envelope is a subsolution and its lower semicontinuous envelope is a supersolution of* (12).

It is easy to verify that any classical subsolution (respectively supersolution) is in particular a viscosity subsolution (respectively supersolution). The fact that $\mathcal{M}_t$ is well defined (that is (12) has a unique solution), and that the definition (5) does not depend on the initial choice of the function u_0, was proven by Imbert in [41], together with the following comparison principle, which will be useful for the result we will present in Section 3.

Proposition 2.1 (Theorem 2 in [41]). *Suppose that the initial datum u_0 is a bounded and Lipschitz continuous function. Let u (respectively v) be a bounded viscosity subsolution (respectively supersolution) of* (12).

If $u(x,0) \le u_0(x) \le v(x,0)$, then $u \le v$ on $\mathbb{R}^n \times (0,+\infty)$.

Let us now describe the MBO approximation in the fractional setting. The idea is to follow the argument described in the previous subsection and replace the heat equation with the fractional heat equation:

$$\partial_t w = -(-\Delta)^{s/2} w,$$

where $(-\Delta)^{s/2}$ denotes the fractional Laplacian, defined as

$$(-\Delta)^{s/2} w(x) := c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x-y|^{n+s}}\, dy, \quad \text{for } s \in (0,2),$$

where $c_{n,s}$ is a constant that depends on s and n and which behaves like $(1-s)$ as $s \uparrow 1$, and the integral has to be taken in the principal value sense.

Using the same notation as above, we can define a sequence of sets E^h_{nh}, of surfaces $\mathcal{M}^h_{nh} = \partial E^h_{nh}$, and of functions $w^h(\cdot, nh)$, where now

$$E^h_{nh} = \{x \in \mathbb{R}^n \,:\, J_h * w(\cdot,(n-1)h)(x) > 0\},$$

being J_h the *fractional heat kernel*. More precisely

$$J_h(x) = p_s(x, \sigma_s(h)),$$

where p_s denotes the fundamental solution of the fractional heat equation, and σ_s denotes the following time-scale

$$\sigma_s(h) = h^{\frac{s}{1+s}} \quad \text{if } s \in (0,1),$$

$$h = \sigma_1(h) |\ln(\sigma_1(h))| \quad \text{if } s = 1,$$

$$\sigma_s(h) = h^{\frac{s}{2}} \quad \text{if } s \in (1,2).$$

We recall that the kernel J_h, differently from the classical case, has polynomial (and not exponential) decay.

With this definition at hand, we can now state the convergence result by Caffarelli and Souganidis.

Theorem 2.3 (Theorem 1 in [13]). *Let $(w^h(\cdot, nh))_{n\in\mathbb{N}}$ be the family of functions defined above.*

Let u be the viscosity solution of

- *the* fractional *level set equation* (12) *if $s \in (0,1)$;*
- *the* classical *level set equation* (7) *if $s \in [1,2)$.*

Let $E_t = \{x \in \mathbb{R}^n \,:\, u(x,t) > 0\}$ and $\mathcal{M}_t = \{x \in \mathbb{R}^n \,:\, u(x,t) = 0\}$.

Then,

$$\liminf_{y\to x\; nh\to t} w^h(y, nh) = 1 \quad \text{in } E_t,$$

$$\limsup_{y\to x\; nh\to t} w^h(y, nh) = -1 \quad \text{in } (E_t \cup \mathcal{M}_t)^c.$$

Again, if we exclude that $\mathcal{M}_t$ develops an interior, we have convergence in the Hausdorff sense:

Theorem 2.4 (Corollary 1 in [13]). *Let $\mathcal{F} := \bigcup_{t>0} \mathcal{M}_t \times \{t\}$ and $\mathcal{F}^h := \bigcup_n \mathcal{M}^h_{nh} \times \{nh\}$. If $\mathcal{F} = \partial\{(x,t) \,:\, u(x,t) > 0\} = \partial\{(x,t) \,:\, u(x,t) < 0\}$, then $\mathcal{F}^h$ converges to $\mathcal{F}$ in the sense of the Hausdorff distance.*

The above results, interestingly, show that the power $s/2 = 1/2$ in the approximation by the fractional heat equation is somehow critical and differentiates between a local and a nonlocal behavior: when $s \in (0,1)$ the discrete approximation leads to motion by fractional mean curvature while when $s \in [1,2)$ it leads to classical mean curvature motion.

3. Neckpinch singularities

In this Section, we describe the result obtained in [23] concerning the formation of neckpinch-type singularities in the fractional mean curvature flow.

We start by recalling some well known facts in the classical setting, for which the formation of singularities has been widely studied in the last decades. One of the most important result in this context, that we already mentioned in the Introduction, was proved by Huisken and concerns the evolution of convex surfaces:

Theorem 3.1 (Theorem 1.1 in [39])**.** *Let $n \geq 3$ and assume that $\mathcal{M}_0$ is uniformly convex, i.e., the eigenvalues of its second fundamental form are strictly positive everywhere. Then the evolution equation* (4) *has a smooth solution on a finite time interval $0 \leq t < T$, and we have that $\mathcal{M}_t$ converge to a single point as $t \to T$.*

To be more precise, Huisken proved that $\mathcal{M}_t$ converge to a *round point*, i.e. that after a suitable rescaling, they converge to a sphere. This result extends to every dimension a result by Gage and Hamilton for $n = 2$ [36].

A natural question is then, what happens if the convexity assumption is removed? In general, other types of singularities may occur, a classical example being the, so-called, neckpinch singularities. The first examples of surfaces developing this type of singularities were provided by Greyson in [37], and later considered also by Angenent and Ecker [4, 32].

Theorem 3.2 (Theorem 2.1 in [37])**.** *There exists an embedded hupersurface $\mathcal{M}_0$ in $\mathbb{R}^n$, $n \geq 3$, which developes a singularity under the mean curvature flow, before it shrinks to a point.*

The idea is to consider a set which is made by two large balls connected by a very thin neck, so that, in dimension $n > 2$, the mean curvature in the neck is much larger than the one in the balls, hence the radius of the neck goes to zero faster than the radius of the balls. Intuitively, here the fact of being in dimension strictly larger than 2 plays a crucial role, indeed if we image the same kind of set in the plane, it is no longer true that the mean curvature (there is just one curvature now!) in the neck is much larger than the one of the balls. Hence, another natural question is whether Huisken's result still holds in the plane removing the convexity assumption. The surprising answer was given by Greyson:

Theorem 3.3 (The Main Theorem in [38])**.** Any *smooth closed embedded curve in the plane shrinks smoothly to a point.*

We would like to understand now what we can say about the existence of surfaces developing neckpinch-type singularities in the nonlocal setting and ask ourselves whether the anoulogue of Greyson's results hold for the fractional mean curvature flow. This is the content of the contribution [23] and, as we are going to show, a different behavior is observed when passing from local to nonlocal evolutions.

The main result in [23] is the following

Theorem 3.4 (Theorem 1 in [23]). *Let $n \geq 2$. There exists an embedded hypersurface $\mathcal{M}_0$ in $\mathbb{R}^n$ such that the viscosity solution of the fractional mean curvature flow* (1) *starting from $\mathcal{M}_0$ does not shrink to a point.*

The important difference with respect to the classical setting is that, here, the construction can be made also in dimension $n = 2$, showing that Greyson's theorem (Theorem 3.3 above) does not hold anymore in the nonlocal case.

In [23], a crucial ingredient in the proof of Theorem 3.4 relies on the following fact: if a set E is contained in a strip and its boundary ∂E has sufficiently small classical curvatures, then the fractional mean curvature of E is positive everywhere. The precise statement is the following:

Proposition 3.1 (Proposition 5 in [23]). *Let $\kappa > 0$. Let E_-, $E_+ \subset \mathbb{R}^n$ be connected sets. Assume that $E_+ \cap E_- = \emptyset$ and that*

$$E_- \supseteq \{x_n \leq -1\} \quad \text{and} \quad E_+ \supseteq \{x_n \geq 1\}.$$

Suppose also that the boundaries of E_- and E_+ are of class C^2, with classical directional curvatures bounded in absolute value by κ.

Let $E := \mathbb{R}^n \setminus (E_- \cup E_+)$. Then, there exist c_0 and $\kappa_0 > 0$, depending on n, s and the C^2 bounds on ∂E_- and ∂E_+, such that for any $x \in \partial E$

$$H^s_E(x) \geq c_0,$$

provided that $\kappa \in [0, \kappa_0]$.

For the details of the proof, we refer to [23]. We emphasize here that the the underlying idea relies crucially on the nonlocal character of the fractional mean curvature: if we sit at a point on the boundary of a thin neck, we see much more complement of E than E itself, and hence, recalling Definition (3), we expect that the fractional mean curvature can be made strictly positive, no matter what the dimension is.

A related easy observation, which can be seen directly again from Definition (3), is that, differently from the local case, a strip has always *strictly positive* fractional mean curvature.

As a corollary of the previous Proposition, we deduce that the set

$$E_\epsilon := \left\{ x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } |x_n| < \epsilon + \frac{1}{\pi} \arctan\left(\delta\, |x'|^2\right) \right\}, \tag{16}$$

has strictly positive fractional mean curvature (uniformly bounded away from zero), if ϵ and δ are sufficiently small.

We can now give a

Sketch of the proof of Theorem 3.4

We start by recalling the evolution, under fractional mean curvature flow, of a sphere, which can be explicitly computed (see Lemma 2 and Corollary 3 in [46]). Indeed, it can be easily seen that the fractional mean curvature of a ball of radius R is constantly equal to $H^s_{B_R}(x) = \bar{\omega} R^{-s}$, where $\bar{\omega}$ is a positive constant depending on s and n. Moreover, if we set $R(t) := (R_0^{s+1} - (\bar{\omega}(1+s))t)^{\frac{1}{s+1}}$, then $B_{R(t)}$ is a solution to the fractional mean curvature flow starting from B_{R_0} and it collapses to a point in the finite time

$$T_{B_{R_0}} = \frac{R_0^{s+1}}{\bar{\omega}(s+1)}. \tag{17}$$

Observe that, while for the classical mean curvature flow, the extinction time of a sphere of radius R_0 is proportional to R_0^2, in the fractional case it is proportional to R_0^{s+1}.

We consider now the set E_ϵ defined in (16); we know that there exists $\bar{\epsilon}$ and $\overline{\delta}$ positive such that, for any $0 < \epsilon \le \bar{\epsilon}$ and $0 < \delta \le \overline{\delta}$

$$\inf_{x \in \partial E_\epsilon} H^s_{E_\epsilon}(x) \ge c_0 > 0, \tag{18}$$

for some c_0 depending only on n and s.

Let now κ and ϵ_0 be two positive parameters such that

$$\kappa < c_0 \quad \text{and} \quad \epsilon_0 < \min\left\{ \bar{\epsilon}, \frac{1}{4}\kappa T_{B_1} \right\}, \tag{19}$$

where T_{B_1} is the extinction time of the ball of radius 1 given in (17).

The idea is to consider the set E_{ϵ_0} and to let it evolve with constant velocity κ in the inner vertical direction. We set

$$\epsilon(t) := \epsilon_0 - \kappa\, t,$$

and, for any t, we consider the set

$$E_{\epsilon(t)} := \left\{ x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } |x_n| < \epsilon(t) + \frac{2}{\pi} \arctan\left(\delta\, |x'|^2\right) \right\}. \tag{20}$$

With this choice, we have that any point $x \in \partial E_{\epsilon(t)}$ satisfies

$$\partial_t x \cdot \nu = V \cdot \nu,$$

where

$$V = \begin{cases} -\kappa e_n & \text{if } x_n > 0 \\ \kappa e_n & \text{if } x_n < 0. \end{cases}$$

From the crucial lower bound (18) and using that $E_{\epsilon(t)} \subset E_{\epsilon_0}$ for any $t > 0$, we deduce that

$$\partial_t x \cdot \nu \geq -\kappa > -c_0 \geq -H^s_{E_{\epsilon(t)}}.$$

Hence, the set $E_{\epsilon(t)}$ is a smooth supersolution (hence also a viscosity supersolution) to the FMCF (1).

By the definition of the set E_{ϵ_0} we see that the minimum distance between the two disconnected components of its boundary is attained at the points $(0, \dots, 0, \epsilon_0)$ and $(0, \dots, 0, -\epsilon_0)$. Since $E_{\epsilon(t)}$ evolves with constant velocity κ along the inner vertical direction, we deduce that the singular time for $E_{\epsilon(t)}$ is given by

$$T_{E_{\epsilon(t)}} = \frac{2\epsilon_0}{\kappa}. \tag{21}$$

To construct a surfaces which developes a neckpinch-type singularity, it is now enough to consider any closed set A_0 with the following properties:

a) A_0 is rotationally symmetric around the x_1 axis;

b) A_0 is symmetric with respect to the $x_1 = 0$ hyperplane;

c) A_0 is contained in E_{ϵ_0};

d) A_0 contains two balls B_1^- and B_1^+ of radius 1 centered at $(-L, 0, \dots, 0)$ and $(L, 0, \dots, 0)$ respectively, where L is chosen large enough so that a) and b) are both satisfied.

Let A_t be the evolution under fractional mean curvature flow starting from A_0. By uniqueness, A_t has the symmetries of A_0. Moreover, using the comparison principle of Proposition 2.1, A_t must be contained in $E_{\epsilon(t)}$ and it must contain the evolutions $B_{1,t}^-$ and $B_{1,t}^+$ of the two balls B_1^- and B_1^+.

Using now the choice of ϵ_0 (19), we deduce that at any time $t > T_A$, where $T_A = \frac{2\epsilon_0}{\kappa} \leq \frac{1}{2} T_{B_1}$ the $x_1 = 0$ cross section of A_t is empty.

But at the same time, A_t contains two balls with positive radius in the $x_1 > 0$ and $x_1 < 0$ half-spaces respectively. This shows that, at some time smaller than T_A, the set A_t splits into two symmetric disconnected components, hence it cannot shrink to a point.

4. The volume preserving case: convergence to a sphere

As mentioned in the previous Section, one of the most well known result on asymptotic convergence in the classical mean curvature flow is Theorem 3.1, due to Huisken, which asserts that convex hypersurfaces remain smooth up to a finite maximal time at which they shrink to a point, and that they converge to a round sphere after rescaling.

We have seen that, in the fractional setting, if we do not assume convexity of the initial datum, the analogue of Huisken result cannot be true in any dimension, due to the existence of surfaces which develope neckpinch-type singularities. Of course, a natural question is whether Huisken's result is still true in the nonlocal setting for convex initial conditions. This problem is still open and no results are available concerning the asymptotic behaviour of surfaces evolving by FMCF.

On the other hand, one could try to investigate convergence results for other particular types of nonlocal geometric evolution, such as, for example, the *volume preserving* version of the fractional mean curvature flow.

The classical volume preserving mean curvature flow has also been well studied: in [40], Huisken obtained again a convergence result, which sates that the solution exists for all times and converges to a sphere as $t \to +\infty$. In later years, many researchers have studied the convergence to a sphere for other kinds of geometric flows, with a speed driven by more general functions of the (classical) principal curvatures. These convergence results are interesting also in relation to possible applications to obtain generalizations or alternative proofs of classical geometric inequalities.

In this Section, we describe the results contained in [24], which investigate the analogue, in the nonlocal setting, of Huisken result concerning convergence to a sphere for the volume preserving flow, under some suitable assumption. They represent a first attempt to establish the asymptotic behavior of surfaces evolving by nonlocal geometric flows.

Let us start by describing the problem. Let $E_0 \subset \mathbb{R}^n$ be a smooth compact convex set, and let $\mathcal{M}_0 = \partial E_0$. We consider now the family of immersions $F : \mathcal{M}_0 \times [0,T) \to \mathbb{R}^n$ which satisfies

$$\begin{cases} \partial_t F(p,t) = [-H_s(p,t) + h(t)]\ \nu(p,t), & p \in \mathcal{M}_0, t \geq 0 \\ F(p,0) = p & p \in \mathcal{M}_0, \end{cases} \tag{22}$$

where $h(t)$ is given by

$$h(t) = \frac{1}{|\mathcal{M}_t|} \int_{\mathcal{M}_t} H_s(x) d\mu, \tag{23}$$

and $d\mu$ denotes the $(n-1)$-dimensional Hausdorff measure. With respect to the FMCF considered up to know, we are just adding a term in the expression of the velocity, which ensure that the volume is preserved. Indeed, an easy computation shows that

$$\frac{d}{dt}|E_t| = \int_{\mathcal{M}_t} (-H_s(p,t) + h(t)) d\mu = 0.$$

Another important feature of this flow is that it decreases the s-perimeter, indeed (see e.g. [24, 47]) the variation of the s-perimeter is given by

$$\begin{aligned} \frac{d}{dt}\mathrm{Per}_s(E_t) &= \int_{\mathcal{M}_t} [-H_s(x) + h(t)] H_s(x) d\mu \\ &= -\int_{\mathcal{M}_t} [H_s(x) - h(t)]^2 d\mu \leq 0. \end{aligned}$$

As a consequence of the two previous properties, we deduce that the *fractional isoperimetric quotient is monotone decreasing* along the volume preserving FMCF. The monotonicity property of the (classical) isoperimetric ratio for the mean curvature flow, was exploited in [3, 48] to give an alternative proof of the convergence result by Huisken. Since this monotonocity is a peculiar and structural property of the volume preserving case, in [24] the authors applied this approach in the fractional setting.

Let us describe, more in details, the results contained in [24]. The main results are some apriori estimates on smooth solutions, which give a uniform control on the geometry of the evolving surfaces, and establish that the fractional curvature remains uniformly bounded along the flow. As a consequence, one can show that any smooth solution, satisfying suitable regularity assumptions, exists for all times and converges to a sphere.

In the following $\underline{\rho_E}$ and $\overline{\rho_E}$ denote, respectively, the inner radius and the outer radius of a set $E \subset \mathbb{R}^n$:

(24)

$$\underline{\rho_E} := \sup\{r > 0 : \exists\, x_o \in \mathbb{R}^n,\ B_r(x_o) \subset E\}, \quad \overline{\rho_E} := \inf\{r > 0 : \exists\, x_o \in \mathbb{R}^n, B_r(x_o) \supset E\}.$$

These are the crucial apriori estimates established in [24].

Theorem 4.1 (Theorem 1.1 in [24]). *Let E_0 be a smooth compact convex set of $\mathbb{R}^n$ and let $\mathcal{M}_0 = \partial E_0$. Let $F : \mathcal{M}_0 \times [0, T) \to \mathbb{R}^n$, with $0 < T \le +\infty$, be a solution of (22) of class $C^{2,\beta}$ for some $\beta > s$. Then there exist positive constants $0 < R_1 \le R_2$, $0 < K_1 \le K_2$, only depending on E_0, such that*

$$R_1 \le \underline{\rho_{E_t}} \le \overline{\rho_{E_t}} \le R_2$$

$$K_1 \le H_s(p, t) \le K_2 \qquad p \in \mathcal{M}_0,$$

for all $t \in [0, T)$.

We give here an idea of the proof of the previous Theorem, the details can be found in [24].

Idea of the Proof. The proof can be divided in three main steps.

STEP 1. For any bounded convex set E, one can show that the following geometric estimate holds true:

$$\frac{\overline{\rho_E}}{\underline{\rho_E}} \le C \left(\frac{(\mathrm{Per}_s(E))^n}{|E|^{n-s}} \right)^{\frac{1}{s}},$$

where the quantity on the right-hand side is the fractional isoperimetric ratio for the set E. The proof of this inequality, which can be found in [24, Proposition 3.1], uses the fractional isoperimetric inequality and some careful integral estimates.

STEP 2. Step 1 implies that

$$R_1 \leq \underline{\rho_{E_t}} \leq \overline{\rho_{E_t}} \leq R_2, \tag{25}$$

for some R_1 and R_2 depending only on the initial datum E_0. Since the proof of this step is based on an easy geometric reasoning and uses crucially the monotonocity of the isoperimetric ratio, we give here the details.

As already mentioned in the Introduction, from [17] we know that the evolution given by (22) preserves convexity, hence we have that E_t is convex for all $0 < t < T$.

By definition of inner and outer radius, we have that

$$\omega_n \underline{\rho_{E_t}}^n \leq |E_t| \leq \omega_n \overline{\rho_{E_t}}^n,$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Since the volume of E_t is preserved, we have automatically an upper bound on $\underline{\rho_{E_t}}$ and a lower bound on $\overline{\rho_{E_t}}$ in terms of $|E_0|$. On the other hand, using that the fractional isoperimetric ratio $(\mathrm{Per}_s(E_t))^n/|E_t|^{n-s}$ is decreasing in time, the inequality in Step 1 gives a uniform bound on the ratio $\overline{\rho_{E_t}}/\underline{\rho_{E_t}}$ in terms of $|E_0|$. Combining these bounds together, we obtain (25).

STEP 3. Some integral estimates and the upper bound on $\overline{\rho_{E_t}}$ from Step 2 gives the lower bound $H_s(p,t) \geq K_1$ (see [24, Corollary 3.2]).

STEP 4. In this last Step, one shows the uniform upper bound for H_s:

$$H_s(p,t) \leq K_2.$$

The proof is more involved and uses crucially the evolution equation satisfied by H_s together with some integral estimates that relates the fractional analogue of the squared second fundamental form and H_s^2. Since it could be of independent interest, we spend a couple of words on this last estimate. In the nonlocal setting, the following integral quantity can be seen as a natural analogue of the squared norm of the second fundamental form:

$$c_s \int_{\mathcal{M}} \frac{1 - \nu(y)\cdot\nu(x)}{|y-x|^{n+s}} d\mu(y) = c_s \int_{\mathcal{M}} \frac{|\nu(x)-\nu(y)|^2}{|y-x|^{n+s}} d\mu(y)$$

It is an analogue of the squared second fundamental form (that we denote by $|A|^2$) for several reasons: it comes out naturally when performing the second variation of the fractional perimeter; it replaces $|A|^2$ in the evolution equations satisfied by geometric quantities, such has the fractional mean curvature; finally it converges to $|A|^2$ as $s \uparrow 1$ (this convergence was established in [30]).

In the classical setting, an easy and very useful estimate which relates H and $|A|^2$ is the following

$$|A|^2 \geq \frac{1}{n} H^2.$$

A natural question is then, whether some kind of inequality of this type holds also in the fractional setting. In this situation, of course, such kind of extension is no obvious due to the integral definition of our geometric quantities. Nevertheless, one can prove that, at least for convex sets, the following holds true (see Proposition 4.2 in [24]):

$$H_s(x) \leq C \left(\operatorname{diam}(E)\right)^{\frac{1-s}{2}} \left((1-s) \int_{\partial E} \frac{1 - \nu(y) \cdot \nu(x)}{|x-y|^{n+s}} \, d\mu(y) \right)^{\frac{1}{2}}.$$

As mentioned above, this integral estimate is crucial in the proof of the upper bound on H_s.

For the remaining details of this last Step 4, we refer to [24, Section 5].

Ones one has the geometric bounds of Theorem 4.1, one can deduce that a solution of (22) exists for all times and converges to a sphere as $t \to +\infty$, provided it satisfies suitable regularity properties. Basically, we need to assume that the solution remains smooth and does not develop singularities as long as the fractional mean curvature remains bounded. More precisely, we assume that there exists a smooth solution of (22) satisfying the following property for some $\beta > s$:

(HP) If H_s is bounded on $\mathcal{M}_t$ for all $t \in [0, T_0)$ for some $T_0 \leq T$, where T is the maximal time of existence, then the $C^{2,\beta}$-norm of $\mathcal{M}_t$, up to translations, is also bounded for $t \in [0, T)$ by a constant only depending on the supremum of H_s. In addition, either $T_0 = T = +\infty$, or $T_0 < T$.

This assumption is a natural analogue of some properties which hold true in the classical case (see [39]), and follow by the standard parabolic theory. In the fractional setting, it is still an open problem whether **(HP)** holds true. The only available result in this direction has been established very recently by Julin and La Manna in [42]. They prove that if the

$C^{1,\beta}$-norm of the solution remains bounded, for some $\beta > s$, then the smooth solution exists for all times. On the other hand, the boundedness of the fractional curvature gives directly $C^{1,\beta}$ bounds only for $\beta < s$. Nevertheless, we can expect that solutions enjoy further regularity properties, as some recent regularity results in nonlocal problems suggest, see e.g. [6, 18, 19, 31].

We can now state the convergence result established in [24]:

Theorem 4.2 (Theorem 1.2 in [24]). *Let E_0 be a smooth compact convex set of $\mathbb{R}^n$ and let $\mathcal{M}_0 = \partial E_0$. Let $F : \mathcal{M}_0 \times [0,T) \to \mathbb{R}^n$, with $0 < T \leq +\infty$, be a solution of* (22) *of class $C^{2,\beta}$ for some $\beta > s$ which satisfies property* **(HP)**. *Then $T = +\infty$, and $\mathcal{M}_t$ converges to a round sphere as $t \to +\infty$ in $C^{2,\beta}$ norm, possibly up to translations.*

Sketch of the proof.

We report here an idea of the proof, whose details can be found in [24]. As one can see, we will exploit the Alexandrov-type result for the classification of surfaces with constant fractional mean curvature, already mentioned in the Introduction. These are the main Steps:

- We first prove that
$$\lim_{t\to\infty} \max_{\mathcal{M}_t} |H_s(p,t) - h(t)| = 0.$$
This follows by our regularity assumption **(HP)**, the uniform bounds on H_s and the fact that
$$\frac{d}{dt}\mathrm{Per}_s(E_t) = -\int_{\mathcal{M}_t} |H_s - h|^2.$$
- Using again our regularity assumptions, we have that E_t converge to a set with constant fractional mean curvature;
- By the Alexandrov-type result in [11, 25], we conclude that the limit set must be a sphere.

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