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Stable s-minimal cones in $\mathbb{R}^3$ are flat for $s \sim 1$

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Abstract. We prove that half spaces are the only stable nonlocal s-minimal cones in $\mathbb{R}^3$, for $s \in (0,1)$ sufficiently close to 1. This is the first classification result of stable s-minimal cones in dimension higher than two. Its proof cannot rely on a compactness argument perturbing from $s = 1$. In fact, our proof gives a quantifiable value for the required closeness of s to 1. We use the geometric formula for the second variation of the fractional s-perimeter, which involves a squared nonlocal second fundamental form, as well as the recent BV estimates for stable nonlocal minimal sets.

1. Introduction

In this paper we prove that half spaces are the only stable nonlocal s-minimal cones – with smooth boundary away from 0 – in dimension $n = 3$ for $s \in (0,1)$ sufficiently close to 1 (see Theorem 1.2). The same classification result for stable s-minimal cones in dimension $n = 2$ for any $s \in (0,1)$ has been established in [15]. For short, we will refer to nonlocal s-minimal cones as s-minimal cones.

For minimizing cones (a stronger assumption than stability) a similar flatness result was proven by Savin and Valdinoci [14] in dimension $n = 2$ for any $s \in (0,1)$, and by Caffarelli and Valdinoci [6] in every dimension $2 \leq n \leq 7$ for $s \in (0,1)$ sufficiently close to 1. The result in [6] relies on the classification of classical ($s = 1$) minimizing cones of Simons [16] and extends it to s sufficiently close to 1 through a compactness argument.

Our statement is also "for s sufficiently close to 1", but – unlike in [6] – it cannot be deduced from the limit case $s = 1$ by some sort of compactness argument. The reason being that – unlike in the framework of minimizers – E_k being stable cones for the s_k-perimeter with $s_k \uparrow 1$ does not guarantee the sequence E_k to be compact. We must rule out, for instance, an hypothetical situation in which the traces of E_k on S^2 were (unions of) curves with their total classical perimeter increasing to infinity. As a matter of fact, and at least in $\mathbb{R}^3$, proving the compactness of sequences E_k of stable cones turns out to be as difficult as proving the flatness result – which then trivially gives the compactness since planes through the origin are compact.

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Let us remark also that the classification of stable cones in low dimensions turns out to be significantly more challenging for $s \in (0,1)$ than in the classical case $s = 1$. Indeed, as mentioned above, when $n = 2$, the classification of stable s-minimal cones – for all $s \in (0,1)$ – requires already a clever idea [7,15]. Moreover, in the case $n = 3$ of this paper, there is an even larger gap of difficulty between $s \in (0,1)$ and $s = 1$. Indeed, in the classical perimeter case $s = 1$, the trace $\partial\Sigma \cap S^2$ on the sphere of every stationary[1)] cone $\Sigma \subset \mathbb{R}^3$ with C^2 boundary away from 0 is immediately a maximal circle – and here the stability assumption is not even required. This is proven just using that the zero mean curvature condition on $\partial\Sigma$ is equivalent to a zero tangential curvature condition for the C^2 curve $\partial\Sigma \cap S^2$. For $s \in (0,1)$, however, the nonlocal character of the equation of s-minimal cones makes it impossible for such sort of "ODE-type" approach.

Before stating precisely our main result, we recall the notion of fractional s-perimeter, which was introduced by Caffarelli, Roquejoffre, and Savin in [5]. Given a set E in $\mathbb{R}^n$ and a bounded open set $\Omega \subset \mathbb{R}^n$, we define the fractional s-perimeter of E in Ω as

$$P_s(E,\Omega) := L(E \cap \Omega, E^c \cap \Omega) + L(E \cap \Omega, E^c \cap \Omega^c) + L(E \cap \Omega^c, E^c \cap \Omega),$$

where E^c denotes the complement of E in $\mathbb{R}^n$ and, for two disjoint measurable sets A and B, $L(A,B)$ denotes the quantity

$$L(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\, dx\, dy.$$

Minimizers for the fractional perimeter, with special interest in their regularity, were studied in several works; see [1, 5–7, 12, 14]. However, to our knowledge, the only available results for stable sets of the s-perimeter have been obtained in [7, 15]. The article [7] includes sharp BV and energy estimates in every dimension $n \geq 2$, and quantitative flatness results in dimension $n = 2$.

Let us state the main result of the current paper. We say that $\Sigma \subset \mathbb{R}^n$ is a cone when $\lambda\Sigma = \Sigma$ for all $\lambda > 0$. We will always take Σ to be an open set. Its boundary $\partial\Sigma$, a hypersurface in $\mathbb{R}^n$, will also be called a cone. The following is the definition of stability that we use.

Definition 1.1. Let $\Sigma \subset \mathbb{R}^n$ be a cone with nonempty boundary of class C^2 away from the origin. We say that Σ is stable if

$$\liminf_{t\to 0} \frac{1}{t^2}\left(P_s(\phi_X^t(\Sigma), B_1) - P_s(\Sigma, B_1)\right) \geq 0$$

for all vector fields $X \in C_c^\infty(B_1 \setminus \{0\}, \mathbb{R}^n)$. Here $\phi_X^t : \mathbb{R}^n \to \mathbb{R}^n$ denotes the integral flow of X at time t (which is a smooth diffeomorphism for t small).

Throughout the paper, Σ being stable as in this definition will also be referred to as Σ being a stable cone for the s-perimeter in $\mathbb{R}^n$, or Σ being a stable s-minimal cone in $\mathbb{R}^n$.

Note that ϕ_X^t is the identity in a neighborhood of 0 and, thus, this is the weakest possible notion of stability of cones that one may assume. In Section 2 we will briefly discuss other notions of stability for the s-perimeter.

It is easy to see that if Σ is as in Definition 1.1 (in particular, $\partial\Sigma$ is C^2 away from 0), then $\partial\Sigma$ is stationary away from 0, and hence it is a solution of the nonlocal minimal surface

[1)] $\partial\Sigma$ has zero mean curvature.

equation (also away from 0). Moreover, using that Σ is a cone, one can show (see the proof of Theorem 1.2 for the details) that $\partial\Sigma$ is a viscosity solution of the nonlocal minimal surface equation also at 0.

The following is our main result.

Theorem 1.2. *There exists* $s_* \in (0,1)$ *such that for every* $s \in (s_*, 1)$ *the following statement holds.*

Let $\Sigma \subset \mathbb{R}^3$ *be a cone with nonempty boundary of class* C^2 *away from* 0. *Assume that* Σ *is stable as in Definition* 1.1. *Then* Σ *is a half space.*

As mentioned before, Theorem 1.2 is the first classification result for stable s-minimal cones in dimension $n = 3$. The analogue result for $n = 2$ and for any $s \in (0,1)$ was established in [15] (see also the quantitative version [7]).

We stress that our result is not a perturbative result from $s = 1$ which could be obtained by some sort of compactness argument. In fact, a careful inspection of our proof gives an explicit (computable) value for s_*, something impossible when using compactness arguments.

A consequence of Theorem 1.2 is the following.

Corollary 1.3. *There exists* $s_* \in (0,1)$ *such that for every* $s \in (s_*, 1)$ *the following statement holds.*

Let E *be an open subset of* $\mathbb{R}^3$. *Assume that* ∂E *is nonempty and of class* C^2, *and that* E *is a stable set for the* s*-perimeter. Then* E *is a half space.*

For the reasons explained next, the proof of Corollary 1.3 will be given in full detail in the forthcoming paper [4]. It follows a rather standard (at least in the context of minimizers) blow-down approach. Besides the classification of stable cones from Theorem 1.2, the proof of Corollary 1.3 needs the following four ingredients, which are known in the setting of stable s-minimal sets provided that their boundaries are C^2:

(i) universal perimeter estimate (established for C^2 stable sets in [7]),

(ii) density estimates (established for C^2 stable sets in [4]),

(iii) monotonicity formula (established for minimizers in [5] with a proof that works also for C^2 stable sets),

(iv) improvement of flatness (established for minimizers in [5] with a proof that works also for C^2 stable – or even stationary – sets).

Note that in the context of classical minimal surfaces, (i), (ii), and (iv) are known for minimizers but not for stable surfaces.

The main obstruction to remove the C^2 assumption is (iv), since the improvement of flatness in [5] has been established for viscosity solutions of the nonlocal minimal surface equation. Although it is obvious that C^2 stable s-minimal sets are viscosity solutions, the same is not known for generic stable sets.

Properties analogous to (i)–(iv) will appear in our forthcoming paper [4] in the context of stable solutions to the fractional Allen–Cahn equation

$$(-\Delta)^{\frac{s}{2}} u = u - u^3, \quad |u| < 1 \quad \text{in } \mathbb{R}^n. \tag{1.1}$$

We will prove there a classification result analogous to Corollary 1.3, but now for equation (1.1). The proofs for solutions to (1.1) and for s-minimal surfaces are roughly the same and, moreover, both use some tools that are not needed in the present paper (e.g., the Caffarelli–Silvestre extension). For these reasons, we have decided to differ the details of the proof of Corollary 1.3 to [4].

The abstract classification result in [4] (which was actually the primary motivation of the present paper) is:

Theorem ([4]). *Assume that for some pair* (n, s)*, with* $s \in (0,1)$*, the half-spaces are the only stable* s*-minimal cones in* $\mathbb{R}^m$ *(which are smooth away from* 0*) for* $2 \le m \le n$*. Then every stable solution of equation* (1.1) *in* $\mathbb{R}^n$ *is a one-dimensional profile, that is,* $u(x) = \phi(e \cdot x)$ *for some increasing function* $\phi : \mathbb{R} \to (-1, 1)$ *and* $e \in S^{n-1}$.

As a consequence of this statement and of Theorem 1.2, we establish in [4]:

(i) one-dimensional profiles are the only *stable* solutions of equation (1.1) when $n = 3$ and $s \in (0,1)$ is sufficiently close to 1,

(ii) one-dimensional profiles are the only *monotone* solutions of equation (1.1) when $n = 4$ and $s \in (0,1)$ is sufficiently close to 1.

Previously, one-dimensional symmetry of stable solutions to (1.1) for $\frac{s}{2} < \frac{1}{2}$ was only known in dimension 2.

The proof in [4] of the classification result for (1.1) establishes that blow down sequences $u(R_k x)$ converge to $\chi_\Sigma - \chi_{\Sigma^c}$, where Σ is a stable minimal cone which, after a dimension reduction, can be assumed to be smooth away from 0. Furthermore, in [4] we prove density estimates ensuring the local uniform convergence of the level sets of u to $\partial\Sigma$ (in the sense of the Hausdorff distance). As a consequence, if we know that the cone is a half space, the improvement of flatness theory for "genuinely nonlocal" phase transitions established in [10] gives that u must be a one-dimensional profile.

Let us finally comment on the proof of Theorem 1.2. It will use three important ingredients from recent works, namely:

(a) The second variation formula for the nonlocal perimeter from [8, 11], which involves a squared nonlocal second fundamental form and that we recall in Theorem 3.1.

(b) The behavior as $s \uparrow 1$ of the optimal constant in the fractional Hardy inequality in dimension two, which can be found for instance in [13], and which we recall in Theorem 3.3.

(c) The universal BV estimate for stable sets of [7], which we recall in Theorem 3.5. In particular, the information that its best constant may be bounded by $\frac{C}{1-s}$ when $s \uparrow 1$.

To prove Theorem 1.2, we plug in the stability inequality given by (a) a radial function that "almost saturates" the Hardy inequality (b) in dimension two. Then we integrate in the radial variable, and appropriately use the universal BV estimate (c) – at every scale – to relate the integrals on $\partial\Sigma$ (a curved two dimensional cone) with the integrals in $\mathbb{R}^2$ appearing in the Hardy inequality. With this, we obtain an integral control on $\partial\Sigma \cap S^2$ for the nonlocal version of the squared second fundamental form of $\partial\Sigma$. This control is given in Proposition 4.3 and is the main goal of Section 4.

Concluding the flatness of the cone from the control in Proposition 4.3 is not a straightforward task. To do it, we need a series of lemmas on curves on S^2 – given in Section 5 – the cornerstone of which is Lemma 5.1. It establishes bounds on the length of a curve on S^2 from an integral control on its squared nonlocal second fundamental form. Interestingly, a crucial ingredient in the proof of Lemma 5.1 is an elementary "topological" observation on closed injective curves in the cylinder $S^1 \times (-1, 1)$, which is given in Lemma 5.2. The application of these lemmas gives that, for s close enough to 1, the curve $\partial\Sigma \cap S^2$ is a simple curve that is very close – in a $C^{1,\frac{1}{4}}$ norm – to a maximal circle. We conclude that the curve must be a maximal circle using the classification of s-minimal Lipschitz graphs of Figalli and Valdinoci [12].

The proof of Theorem 1.2 is given in Section 6 by combining all the previous results.

2. On the notion of stable sets for the s-perimeter

Throughout the paper the notion of stability that we consider is the one of Definition 1.1, which is given specifically in the context of cones in $\mathbb{R}^3$ with C^2 boundary away from 0. In this setting, Definition 1.1 is the weakest notion of stability one can think of – note that we do not need to allow perturbations that affect the vertex of the cone.

For the sake of clarity, we recall now the notion of stability that was introduced and used in [7], and we explain below why this was done. It applies to general sets of finite s-perimeter.

Definition 2.1 ([7], stability). A set $E \subset \mathbb{R}^n$ with $P_s(E, \Omega) < \infty$ is said to be *stable in* Ω if for every given vector field $X = X(x, t) \in C_c^\infty(\Omega \times (-1, 1); \mathbb{R}^n)$ we have

$$\liminf_{t\to 0} \frac{1}{t^2}(P_s(\phi_X^t(E) \cap E, \Omega) - P_s(E, \Omega)) \geq 0$$

and

$$\liminf_{t\to 0} \frac{1}{t^2}(P_s(\phi_X^t(E) \cup E, \Omega) - P_s(E, \Omega)) \geq 0,$$

where ϕ_X^t is the integral flow of X at time t.

Another possible notion of stability, which is weaker than the one given in Definition 2.1 above, is the following:

Definition 2.2. A set $E \subset \mathbb{R}^n$ with $P_s(E, \Omega) < \infty$ is said to be *weakly stable in* Ω if for every given vector field $X = X(x, t) \in C_c^\infty(\Omega \times (-1, 1); \mathbb{R}^n)$ we have

$$\liminf_{t\to 0} \frac{1}{t^2}(P_s(\phi_X^t(E), \Omega) - P_s(E, \Omega)) \geq 0,$$

where ϕ_X^t is the integral flow of X at time t.

Notice that every stable set E (i.e., satisfying Definition 2.1) is also weakly stable (in the sense of Definition 2.2), as it is immediately shown using the inequality

$$P_s(\phi_X^t(E), \Omega) + P_s(E, \Omega) \geq P_s(\phi_X^t(E) \cap E, \Omega) + P_s(\phi_X^t(E) \cup E, \Omega).$$

For $s \in (0, 1]$, both definitions are known to be equivalent[2] when applied to sets E with C^2 boundary in Ω. Thus, our stability assumption in Definition 1.1 and Theorem 1.2 is equiv-

[2] See Remark 3.2.

alent to the cone with C^2 boundary away from the origin being stable in $\mathbb{R}^n \setminus \{0\}$ in the sense of Definition 2.1, and also to being weakly stable in $\mathbb{R}^n \setminus \{0\}$ as in Definition 2.2.

Note that, for the classical perimeter ($s = 1$), Definition 2.1 – and not Definition 2.2 – is the correct notion of stability in order to rule out cones such as "the cross"

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 > 0\}$$

to be stable. The reason is that Definition 2.1 allows "infinitesimal" perturbations that "break the topology" of E – while Definition 2.2 does not. In Definition 2.1 the topology may be changed, for instance, when E is the above "cross" and ϕ_X^t is a translation near the origin. As an example, $\phi_X^t(E) \cup E$ could be given by $\{(x_1 - t)(x_2 - t) > 0\} \cup \{x_1 x_2 > 0\}$, which is a connected set, while E is disconnected.

The previous example shows that the two notions of stability are indeed different in the limit case $s = 1$ of the classical perimeter. For $s \in (0, 1)$, however, some heuristics seem to suggest that the two definitions might be equivalent. For instance, "the cross" is no longer weakly stable, due to nonlocal effects. However, it is an open question whether (or not), in the nonlocal case $s \in (0, 1)$, every weakly stable set is stable. This statement, if true, would be very useful to obtain – using the BV estimates of [7] – clean compactness results for stable sets for the s-perimeter – with $s \in (0, 1)$ fixed –, since weak stability is better suited for passages to the limit.

3. Previously known ingredients that our proof uses

As explained in the introduction, the proof of Theorem 1.2 uses three main ingredients from previous works, which we gather here.

First, we will use a formula, found in [8, 11], for the second (normal) variation of the fractional perimeter. We state it in $\mathbb{R}^3$ but an analogue in $\mathbb{R}^n$ also holds true.

Theorem 3.1 ([8, 11]). *Let $\Sigma \subset \mathbb{R}^3$ be a stable cone for the s-perimeter. Assume that $\partial\Sigma$ is C^2 away from 0. Then, for every $\zeta \in C_c^2(\mathbb{R}^3 \setminus \{0\})$, we have*

$$\int_{\partial\Sigma} c_{s,\partial\Sigma}^2(x) |\zeta(x)|^2 \, dH^2(x) \le \iint_{\partial\Sigma \times \partial\Sigma} \frac{|\zeta(x) - \zeta(y)|^2}{|x - y|^{3+s}} \, dH^2(x) \, dH^2(y),$$

where

$$c_{s,\partial\Sigma}^2(x) := \int_{\partial\Sigma} \frac{|\nu_\Sigma(x) - \nu_\Sigma(y)|^2}{|x - y|^{3+s}} \, dH^2(y)$$

and $\nu_\Sigma(x)$ denotes the outward normal vector to Σ at $x \in \partial\Sigma$.

Remark 3.2. Theorem 3.1 is an application (to the case of cones in $\mathbb{R}^3$) of a second variation formula found in [8, 11] for stationary sets $E \subset \mathbb{R}^n$ with C^2 boundaries. Namely, if X is a smooth vector field and ∂E is stationary and C^2, we have

$$\begin{aligned} \text{(3.1)} \qquad & \lim_{t \to 0} \frac{1}{t^2} \big(P_s(\phi_X^t(E), \Omega) - P_s(E, \Omega)\big) \\ &= \iint_{\partial E \times \partial E} \frac{|\zeta(x) - \zeta(y)|^2}{|x - y|^{n+s}} \, dH^{n-1}(x) \, dH^{n-1}(y) \\ &\quad - \int_{\partial\Sigma} c_{s,\partial\Sigma}^2(x) |\zeta(x)|^2 \, dH^{n-1}(x), \end{aligned}$$

where $\zeta = X \cdot \nu_E$. A standard approximation argument then shows that (3.1) holds for all ζ Lipschitz and compactly supported on ∂E.

Using this formula, we can show that, in the class of C^2 sets, the two notions of stability in Definitions 2.1 and 2.2 are equivalent. Indeed, when ∂E is C^2 we have

$$\begin{aligned}\liminf_{t\to 0} \frac{1}{t^2}(P_s(\phi_X^t(E) \cap E, \Omega) - P_s(E,\Omega)) \\ = \iint_{\partial\Sigma\times\partial\Sigma} \frac{|\zeta^-(x) - \zeta^-(y)|^2}{|x-y|^{n+s}}\, dH^{n-1}(x)\, dH^{n-1}(y) \\ - \int_{\partial\Sigma} c^2_{s,\partial\Sigma}(x)|\zeta^-(x)|^2\, dH^{n-1}(x),\end{aligned}$$

where ζ^- denotes the negative part of $\zeta = X \cdot \nu_E$. The same holds with $\cap$ replaced by $\cup$ and the negative part replaced by the positive part. From these observations, it follows that the stronger notion of stability (Definition 2.1) holds whenever the weaker definition of stability (Definition 2.2) holds.

We recall now the precise dependence on the power σ as $\sigma \uparrow 1$ in the definition of the fractional Laplacian in $\mathbb{R}^d$:

$$(-\Delta)^\sigma \zeta(x) = c_{d,\sigma} \int_{\mathbb{R}^d} \frac{\zeta(x) - \zeta(y)}{|x-y|^{d+2\sigma}}\, dy,$$

where

$$c_{d,\sigma} = 2^{2\sigma}\pi^{-\frac{d}{2}} \frac{\Gamma(\frac{d}{2}+\sigma)}{-\Gamma(-\sigma)} = 2^{2\sigma}\pi^{-\frac{d}{2}} \frac{\Gamma(\frac{d}{2}+\sigma)}{\Gamma(2-\sigma)}\sigma(1-\sigma). \tag{3.2}$$

In particular, we observe that, up to a positive multiplicative constant, $c_{d,\sigma}$ behaves like $1-\sigma$ as $\sigma \uparrow 1$. Note also that integration by parts yields

$$\begin{aligned}\int_{\mathbb{R}^d} \zeta(x)(-\Delta)^\sigma \zeta(x)\, dx &= c_{d,\sigma} \int_{\mathbb{R}^d} \zeta(x)\left(\int_{\mathbb{R}^d} \frac{\zeta(x)-\zeta(y)}{|x-y|^{d+2\sigma}}\, dy\right) dx \\ &= \frac{c_{d,\sigma}}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^{d+2\sigma}}\, dx\, dy.\end{aligned} \tag{3.3}$$

Our proof requires the knowledge of the behavior as $\sigma \uparrow 1$ of the best constant in the Hardy–Rellich inequality involving the H^σ seminorm – see[3)] for instance [13]. We will also use the fact that the inequality is (almost) saturated by radial $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ functions. That radial functions saturate the inequality is proved in [13, Section 3.3]. Moreover, by a standard approximation argument, we can choose these radial functions to be smooth and identically zero in a neighborhood of the origin, since points have zero H^σ capacity in $\mathbb{R}^d$ for $d \ge 2$.

Theorem 3.3 (see [13]). *Given $d \ge 2$ and $0 < \sigma < \frac{d}{2}$, the inequality*

$$\mathcal{H}_{d,\sigma} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^{2\sigma}}\, dx \le \int_{\mathbb{R}^d} u(x)(-\Delta)^\sigma u(x)\, dx$$

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3) Note that there is a typo in the expression for the optimal constant in in [13, formula (1.6)].

holds for every $u \in H^\sigma(\mathbb{R}^d)$ *with optimal constant*

$$\mathcal{H}_{d,\sigma} = 2^{2\sigma} \frac{\Gamma^2(\frac{d}{4}+\frac{\sigma}{2})}{\Gamma^2(\frac{d}{4}-\frac{\sigma}{2})} = 2^{2\sigma-2}(\frac{d}{2}-\sigma)^2 \frac{\Gamma^2(\frac{d}{4}+\frac{\sigma}{2})}{\Gamma^2(\frac{d}{4}-\frac{\sigma}{2}+1)}, \tag{3.4}$$

where Γ *is the Gamma function.*

Moreover, for every $\epsilon > 0$ *there exists a nontrivial (not identically zero) radial function* $\zeta = \zeta(|x|) \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ *such that*

$$\int_{\mathbb{R}^d} \zeta(x)(-\Delta)^\sigma \zeta(x)\, dx \le (\mathcal{H}_{d,\sigma} + \epsilon) \int_{\mathbb{R}^d} \frac{|\zeta(x)|^2}{|x|^{2\sigma}}\, dx.$$

Next, we rewrite in polar coordinates the last inequality of the theorem, for $d = 2$.

Corollary 3.4. *Let* $\sigma \in (\frac{1}{2}, 1)$. *There exists a radial function* $\zeta \in C_c^\infty((0,+\infty))$, $\zeta \not\equiv 0$, *such that*

$$I[\zeta,\sigma] \le C(1-\sigma) J[\zeta,\sigma],$$

where

$$I[\zeta,\sigma] := \int_0^\infty dr\, r^{1-2\sigma} \int_1^\infty d\tau\, \tau |\zeta(r) - \zeta(r\tau)|^2 \iint_{S^1\times S^1} \frac{dH^1(\hat{X})\, dH^1(\hat{Y})}{|\hat{X} - \tau\hat{Y}|^{2+2\sigma}}, \tag{3.5}$$

$$J[\zeta,\sigma] := \int_0^\infty dr\, r^{1-2\sigma} |\zeta(r)|^2, \tag{3.6}$$

and C *is a universal constant (in particular, independent of* σ*).*

Proof. First, we observe that the best constant in Theorem 3.3 for $d = 2$ satisfies

$$\mathcal{H}_{2,\sigma} \le C(1-\sigma)^2$$

(where C is a positive universal constant) as one can see from the last expression in (3.4). Combining equality (3.3), where $c_{d,\sigma}$ is given by (3.2), and the second inequality of Theorem 3.3 with the choice $\epsilon = C(1-\sigma)^2$, we deduce that there is a radial function

$$\zeta = \zeta(|x|) \in C_c^\infty(\mathbb{R}^d \setminus \{0\}),$$

with $\zeta \not\equiv 0$, satisfying

$$C(1-\sigma)^2 \int_{\mathbb{R}^2} \frac{|\zeta(|x|)|^2}{|x|^{2\sigma}}\, dx \ge (1-\sigma) \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|\zeta(|x|) - \zeta(|y|)|^2}{|x-y|^{2+2\sigma}}\, dx\, dy. \tag{3.7}$$

Finally, we use polar coordinates $x = r\hat{X}$, $y = t\hat{Y}$, where $r, t \in (0,+\infty)$ and $\hat{X}, \hat{Y} \in S^1$, and we integrate only in the set $\{|x| \le |y|\}$ on the right-hand side of (3.7), to get

$$\begin{aligned} C(1-\sigma) &\int_0^\infty dr\, r^{1-2\sigma} |\zeta(r)|^2 \\ &\ge \int_0^\infty dr\, r \int_r^\infty dt\, t \int_{S^1} dH^1(\hat{X}) \int_{S^1} dH^1(\hat{Y}) \frac{|\zeta(r) - \zeta(t)|^2}{r^{2+2\sigma} |\hat{X} - \frac{t}{r}\hat{Y}|^{2+2\sigma}} \\ &= \int_0^\infty dr\, r^{1-2\sigma} \int_1^\infty d\tau\, \tau \int_{S^1} dH^1(\hat{X}) \int_{S^1} dH^1(\hat{Y}) \frac{|\zeta(r) - \zeta(r\tau)|^2}{|\hat{X} - \tau\hat{Y}|^{2+2\sigma}}, \end{aligned}$$

where in the last equality we have used the change of variables $t = r\tau$. This concludes the proof of the corollary. □

Finally, we state the estimate established in [7] for the classical perimeter of stable sets for the s-perimeter, keeping track on how the constant in the estimate blows up as $s \uparrow 1$.

Theorem 3.5 ([7]). *Let $E \subset \mathbb{R}^n$ be a stable set for the s-perimeter in $B_r(z)$, where $z \in \mathbb{R}^n$, $r > 0$. Assume that ∂E is C^2 in that ball. Then*

$$\mathrm{Per}_{B_{\frac{r}{2}}(z)}(E) \le \frac{C}{1-s} r^{n-1},$$

where $\mathrm{Per}_{B_{\frac{r}{2}}}(E)$ denotes the relative (classical) perimeter of E in $B_{\frac{r}{2}}(z)$ and $C = C(n,s)$ is bounded as $s \uparrow 1$.

Proof. The theorem follows by inspection of the proof of [7, Theorem 1.7], taking into account the explicit dependence of the constants on s as $s \uparrow 1$. For the sake of clarity, we write here below the crucial estimates in [7, proof of Theorem 1.7], with the precise dependence of all constants on s, as $s \uparrow 1$. In the sequel C will denote positive constants depending only on n and s (possibly different ones) which remain bounded as $s \uparrow 1$. Note that [7, Theorem 1.9], applied to the kernel

$$K(z) = |z|^{-n-s},$$

gives that

$$\mathrm{Per}_{B_1}(E) \le C\big(1 + \sqrt{P_s(E, B_4)}\big) \tag{3.8}$$

if E is a stable set in B_4, where C only depends on n. We rewrite now [7, inequalities (3.8) and (3.9)] keeping track of the dependence on s of all constants. We have

$$\begin{aligned}(1-s)P_s(E,B_4) &\le (1-s)\iint_{B_4\times B_4} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy \\ &\quad + 2(1-s)\iint_{B_4\times B_4^c} \frac{1}{|x-y|^{n+s}}\,dx\,dy \\ &\le (1-s)\iint_{B_4\times B_4} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy + C \\ &\le C(\mathrm{Per}_{B_4}(E)+1),\end{aligned} \tag{3.9}$$

with $C = C(n,s)$ bounded as $s \uparrow 1$, where for the last inequality we refer to [2, Theorem 1 and Remark 5] or to [9, proof of Proposition 2.2]. Hence, (3.8), (3.9), and Young inequality lead to

$$\begin{aligned}\mathrm{Per}_{B_1}(E) &\le C\left(1 + \frac{1}{(1-s)^{\frac{1}{2}}}(1+\mathrm{Per}_{B_4}(E))^{\frac{1}{2}}\right) \\ &\le C\left(1 + \frac{1}{\delta(1-s)} + \delta\right) + \delta \mathrm{Per}_{B_4}(E)\end{aligned}$$

for all $\delta > 0$. Arguing exactly as in [7, end of the proof of Theorem 1.7] (that is, rescaling and using [7, Lemma 3.1]), we deduce that

$$\mathrm{Per}_{B_1}(E) \le \frac{C}{1-s}.$$

Thus, after rescaling, we conclude the statement of the theorem. □

4. Bounding the squared nonlocal second fundamental form

In this section we denote by γ the intersection of the boundary of the cone $\partial\Sigma \subset \mathbb{R}^3$ and the sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$. Note that γ is a finite union of C^2 simple curves on S^2.

In the following lemma we compute the stability formula of Theorem 3.1 for a radial test function $\zeta = \zeta(r)$, where $r = |x|$.

Lemma 4.1. *Let Σ be a stable cone for the s-perimeter in $\mathbb{R}^3$. Assume that $\partial\Sigma$ is C^2 away from 0. Let us call $\gamma := \partial\Sigma \cap S^2$, where $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$. Then, for every $\zeta \in C_c^2((0, +\infty))$, we have*

$$A\, J[\zeta, \tfrac{1+s}{2}] \le \int_0^\infty dr\, r^{-s} \int_1^\infty d\tau\, \tau |\zeta(r) - \zeta(r\tau)|^2 \iint_{\gamma\times\gamma} \frac{dH^1(\hat{x}) dH^1(\hat{y})}{|\hat{x} - \tau\hat{y}|^{3+s}},$$

where

$$A := \int_\gamma dH^1(\hat{x})\, c^2_{s,\partial\Sigma}(\hat{x})$$

and $J[\zeta, \frac{1+s}{2}]$ is given by (3.6).

Proof. We take the radial test function $\zeta = \zeta(|x|)$ in the stability inequality of Theorem 3.1, and we use polar coordinates $x = r\hat{x}$, $y = t\hat{y}$ to obtain

$$\begin{aligned}\int_\gamma dH^1(\hat{x}) \int_\gamma dH^1(\hat{y}) \int_0^\infty dr\, r \int_0^\infty dt\, t \frac{|\zeta(r) - \zeta(t)|^2}{|r\hat{x} - t\hat{y}|^{3+s}} \\ \ge \int_\gamma dH^1(\hat{x}) \int_0^\infty dr\, r\, c^2_{s,\partial\Sigma}(r\hat{x}) |\zeta(r)|^2.\end{aligned}$$

We observe that since Σ is a cone, $\nu_\Sigma(r\hat{x}) = \nu_\Sigma(\hat{x})$ for all $r > 0$ and thus, denoting $\hat{y} = \frac{y}{|y|}$,

$$\begin{aligned}c^2_{s,\partial\Sigma}(r\hat{x}) &= \int_{\partial\Sigma} \frac{|\nu_\Sigma(\hat{x}) - \nu_\Sigma(\hat{y})|^2}{|r\hat{x} - y|^{3+s}}\, dH^2(y) = \int_{\partial\Sigma} \frac{|\nu_\Sigma(\hat{x}) - \nu_\Sigma(\hat{z})|^2}{|r\hat{x} - r\hat{z}|^{3+s}}\, r^2\, dH^2(\hat{z}) \\ &= \frac{c^2_{s,\partial\Sigma}(\hat{x})}{r^{1+s}}.\end{aligned}$$

Hence, using that $\partial\Sigma \times \partial\Sigma = \{|x| > |y|\} \cup \{|y| > |x|\}$ up to measure zero sets, and the symmetry of the integrand with respect to interchanging x, y, we obtain

$$2\int_\gamma dH^1(\hat{x}) \int_\gamma dH^1(\hat{y}) \int_0^\infty dr\, r \int_r^\infty dt\, t \frac{|\zeta(r) - \zeta(t)|^2}{|r\hat{x} - t\hat{y}|^{3+s}} \ge A \int_0^\infty \frac{dr}{r^s} |\zeta(r)|^2,$$

where

$$A = \int_\gamma dH^1(\hat{x})\, c^2_{s,\partial\Sigma}(\hat{x}).$$

Doing the change of variables $t = r\tau$, we obtain

$$2\int_\gamma dH^1(\hat{x}) \int_\gamma dH^1(\hat{y}) \int_0^\infty dr\, r \int_1^\infty d\tau \frac{r^2}{r^{3+s}} \tau \frac{|\zeta(r) - \zeta(r\tau)|^2}{|\hat{x} - \tau\hat{y}|^{3+s}} \ge A \int_0^\infty \frac{dr}{r^s} |\zeta(r)|^2,$$

and the lemma follows recalling the definition of $J[\zeta, \frac{1+s}{2}]$ in (3.6). □

The following lemma allows to estimate the integral on $\gamma \times \gamma$, appearing in the previous lemma, by the integral on $S^1 \times S^1$ that appears in $I[\zeta, \sigma]$ of Corollary 3.4 for $\sigma = \frac{1+s}{2}$. Here it is crucial to use in every ball $B_r(\hat{x})$, with $\hat{x} \in \gamma$ and $r \in (0, \frac{1}{2})$ the universal perimeter estimate from Theorem 3.5.

Lemma 4.2. *Let Σ be a stable cone for the s-perimeter in $\mathbb{R}^3$, of class C^2 away from* 0*. Let us call $\gamma := \partial\Sigma \cap S^2$, where $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$. Then, for all $\tau > 1$, we have*

$$\iint_{\gamma\times\gamma} \frac{dH^1(\hat{x})dH^1(\hat{y})}{|\hat{x} - \tau\hat{y}|^{3+s}} \le \frac{CH^1(\gamma)}{1-s} \int_{S^1\times S^1} \frac{dH^1(\hat{X})\,dH^1(\hat{Y})}{|\hat{X} - \tau\hat{Y}|^{3+s}},$$

where $S^1 := \{X \in \mathbb{R}^2 : |X| = 1\}$ and $C = C(s)$ is bounded as $s \uparrow 1$.

Proof. Applying Theorem 3.5 to the stable cone Σ, we obtain that, for all $\hat{x} \in \gamma$ and $r \in (0, \frac{1}{2})$, we have

$$H^1(\gamma \cap B_r(\hat{x})) \le \frac{C}{1-s} r, \tag{4.1}$$

where C denotes a constant depending only on s which is bounded as $s \uparrow 1$. In particular, by a covering argument we obtain $H^1(\gamma) \le \frac{C}{1-s}$.

We now take any couple of points $\hat{x} \in \gamma$ and $\hat{X} \in S^1 := \{X \in \mathbb{R}^2 : |X| = 1\}$. Let us show that, for all $\tau > 1$,

$$\int_\gamma dH^1(\hat{y})\,\frac{1}{|\hat{x} - \tau\hat{y}|^{3+s}} \le \frac{C}{1-s} \int_{S^1} dH^1(\hat{Y})\,\frac{1}{|\hat{X} - \tau\hat{Y}|^{3+s}}. \tag{4.2}$$

Indeed, we use a dyadic ring decomposition

$$\gamma \setminus \{\hat{x}\} = \bigcup_{-\infty\le k\le 1} \mathcal{A}_k, \quad \text{where } \mathcal{A}_k = \gamma \cap (B_{2^k}(\hat{x}) \setminus B_{2^{k-1}}(\hat{x})).$$

Using (4.1), we obtain

$$H^1(\mathcal{A}_k) \le \frac{C}{1-s} 2^k.$$

Then, using that

$$|\hat{x} - \tau\hat{y}|^2 = 1 + \tau^2 - 2\tau\hat{x}\cdot\hat{y} = (\tau-1)^2 + 2\tau(1 - \hat{x}\cdot\hat{y})$$

and that $2^{-3}2^{2k} \le 2^{-1}|\hat{x} - \hat{y}|^2 = 1 - \hat{x}\cdot\hat{y} \le 2^{-1}2^{2k}$ for $y \in \mathcal{A}_k$, we obtain

$$\begin{aligned}
\int_\gamma dH^1(\hat{y})\,\frac{1}{|\hat{x} - \tau\hat{y}|^{3+s}} &= \sum_{-\infty\le k\le 1} \int_{\mathcal{A}_k} dH^1(\hat{y})\,\frac{1}{|\hat{x} - \tau\hat{y}|^{3+s}} \\
&\le \sum_{-\infty\le k\le 1} H^1(\mathcal{A}_k) \frac{C}{((\tau-1)^2 + \tau 2^{2k})^{\frac{3+s}{2}}} \\
&\le \sum_{-\infty\le k\le 1} \frac{C}{1-s} 2^k \frac{1}{((\tau-1)^2 + \tau 2^{2k})^{\frac{3+s}{2}}} \\
&\le \frac{C}{1-s} \int_{S^1} dH^1(\hat{Y})\,\frac{1}{|\hat{X} - \tau\hat{Y}|^{3+s}},
\end{aligned}$$

where the last inequality follows from the previous considerations applied with (Σ, γ) replaced by $(\mathbb{R}^3_+, S^1)$. The lemma then follows integrating (4.2) with respect to $\hat{x}$ and $\hat{X}$. □

We can now give the proof of the key integral estimate on γ of the squared nonlocal second fundamental form of $\partial\Sigma$.

Let us compute $c^2_{s,\partial\Sigma}(\hat{x})$ in terms of only the trace $\gamma = \partial\Sigma \cap S^2$. Recall that $c^2_{s,\partial\Sigma}$ was defined in Theorem 3.1. For this, we introduce the kernel

$$k_s(\hat{x}, \hat{y}) := \int_0^\infty \frac{t\,dt}{|\hat{x} - t\hat{y}|^{3+s}} = \int_0^\infty \frac{t\,dt}{(t^2 + 1 - 2t\hat{x}\cdot\hat{y})^{\frac{3+s}{2}}},$$

and we note that, since Σ is a cone,

$$\begin{aligned} c^2_{s,\partial\Sigma}(\hat{x}) &= \int_{\partial\Sigma} \frac{|\nu_\Sigma(\hat{x}) - \nu_\Sigma(y)|^2}{|\hat{x} - y|^{3+s}}\, dH^2(y) \\ &= \int_\gamma dH^1(\hat{y}) \int_0^\infty dt\, t\, \frac{|\nu_\Sigma(\hat{x}) - \nu_\Sigma(\hat{y})|^2}{|\hat{x} - t\hat{y}|^{3+s}} \\ &= \int_\gamma |\nu_\Sigma(\hat{x}) - \nu_\Sigma(\hat{y})|^2 k_s(\hat{x}, \hat{y})\, dH^1(\hat{y}), \end{aligned}$$

where ν_Σ is the exterior normal vector to $\partial\Sigma$.

We can now state the key integral estimate from which we will deduce our main theorem.

Proposition 4.3. *Let Σ be a stable cone for the s-perimeter in $\mathbb{R}^3$, and of class C^2 away from* 0. *Let us call $\gamma := \partial\Sigma \cap S^2$, where $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$. Then*

$$\int_\gamma c^2_{s,\partial\Sigma}(\hat{x})\, dH^1(\hat{x}) \le CH^1(\gamma),$$

that is,

$$\iint_{\gamma\times\gamma} |\nu_\Sigma(\hat{x}) - \nu_\Sigma(\hat{y})|^2 k_s(\hat{x}, \hat{y})\, dH^1(\hat{x})\, dH^1(\hat{y}) \le CH^1(\gamma),$$

where $C = C(s)$ is bounded as $s \uparrow 1$.

Proof. Let $\zeta = \zeta(|x|)$ be a radial $C^2_c((0, +\infty))$ test function. Using Lemma 4.1, we obtain

$$AJ[\zeta, \tfrac{1+s}{2}] \le \int_0^\infty dr\, r^{-s} \int_1^\infty d\tau\, \tau |\zeta(r) - \zeta(r\tau)|^2 \iint_{\gamma\times\gamma} \frac{dH^1(\hat{x})\, dH^1(\hat{y})}{|\hat{x} - \tau\hat{y}|^{3+s}},$$

where

$$A = \int_\gamma dH^1(\hat{x})\, c^2_{s,\partial\Sigma}(\hat{x})$$

and $J[\zeta, \frac{1+s}{2}]$ is given by (3.6). Next, applying Lemma 4.2, we deduce that

$$\begin{aligned} &\int_0^\infty dr\, r^{-s} \int_1^\infty d\tau\, \tau |\zeta(r) - \zeta(r\tau)|^2 \iint_{\gamma\times\gamma} \frac{dH^1(\hat{x})\, dH^1(\hat{y})}{|\hat{x} - \tau\hat{y}|^{3+s}} \\ &\quad\le \frac{CH^1(\gamma)}{1-s} \int_0^\infty dr\, r^{-s} \int_1^\infty d\tau\, \tau |\zeta(r) - \zeta(r\tau)|^2 \iint_{S^1\times S^1} \frac{dH^1(\hat{X})\, dH^1(\hat{Y})}{|\hat{X} - \tau\hat{Y}|^{3+s}} \\ &\quad= \frac{CH^1(\gamma)}{1-s} I[\zeta, \tfrac{1+s}{2}], \end{aligned}$$

where $I[\zeta, \sigma]$ is as in (3.5). Therefore, we have

$$A\, J[\zeta, \tfrac{1+s}{2}] \le \frac{CH^1(\gamma)}{1-s}\, I[\zeta, \tfrac{1+s}{2}].$$

Finally, choosing $\zeta \not\equiv 0$ as in Corollary 3.4 (with $\sigma = \frac{1+s}{2}$) we have

$$I[\zeta, \tfrac{1+s}{2}] \le C(1-s) J[\zeta, \tfrac{1+s}{2}].$$

Since $\zeta \not\equiv 0$, the proposition follows combining the last two inequalities. □

The following lemma gives a lower bound for k_s.

Lemma 4.4. *For $s \in (\frac{1}{2}, 1)$, we have*

$$k_s(\hat{x}, \hat{y}) \ge c \frac{1}{|\hat{x} - \hat{y}|^{2+s}} \quad \text{for all } \hat{x}, \hat{y} \in S^2$$

and for some universal constant $c > 0$.

Proof. Let us call $b^2 := 1 - \hat{x} \cdot \hat{y} = \frac{1}{2}|\hat{x} - \hat{y}|^2$. Note that $b \in (0, \sqrt{2})$. We have

$$\begin{aligned} k_s(\hat{x}, \hat{y}) &= \int_0^\infty \frac{t\, dt}{((t-1)^2 + 2t(1 - \hat{x} \cdot \hat{y}))^{\frac{3+s}{2}}} \ge \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{(\frac{1}{2})\, dt}{((t-1)^2 + 3b^2)^{\frac{3+s}{2}}} \\ &\ge \frac{1}{2} \int_{-\frac{1}{2b}}^{\frac{1}{2b}} \frac{b\, d\bar{t}}{((b\bar{t})^2 + 3b^2)^{\frac{3+s}{2}}} \ge \frac{1}{2b^{2+s}} \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{d\bar{t}}{(\bar{t}^2 + 3)^{\frac{3+s}{2}}} \\ &\ge \frac{c}{|\hat{x} - \hat{y}|^{2+s}}, \end{aligned}$$

where, in the second inequality, we have used the change of variables $\bar{t} = \frac{t-1}{b}$. This concludes the proof of the lemma. □

We observe that, if a connected component γ_0 of γ is parametrized by arc length, then

$$k_s(\gamma_0(t), \gamma_0(\bar{t})) \ge \frac{c}{|\gamma_0(t) - \gamma_0(\bar{t})|^{2+s}} \ge \frac{c}{|t - \bar{t}|^{2+s}}, \tag{4.3}$$

where we have used Lemma 4.4 for the first inequality and that $|\gamma_0(t) - \gamma_0(\bar{t})| \le |t - \bar{t}|$ for the second inequality.

We conclude this section with the following embedding.

Lemma 4.5. *Let $\sigma \in [\frac{3}{4}, 1)$ and $I = [0, 5\pi]$. Given $f : I \to \mathbb{R}$, we have*

$$\|f - \overline{f}\|_{C^{\frac{1}{4}}(I)} \le C[f]_{H^\sigma(I)},$$

where $\overline{f} = \frac{1}{5\pi} \int_I f$,

$$[f]_{H^\sigma(I)} := \left((1-\sigma) \int_I \int_I \frac{|f(t) - f(\bar{t})|^2}{|t - \bar{t}|^{1+2\sigma}}\, dt\, d\bar{t} \right)^{\frac{1}{2}},$$

and C is a universal constant.

Proof. Let us denote

$$\|f\|_{H^\sigma(I)} = \|f\|_{L^2(I)} + [f]_{H^\sigma(I)}.$$

Since $\sigma \geq \frac{3}{4}$, we have that $H^\sigma(I)$ is continuously embedded in $C^{\frac{1}{4}}(I)$. Then, using the fractional Poincaré inequality (see e.g. [3, 'Fact" stated on p. 80]) in the interval I, we obtain

$$\begin{aligned}\|f - \overline{f}\|_{C^{\frac{1}{4}}(I)} &\leq C\|f - \overline{f}\|_{H^{\frac{3}{4}}(I)} \\ &= C\left\{\|f - \overline{f}\|_{L^2(I)} + [f]_{H^{\frac{3}{4}}(I)}\right\} \\ &\leq C[f]_{H^{\frac{3}{4}}(I)} \leq C[f]_{H^\sigma(I)},\end{aligned}$$

with C universal. We have used, in the last inequality, Remark 5 in [2]. □

5. Auxiliary results on curves of S^2

In this section we prove geometric estimates for a simple curve γ_0 in S^2 satisfying the curvature bounds from Proposition 4.3.

Recall that, throughout the paper, the trace on S^2 of $\partial\Sigma$, which we call γ, is (since Σ is C^2 away from 0) a finite union of C^2 simple closed curves on S^2. Moreover, by the perimeter estimate of Theorem 3.5 we know that the total length of γ is bounded by $C(1-s)^{-1}$. In addition, we obtained in Proposition 4.3 a certain integral control on the squared nonlocal second fundamental form of $\partial\Sigma$.

Lemmas 5.1 and 5.3 below contain geometric estimates for a closed simple curve (i.e., without self-intersections) γ_0 in S^2, whose length is bounded by $C(1-s)^{-1}$ and satisfying an integral control on its squared nonlocal second fundamental form. A crucial point is that the constants in these estimates do not blow up as $s \uparrow 1$. In the proof of Theorem 1.2 these lemmas will be applied to the connected components of γ.

The first and most important estimate is the following bound, uniform as $s \uparrow 1$, for the length of γ_0.

Lemma 5.1. *Let $s \in (\frac{1}{2}, 1)$, $L > 0$, and let $\gamma_0 = \gamma_0(t) : [0, L] \to S^2$ be some C^2 closed curve without self-intersections and parametrized by arc length – thus $L =$ length(γ_0). Let $\nu = \gamma_0 \wedge \gamma_0'$ be the "clockwise" normal vector (which is tangent to the sphere).*

Assume that, for some positive constant C_0,

$$\int_0^L \int_0^L |\nu(t) - \nu(\bar{t})|^2 k_s(\gamma_0(t), \gamma_0(\bar{t}))\, dt\, d\bar{t} \leq C_0 L < +\infty. \tag{5.1}$$

Assume in addition that

$$0 < L \leq \frac{C_0}{1-s}.$$

Then

$$L \leq C$$

for some constant C depending only on C_0.

To prove Lemma 5.1, the following "topological" observation will be crucial.

Lemma 5.2. *Let $S^1 \times (-1, 1)$ be the cylinder*

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ |z| < 1\}.$$

Let $\theta \in \mathbb{R}$ (mod $2\pi\mathbb{Z}$) and $z \in (-1, 1)$ be the standard cylindrical coordinates.

Assume that $\omega : [0, 4\pi] \to S^1 \times (-1, 1)$ is a C^1 curve of the type

$$\omega = \omega(\theta) = (\cos\theta, \sin\theta, z(\theta))$$

and satisfying, for some $b \in (0, \frac{1}{100})$,

$$|z(0)| \le \frac{b}{2} \qquad \text{and} \qquad |z'(\theta)| \le \frac{b}{8\pi} \quad \text{for all } \theta \in [0, 4\pi].$$

Assume in addition that ω is injective – i.e., it does not have self intersections – and that $z(0) < z(4\pi)$.

Let $\tilde{\omega} : [t_1, t_2] \to S^1 \times (-1, 1)$ be any C^1 curve such that $\tilde{\omega}(t_1) = \omega(4\pi)$ and $\tilde{\omega}(t_2) = \omega(0)$ such that $\tilde{\omega}((t_1, t_2))$ and $\omega((0, 4\pi))$ are disjoint. Assume that $\tilde{\omega}$ is parametrized by the arc length. Let us denote

$$\tilde{\omega}(t) = (\cos\tilde{\theta}(t), \sin\tilde{\theta}(t), \tilde{z}(t)).$$

Then, for each $\theta_0 \in (0, 2\pi)$, there is at least one $t \in (t_1, t_2)$ such that

$$\tilde{\theta}(t) = \theta_0 \ (\mathrm{mod}\, 2\pi), \quad -b \le z(\theta_0) \le \tilde{z}(t) \le z(\theta_0 + 2\pi) \le b, \quad \text{and} \quad \tilde{\theta}'(t) \le 0.$$

As a consequence, using that $\tilde{\omega}$ is parametrized by the arc length and defining

$$A := \{t \in (t_1, t_2) : |\tilde{z}(t)| \le b,\ \tilde{\theta}'(t) \le 0\},$$

we have

$$H^1(A) \ge 2\pi.$$

Proof. Note that

$$|z(4\pi)| \le \frac{b}{2} + 4\pi\frac{b}{8\pi} \le b.$$

Let us call $P = \omega(0) = (1, 0, z(0))$ and $Q = \omega(4\pi) = (1, 0, z(4\pi))$.

For each $\theta_0 \in (0, 2\pi)$ the open set

$$(S^1 \times (-1, 1)) \setminus (\omega([0, 4\pi]) \cup \{(\cos\theta_0, \sin\theta_0)\} \times [-b, b])$$

has exactly two connected components. The curve $\tilde{\omega}$, which connects the points Q and P without intersecting $\omega((0, 4\pi))$ starts in the upper connected component (the one containing a neighborhood of Q) and finishes in the lower connected component (the one containing a neighborhood of P). Hence there is at least one time $t_{\theta_0} \in (t_1, t_2)$ at which $\tilde{\omega}$ intersects the segment $\{(\cos\theta_0, \sin\theta_0)\} \times [-b, b]$ to go from the upper to the lower components. It easily follows that t_{θ_0} in A.

For the last inequality in the statement, we use that, as shown above, the image of $\tilde{\omega}(A)$ under the projection of $S^1 \times (-1, 1) \to S^1$ has length 2π. It follows that the length of $\tilde{\omega}(A)$ is at least 2π, and thus also the length of A (since $\tilde{\omega}$ is parametrized by the arc length). □

We can now give the proof of Lemma 5.1.

Proof of Lemma 5.1. Let us assume that $5\pi N \le L < 5\pi(N+1)$, where $N > 0$ is an integer. We need to bound N. Hence, we may clearly assume that N is large enough.

Let us consider the N disjoint intervals $I_j := [5\pi(j-1), 5\pi j)$, $1 \le j \le N$, which are subsets of $[0, L)$. Let

$$\kappa_j := (1-s)\int_{I_j}\int_0^L |\nu(t)-\nu(\bar t)|^2 k_s(\gamma_0(t),\gamma_0(\bar t))\, dt\, d\bar t$$

and let $j_1, j_2, \dots, j_N$ be an ordering for which

$$\kappa_{j_1} \le \kappa_{j_2} \le \cdots \le \kappa_{j_N}.$$

Choose $M := \lfloor \frac{N}{2} \rfloor$ and notice that

$$\begin{aligned}
(5.2)\quad \max_{1\le i\le M} \kappa_{j_i} = \kappa_{j_M} &\le \frac{1}{N-M}\sum_{i=M+1}^{N} \kappa_{j_i}\\
&= \frac{1}{N-M}(1-s)\sum_{i=M+1}^{N}\int_{I_{j_i}}\int_0^L |\nu(t)-\nu(\bar t)|^2 k_s(\gamma_0(t),\gamma_0(\bar t))\, dt\, d\bar t\\
&\le \frac{1}{N-M}(1-s)\int_0^L\int_0^L |\nu(t)-\nu(\bar t)|^2 k_s(\gamma_0(t),\gamma_0(\bar t))\, dt\, d\bar t\\
&\le 2\frac{C_0(1-s)L}{N},
\end{aligned}$$

where, in the last inequality, we have used assumption (5.1).

For the sake of clarity, we split the proof into four steps.

Step 1. Let us prove that for $I = I_{j_i}$, where $1 \le i \le M = \lfloor \frac{N}{2} \rfloor$, we have

$$\|\nu(t) - e\|_{C^{\frac14}(I)} \le C[\nu]_{H^\sigma(I)} \le C\delta^{\frac12} \quad \text{for some } e \in S^2,$$

$\sigma = \frac12(1+s)$ and

$$\delta := 2\frac{C_0^2}{N}.$$

Indeed, using in (5.2) the assumption $(1-s)L \le C_0$ we have that, for $1 \le i \le M = \lfloor \frac{N}{2} \rfloor$, the interval $I = I_{j_i}$ has length 5π and satisfies

$$(5.3)\qquad (1-s)\int_I\int_0^L |\nu(t)-\nu(\bar t)|^2 k_s(\gamma_0(t),\gamma_0(\bar t))\, dt\, d\bar t \le 2\frac{C_0^2}{N} = \delta.$$

Now using (4.3), we deduce that

$$(1-s)\int_I\int_I \frac{|\nu(t)-\nu(\bar t)|^2}{|t-\bar t|^{2+s}}\, dt\, d\bar t \le C\delta,$$

where C is universal. That is, for $\sigma = \frac{1+s}{2}$ and $j = 1, 2, 3$, we have (recall the definition of the H^σ-seminorm in Lemma 4.5)

$$[\nu^j]_{H^\sigma(I)} \le C\delta^{\frac12}, \quad \text{where } \nu = (\nu^1, \nu^2, \nu^3).$$

Using Lemma 4.5, we obtain

$$\|\nu(t) - e\|_{C^{\frac{1}{4}}(I)} \le C[\nu]_{H^\sigma(I)} \le C\delta^{\frac{1}{2}} \quad \text{for some } e \in \mathbb{R}^3.$$

Note that (for δ small enough and up to changing C) we may assume that $e \in S^2$ since $\nu \in S^2$. This proves that the velocity (or tangent) vector γ_0' is almost perpendicular to e in all of I with a very small error in the angle controlled by $C\delta^{\frac{1}{2}}$. Recall that we may assume δ to be sufficiently small since, as mentioned in the beginning of the proof, N may be assumed to be large enough.

Step 2. We have proven in Step 1 that the restriction of γ_0 to I is $C\delta^{\frac{1}{2}}$ close to tracing a maximal circle. As pointed out in the beginning of the proof, we may assume that N is large enough and thus that $\delta = \frac{2C_0^2}{N}$ is small enough.

Note that since I has length $5\pi > 4\pi$, for δ small enough the curve $\gamma_0|_I$ makes two loops at the (topological) cylinder $S^2 \cap \{-\frac{1}{4} < e \cdot x < \frac{1}{4}\}$, and these loops are $C\delta^{\frac{1}{2}}$ close to the "equator" $S^2 \cap \{e \cdot x = 0\}$. In particular, the "vertical" displacement is less than $C\delta^{\frac{1}{2}}$. Intuitively, since γ_0 is a closed curve, it will have to come back again to the starting point of two loops, and since it does not have self-intersections, the only way this may happen is with γ_0 passing again between the two loops with the opposite orientation (i.e., "undoing" the loop). More precisely, let us prove that

$$\bar{A} := \left\{\bar{t} \in [0, L] \setminus I_\circ : |e \cdot \gamma_0(\bar{t})| \le C\delta^{\frac{1}{2}} \text{ and } e \cdot \nu(\bar{t}) \le \frac{1}{100}\right\}, \tag{5.4}$$

where $I_\circ \subset I$ is an interval to be defined next and with $|I_0| \ge 3\pi$, satisfies

$$H^1(\bar{A}) \ge \frac{19}{10}\pi. \tag{5.5}$$

Indeed, let us choose an orthonormal coordinate frame X, Y, Z with origin at 0 and with Z directed as e. Let us define "cylindrical" coordinates in $S^2 \cap \{-\frac{1}{4} < e \cdot x < \frac{1}{4}\}$ as follows:

$$X = \cos\theta \cos z, \quad Y = \sin\theta \cos z, \quad Z = \sin z.$$

Since γ_0 is a closed curve without self-intersections, we may apply Lemma 5.2 with

$$\omega = \gamma_0|_{I_\circ}, \quad \tilde{\omega} = \gamma_0|_{([0,L]/\{0,L\})\setminus I_\circ}, \quad \text{and} \quad b = C\delta^{\frac{1}{2}},$$

where $I_\circ \subset I$ is an interval for which

$$\int_{I_\circ} \theta'(\gamma_0(t))\, dt = 4\pi$$

as in Lemma 5.2 – here we abuse notation and omit the fact that ω and $\tilde{\omega}$ would need to reparametrized by the angle θ and by the arc length of the cylinder respectively. From the last equality we deduce, using

$$1 = |\gamma_0'| = (\theta')^2 \cos^2 z + (z')^2 \ge (\theta')^2 \left(\frac{3}{4}\right)^2$$

if δ is small enough, that $|I_\circ| \ge 3\pi$.

Applying Lemma 5.2, the set

$$A := \{\bar{t} \in ([0,L]/\{0,L\}) \setminus I_\circ : |\tilde{z}(\bar{t})| \le C\delta^{\frac{1}{2}},\ \tilde{\theta}'(\bar{t}) \le 0\}$$

– in the notation of Lemma 5.2 – satisfies $H^1(A) \ge \frac{19}{10}\pi$. Here, on the right-hand side we need to choose a number slightly smaller than 2π due to the fact that $\tilde{\omega}$ needs to be reparametrized by the arc length of the cylinder in order to apply Lemma 5.2 (understanding that δ is chosen accordingly small enough so that the arc lengths on the sphere near the equator and on cylinder are almost the same).

Observe also that for every $\bar{t} \in A$ we have that $|\tilde{z}(\bar{t})|$ is very small (for δ small enough) and that $\tilde{\theta}'(\bar{t}) \le 0$. As a consequence[4)]

$$e \cdot \nu(\bar{t}) \le \frac{1}{100},$$

as before in (5.4), provided that δ is small enough. In other words, the normal vector to γ_0 at $\bar{t}$, which is tangent to S^2, can only have, at most, a tiny positive projection in the "vertical" direction e. Hence, $A \subset \bar{A}$ and (5.5) follows.

Step 3. For each given $\bar{t} \in \bar{A}$ there exists $t' \in I_\circ$ such that $|\gamma_0(\bar{t}) - \gamma_0(t')| \le C\delta^{\frac{1}{2}}$, with C universal, since $\gamma_0|_{I_\circ}$ makes two full loops to the equator. Hence, we deduce that

$$(5.6)\qquad \begin{aligned}\int_{I_0} \frac{1}{|\gamma_0(t) - \gamma_0(\bar{t})|^{2+s}}\,dt &\ge c\int_{I_0} \frac{1}{(\delta^{\frac{1}{2}} + |\gamma_0(t) - \gamma_0(t')|)^{2+s}}\,dt \\ &\ge c\int_{I_0} \frac{1}{(\delta^{\frac{1}{2}} + |t - t'|)^{2+s}}\,dt \\ &\ge c\int_0^{\frac{3\pi}{2}} \frac{1}{(\delta^{\frac{1}{2}} + \tau)^{2+s}}\,d\tau \ge c(\delta^{\frac{1}{2}})^{-1-s},\end{aligned}$$

where in the third inequality we have used that $t' \in I_0$ and that I_0 is an interval of length at least 3π. Now, notice that for all $t \in I_0$ and $\bar{t} \in \bar{A}$ we have $|\nu(t) - \nu(\bar{t})| \ge 1$ – since the angle between $\nu(t)$ and $\nu(\bar{t})$ is at least of 85°. In addition, recall (5.3) and (5.5) to obtain

$$(5.7)\qquad \begin{aligned}\delta &\ge (1-s)\int_{I_0}\int_{\bar{A}} |\nu(t) - \nu(\bar{t})|^2 k_s(\gamma_0(t), \gamma_0(\bar{t}))\,d\bar{t}\,dt \\ &\ge (1-s)c\int_{\bar{A}}\int_{I_0} \frac{1}{|\gamma_0(t) - \gamma_0(\bar{t})|^{2+s}}\,dt\,d\bar{t} \\ &\ge (1-s)\frac{cH^1(\bar{A})}{(\delta^{\frac{1}{2}})^{1+s}} \\ &\ge (1-s)\frac{c}{(\delta^{\frac{1}{2}})^{1+s}}\end{aligned}$$

for different universal constants $c > 0$. It follows that

$$\left(2\frac{C_0^2}{N}\right)^{-1-\frac{1+s}{2}} = \delta^{-1-\frac{1+s}{2}} \le \frac{C}{1-s}.$$

4) Note that if it was $|z(\bar{t})| = 0$ at some $\bar{t}$, the condition $\tilde{\theta}'(\bar{t}) \le 0$ would be exactly equivalent to $\nu(\bar{t}) \cdot e \le 0$. Therefore, if $|z(\bar{t})|$ is very small, $e \cdot \nu(\bar{t})$ cannot be too positive.

Hence, using that $s \geq \frac{1}{2}$, we obtain

$$L \leq 5\pi(N+1) \leq \frac{C}{(1-s)^{\frac{4}{7}}},$$

where C depends only on C_0.

Step 4. Next we repeat exactly the same argument as in Steps 1, 2, and 3 but now using (5.2) together with the improved estimate $L \leq C(1-s)^{-\frac{4}{7}}$ instead of $L \leq C_0(1-s)^{-1}$. We now have that, for $1 \leq i \leq M = \lfloor \frac{N}{2} \rfloor$, the interval $I = I_{j_i}$ has length 5π and satisfies

$$\begin{aligned}(1-s)\int_I \int_0^L |\nu(t)-\nu(\bar{t})|^2 k_s(\gamma_0(t),\gamma_0(\bar{t}))\,dt\,d\bar{t} &\leq 2\frac{C_0(1-s)L}{N}\\ &\leq \frac{C(1-s)^{\frac{3}{7}}}{N} =: \delta'.\end{aligned}$$

Therefore, arguing exactly as above, we obtain

$$\left(\frac{C(1-s)^{\frac{3}{7}}}{N}\right)^{-1-\frac{1+s}{2}} = (\delta')^{-1-\frac{1+s}{2}} \leq \frac{C}{1-s},$$

where C depends only on C_0. Hence

$$\frac{N}{(1-s)^{\frac{3}{7}}} \leq \frac{C}{(1-s)^{\frac{4}{7}}}$$

and thus

$$L \leq 5\pi(N+1) \leq \frac{C}{(1-s)^{\frac{1}{7}}},$$

where C depends only on C_0.

Finally, we repeat exactly the same argument once more but now using (5.2) together with the improved estimate $L \leq C(1-s)^{-\frac{1}{7}}$ instead of $L \leq C(1-s)^{-\frac{4}{7}}$. We now have that, for $1 \leq i \leq M = \lfloor \frac{N}{2} \rfloor$, the interval $I = I_{j_i}$ satisfies

$$\begin{aligned}(1-s)\int_I \int_0^L |\nu(t)-\nu(\bar{t})|^2 k_s(\gamma_0(t),\gamma_0(\bar{t}))\,dt\,d\bar{t} &\leq 2\frac{C_0(1-s)L}{N}\\ &\leq \frac{C(1-s)^{\frac{6}{7}}}{N} =: \delta''.\end{aligned}$$

Therefore,

$$\left(\frac{C(1-s)^{\frac{6}{7}}}{N}\right)^{-1-\frac{1+s}{2}} = (\delta'')^{-1-\frac{1+s}{2}} \leq \frac{C}{1-s},$$

and we conclude

$$\frac{N}{(1-s)^{\frac{6}{7}}} \leq \frac{C}{(1-s)^{\frac{4}{7}}}$$

and

$$L \leq 5\pi(N+1) \leq C(1-s)^{\frac{2}{7}},$$

where C depends only on C_0.

Note that when $s \uparrow 1$, the previous inequality does not really lead to a contradiction since, to obtain it, we assumed that $L \geq 5\pi N$ with $N \geq 1$ large enough (depending on C_0). It follows from this observation that $L \leq C$ for s close to 1, where C depends only on C_0. □

Finally, once we know that $L \le C$, with C universal and in particular independent of s for $s \in (\frac{1}{2}, 1)$, we conclude from the integral control on the squared nonlocal second fundamental form that γ_0 converges in $C^{1,\frac{1}{4}}$ norm to a maximal circle as $s \uparrow 1$. This is the content of the next result.

Lemma 5.3. *Let $s \in (\frac{1}{2}, 1)$, $L > 0$, γ_0, and ν be as in Lemma* 5.1. *In particular, we assume that, for some constant C_0,*

$$\int_0^L \int_0^L |\nu(t) - \nu(\bar{t})|^2 k_s(\gamma_0(t), \gamma_0(\bar{t}))\, dt\, d\bar{t} \le C_0 L.$$

Assume in addition that

$$0 < L \le C_0.$$

Then $L \to 2\pi$ as $s \uparrow 1$ and, for some $e \in S^2$, we have

$$\|\nu - e\|_{C^{\frac{1}{4}}([0,L]/\{0,L\})} \le C(1-s)^{\frac{1}{2}},$$

where C is some constant depending only on C_0.

In, particular, for s close enough to 1, *the cone generated by (the image of) γ_0 is a very flat Lipschitz graph.*

Proof. Since γ_0 is a closed curve, let us reparametrize it as follows:

$$\hat{\gamma}_0 : S^1 \to S^2, \quad \text{where } \hat{\gamma}_0(\theta) = \gamma_0\left(\frac{L}{2\pi}\theta\right),\ \theta \in S^1 \cong \mathbb{R}/\{2\pi\mathbb{Z}\},$$

where we are identifying $\mathbb{R}/\{2\pi\mathbb{Z}\}$ and $\theta \in S^1$ via the isometry $\theta \mapsto (\cos\theta, \sin\theta)$. Similarly as in the proof of Lemma 5.1, defining $\sigma = \frac{1+s}{2}$ and using now that $L \le C_0$, we obtain

$$\begin{aligned}
[\hat{\nu}^i]^2_{H^\sigma(S^1)} &= (1-\sigma)\int_{S^1}\int_{S^1} \frac{|\hat{\nu}(\theta) - \hat{\nu}(\bar{\theta})|^2}{|(\cos\theta, \sin\theta) - (\cos\bar{\theta}, \sin\bar{\theta})|^{1+2\sigma}}\, d\theta\, d\bar{\theta}\\
&\le C(1-s)L^s \int_0^L \int_0^L |\nu(t) - \nu(\bar{t})|^2 k_s(\gamma_0(t), \gamma_0(\bar{t}))\, dt\, d\bar{t}\\
&\le C(1-s),
\end{aligned}$$

where $\hat{\nu}(\theta) = \nu(\frac{L}{2\pi}\theta)$ is the normal vector accordingly reparametrized, and where C depends only on C_0. Here we have used that

$$t = \frac{L}{2\pi}\theta, \quad \bar{t} = \frac{L}{2\pi}\bar{\theta}, \quad \hat{\gamma}_0'(\theta) = \frac{L}{2\pi},$$
$$\frac{L}{2\pi}|(\cos\theta, \sin\theta) - (\cos\bar{\theta}, \sin\bar{\theta})| \ge |\hat{\gamma}_0(\theta) - \hat{\gamma}_0(\bar{\theta})| = |\gamma_0(t) - \gamma_0(\bar{t})|,$$

and (4.3). Using a small variation of Lemma 4.5 – for S^1 instead of an interval –, we obtain

$$\|\hat{\nu}(t) - e\|_{C^{\frac{1}{4}}(S^1)} \le C[\hat{\nu}]_{H^\sigma(S^1)} \le C(1-s)^{\frac{1}{2}}$$

for some $e \in S^2$. It follows that $\hat{\nu}(t)$ is almost parallel to e and thus γ_0 is a small perturbation of a maximal circle. In particular, $L \to 2\pi$ as $s \uparrow 1$. Note also that the cone generated by γ_0 is a Lipschitz graph in the direction e for s sufficiently close to 1. □

6. Proof of main theorem

In this section we finally give the proof of Theorem 1.2.

Proof of Theorem 1.2. Recall that by assumption Σ is a stable minimal cone for the s-perimeter in $\mathbb{R}^3$. Let us call $\gamma := \partial\Sigma \cap S^2$, where $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$. The curve γ can be written as a disjoint union $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_J$, where γ_i are closed C^2 oriented curves, each of them connected and without self-intersections.

Let L_i denote the length of γ_i and $L = \sum_{1\le i\le J} L_i$. Applying Theorem 3.5 to the cone $\partial\Sigma$, we deduce that

$$L \le \frac{C}{1-s}. \tag{6.1}$$

Throughout the proof C will denote, possibly different, positive constants depending only on s and bounded as $s \uparrow 1$.

Step 1. Let us consider first the case $J = 1$. In this case γ is a connected closed curve. By Proposition 4.3 we have

$$\iint_{\gamma\times\gamma} |\nu(\hat{x}) - \nu(\hat{y})|^2 k_s(\hat{x},\hat{y})\, dH^1(\hat{x})\, dH^1(\hat{y}) = \int_\gamma dH^1(\hat{x}) c^2_{\partial\Sigma}(\hat{x}) \le CL, \tag{6.2}$$

where L is the length of γ. By Lemma 5.1, we know that $L \le C$. Therefore, using Lemma 5.3 we prove that, if $s \in (0,1)$ is close enough to 1, then γ is a small $C^{1,\frac{1}{4}}$ deformation of a maximal circle and thus $\partial\Sigma$ is a Lipschitz graph.

Since $\partial\Sigma$ is C^2 and stable away from 0, it is a viscosity solution of the fractional minimal surface equation in $\partial\Sigma \setminus \{0\}$. Then, since $\partial\Sigma$ is a cone, it must be[5)] a viscosity solution also at 0. As a consequence, using the $C^{1,\alpha}$ regularity of sufficiently flat viscosity solutions of the nonlocal minimal surface equation (see [5, Theorem 6.1] and its proof[6)]), we conclude that $\partial\Sigma$ is $C^{1,\alpha}$ (also at 0) and hence – since it is a cone – it must be a hyperplane. Alternatively, one could use a standard foliation argument to prove that since $\partial\Sigma$ is a graph (globally) then it must be minimizer of the s-perimeter (and not just a stable set) on every compact set. Therefore, since $\partial\Sigma$ is a Lipschitz s-minimal graph, we can also apply [12, Theorem 1.1] and deduce that Σ is C^∞ and hence it must be a hyperplane.

Since now we know that $\partial\Sigma$ is a Lipschitz (minimizing) s-minimal graph, we can apply [12, Theorem 1.1] and deduce that Σ is C^∞ and hence, being a cone, it is necessarily a hyperplane.

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5) If a C^2 surface touches a cone at 0, the cone is contained in a half-space. Thus, the convex envelope of the cone is a subsolution (of the fractional minimal surface equation) that touches $\partial\Sigma$ by below along generatrices. Since $\partial\Sigma$ is s-minimal away from 0, the strong maximum principle yields that $\partial\Sigma$ must be a plane in such a situation.

6) Although [5, Theorem 6.1] is stated for simplicity for minimizers, its proof is really for viscosity solutions (every minimizer is a viscosity solution as proved in [5]). The fact that the improvement of flatness result from which [5, Theorem 6.1] holds true for any viscosity solution and not just for minimizers is an interesting nonlocal feature – this is not true in the local case. As well known to experts, the crucial difference lies in the short nonlocal proof of the "Harnack inequality" [5, Lemma 6.9], which applies verbatim to viscosity solutions.

Step 2. Let us now assume that $J > 1$ and reach a contradiction. Now, (6.2) reads

$$\sum_{1\le i\le J}\int_{\gamma_i} dH^1(\hat{x})\int_{\gamma} dH^1(\hat{y})\,|\nu_\Sigma(\hat{x})-\nu_\Sigma(\hat{y})|^2\,k_s(\hat{x},\hat{y}) \le C \sum_{1\le i\le J} L_i. \tag{6.3}$$

For each i let

$$q_i := \frac{1}{L_i}\int_{\gamma_i} dH^1(\hat{x})\int_{\gamma} dH^1(\hat{y})\,|\nu_\Sigma(\hat{x})-\nu_\Sigma(\hat{y})|^2\,k_s(\hat{x},\hat{y}).$$

Without loss of generality let us assume that $q_1 \le q_2 \le \cdots \le q_J$, after relabeling the indexes.

By (6.3) we have

$$\frac{\sum_{1\le i\le J} L_i q_i}{\sum_{1\le i\le J} L_i} \le C$$

and hence,

$$q_1 \le C.$$

Then Lemmas 5.1 and 5.3 yield

$$0 < \pi \le L_1 \le C,$$

with C universal, for s sufficiently close to 1.

It follows, by (6.3), that

$$\sum_{i=2}^{J} L_i q_i \ge C\left(C + \sum_{i=2}^{J} L_i\right). \tag{6.4}$$

Note that we have $\sum_{2\le i\le J} L_i \ge \pi$. Indeed, if this were not true, we would have

$$\int_{\gamma_2} dH^1(\hat{x})\int_{\gamma_2} dH^1(y)\,|\nu_\Sigma(\hat{x})-\nu_\Sigma(\hat{y})|^2\,k_s(\hat{x},\hat{y}) \le C$$

and $L_2 < \pi$. The proof of Lemma 5.3 then gives that L_2 is close to 2π if s is sufficiently close to 1 – a contradiction with $L_2 < \pi$. Therefore, (6.4) yields

$$\sum_{i=2}^{J} L_i q_i \le C\sum_{i=2}^{J} L_i$$

and thus

$$q_2 \le C.$$

Then, using again Lemma 5.1, we find that

$$L_2 \le C,$$

with C universal.

Next, using Lemma 5.3, we have

$$\|\nu_i(t) - e_i\|_{C^{\frac{1}{4}}(I_i)} \le C(1-s)^{\frac{1}{2}} \quad \text{for some } e_i \text{ in } S^2$$

and $i = 1, 2$, where $\nu_i(t)$ is the normal to γ_i at $\gamma_i(t)$ – recall that γ_i are parametrized by the arc length in an interval I_i. Since the two curves do not intersect and are $C(1-s)^{\frac{1}{2}}$ close to maximal circles, we must have either

$$|e_1 - e_2| \le C(1-s)^{\frac{1}{2}}$$

or

$$|e_1 + e_2| \le C(1-s)^{\frac{1}{2}}.$$

In other words, the two curves are very close to the same maximal circle (in $C^{1,\frac{1}{4}}$ norm), but they may have either the same or opposite orientation.

In the second case (opposite orientations) we use that $q_1 L_1 \le C$ and reason exactly as in Step 3 of the proof of Lemma 5.1 – more precisely, as in (5.6) and (5.7) – to obtain

$$\begin{aligned}
\frac{1}{C((1-s)^{\frac{1}{2}})^{1+s}} &\le \int_{\gamma_1} dH^1(\hat{x}) \int_{\gamma_2} dH^1(\hat{y})\, \frac{1}{|\hat{x}-\hat{y}|^{2+s}} \\
&\le \int_{\gamma_1} dH^1(\hat{x}) \int_{\gamma_2} dH^1(\hat{y})\, \frac{|\nu_\Sigma(\hat{x}) - \nu_\Sigma(\hat{y})|^2}{|\hat{x}-\hat{y}|^{2+s}} \\
&\le C \int_{\gamma_1} dH^1(\hat{x}) \int_{\gamma_2} dH^1(\hat{y})\, |\nu_\Sigma(\hat{x}) - \nu_\Sigma(\hat{y})|^2\, k_s(\hat{x},\hat{y}) \\
&\le q_1 L_1 \le C.
\end{aligned} \tag{6.5}$$

This yields a contradiction if s is close to 1.

In the first case, if the two curves γ_1 and γ_2 happen to have the same orientation, since $\gamma \subset S^2$ is a boundary (of the set $\Sigma \cap S^2$), then there must be a third curve γ_{j_*} with the opposite orientation and trapped between γ_1 and γ_2. In this case, reasoning as in (6.5) with γ_2 replaced by γ_{j_*} we reach a contradiction if s is close to 1. Note though that here we need to be a bit more careful since we have not proven that γ_{j_*} is very close to the maximal circle in $C^{1,\frac{1}{4}}$ norm but just in Hausdorff distance (we know that it is trapped between two small perturbations of a maximal circle). However, we can proceed exactly as we did in (5.4): define the set $\bar{A}$ of times $\bar{t}$ such that $e_1 \cdot \nu_{j_*}(\bar{t}) \le \frac{1}{100}$, which will satisfy (5.5), and repeat (6.5) but integrating only on the set $\{\hat{y} \in \gamma_{j_*}(\bar{A})\}$ and not along the whole γ_{j_*}. Doing so we guarantee that $|\nu_\Sigma(\hat{x}) - \nu_\Sigma(\hat{y})| \ge 1$ and the computation would be again identical as in (5.6) and (5.7). □

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