



# Generalized Uncertainty Principle, Classical Mechanics, and General Relativity



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## ABSTRACT

The Generalized Uncertainty Principle (GUP) has been directly applied to the motion of (macroscopic) test bodies on a given space-time in order to compute corrections to the classical orbits predicted in Newtonian Mechanics or General Relativity. These corrections generically violate the Equivalence Principle. The GUP has also been indirectly applied to the gravitational source by relating the GUP modified Hawking temperature to a deformation of the background metric. Such a deformed background metric determines new geodesic motions without violating the Equivalence Principle. We point out here that the two effects are mutually exclusive when compared with experimental bounds. Moreover, the former stems from modified Poisson brackets obtained from a wrong classical limit of the deformed canonical commutators.

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## 1. Equivalence Principle and diffeomorphism invariance

It is well known [1] that the (weak) Equivalence Principle (EP; namely the equality between gravitational and inertial mass) dictates that the equation of motion of test particles in a gravitational field be of the form

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma_{\mu\nu}^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \quad (1)$$

On the other hand, Eq. (1) turns *also* out to describe geodesics in a manifold with metric  $g_{\mu\nu}$  and the Levi-Civita connection  $\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})$ .<sup>1</sup> In his foundational paper [2] of General Relativity (GR), Albert Einstein proposed that the geodesic equation (1) played the role of the equation of motion for a point particle in the gravitational field  $g_{\mu\nu}$ , which in turn should obey the celebrated field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \quad (2)$$

where  $T_{\mu\nu}$  is the energy-momentum tensor of the matter source. In that original formulation, the identification of the equation of motion with the geodesic equation was seen as an independent axiom of the theory, in particular *independent* from the field equations (2). From this point of view, one can say that the content of the EP is precisely that *the equation of motion is the geodesic equation*.

In successive studies [3,4], Einstein and collaborators obtained a result of considerable importance: the equation of motion of point particles, that is the geodesic equation (1), can in fact be derived from the gravitational field equations (2).<sup>2</sup> In other words, the field equations determine uniquely the equation of motion for bodies in a gravitational field which are not subjected to other forces, and the ensuing trajectories are geodesics of the corresponding metric. This finding is in full agreement with the postulate of geodesic motion, which therefore appears as a consequence of the field equations, and not as an independent axiom of the theory.

An explicit derivation can be found for instance in Refs. [6,7]. It is important here to remark that the starting point is the conservation of the energy-momentum tensor, to wit

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<sup>1</sup> As usual, commas denote partial derivatives w.r.t. the coordinates  $x^\mu$  and semi-colons the covariant derivatives in the metric  $g_{\mu\nu}$ ;  $R_{\mu\nu}$  is the Ricci tensor and  $R$  the Ricci scalar; we shall also use units with  $c = 1$  but display the Boltzmann constant  $k_B$ , the Planck constant  $\hbar$ , the Newton constant  $G_N$  and the Planck mass  $m_p = \sqrt{\hbar/G_N}$  explicitly.

<sup>2</sup> Strictly speaking, the argument applies to dust (a smooth fluid with zero pressure), since point-like sources are known to be mathematically incompatible with Eq. (2) [5].

$$T^{\mu\nu}_{;\nu} = 0. \quad (3)$$

This continuity condition can be obtained directly from Eq. (2), using the Bianchi identity for the Einstein tensor,  $0 = G^{\mu\nu}_{;\nu} = 8\pi G_N T^{\mu\nu}_{;\nu}$ . In this way, it appears as a consistency condition for the field equations. More generally, Eq. (3) can be derived by requiring the diffeomorphism invariance of the matter action [1,8]. In fact, under a generic (infinitesimal) change of coordinates,  $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$ , the metric tensor changes by  $\delta g_{\mu\nu} = -(\xi_{\mu;\nu} + \xi_{\nu;\mu})$ , and the matter action varies as

$$\begin{aligned} \delta S_M &= \frac{1}{2} \int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} \\ &= - \int d^4x \sqrt{-g} T^{\mu\nu} \xi_{\mu;\nu} \\ &= \int d^4x \sqrt{-g} T^{\mu\nu}_{;\nu} \xi_{\mu}. \end{aligned} \quad (4)$$

Since the variation  $\xi_{\mu}$  is arbitrary, requiring that  $\delta S_M = 0$  is equivalent to require Eq. (3). In conclusion, geodesic motion and the EP are deeply rooted into the field equations of GR and, even more fundamentally, they stem from the diffeomorphism invariance of the matter action (which is demanded by the Principle of GR). One therefore cannot modify or renounce to either of them easily.

## 2. Generalized Uncertainty Principle

Much effort has been put into trying to incorporate the effects of gravity in quantum physics by means of a GUP of the form [9–17]

$$\Delta x \Delta p \geq \frac{\hbar}{2} (1 + \beta_0 \Delta p^2), \quad (5)$$

where  $x$  and  $p$  are the position and conjugate momentum of a particle, with the corresponding quantum observables denoted by  $\hat{x}$  and  $\hat{p}$ ,  $\Delta O^2 \equiv \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$  for any operator  $\hat{O}$ , and  $\beta_0 = \beta/m_p^2$  is a deforming parameter expected to emerge from candidate theories of quantum gravity. Uncertainty relations can be associated with (fundamental) commutators by means of the general inequality

$$\Delta A \Delta B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|. \quad (6)$$

For instance, one can derive Eq. (5) from the commutator

$$[\hat{x}, \hat{p}] = i\hbar (1 + \beta_0 \hat{p}^2), \quad (7)$$

for which Eq. (6) yields

$$\begin{aligned} \Delta x \Delta p &\geq \frac{\hbar}{2} (1 + \beta_0 \langle \hat{p}^2 \rangle) \\ &= \frac{\hbar}{2} [1 + \beta_0 (\Delta p^2 + \langle \hat{p} \rangle^2)]. \end{aligned} \quad (8)$$

This immediately implies that the GUP (5) holds for any quantum state, since  $\langle \hat{p}^2 \rangle \geq 0$  always. In particular for mirror-symmetric states  $\psi_{ms}$  satisfying

$$\langle \psi_{ms} | \hat{p} | \psi_{ms} \rangle = 0, \quad (9)$$

one has  $\Delta p^2 = \langle \psi_{ms} | \hat{p}^2 | \psi_{ms} \rangle$  and the inequality (8) coincides with the GUP (5). We also recall that Eq. (5) implies the existence of a minimum length  $\ell = \hbar \sqrt{\beta_0}$  which one expects of the order of the Planck length.

Theoretical consequences of the GUP on quantum (microscopic) systems have been extensively investigated by various authors (see e.g. [18–20]). In addition, several experiments have been proposed

to test different GUP's in the laboratory [21–23], as well as some ground and space based experiments could also be able to reveal GUP effects (see e.g. [24]). It is very important that the size of such modifications can be constrained also with macroscopic test bodies by existing astronomical data employed for the standard tests of GR. Constraining the deforming parameter  $\beta$  using astronomical data requires to estimate the effect of the GUP (5) in the classical limit. In the existing literature, this has been done in different ways. In the following, we critically review and compare two complementary approaches of particular relevance.

## 3. GUP and classical mechanics

Works devoted to evaluate the impact of the GUP on the motion of classical (macroscopic) bodies usually employ a modification of the classical Poisson brackets which resembles the deformed quantum commutator (7) (see, e.g. [25–31]). They essentially implement the classical limit as the formal mapping into Poisson brackets

$$\frac{1}{i\hbar} [\hat{x}, \hat{p}] = (1 + \beta_0 \hat{p}^2) \rightarrow \{x, p\} = (1 + \beta_0 p^2). \quad (10)$$

Such deformed Poisson brackets are then used to determine orbits in the Solar system and derive perturbative corrections to the Newtonian trajectories.

The typical form for the correction coming from Eq. (10) can be found in Appendix A of Ref. [32]. To keep the calculation transparent and focus on the concepts, we just consider a point-like mass  $m$  falling radially towards a mass  $M \gg m$ . From the Newtonian Hamiltonian

$$H = \frac{p^2}{2m} - \frac{G_N M m}{r} \equiv \frac{p^2}{2m} + m V_N \quad (11)$$

and the Poisson brackets (10) with  $x = r$ , the canonical equations read

$$\dot{r} = \{r, H\} = (1 + \beta_0 p^2) \frac{p}{m} \quad (12)$$

$$\dot{p} = \{p, H\} = - (1 + \beta_0 p^2) \frac{G_N M m}{r^2}, \quad (13)$$

where a dot stands for the time derivative. To first order in  $\beta$ , one then obtains the equation of motion

$$\ddot{r} \simeq - \frac{G_N M}{r^2} \left( 1 + 4\beta \frac{m^2}{m_p^2} \dot{r}^2 \right). \quad (14)$$

Equivalently, one can proceed like in Ref. [25], starting from Eq. (12). The conservation of the total energy  $E = m\mathcal{E}$  then implies  $p^2 = 2m^2 (\mathcal{E} - V_N)$ , using which one can finally write (for a particle with zero angular momentum)

$$\dot{r}^2 \simeq 2 (\mathcal{E} - V_N) \left[ 1 + 4\beta \frac{m^2}{m_p^2} (\mathcal{E} - V_N) \right], \quad (15)$$

again to first order in  $\beta$ .

The terms of order  $\beta$  in both Eqs. (14) and (15) depend on the mass  $m$  of the test body and on its velocity  $\dot{r} \sim (\mathcal{E} - V_N)^{1/2}$ . It is therefore clear that the GUP correction obtained in this approach will correspond to a deviation from the geodesic motion (in a reference Schwarzschild space-time), thus leading to a violation of the EP in general. Moreover, and even worse, the size of this correction grows quadratically with the mass  $m$  of the test body in units of the Planck mass. This would inevitably lead to huge departures from GR (and violations of the EP) for any astronomical object, unless  $\beta$  is vanishingly small, like it was indeed argued in Ref. [25].

Difficulties as the above are fully confirmed also when the modified classical Poisson brackets are formulated in a covariant way, on a fixed background metric [29,30]. A slightly different path is followed in Ref. [28], where the EP is recovered even for the GUP modified classical mechanics, by considering composite bodies and postulating that the kinetic energy is additive. The price to pay in this case is a different deformation parameter  $\beta_{0i}$  for each specie  $i$  of (elementary) particles of mass  $m_i$  composing the macroscopic body. Correspondingly, there would exist a different minimal length  $\ell_i = \hbar \sqrt{\beta_{0i}}$  for each elementary particle. For instance, the minimal length that can be probed by a proton should be smaller than that probed by an electron. This feature is clearly at odd with the universality of gravitation, and with the fact that the Planck length can be computed in a way that does not depend at all on the particle considered (see e.g. [33]).

What is the origin of such blatantly unphysical predictions and potential violation of the EP? The error can be traced back to the implementation of the classical limit in Eq. (10) for objects with strictly non-vanishing momentum. In fact, for a generic (normalized) state  $\psi$  with  $\langle \hat{p} \rangle \neq 0$ , the classical limit of the commutator (7) is formally given by

$$\begin{aligned} \{x, p\} &= \lim_{\hbar \rightarrow 0} \frac{\langle \psi | [\hat{x}, \hat{p}] | \psi \rangle}{i \hbar} \\ &= \lim_{\hbar \rightarrow 0} \left[ 1 + \beta \frac{G_N}{\hbar} \left( \langle \hat{p} \rangle^2 + \Delta p^2 \right) \right]. \end{aligned} \quad (16)$$

However, classical (macroscopic) bodies with non-vanishing momentum should be more precisely represented by semiclassical states  $\psi_{cl}$ , for which we expect the classical limit can be generically defined by the two properties<sup>3</sup>

$$\lim_{\hbar \rightarrow 0} \langle \psi_{cl} | \hat{p} | \psi_{cl} \rangle = p, \quad (17)$$

where  $p$  is the classical momentum, and

$$\begin{aligned} \lim_{\hbar \rightarrow 0} \Delta p^2 &\equiv \lim_{\hbar \rightarrow 0} \left( \langle \psi_{cl} | \hat{p}^2 | \psi_{cl} \rangle - \langle \psi_{cl} | \hat{p} | \psi_{cl} \rangle^2 \right) \\ &= 0. \end{aligned} \quad (18)$$

Therefore, even under the stronger condition  $\Delta p^2/\hbar \rightarrow 0$ , the limit (16) becomes

$$\{x, p\} = \lim_{\hbar \rightarrow 0} \left( 1 + \beta \frac{G_N p^2}{\hbar} \right), \quad (19)$$

which diverges badly like  $\hbar^{-1}$ .<sup>4</sup> Of course, this divergence does not occur for mirror symmetric states, for which Eq. (9) implies that the classical momentum  $p = 0$ . In fact Eq. (19) yields the standard Poisson brackets without corrections if we set  $p = 0$  before taking the limit. In other words, since mirror symmetric states can only represent objects with zero momentum, the commutator (7) and the corresponding Poisson brackets (10) should be applied only to classical bodies strictly at rest. It is then obvious why Eq. (10) cannot describe the dynamics of planets orbiting the Sun!

A possible way out of this conundrum is to derive the GUP (5) from the (explicitly state dependent) deformed commutator

$$[\hat{x}, \hat{p}]_{\Delta} = i \hbar \left[ 1 + \beta_0 \left( \hat{p}^2 - \langle \hat{p} \rangle^2 \right) \right], \quad (20)$$

which indeed leads to the GUP (5) for any quantum state via the inequality (6), and it further reduces to the commutator (7) for mirror symmetric states. The commutator (20), for semiclassical states satisfying the conditions (17) and (18), implies

$$\begin{aligned} \{x, p\} &= \lim_{\hbar \rightarrow 0} \frac{\langle \psi_{cl} | [\hat{x}, \hat{p}]_{\Delta} | \psi_{cl} \rangle}{i \hbar} \\ &= 1 + \beta G_N \Delta_0, \end{aligned} \quad (21)$$

where  $\Delta_0 \equiv \lim_{\hbar \rightarrow 0} (\Delta p^2/\hbar)$  depends on the state  $\psi_{cl}$  and can take the following values:

- i)  $\Delta_0 = 0$  and the classical limit (21) yields the standard Poisson brackets with  $\{x, p\} = 1$ ;
- ii)  $\Delta_0 > 0$  and finite. The limit in Eq. (21) then yields the constant  $C_0^2 = 1 + \beta G_N \Delta_0$ , which can be simply used to rescale  $x$  and  $p$  so that the standard Poisson brackets are again recovered;
- iii)  $\Delta_0 = \infty$  and the commutator (20) does not yield a consistent classical limit. Hence, the corresponding states  $\psi_{cl}$  should be avoided.

Summarizing: the classical limit is either badly defined [because Eqs. (19) or (21) diverge], or is just given by the classical Poisson brackets with  $\{x, p\} = 1$  without corrections. Therefore, along this way, it is clearly impossible to estimate any effect of the GUP on macroscopic bodies. To this aim, we should follow a completely different path.

#### 4. GUP and General Relativity

In order to compute GUP effects on macroscopic bodies, we may rely on the indirect argument illustrated in Ref. [32]. Let us consider a Schwarzschild black hole of mass  $M$ , whose metric is given by

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (22)$$

with  $f(r) = 1 - 2G_N M/r$ . From the inequality (5), one can derive a modified Hawking temperature which, to first order in  $\beta$ , reads [33,35–37]

$$T \simeq \frac{\hbar}{8\pi G_N k_B M} \left( 1 + \frac{\beta m_p^2}{4\pi^2 M^2} \right). \quad (23)$$

We then introduce a modified metric function

$$f(r) + \delta f(r) = 1 - \frac{2G_N M}{r} + \varepsilon \frac{G_N^2 M^2}{r^2}, \quad (24)$$

and compute the correction  $\delta f(r)$  which can reproduce the result (23) by means of a standard Quantum Field Theory calculation. We thus find a relation between the deformation parameter  $\varepsilon$  of the metric and the deformation parameter  $\beta$  of the GUP as

$$\beta \simeq -\frac{M^2}{m_p^2} \varepsilon^2. \quad (25)$$

A negative  $\beta$  should not surprise, as it was also found in different contexts, e.g. when uncertainty relations are formulated on a lattice of finite size [38], when the Chandrasekhar limit for white dwarfs is computed with the GUP [39], or when GUP results are compared with corpuscular gravity models [40]. If we now study the geodesic motion of test bodies on this deformed

<sup>3</sup> Of course, the whole topic of how the classical behavior emerges in quantum physics is far richer than what we need to discuss here (for a recent review, see Ref. [34]). For instance, the condition (18) for the states  $\psi_{cl}$  could be implemented by requiring  $\Delta p \sim \hbar^\alpha$ , with  $\alpha > 0$ . Since for such semiclassical states we can also assume  $\Delta x \sim \hbar^\gamma$ , with  $\gamma > 0$ , then Heisenberg uncertainty relation  $\Delta x \Delta p \sim \hbar^{\alpha+\gamma} \geq \hbar/2$  would continue to hold throughout the limiting process for  $\hbar \rightarrow 0$  if  $\alpha + \gamma \leq 1$ . However, this is only a naive way to enforce Eqs. (17) and (18) and not necessarily a useful one.

<sup>4</sup> The divergence obviously disappears when gravity is switched off ( $G_N = 0$ ) before taking the limit.

background metric,<sup>5</sup> we expect no violation of the EP *by construction*, and obtain a typical correction to the Newtonian potential of the form [32]<sup>6</sup>

$$\Delta V_{\text{GUP}} = \varepsilon \frac{G_{\text{N}}^2 M^2}{2r^2} \simeq \sqrt{|\beta|} \frac{m_{\text{p}}}{M} V_{\text{N}}^2. \quad (28)$$

Unlike Eqs. (14) and (15), this correction does not depend on the mass or speed of the orbiting object at all, in full agreement with the EP. Moreover, it becomes vanishingly small for macroscopic sources of mass  $M \gg m_{\text{p}}$  (as one should reasonably expect). For the sake of completeness, we remark that there are other approaches which avoid any EP violation by construction, like that of Ref. [43], where gravitational waves are used for constraining a GUP-modified dispersion relation for gravitons, and that of Ref. [44], where a GUP-deformed background metric is used to compute corrections to the black hole shadow. Furthermore, extensive discussions of precision tests of the EP, and its possible violations, in different contexts (e.g. in scalar-tensor gravity and at finite temperature) can be found in Refs. [45,46].

## 5. Experimental bounds and conclusions

Aside from the previous considerations on the EP and the classical limit, the correction term proportional to  $\beta$  in Eq. (15) can also be quantitatively confronted with the correction (28), assuming of course that the deforming parameter  $\beta$  is universal and applies to both test bodies and gravitational sources of any scale. For macroscopic objects and, in particular, for consistency with Solar System tests, the correction in Eq. (15) requires an incredibly small GUP parameter  $\beta \lesssim 10^{-66}$  [25,29]. Consequently, using this bound in the correction (28) for the extreme case of a Planck size source of mass  $M \simeq m_{\text{p}}$ , one finds  $\Delta V_{\text{GUP}} \simeq 10^{-33} V_{\text{N}}^2$ , which is essentially zero. This appears rather odd, since one introduces the GUP (5) precisely for describing quantum gravity effects at the Planck scale. For instance, one expects a minimum measurable length  $\ell \sim \ell_{\text{p}} \sqrt{\beta}$  comparable to the Planck length, rather than many orders of magnitude shorter. On the other hand, if one accepts the Solar System bounds on  $\beta$  coming from  $\Delta V_{\text{GUP}}$  in Eq. (28), that is  $\beta \lesssim 10^{69}$  [32,42], the correction for a hypothetical Planck size source can still be very relevant (as expected).

Since the corrections of the form in Eq. (15) are irrelevant at the Planck scale, violate the EP, grow larger and larger for planets in the Solar System, moreover they stem from a commutator which is incompatible with the proper classical limit for any state with non-vanishing classical momentum, we conclude that the dynamical equations (14) and (15), and the modified Poisson brackets (10) should be viewed as both conceptually wrong and phenomenologically unviable.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

<sup>5</sup> For details about orbits in GR, see e.g. Ref. [41].

<sup>6</sup> A deformation of the metric function of the form

$$\delta f(r) = \varepsilon f(r) \left( \frac{2G_{\text{N}} M}{r} \right)^2 \quad (26)$$

was used in Ref. [42], where the authors obtain a GUP parameter

$$\alpha_0 \simeq -\frac{M}{m_{\text{p}}} \varepsilon, \quad (27)$$

which is related to  $\beta$  by  $\beta \simeq \alpha_0^2$ . The experimental bounds on  $\alpha_0$  obtained in Ref. [42] are therefore equivalent to those on  $\beta$  derived in Ref. [32].

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