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RESEARCH ARTICLE

Output Regulation by Post-Processing Internal Models for a Class of Multivariable Nonlinear Systems[†]

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Pepoli 3/2, 40123 Bologna**Summary**

In this paper we propose a new design paradigm, which employing a post-processing internal model unit, to approach the problem of output regulation for a class of multivariable minimum-phase nonlinear systems possessing a partial normal form. Contrary to previous approaches, the proposed regulator handles control inputs of dimension larger than the number of regulated variables, provided that a controllability assumption holds, and can employ additional measurements that need not to vanish at the ideal error-zeroing steady state, but that can be useful for stabilization purposes or to fulfill the minimum-phase requirement. Conditions for practical and asymptotic output regulation are given, underlying how in post-processing schemes the design of internal models is necessarily intertwined to that of the stabilizer.

KEYWORDS:

Output Regulation, Multivariable Systems, Nonlinear Control, Internal Model, Stabilization

1 | INTRODUCTION

Output regulation is the branch of control theory studying the design of control systems making the plant follow some desired reference trajectories while rejecting at the same time unknown disturbances. The output regulation problem for linear systems was elegantly solved in the mid 70s in the seminal works of Francis, Wonham and Davison^{1,2} under the assumption that the all the exogenous signals, i.e. references and disturbances, are generated by a known autonomous linear process, called the *exosystem*. The key result presented in that works was that a necessary¹ and sufficient² condition for a regulator to solve the linear output regulation problem *robustly* (i.e. despite model uncertainties), is that the regulator must suitably embed, in the control loop, an *internal model* of the exosystem. While the concept of internal model led to a definite answer to the linear regulation problem, the situation is quite different when nonlinear systems are concerned³, and nonlinear output regulation is still a quite open problem⁴. Without restrictive *immersion* assumptions^{5,6,7,8,9}, indeed, the knowledge of the exosystem alone is neither sufficient nor necessary for the solvability of the problem^{3,4,10}, and its role in conditioning the asymptotic behavior of the regulator mixes up with the plant's residual dynamics, thus making the celebrated robustness property of the linear regulator hard to imagine in a general nonlinear context¹¹.

Under a control design perspective, nonlinear output regulation has reached a mature state, and many regulators have been proposed in the last decades^{6,8,10,12,13,14}. Nevertheless, the existing approaches that can guarantee an asymptotically exact regulation mostly remain limited to minimum-phase single-input-single-output (partial) normal forms, and their immediate square multivariable extensions^{15,16,17,18} (i.e. having the same number of inputs and regulation errors), where the only plant's measurements exploitable by the regulator are the *regulation errors* themselves. The source of such limitation was recently sought in the common structure shared by the largest part of the nonlinear designs, which is somewhat complementary to those possessed

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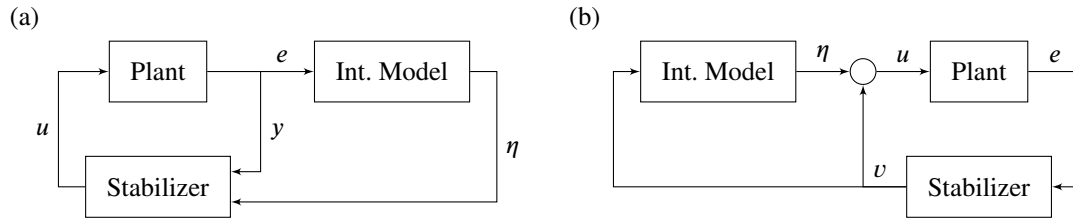


FIGURE 1 Post-processing (a) and Pre-Processing (b) Internal Models.

by the original linear regulator of Davison², and which induces some conceptual problems when extensions are sought. The linear regulator² is obtained by first augmenting the plant with a properly defined *internal model unit*, which is driven by the regulation errors, and then by closing the loop by means of a *stabilizer*, which employs all the available measurements to stabilize the resulting closed-loop system. Due to the fact that the internal model directly processes the regulation errors, this design paradigm is called *post-processing*. Most of the nonlinear approaches, instead, show a complementary design paradigm, where the internal model unit is driven by the control input produced by the stabilizer. A block-diagram representation of the two control structures is depicted in Figure 1. A part from their structural differences, pre and post-processing schemes also differ in terms of “design philosophy”: in the post-processing paradigm the plant is augmented with the internal model unit, the stabilizer is designed to stabilize the resulting cascade, so as to guarantee that the closed-loop system has a well-defined steady state (not fixed in advance) and, finally, the properties of the steady-state regulation error are inferred by the structure of the internal model. In pre-processing regulators, instead, the internal model unit is designed in advance to be able to generate the ideal steady-state control action needed to keep the regulation errors to zero (previously computed by exploiting the known structure of the plant), and the stabilizer is then designed to ensure the asymptotic stability of such ideal steady state. Thus, while in pre-processing schemes the ideal steady states of the internal model unit and of the stabilizer are fixed a priori on the basis of the plant’s data, in post-processing regulators the ideal steady state for the internal model unit and for the stabilizer cannot be fixed a priori. As a matter of fact, since the state of the internal model unit is used for stabilization purposes, and it is thus processed by the stabilizer, its ideal steady state is strongly dependent on the choice of the stabilizer itself.

The pre-processing schemes have the interesting property that the roles of the internal model unit and the stabilizer are neatly separated, and the ideal steady state of the closed-loop system is given by the problem statement. This larger conceptual simplicity is, perhaps, the reason why most of the existing designs are of this kind. Nevertheless, pre-processing regulators have some structural limitations that prevent their extension to larger classes of systems of those mentioned before. In particular, it is not clear, at a conceptual level, how a pre-processing regulator could handle in a systematic way additional measured outputs that are necessary to obtain closed-loop stability (or even minimum-phase), but that need not to vanish at the ideal error-zeroing steady state, unless filtering them out at the steady state employing *redundant* internal models, as recently proposed by Wang and Marconi¹⁹. On the other hand, there is not even a clear road map on how to handle systems having more inputs than errors. If more inputs than errors are present, indeed, a pre-processing solution would lead to the employment of a number of internal models equal to the input dimension, thus yielding a redundant design, not stabilizable by error feedback (as a simple linear example would show).

These conceptual problems, in principle not present in regulators of the post-processing type, recently motivated the community to look for post-processing alternatives to the existing regulators. In particular, the fundamental designs of Byrnes and Isidori¹² and Marconi, Praly and Isidori¹⁰ were “shifted” to an equivalent post-processing design^{20,21}. Nevertheless, no conceptual progress has been made in terms of extensions to larger classes of systems compared to their pre-processing counterparts. A different approach to the design of post-processing regulators was recently pursued by Astolfi, Praly and Marconi^{22,23}, where the linear regulator is attached to a class of nonlinear systems. In particular, the authors have shown that the output regulation problem can be solved *robustly*²⁴ by a post-processing integral action whenever the steady state is made of equilibria²³, and then Astolfi, Praly and Marconi²² have extended the results to the case in which the steady-state signals are periodic, obtaining, however, only an approximate result stating that the Fourier coefficients in the regulation errors corresponding to the frequencies embedded in the internal model vanish at the steady state.

In this paper, for a class of nonlinear systems possessing a partial normal form, we investigate the existence of a post-processing regulator handling additional non-vanishing measurements and input dimensions larger than the number of regulation errors. We give conditions for asymptotic, practical and approximate regulation results of the same kind of those

proved⁹ for the “regression-like” pre-processing regulator originally proposed by Byrnes and Isidori¹². The proposed design shows a feature that is characteristic of post-processing schemes (even if hidden by linearity in the linear case): the design of the stabilizer and those of the internal model unit are intertwined, and the two modules need to be co-designed (this property is known as the “chicken-egg dilemma” of output regulation⁴). Compared to previous multivariable approaches^{16,17,18}, other than handling non-vanishing outputs and additional inputs, we develop the result without the quite restrictive assumption of the existence of a single-valued steady-state map defining the attractor of the zero dynamics, and we thus collocate in a broader “non-equilibrium” setting³. The proposed design thus enlarge the class of multivariable systems for which the output regulation problem can be solved. Nevertheless, as main drawbacks, the proposed approach still limits to a design procedure strongly based on a high-gain perspective and on a minimum-phase assumption with respect to the full set of measured outputs, although the latter requirement is mitigated by the availability of additional measurements. Furthermore, due to the chicken-egg dilemma mentioned before, the proposed conditions for asymptotic regulation become easily not constructive when the complexity of the problem increases. Nevertheless, the “structure” of the regulator remain fixed, and approximate, possibly practical, regulation is guaranteed.

Notation: \mathbb{R} denotes the set of real numbers, \mathbb{N} the set of naturals and $\mathbb{R}_{\geq 0} := [0, \infty)$. With $r > 0$, $\mathbb{B}_r(\mathbb{R}^n)$ denotes the open ball of radius r on \mathbb{R}^n . If n is clear, we omit \mathbb{R}^n and we write \mathbb{B}_r . If $S \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a set, we denote by S° its interior, by \bar{S} its closure, and we let $|S| := \sup_{s \in S} |s|$. With $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $1 \leq i < j \leq n$, we let $x_{[i,j]} := (x_i, \dots, x_j)$. If A_1, \dots, A_n are matrices, we let $\text{col}(A_1, \dots, A_n)$ and $\text{diag}(A_1, \dots, A_n)$ be their column and block-diagonal concatenations, whenever they make sense. If S is a closed set, $|x|_S := \inf_{s \in S} |x - s|$ denotes the distance of $x \in \mathbb{R}^n$ to S . If f is a function defined on \mathbb{R}^n , $f|_S$ denotes the restriction of f on S . C^1 denotes the set of continuously differentiable functions. A continuous function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class- K if it is strictly increasing and $f(0) = 0$. It is of class- K_∞ if f is of class- K and $f(s) \rightarrow \infty$ as $s \rightarrow \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}_{\geq 0}$ is of class- KL if $\beta(\cdot, t)$ is of class- K for each $t \in \mathbb{R}_{\geq 0}$, and $\beta(s, \cdot)$ is strictly decreasing to zero for each $s \in \mathbb{R}_{\geq 0}$. With $h : \mathbb{R}^n \rightarrow \mathbb{R}$ a C^1 function in the arguments x_1, \dots, x_n , and $f : \mathbb{R}^n \rightarrow \mathbb{R}$, for each $i \in \{1, \dots, n\}$ we denote by $L_f^{(x_i)} h$ the map

$$x \mapsto L_{f(x)}^{(x_i)} h(x) := \frac{\partial h(x)}{\partial x_i} f(x).$$

If $x : \mathbb{R} \rightarrow \mathbb{R}^n$ is a locally bounded function, we let $|x|_{[0,t]} := \sup_{s \in [0,t]} |x(s)|$. In this paper we consider differential equations of the form

$$\Sigma : \dot{x} = f(x), \quad x \in \mathbb{R}^n.$$

With $X \subset \mathbb{R}^n$ and $\tau \in \mathbb{R}_{\geq 0}$, we define the τ -reachable set from X as

$$\mathcal{R}_\Sigma^\tau(X) := \left\{ \xi \in \mathbb{R}^n : \xi = x(t), x(0) \in X, t \geq \tau \right\}. \quad (1)$$

Clearly, $\tau_1 \geq \tau_2$ implies $\mathcal{R}_\Sigma^{\tau_2}(X) \subseteq \mathcal{R}_\Sigma^{\tau_1}(X)$, so that the (possibly empty) Ω -limit set

$$\Omega_\Sigma(X) := \lim_{\tau \rightarrow \infty} \mathcal{R}_\Sigma^\tau(X) = \bigcap_{\tau \geq 0} \overline{\mathcal{R}_\Sigma^\tau(X)}$$

is well-defined. The following theorem³, which follows directly by the definition of Ω_Σ and the group property of the solutions of (1), summarizes the main properties of $\Omega_\Sigma(X)$.

Theorem 1. $\Omega_\Sigma(X)$ exists and is closed. If there exists $\tau \geq 0$ such that $\mathcal{R}_\Sigma^\tau(X)$ is bounded, then $\Omega_\Sigma^\tau(X)$ is compact, non empty, invariant and uniformly attractive from X . If in addition $\Omega_\Sigma(X) \subset X^\circ$, then $\Omega_\Sigma(X)$ is stable, and hence it is asymptotically stable.

2 | THE FRAMEWORK

We consider nonlinear systems of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w, x) + b(w, x)u \\ y &= \begin{pmatrix} e \\ y_a \end{pmatrix} = \begin{pmatrix} h_e(w, x) \\ h_a(w, x) \end{pmatrix} \end{aligned} \quad (2)$$

where $x \in \mathbb{R}^{n_x}$ is the state of the plant, $y \in \mathbb{R}^{n_y}$ the measured output, $u \in \mathbb{R}^{n_u}$ is the control input, with $n_u \geq n_y$, and w is an *exogenous* signal modeling disturbances acting on the plant and references to be tracked (as customary in the literature of

output regulation we refer to the subsystem w as the *exosystem*). The output y is subdivided into two components, $e \in \mathbb{R}^{n_e}$ and $y_a \in \mathbb{R}^{n_a}$ ($n_y = n_e + n_a$). The outputs e are called the *regulation errors*, and they represent the quantities that we aim to drive to zero asymptotically. The outputs y_a represent instead additional measurable outputs that need not to vanish at the steady state, but that are available for stabilization purposes. More precisely, we consider the problem of *approximate output regulation* for system (2), which consists in finding an output-feedback regulator ensuring boundedness of the closed-loop trajectories and that

$$\limsup_{t \rightarrow \infty} |e(t)| \leq \varepsilon,$$

with $\varepsilon \geq 0$ possibly a small number. In particular, we say that the problem is solved *asymptotically* if $\varepsilon = 0$, or *practically* if $\varepsilon > 0$ can be made arbitrarily small by opportunely tuning the regulator.

In the following we construct a regulator dealing with the problem of output regulation for (2) under a number of assumption detailed hereafter. We first assume that there exists a compact set $W \subset \mathbb{R}^{n_w}$ which is forward invariant for the exosystem, and we limit our analysis to the set of w originating (and thus staying) in W . We also assume that the functions s , f and b are locally Lipschitz, and sufficiently smooth so that there exist $r > 0$, a set of integers $p_1, \dots, p_r > 0$ satisfying $p_1 + \dots + p_r = n_y$, a set of integers $N_1, \dots, N_r > 0$ satisfying $p_1 N_1 + \dots + p_r N_r =: N \leq n_x$, and, for $i = 1, \dots, r$, a set of \mathbb{R}^{p_i} -valued smooth functions¹ $\{\xi_1^i(w, x), \dots, \xi_{N_i-1}^i(w, x), \zeta^i(w, x)\}$ with linearly independent differentials, such that, by letting $\xi := \text{col}(\xi^1, \dots, \xi^r) \in \mathbb{R}^{N-n_y}$, $\xi^i := \text{col}(\xi_1^i, \dots, \xi_{N_i-1}^i) \in \mathbb{R}^{p_i(N_i-1)}$, and $\zeta := \text{col}(\zeta^1, \dots, \zeta^r) \in \mathbb{R}^{n_y}$, we have that

$$L_{b(w,x)}^{(x)} \xi(w, x) = 0$$

for all $(w, x) \in \mathbb{R}^{n_w} \times \mathbb{R}^{n_x}$ and that, along the solutions to (2), ξ and ζ satisfy

$$\dot{\xi} = F\xi + H\zeta \tag{3a}$$

$$\dot{\zeta} = q(w, x) + B(w, x)u \tag{3b}$$

$$y = TC\xi, \tag{3c}$$

for some locally Lipschitz functions $q : \mathbb{R}^{n_w} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_y}$ and $B : \mathbb{R}^{n_w} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_y \times n_u}$ with $q(0, 0) = 0$, and in which $C := \text{diag}(C_1, \dots, C_r)$ with $C_i := \begin{pmatrix} I_{p_i} & 0_{p_i \times p_i(N_i-2)} \end{pmatrix}$, $T \in \mathbb{R}^{n_y \times n_y}$ is a known permutation matrix, and $F \in \mathbb{R}^{(N-n_y) \times (N-n_y)}$ and $H \in \mathbb{R}^{(N-n_y) \times n_y}$ are block lower-triangular matrices whose diagonal blocks are given respectively by

$$F_{ii} := \begin{pmatrix} 0_{p_i(N_i-2) \times p_i} & I_{p_i(N_i-2)} \\ 0_{p_i} & 0_{p_i \times p_i(N_i-2)} \end{pmatrix}, \quad H_{ii} := \begin{pmatrix} 0_{p_i(N_i-2) \times p_i} \\ I_{p_i} \end{pmatrix}.$$

According to the partition $y = \text{col}(e, y_a)$, we let $C_e \in \mathbb{R}^{n_e \times (N-n_y)}$ and $C_a \in \mathbb{R}^{n_a \times (N-n_y)}$ be such that

$$TC = \begin{pmatrix} C_e \\ C_a \end{pmatrix}.$$

For simplicity, we develop here the case in which $T = I_{n_y}$, i.e. we assume that

$$e = C_e \xi = \text{col}(\xi_1^i : i = 1, \dots, r_e)$$

$$y_a = C_a \xi = \text{col}(\xi_1^i : i = r_e + 1, \dots, r)$$

where r_e is such that $p_1 + \dots + p_{r_e} = n_e$ and $r_a := r - r_e$ (we also let $N_e = p_1 N_1 + \dots + p_{r_e} N_{r_e}$ and $N_a = N - N_e$). We note though, that the result can be easily extended to arbitrary transformations T by means of more involved technical treatise. In the following we also let

$$\begin{aligned} \xi^e &:= \text{col}(\xi^i : i = 1, \dots, r_e), & \xi^a &:= \text{col}(\xi^i : i = r_e + 1, \dots, r), \\ \zeta^e &:= \text{col}(\zeta^i : i = 1, \dots, r_e), & \zeta^a &:= \text{col}(\zeta^i : i = r_e + 1, \dots, r). \end{aligned} \tag{4}$$

Remark 1. The class of systems considered includes multivariable normal forms and partial normal forms²⁵, with the latter that always exist locally whenever (possibly after a preliminary feedback) the system (2) is *a*) strongly invertible in the sense of Hirschorn and Singh^{26,27}, and *b*) input-output linearizable². The dimension n_u of the input is not constrained to be equal to the dimension n_e of the regulation errors nor the dimension n_y of the overall measured outputs, i.e. we consider possibly *non-square* systems in which $n_u \geq n_y$.

¹With slight abuse of notation, in the following we will call with the same symbols ξ and ζ both the functions $\xi(w, x)$ and $\zeta(w, x)$ and the functions $t \mapsto \xi(w(t), x(t))$ and $t \mapsto \zeta(w(t), x(t))$.

²That is²⁸, there exists a state-feedback control of the form $u = \alpha(w, x) + G(w, x)v$, with $v \in \mathbb{R}^{n_u}$ an auxiliary input and G full rank, such that the resulting system has linear input-output behavior from v to y .

On system (22) we then make the following strong minimum-phase and controllability assumptions.

Assumption 1. There exist a class- KL function β , and class- K functions ρ_0 and ρ_1 , with ρ_1 locally Lipschitz, such that all the solution pairs (w, x, u) to (2), with u locally bounded, satisfy

$$|x(t)| \leq \beta(|x(0)|, t) + \rho_0(|w|_{[0,t]}) + \rho_1(|(\xi, \zeta)|_{[0,t]}),$$

for all $t \in \mathbb{R}_{\geq 0}$ for which they are defined.

Assumption 2. There exists a C^1 map $\mathcal{P} : \mathbb{R}^{n_w} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_y \times n_y}$, constants $\underline{\lambda}, \bar{\lambda} > 0$, and a full-rank matrix $\mathcal{L} \in \mathbb{R}^{n_u \times n_y}$ such that:

a. for all $(w, x) \in \mathcal{W} \times \mathbb{R}^{n_x}$

$$\underline{\lambda}I \leq \mathcal{P}(w, x) \leq \bar{\lambda}I.$$

b. For all $(w, x) \in \mathcal{W} \times \mathbb{R}^{n_x}$ and all $u \in \mathbb{R}^{n_u}$,

$$L_{b(w,x)u}^{(x)} \mathcal{P}(w, x) = 0.$$

c. for all $(w, z) \in \mathcal{W} \times \mathbb{R}^{n_x}$

$$\mathcal{L}^\top B(w, x)^\top \mathcal{P}(w, x) + \mathcal{P}(w, x) B(w, x) \mathcal{L} \geq I.$$

Remark 2. As in the context of normal forms and partial normal forms, ξ and ζ are combinations of derivatives of the output y . Hence, Assumption 1 can be seen as an uniform (in u) “output-input stability” (OIS) property^{29,30} of x , that here plays the role of a *minimum-phase* assumption. Similar minimum-phase assumptions appeared in many state-of-art frameworks^{31,17,18}, in which, however, the OIS property is asked with respect to a compact attractor for (w, x) , typically coincident with the graph of a single-valued steady-state map $\pi : \mathbb{R}^{n_w} \rightarrow \mathbb{R}^{n_x}$ whose existence must be assumed. We stress, moreover, that here the minimum phase is asked with respect to the *whole set of outputs* (included those that do not need to vanish at the steady state) and, thus, Assumption 1 is milder than usual minimum-phase assumptions, and it can be possibly obtained by adding further measurements.

Remark 3. Assumption 2 is a controllability condition needed here to employ the proposed high-gain stabilization technique. We underline that, while \mathcal{L} will be used in the definition of the regulator, \mathcal{P} needs only to exist, and it needs not to be known by the designer. As shown in Section 6 below, this assumption is implicated by many customary assumptions made in the context of regulation and stabilization of partial normal forms^{15,16,17,18,31,32,33}.

3 | A POST-PROCESSING REGULATOR

3.1 | The Regulator Structure

With $d \in \mathbb{N}$ an arbitrary index, the proposed regulator is a system with state $\eta \in \mathbb{R}^{d n_e}$ whose dynamics is described by the following equations

$$\begin{aligned} \dot{\eta} &= \Phi(\eta) + Ge \\ u &= \mathcal{L}(\mathcal{K}_\xi \xi + \mathcal{K}_\zeta \zeta + \mathcal{K}_\eta \eta_1), \end{aligned} \quad (5)$$

with Φ and G having the form

$$\Phi(\eta) := \begin{pmatrix} 0 & I_{n_e} & 0 & \cdots & 0 \\ 0 & 0 & I_{n_e} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & I_{n_e} \\ & & \phi(\eta) & & \end{pmatrix}, \quad G := \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_d \end{pmatrix}$$

in which $\phi : \mathbb{R}^{n_e d} \rightarrow \mathbb{R}^{n_e}$ is a Lipschitz function satisfying

$$|\phi(\eta_1) - \phi(\eta_2)| \leq L_\phi |\eta_1 - \eta_2|, \quad \forall \eta_1, \eta_2 \in \mathbb{R}^{d n_e} \quad (6)$$

for some $L_\phi > 0$, and in which $G_i \in \mathbb{R}^{n_e \times n_e}$, $\mathcal{K}_\xi \in \mathbb{R}^{n_y \times (N - n_y)}$, $\mathcal{K}_\zeta \in \mathbb{R}^{n_y \times n_y}$ and $\mathcal{K}_\eta \in \mathbb{R}^{n_y \times n_e}$ are control gains to be designed, with \mathcal{K}_η that has the form

$$\mathcal{K}_\eta = \begin{pmatrix} \mathcal{K}'_\eta \\ 0_{n_e \times n_e} \end{pmatrix}, \quad (7)$$

for some $\mathcal{K}'_\eta \in \mathbb{R}^{n_e \times n_e}$. All these degrees of freedom have to be fixed according to the forthcoming Proposition 1 and Theorem 2.

Remark 4. We give here a *partial state feedback* result employing the auxiliary variables ξ and ζ (i.e. combinations of derivatives of the measured output y). We note, however, that a purely output-feedback regulator can be easily obtained by augmenting (5) with a partial-state observer^{31,34,35}.

As mentioned in the introduction, the chicken-egg dilemma⁴ prevents a sequential design of the different parts of regulator, leading us to a more complicated construction. Since the ideal choice of ϕ in (5) depends on the actual choice of the matrices \mathcal{L} , \mathcal{K}_ξ , \mathcal{K}_ζ and \mathcal{K}_η , before discussing the choice of ϕ , we state a preliminary uniform boundedness result, which allows us to individuate a class of possible steady-state trajectories on which ϕ can be tuned.

3.2 | Existence of a Steady State

The closed-loop system reads as follows

$$\Sigma_{cl} : \begin{cases} \dot{w} = s(w) \\ \dot{x} = f(w, x) + b(w, x)\mathcal{L}(\mathcal{K}_\xi\xi(w, x) + \mathcal{K}_\zeta\zeta(w, x) + \mathcal{K}_\eta\eta_1) \\ \dot{\eta} = \Phi(\eta) + Gh_e(w, x). \end{cases} \quad (8)$$

The following proposition, proved in Section 5.1, shows that the parameters of the regulator (3) can be chosen to ensure (semi-global) uniform boundedness of the closed-loop trajectories, and hence the existence of a steady state.

Proposition 1. Suppose that Assumptions 1 and 2 hold and consider the regulator (5), with $d > 1$ and ϕ satisfying (6) for some $L_\phi > 0$ and \mathcal{L} given by Assumption 2. Then there exists a class- K functions ρ_{cl} and, for each pair of compact sets $X \subset \mathbb{R}^{n_x}$ and $H \subset \mathbb{R}^{d n_e}$, a class- KL function β_{cl} and matrices \mathcal{K}_ξ , \mathcal{K}_ζ , \mathcal{K}_η and G , with \mathcal{K}_η of the form (7) with \mathcal{K}'_η invertible, such that all the solutions to Σ_{cl} originating in $W \times X \times H$ are complete and satisfy

$$|(x(t), \eta(t))| \leq \beta_{cl}(|(x(0), \eta(0))|, t) + \rho_{cl}(|w|_{[0,t]}), \quad (9)$$

for all $t \geq 0$.

The claim of Proposition 1 shows that, despite the particular ϕ implemented in (5), if it satisfies (6), then for each compact set of initial conditions the other degrees of freedom of the regulator can be chosen to ensure that the closed-loop system Σ_{cl} is uniformly bounded³ from $W \times X \times H$ and hence, by Theorem 1, there exists a compact invariant attractor

$$\mathcal{A} := \Omega_{\Sigma_{cl}}(W \times X \times H) \quad (10)$$

which is invariant and uniformly attractive from $W \times X \times H$. Moreover, since ρ_{cl} does not depend on X and H , the latter sets can be thought, without loss of generality, to be large enough so that $\mathcal{A} \subset (W \times X \times H)^\circ$, which implies that \mathcal{A} is also asymptotically stable.

3.3 | Tuning the Internal Model

If (6) holds, Proposition 1 ensures the existence of G , \mathcal{K}_ξ , \mathcal{K}_ζ , \mathcal{K}_η , such that there exists a compact attractor \mathcal{A} for the trajectories closed-loop system Σ_{cl} originating in $W \times X \times H$. In this section, we consider the restriction of Σ_{cl} on \mathcal{A} , i.e. the system⁴

$$\Sigma_{cl}^{\mathcal{A}} : \begin{cases} \dot{w} = s(w) \\ \dot{x} = f(w, x) + b(w, x)\mathcal{L}(\mathcal{K}_\xi\xi(w, x) + \mathcal{K}_\zeta\zeta(w, x) + \mathcal{K}_\eta\eta_1) \\ \dot{\eta} = \Phi(\eta) + Gh_e(w, x) \end{cases} \quad (w, x, \eta) \in \mathcal{A}, \quad (11)$$

and we investigate the conditions that ϕ must satisfy on \mathcal{A} to guarantee a given asymptotic performance, expressed in terms of the bound (ideally zero) on the regulation error $e = h_e(w, x)$ along the solutions to (11). This conditions are then used in the forthcoming Theorem 2 to complement the claim of Proposition 1 with an asymptotic bound on the regulation error depending on “how well” the function ϕ is chosen.

³That is, there exists $M > 0$ such that every trajectory (w, x, η) of the closed-loop system (2), (5) originating in $W \times X \times H$ satisfies $|(w(t), x(t), \eta(t))| \leq M$ for each $t \in \mathbb{R}_{\geq 0}$.

⁴We remark that, being \mathcal{A} invariant, all the solutions to $\Sigma_{cl}^{\mathcal{A}}$ are complete.

Substituting the expression of u into (3b) yields

$$\dot{\zeta} = q(w, x) + B(w, x)\mathcal{L}(\mathcal{K}_\xi \dot{\xi} + \mathcal{K}_\zeta \dot{\zeta} + \mathcal{K}_\eta \eta_1). \quad (12)$$

Let us define the projected set \mathcal{O} as

$$\mathcal{O} := \left\{ (w, x) \in W \times \mathbb{R}^{n_x} : (w, x, \eta) \in \mathcal{A} \right\},$$

let q^e and q^a be functions with values in \mathbb{R}^{n_e} and \mathbb{R}^{n_a} respectively such that $q(w, x) = \text{col}(q^e(w, x), q^a(w, x))$, and let denote for brevity

$$D(w, x) := B(w, x)\mathcal{L} \in \mathbb{R}^{n_y \times n_y}.$$

Let partition $D(w, x)$, \mathcal{K}_ξ and \mathcal{K}_ζ as:

$$D := \begin{pmatrix} D^{e,e} & D^{e,a} \\ D^{a,e} & D^{a,a} \end{pmatrix}, \quad \mathcal{K}_\xi := \begin{pmatrix} \mathcal{K}_\xi^{e,e} & \mathcal{K}_\xi^{e,a} \\ \mathcal{K}_\xi^{a,e} & \mathcal{K}_\xi^{a,a} \end{pmatrix}, \quad \mathcal{K}_\zeta := \begin{pmatrix} \mathcal{K}_\zeta^{e,e} & \mathcal{K}_\zeta^{e,a} \\ \mathcal{K}_\zeta^{a,e} & \mathcal{K}_\zeta^{a,a} \end{pmatrix} \quad (13)$$

for some $D^{i,j}(w, x), \mathcal{K}_\xi^{i,j}, \mathcal{K}_\zeta^{i,j} \in \mathbb{R}^{n_i \times n_j}$, $i, j \in \{e, a\}$. Then, in view of (7), equation (12) gives

$$\dot{\zeta}^e = q^e(w, x) + D^{e,e}(w, x) \left(\mathcal{K}_\xi^{e,e} \dot{\xi}^e + \mathcal{K}_\xi^{e,a} \dot{\xi}^a + \mathcal{K}_\zeta^{e,e} \dot{\zeta}^e + \mathcal{K}_\zeta^{e,a} \dot{\zeta}^a + \mathcal{K}'_\eta \eta_1 \right) + D^{e,a}(w, x) \left(\mathcal{K}_\xi^{a,e} \dot{\xi}^e + \mathcal{K}_\xi^{a,a} \dot{\xi}^a + \mathcal{K}_\zeta^{a,e} \dot{\zeta}^e + \mathcal{K}_\zeta^{a,a} \dot{\zeta}^a \right). \quad (14)$$

In view of (14), the regulation error e , along with its derivatives, vanish on the attractor \mathcal{A} of and only if $\dot{\xi}^e = 0, \dot{\zeta}^e = 0$ (both implying $\dot{\xi}^e = 0$), and

$$0 = q^e(w, x) + D^{e,e}(w, x) \left(\mathcal{K}_\xi^{e,a} \dot{\xi}^a + \mathcal{K}_\zeta^{e,a} \dot{\zeta}^a + \mathcal{K}'_\eta \eta_1 \right) + D^{e,a}(w, x) \left(\mathcal{K}_\xi^{a,a} \dot{\xi}^a + \mathcal{K}_\zeta^{a,a} \dot{\zeta}^a \right) \quad (15)$$

for all $(w, x) \in \mathcal{O}$. By solving (15) for η_1 , we thus obtain an ideal steady-state value for η_1 , given by the function $\eta_1^* : \mathcal{O} \rightarrow \mathbb{R}^{n_e}$ satisfying

$$D^{e,e}(w, x) \mathcal{K}'_\eta \eta_1^*(w, x) = -q^e(w, x) - D^{e,e}(w, x) \left(\mathcal{K}_\xi^{e,a} \dot{\xi}^a(w, x) + \mathcal{K}_\zeta^{e,a} \dot{\zeta}^a(w, x) \right) - D^{e,a}(w, x) \left(\mathcal{K}_\xi^{a,a} \dot{\xi}^a(w, x) + \mathcal{K}_\zeta^{a,a} \dot{\zeta}^a(w, x) \right) \quad (16)$$

for all $(w, x) \in \mathcal{O}$. Since η_1 is the first component of the internal model unit η of (5), and since in the ideal condition in which $e = 0$ the system η coincides with the autonomous equation

$$\dot{\eta} = \Phi(\eta), \quad (17)$$

then for each solution to Σ_{cl}^A , (16) individuates an ideal steady-state value for η given by

$$\eta^*(t) := \text{col} \left(\eta_1^*(t), \dot{\eta}_1^*(t), \dots, \eta_1^{*(d-1)}(t) \right) \quad (18)$$

in which we denoted for brevity $\eta_1^*(t) = \eta_1^*(w(t), x(t))$. In view of the structure of the map Φ in (5), such an ideal steady state trajectory η^* may be a solution of (17) only if $\eta_1^*(t)$ satisfies

$$\eta_1^{*(d)}(t) = \phi \left(\eta_1^*(t), \dot{\eta}_1^*(t), \dots, \eta_1^{*(d-1)}(t) \right) \quad (19)$$

almost everywhere. Condition (19) expresses indeed the *internal model property*, that is, the property of the regulator to generate all the ideal steady-state control actions $\eta_1^*(t)$ needed to keep the regulation error e to zero. We also observe that the chicken-egg dilemma is strongly present in (19), since η_1^* and its derivatives, and thus the correct value of ϕ to be implemented, depend on the closed-loop trajectories and thus, in particular, on stabilization gains $\mathcal{K}_\xi, \mathcal{K}_\zeta$, and \mathcal{K}_η , the latter dependent from the Lipschitz constant L_ϕ of ϕ .

Remark 5. We observe that in the linear case the chicken-egg dilemma is broken by the fact that, no matter how the stabilization gains are chosen, if a compact attractor \mathcal{A} exists then all the closed-loop trajectories have the same modes of the exosystem (the closed-loop system (8), indeed, is a linear stable system driven by the exosystem). Therefore, a function ϕ satisfying (6) and (19) can be fixed a priori according to the knowledge of the exosystem dynamics. A similar situation also takes place in a nonlinear setting if $\dot{w} = 0$. If d and ϕ are taken as $d = 1$ and $\phi(\eta) = 0$, and if the closed-loop system Σ_{cl} can be made “contractive” via stabilization, then the steady-state trajectories (i.e. the solutions to Σ_{cl}^A) are constant, and thus (19) holds. This is in fact the well-known integral action²³.

The existence of a functions $\eta_1^*(w, x)$ solving (16) in \mathcal{O} and such that $\eta_1^*(t)$ is d -times differentiable, and thus the existence of the ideal steady-state η^* for η , follows by the assumptions below as stated in the forthcoming Remark 7.

Assumption 3. The functions s, f and b are C^d .

Assumption 4. There exists a map $\mathcal{M} : \mathbb{R}^{n_w} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_e \times n_e}$, C^1 on an open set including \mathcal{O} , such that:

- $\mathcal{M}(w, x) > 0$ in \mathcal{O} .
- For all $(w, x) \in \mathcal{O}$ and all $u \in \mathbb{R}^{n_u}$,

$$L_{b(w,x)u}^{(x)} \mathcal{M}(w, x) = 0.$$

- for all $(w, x) \in \mathcal{O}$

$$D^{e,e}(w, x)^\top \mathcal{M}(w, x) + \mathcal{M}(w, x) D^{e,e}(w, x) \geq I.$$

Remark 6. Assumption 4 is a controllability condition resembling those of Assumption 2, but limited to the submatrix $D^{e,e}(w, x)$ inside the attractor \mathcal{O} . In addition to Assumption 2, this condition ensures that the regulation error contracts to zero when the internal model reaches its ideal steady-state. While in the case in which $y = e$ this condition is directly implicated by Assumption 2, in general it is not and it has to be assumed. We stress that, as it is the case for \mathcal{P} in Assumption 2, the matrix \mathcal{M} is not used in the construction of the control law, and it is thus not required to be known by the designer. In the forthcoming Section 6, we complement this assumption by showing that it holds, together with Assumption 2, in many state-of-art relevant frameworks of high-gain regulation and stabilization of multivariable systems.

Remark 7. Assumption 4 implies that $D^{e,e}(w, x)$ is non-singular in \mathcal{O} . This, together with Assumption 3 and the fact that \mathcal{K}'_η can be taken invertible, implies that (16) admits a unique solution in \mathcal{O} such that $\eta_1^*(t)$ is C^d for each solution (w, x, η) of Σ_{cl}^A .

3.4 | Asymptotic Performances

Apart from the chicken-egg dilemma, fulfilling condition (19) in general requires a very precise knowledge of the function η_1^* and its derivatives, which strongly depend on the plant and exosystem dynamics. Therefore, uncertainties in the model of the plant and exosystem strongly reflect into uncertainties in the right internal model function to implement, potentially ruining the asymptotic performance of the regulator²⁴. This motivates introducing, for each solution (w, x, η) to Σ_{cl}^A , the following quantity

$$\delta(t) := \phi(\eta^*(t)) - \eta_1^{*(d)}(t),$$

representing the *internal model mismatch* along (w, x, η) , i.e. the error that the system η of (5) attains in modeling the process that generates the ideal control action $\eta_1^*(t)$. We also define the worst-case mismatch as the quantity

$$\bar{\delta} := \sup \{ |\delta|_\infty : (w, x, \eta) \text{ solution to } \Sigma_{cl}^A \}, \quad (20)$$

which, we stress, in general depends also on the control gains \mathcal{K}_ξ , \mathcal{K}_ζ and \mathcal{K}_η .

The following theorem, which represents the main result of the paper, characterizes the asymptotic properties of the proposed regulator (5) and, in particular, relates the worst-case internal model mismatch (20) to the asymptotic bound on the regulation error.

Theorem 2. Suppose that Assumptions 1, 2, 3 and 4 hold, and consider the regulator (5) with $d > 1$ and ϕ satisfying (6) for some $L_\phi > 0$, and with \mathcal{L} given by Assumption 2. Then, for each pair of compact sets $X \subset \mathbb{R}^{n_x}$ and $H \subset \mathbb{R}^{d n_e}$ of initial conditions and each $\varepsilon > 0$, there exist matrices \mathcal{K}_ξ , \mathcal{K}_ζ , \mathcal{K}_η and G , and a compact set \mathcal{A} of the form (10), such that every solution to the closed-loop system Σ_{cl} originating in $W \times X \times H$ is uniformly attracted by \mathcal{A} and satisfies

$$\limsup_{t \rightarrow \infty} |e(t)| \leq \varepsilon \bar{\delta} \quad (21)$$

uniformly in the initial conditions.

Theorem 2, proved in Section 5.2, claims that the stabilization parameters in (5) can be chosen to ensure the existence of a compact attractor for the closed-loop system (8), and that the asymptotic bound on the regulation error is proportional to the worst-case internal model mismatch on the attractor. The claim of Theorem 2 is thus an *approximate* regulation result. Furthermore, if the worst-case internal model mismatch (20) does not depend on the control parameters \mathcal{L} , \mathcal{K}_ξ , \mathcal{K}_ζ , \mathcal{K}_η and G , then the asymptotic bound on the error can be reduced arbitrarily by opportunely choosing the control parameters to lower the proportionality constant ε in (21). This, in turn, makes the claim of the theorem a *practical* output regulation result. Since X and H are arbitrary, moreover, the result is *semiglobal* in the plant's and internal model's initial conditions. We remark that, as evident in (16), in the canonical ‘‘square’’ setting in which $b(w, x) = 0$, $n_a = 0$ (i.e. $y = e$) and $n_u = n_e$, the map $\eta_1^*(w, x)$,

its derivatives, and hence $\bar{\delta}$, can be always bounded uniformly in the control parameters, thus making the result of Theorem 1 always practical.

Finally, we observe that if the worst-case internal model mismatch (20) is zero, i.e. the internal model includes a copy of the process generating the ideal steady-state η^* for (5), then *asymptotic regulation* is achieved, that is

$$\lim_{t \rightarrow \infty} e(t) = 0.$$

holds uniformly. The intertwining between internal model and the stabilizer and the possible presence of model's uncertainties make $\bar{\delta} = 0$ very difficult to satisfy, thus making the conditions of asymptotic regulation de facto not constructive. Nevertheless, Theorem 2 individuates a clear sufficient condition for asymptotic regulation, given by equation (19), which expresses in this setting the nonlinear version of the *internal model principle*: the regulator must embed a copy of the process that generates the ideal steady state control action (i.e. η_1^* above) that makes the set in which the error vanishes invariant.

Finally, we observe that this chicken-egg dilemma is way more evident when non-vanishing outputs are used for stabilization, as they need to be compensated at the steady state by the output of the internal model (see (16)). In this respect we also note that, as it is the case of the linear regulator, the feedback of auxiliary outputs might also have a simplifying effect on the internal model.

4 | EXAMPLE

In this section we present an example showing how the post-processing paradigm presented in the previous sections can be used to approach a problem for which no solution of the pre-processing type is known. We consider the system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x}_1 &= f_1(x_1) + \gamma_1(w, x_2) + x_3 \\ \dot{x}_2 &= \gamma_2(w, x) + u_1 + u_2 \\ \dot{x}_3 &= \gamma_3(w, x) - b(w)u_1 + (1 - b(w))u_2 \end{aligned} \quad (22)$$

with regulation error

$$e := x_2,$$

in which all the functions f_1 , γ_i and b are smooth, and where w ranges in a compact invariant set $\mathcal{W} \subset \mathbb{R}^{n_w}$. We observe that if the function γ_2 satisfies $\partial\gamma_2(0)/\partial x_i = 0$ for $i = 1, 3$, the linear approximation of (22) at 0 is not detectable from e . Thus, e is not enough to stabilize (22), and additional outputs are needed. We specifically assume to have available for feedback the other two variables, i.e. the additional output

$$y_a := \text{col}(x_1, x_3).$$

We observe that y_a does not necessarily vanish at the ideal steady state in which $e = 0$. We also observe that a control strategy based on a preliminary ‘‘pre-stabilizing’’ inner-loop employing y_a to reduce to a canonical case in which the pre-stabilized plant is stabilizable from e is hard to imagine. In fact the inputs u_1 and u_2 affect both the equations of \dot{x}_2 and \dot{x}_3 , and x_3 has no relative degree with respect to any of the two inputs, since both $b(w)$ and $1 - b(w)$ may vanish. Therefore, both u_1 and u_2 must be used in case x_3 is pre-stabilized, thus leaving no further degree of freedom to deal with e . As a consequence, this case does not fit into any of the previous pre-processing frameworks in which only e can be used for feedback.

In the rest of the section we build a regulator of the form (5) dealing with this case. For, we suppose to know a function κ such that⁵

$$s(f_1(s) - \kappa(s)) \leq 0, \quad \forall s \in \mathbb{R} \quad (23)$$

and we define the variable $\zeta := \text{col}(\zeta_1, \zeta_2)$ as

$$\zeta_1 := x_2, \quad \zeta_2 := x_3 + x_1 + \kappa(x_1),$$

which satisfy

$$\dot{\zeta} = q(w, x) + B(w, x)u \quad (24)$$

with

$$q(w, x) := \begin{pmatrix} \gamma_2(w, x) \\ \gamma_3(w, x) + (1 + \kappa'(x_1))(f_1(x_1) + \gamma_1(w, x_2) + x_3) \end{pmatrix}, \quad B(w, x) := \begin{pmatrix} 1 & 1 \\ -b(w) & 1 - b(w) \end{pmatrix}.$$

⁵We observe that if f_1 is locally Lipschitz, then κ can be taken linear over compact sets.

Hence, (22)-(24) has the form (2), (3b). We further notice that $|(x_2, x_3)| \leq |\zeta| + |x_1 + \kappa(x_1)|$, with x_1 that fulfills

$$\dot{x}_1 = -x_1 + f_1(x_1) - \kappa(x_1) + \gamma_1(w, \zeta_1) + \zeta_2.$$

In view of (23), simple computations show that the x_1 subsystem is input-to-state stable relative to the origin and with respect to the input $\gamma_1(w, \zeta_1) + \zeta_2$, and hence Assumption 1 is fulfilled.

With $\alpha > 0$ satisfying $4\alpha > 1 + \sup_{w \in W} (1 + b(w))^2$, let

$$\mathcal{L}_0 := \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha \end{pmatrix}, \quad \mathcal{P}(w, x) := \begin{pmatrix} 1 + b(w)^2 & b(w) \\ b(w) & 1 \end{pmatrix}.$$

Then the high-frequency matrix $B(w, x)$ fulfills

$$\mathcal{L}_0^\top B(w, x)^\top \mathcal{P}(w, x) + \mathcal{P}(w, x) B(w, x) \mathcal{L}_0 = \begin{pmatrix} 2\alpha^2 & \alpha(1 + b(w)) \\ \alpha(1 + b(w)) & 2\alpha \end{pmatrix} =: M(w).$$

As $2\alpha^2 > 0$ and

$$\det M(w) = 4\alpha^3 - \alpha^2(1 + b(w))^2 = \alpha^2(4\alpha - (1 + b(w))^2) > 4\alpha^2 + \alpha^2(4 \sup_{w \in W} (1 + b(w))^2 - (1 + b(w))^2) > 4\alpha^2,$$

$M(w)$ is positive definite and there exists $m > 0$ such that, for all $x \in \mathbb{R}^2$,

$$x^\top M(w)x \geq m|x|^2. \quad (25)$$

Therefore the pair $(\mathcal{L}, \mathcal{P})$, with $\mathcal{L} := \mathcal{L}_0/m$, satisfy Assumption 2 for each compact subset $X \subset \mathbb{R}^{n_x}$.

Moreover, Assumption 3 holds by construction, and Assumption 4 holds with $\mathcal{M} = 1$, since the quantity $D(w, x) = B(w, x)\mathcal{L}$ defined as in (13) satisfies

$$D^{e,e}(w, x) = \alpha^2.$$

According to Section 3, and by following the proof of Proposition 1 and Theorem 2, the regulator has the form (5), with the stabilizing action given by

$$u = -\ell \frac{\alpha}{m} \begin{pmatrix} \alpha(e - \eta_1) \\ x_3 + x_1 + \kappa(x_1) \end{pmatrix}$$

in which $\ell > 0$ is a design parameter to be taken sufficiently large and η_1 is the first component of the internal model unit

$$\begin{aligned} \dot{\eta}_i &= \eta_{i+1} + g^i h_i e, \quad i = 1, \dots, d-1 \\ \dot{\eta}_d &= \phi(\eta) + g^d h_d e \end{aligned}$$

in which the parameters h_i are the coefficients of a Hurwitz polynomial, $g > 0$ is a high-gain parameter, and the dimension d and the ϕ are degrees of freedom to be fixed according to the available knowledge of $s(w)$ and of the ideal steady-state of the internal model given by (18). According to Proposition 1, for all sufficiently large ℓ and g the closed-loop system has a compact attractor, on which d and ϕ must be tuned. In particular, η_1^* is obtained as in (16) and by imposing in (22) $e = 0$ and $\eta = \eta^*$, thus obtaining

$$\begin{aligned} \eta_1^* &= -\frac{m}{\ell \alpha^2} \gamma_2(w, (x_1, 0, x_3)) + \frac{1}{\alpha} (x_3 + x_1 + \kappa(x_1)) \\ \dot{x}_1 &= f_1(x_1) + \gamma_1(w, 0) + x_3 \\ \dot{x}_3 &= \gamma_3(w, (x_1, 0, x_3)) - b(w) \gamma_2(w, (x_1, 0, x_3)). \end{aligned} \quad (26)$$

In the following simulations we set

$$\begin{aligned} n_w &= 2, & s(w) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w, & W &= \mathbb{B}_3, & f_1(x_1) &= 0, & \kappa(x_1) &= 0, \\ \gamma_1(w, x_2) &= 0, & \gamma_2(w, x) &= q \exp(2w_1^2), & \gamma_3(w, x) &= w_1^2, & b(w) &= w_1, & \alpha &= 5 \end{aligned}$$

with $q \geq 0$. Then (25) holds with $m = 1$, and equations (26) reduce to

$$\eta_1^* = -\frac{1}{\ell \alpha^2} q \exp(w_1^2) + \frac{1}{\alpha} (x_3 + x_1), \quad \dot{x}_1 = x_3, \quad \dot{x}_3 = w_1^2 - q w_1 \exp(2w_1^2).$$

We first consider the case in which $q = 0$. It can be shown that, in this case, η_1^* can be generated by a linear system of dimension $d = 5$ and of the form (17) in which

$$\phi(\eta) = -4\eta_4.$$

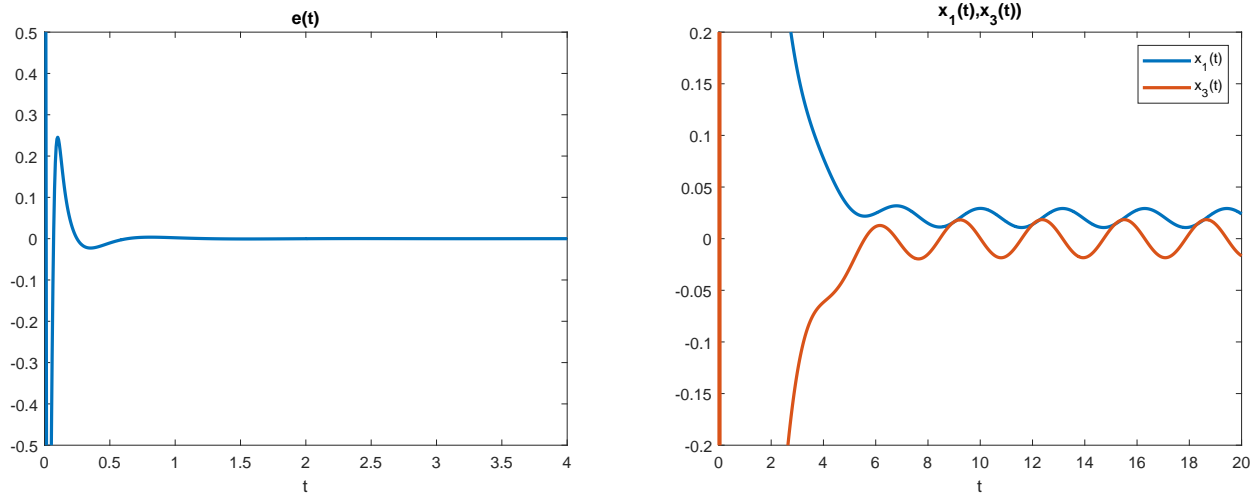


FIGURE 2 The left plot shows the time evolution of the regulation error $e(t) = x_2(t)$ which is asymptotically vanishing. The right plot shows the time evolution of the other two measurements $y_a(t) = (x_1(t), x_3(t))$ that have a non-zero steady state.

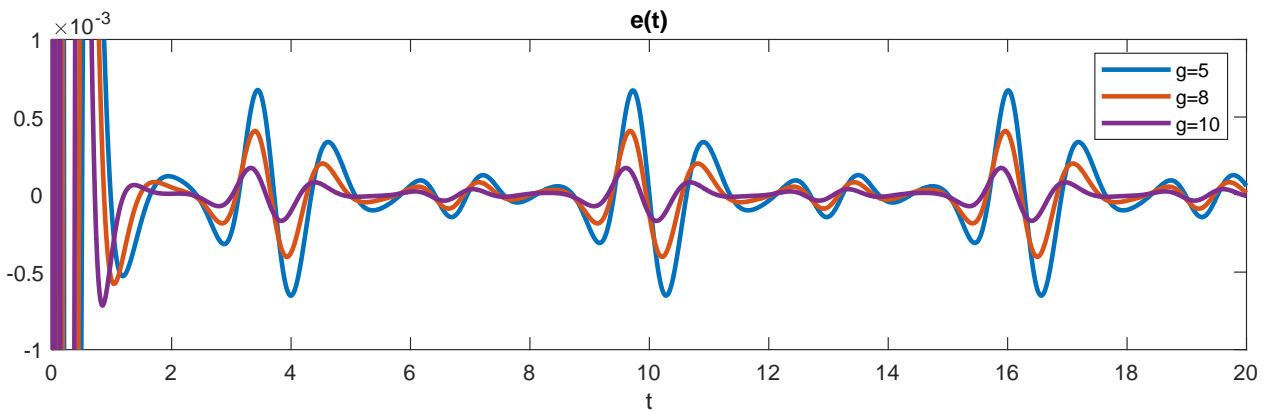


FIGURE 3 Time evolution of the regulation error $e(t) = x_2(t)$ for $g = 5, 8, 10$.

Figure 2 shows a simulation in this first case in which $q = 0$, $d = 5$ and ϕ is chosen as above. The system originates at $w(0) = (1, 0)$, $x(0) = (3, 5, -2)$ and $\eta(0) = 0$, and the control gains are chosen as $g = 5$ and $\ell = 5$.

In the second case in which q is set to 1, η_1^* cannot be generated by a linear system of dimension 5 and thus asymptotic regulation is not achieved. In this setting, however, the internal model mismatch (20) can be bounded uniformly in the control parameters, and practical regulation is achieved. Figure 3 thus shows a simulation obtained with the same regulator in the case in which $q = 1$ and $g = 5, 8, 10$, by showing that the asymptotic error can be reduced by increasing g .

5 | PROOFS

5.1 | Proof of Proposition 1

Let partition η as $\eta = \text{col}(\eta_1, \dots, \eta_d)$ and, for each $\eta_i \in \mathbb{R}^{n_e}$, let partition η_i as $\eta_i = \text{col}(\eta_i^1, \dots, \eta_i^{r_e})$ with $\eta_i^j \in \mathbb{R}^{p_j}$. Consider the change of coordinates:

$$\begin{aligned} \forall i = 1, \dots, r_e : & \quad \begin{cases} \xi_1^i \mapsto \chi_1^i := \xi_1^i + \eta_1^i, \\ \xi_j^i \mapsto \chi_j^i := \xi_j^i, \quad j = 2, \dots, N_i - 1 \end{cases} \\ \forall i = r_e, \dots, r : & \quad \xi^i \mapsto \chi^i := \xi^i. \\ & \quad e \mapsto \bar{e} := e + \eta_1. \end{aligned} \quad (27)$$

By letting $\chi := \text{col}(\chi^1, \dots, \chi^r)$, with $\chi^i := \text{col}(\chi_1^i, \dots, \chi_{N_i-1}^i)$, the change of variables (27) can be compactly rewritten as

$$\begin{aligned} \chi &:= \xi + C_e^\top \eta_1 \\ \bar{e} &= C_e \chi. \end{aligned}$$

From (5), and since by construction $FC_e^\top = 0$, we obtain

$$\dot{\chi} = (F + C_e^\top G_1 C_e) \chi + H \zeta + C_e^\top (\eta_2 - G_1 \eta_1). \quad (28)$$

We state now the following lemma, proved in the Appendix.

Lemma 1. For any $\epsilon > 0$, there exists $K \in \mathbb{R}^{n_e \times (N-n_e)}$, class- KL functions β_χ, β_e and $a_1 > 0$, such that the system (28) with output \bar{e} and with input $\zeta = \bar{\zeta} + K \chi$, being $\bar{\zeta} \in \mathbb{R}^{n_e}$ an auxiliary input, satisfies

$$|\chi^i(t)| \leq \beta_\chi(|\chi(0)|, t/\epsilon) + a_1 \left(|\bar{\zeta}_{[1,i]}|_{[0,t]} + |G_1| |\eta_1|_{[0,t]} + |\eta_2|_{[0,t]} \right) \quad (29)$$

$$|\bar{e}(t)| \leq \beta_e(|\chi(0)|, t/\epsilon) + \epsilon |\bar{\zeta}^e|_{[0,t]} + \epsilon \left(|G_1| |\eta_1|_{[0,t]} + |\eta_2|_{[0,t]} \right) \quad (30)$$

for all $i = 1, \dots, r$ and with $\bar{\zeta}^e := \bar{\zeta}_{[1,r_e]}$.

Pick $(h_1, \dots, h_d) \in \mathbb{R}^d$ such that the polynomial $\lambda^d + h_1 \lambda^{d-1} + \dots + h_{d-1} \lambda + h_d$ has d roots with negative real part and, with $g > 0$ a control parameter, let

$$G_i := g^i h_i I_{n_e}. \quad (31)$$

Let $\Delta(g) := \text{diag}(1, g, \dots, g^{d-1})$ and change variables as

$$\eta \mapsto \mu := \Delta(g)^{-1} \eta.$$

In the new variables we obtain

$$\dot{\mu} = \Delta(g)^{-1} \left(A \Delta(g) \mu + E \phi(\Delta(g) \mu) + G(\bar{e} - \Gamma \mu_1) \right)$$

with $A \in \mathbb{R}^{d n_e \times d n_e}$, $E \in \mathbb{R}^{d n_e \times n_e}$ and $\Gamma \in \mathbb{R}^{n_e \times d n_e}$ defined as

$$A := \begin{pmatrix} 0_{n_e} & I_{n_e} & 0_{n_e} & \dots & 0_{n_e} \\ 0_{n_e} & 0_{n_e} & I_{n_e} & \dots & 0_{n_e} \\ \vdots & & & \ddots & \vdots \\ 0_{n_e} & \dots & & & I_{n_e} \\ 0_{n_e} & \dots & & & 0_{n_e} \end{pmatrix}, \quad E := \begin{pmatrix} 0_{n_e} \\ 0_{n_e} \\ \vdots \\ 0_{n_e} \\ I_{n_e} \end{pmatrix}, \quad \Gamma := (I_{n_e} \ 0_{n_e} \ \dots \ 0_{n_e}).$$

Noting that:

$$\Delta(g)^{-1} A \Delta(g) = gA, \quad \Delta(g)^{-1} E = g^{1-d} E, \quad \Delta(g)^{-1} G = gR, \quad \text{with } R := \text{col}(h_i I_{n_e} : i = 1, \dots, d),$$

then, by letting $M := A - R\Gamma$, we obtain

$$\dot{\mu} = gM \mu + g^{1-d} E \phi(\Delta(g) \mu) + gR C_e \chi. \quad (32)$$

Let $L_\phi > 0$ such that (6) holds. Then

$$|g^{1-d} E \phi(\Delta(g) \mu)| \leq L_\phi |\mu| \quad (33)$$

holds for all $\mu \in \mathbb{R}^{d n_e}$. By definition of (h_1, \dots, h_d) , M is Hurwitz. Then, in view of (33), standard high-gain arguments³⁶ can be used to show that there exists $g^* > 0$, a class- KL function β_μ and $a_2 > 0$, such that, for all $g > g^*$, the system μ fulfills

$$|\mu(t)| \leq \beta_\mu(|\mu(0)|, gt) + a_2 |\bar{e}|_{[0,t]}.$$

for all $t \geq 0$. Thus, by letting $\beta_\eta = g^{d-1}\beta_\mu$, using the fact that $|\eta_i| \leq g^{i-1}|\mu_i|$, we also obtain that the system η fulfills

$$|\eta_i(t)| \leq \beta_\eta(|\eta(0)|, t) + a_2 g^{i-1} |\bar{e}|_{[0,t]} \quad (34)$$

for all $i = 1, \dots, d$. With ϵ be any small number so that

$$\epsilon < \epsilon_1^*(g) := \frac{1}{a_2(|G_1| + g)},$$

being a_2 the same as in (34), let $K = K(g)$ be the corresponding matrix produced by Lemma 1, and change variables as

$$\zeta \mapsto \bar{\zeta} := \zeta - K\chi. \quad (35)$$

Then, the bounds (29)-(30) hold with $|\eta_2|_{[0,t]} \leq g|\mu|_{[0,t]}$ and, since $\epsilon a_2(g + |G_1|) < 1$, standard small-gain arguments³⁷ imply the existence of a class- KL function $\beta_{\chi,\eta}$ and $a_3 > 0$ such that the following bound holds

$$|(\chi(t), \eta(t))| \leq \beta_{\chi,\eta}(|(\chi(0), \eta(0))|, gt) + a_3 |\bar{\zeta}|_{[0,t]}. \quad (36)$$

Moreover, since for any class- K function ρ , $\rho(a + b) \leq \rho(2a) + \rho(2b)$, Assumption 1 yields

$$\begin{aligned} |x(t)| &\leq \beta(|x(0)|, t) + \rho_0(|w|_{[0,t]}) + \rho_1(|(\xi, \zeta)|_{[0,t]}) \\ &\leq \beta(|x(0)|, t) + \rho_0(|w|_{[0,t]}) + \rho_1((1 + |K|)|\chi|_{[0,t]} + |\eta_1|_{[0,t]} + |\bar{\zeta}|_{[0,t]}) \\ &\leq \beta(|x(0)|, t) + \rho_0(|w|_{[0,t]}) + \rho'_1(|(\chi, \eta)|_{[0,t]}) + \rho''_1(|\bar{\zeta}|_{[0,t]}) \end{aligned} \quad (37)$$

with $\rho'_1(s) := \rho_1(2(2 + |K|)s)$ and $\rho''_1(s) := \rho_1(2s)$. Thus, we conclude by (36) and (37) that there exist $\beta_{x\chi\eta}$ of class- KL and ρ_3 of class- K such that every solution of Σ_{cl} satisfies

$$|(x(t), \chi(t), \eta(t))| \leq \beta_{x\chi\eta}(|(x(0), \chi(0), \eta(0))|, t) + \rho_0(|w|_{[0,t]}) + \rho_3(|\bar{\zeta}|_{[0,t]}) \quad (38)$$

for all $t \geq 0$.

In view of (3b) and (28), $\bar{\zeta}$ fulfills

$$\dot{\bar{\zeta}} = \varrho(\eta, \chi, \bar{\zeta}) + q(w, x) + B(w, x)u \quad (39)$$

where

$$\varrho(\eta, \chi, \bar{\zeta}) := -K \left((F + C_e^\top G_1 C_e) \chi + H \zeta + C_e^\top (\eta_2 - G_1 \eta_1) \right). \quad (40)$$

With $\ell > 0$ a design parameter to be fixed, let in (5)

$$\mathcal{K}_\xi := \ell K, \quad \mathcal{K}_\zeta := -\ell I_{n_e}, \quad \mathcal{K}_\eta := \ell K C_e^\top. \quad (41)$$

The fact that \mathcal{K}_η can be taken of the form (7), with \mathcal{K}'_η invertible follows by the presence of C_e^\top in (41), and by the construction of K in Lemma 1. In the new coordinates we then have

$$u = \ell \mathcal{L} (K(\xi + C_e^\top \eta_1) - \zeta) = -\ell \mathcal{L} \bar{\zeta}$$

and thus (39) yields

$$\dot{\bar{\zeta}} = \varrho(\eta, \chi, \bar{\zeta}) + q(w, x) - \ell B(w, x) \mathcal{L} \bar{\zeta}. \quad (42)$$

We fix ℓ on the basis of the following Lemma, whose proof is in the appendix.

Lemma 2. Consider an equation of the form (42), with ϱ and q locally Lipschitz. Under Assumption 2 there exist $\beta_{\bar{\zeta}}$ of class- KL and, for each compact set $\bar{X} \subset \mathbb{R}^{n_x}$, constants $a_4 > 0$, $\ell_0^*(g, K) > 0$ such that, for all $\ell > \ell_0^*(g, K)$ the following holds

$$|\bar{\zeta}(t)| \leq \beta_{\bar{\zeta}}(|\bar{\zeta}(0)|, \ell t) + \frac{a_4}{\ell} (|(w, x, \chi, \eta)|_{[0,t]}), \quad (43)$$

for all $t \in \mathbb{R}_{\geq 0}$ such that $(w(s), x(s)) \in W \times \bar{X}$ for all $s \in [0, t]$.

Pick a compact set of initial conditions for (x, η) of the form $X \times H$ and let $\pi_1, \pi_2 > 0$ be such that $(w, x, \eta) \in W \times X \times H$ implies

$$\begin{aligned} |\chi| &= |\xi(w, x) + C_e^\top \eta| \leq \pi_1 \\ |\bar{\zeta}| &= |\zeta(w, x) - K(\xi(w, x) + C_e^\top \eta)| \leq \pi_2. \end{aligned}$$

Pick $\varpi_1 > 0$ so that

$$\varpi_1 \geq 1 + \max \left\{ \max_{(x,\eta) \in X \times H} |(x, \eta)| + \pi_1, \sup_{(x,\eta) \in X \times H} \beta_{x\chi\eta}(|(x, \eta)| + \pi_1, 0) + \sup_{w \in W} \rho_0(|w|) + \rho_3(\beta_{\bar{\zeta}}(\pi_2, 0) + 1) \right\}, \quad (44)$$

which can be fixed at this stage since, according to Lemma 2, $\beta_{\bar{\zeta}}$ does not depend on \bar{X} . With $\varpi_2 > \varpi_1$ define

$$\bar{X} := \overline{\mathbb{B}_{\varpi_2}}. \quad (45)$$

let $\ell_0^*(g, K)$ and a_4 be given by Lemma 2 for \bar{X} chosen as in (45), and pick

$$\ell > \ell_1^*(g, K) := \max \left\{ \ell_0^*(g, K), a_4 \left(\max_{w \in W} |w| + \varpi_2 \right) \right\}.$$

Then, for any solution to Σ_{cl} originating in $W \times X \times H$ it holds that

$$|\bar{\zeta}(t)| \leq \beta_{\bar{\zeta}}(\pi_2, 0) + 1. \quad (46)$$

for all $t \geq 0$ such that $|(x, \chi, \eta)|_{[0,t]} < \varpi_2$, as the latter condition implies $x(s) \in \bar{X}$ for all $s \in [0, t]$.

Now, suppose that a solution to Σ_{cl} exists that originates in $W \times X \times H$ and for some $\bar{t} > 0$ satisfies $|(x(\bar{t}), \chi(\bar{t}), \eta(\bar{t}))| > \varpi_2$. By definition of ϖ_1 , $|x(0), \chi(0), \eta(0)| < \varpi_1 < \varpi_2$, so that by continuity there exists $\bar{s} \in (0, \bar{t})$ such that $|x(\bar{s}), \chi(\bar{s}), \eta(\bar{s})| = \varpi_1$ and $|(x, \chi, \eta)|_{[0,\bar{s}]} \leq \varpi_1$. However, by (38), (44) and (46), this implies that

$$|\bar{\zeta}|_{[0,\bar{s}]} \leq \beta_{\bar{\zeta}}(\pi_2, 0) + 1$$

and hence

$$\begin{aligned} \varpi_1 &= |(x(\bar{s}), \chi(\bar{s}), \eta(\bar{s}))| \\ &\leq \beta_{x\chi\eta}(|x(0), \chi(0), \eta(0)|, 0) + \rho_0(|w|_{[0,\bar{s}]}) + \rho_3(|\bar{\zeta}|_{[0,\bar{s}]}) \\ &\leq \sup_{(x,\eta) \in X \times H} \beta_{x\chi\eta}(|(x, \eta)| + \pi_1, 0) + \sup_{w \in W} \rho_0(|w|) + \rho_3(\beta_{\bar{\zeta}}(\pi_2, 0) + 1) \\ &\leq \varpi_1 - 1 \end{aligned}$$

which is a contradiction. Thus, we claim that every solution to Σ_{cl} originating in $W \times X \times H$ is complete and satisfies $|(x(t), \chi(t), \eta(t))| \leq \varpi_2$ and $|\bar{\zeta}(t)| \leq \beta_{\bar{\zeta}}(\pi_2, 0) + 1$ for all $t \geq 0$. As ρ_3 is locally Lipschitz, then this implies that there exists $a_5 > 0$ such that every trajectory originating in $W \times X \times H$ satisfies

$$|(x(t), \chi(t), \eta(t))| \leq \beta_{x\chi\eta}(|(x(0), \chi(0), \eta(0))|, t) + \rho_0(|w|_{[0,t]}) + a_5 |\bar{\zeta}|_{[0,t]}$$

for all $t \geq 0$. In view of (43), which holds for every $t \geq 0$, standard small-gain arguments show that there exists β_{cl} of class- KL such that, with $\rho_{cl}(\cdot) = \rho_0(\cdot) + 1$ and for all

$$\ell > \ell^*(g, K) := \max \{ \ell_1^*(g, K), a_4 a_5 \},$$

the claim (9) of the proposition holds. \square

5.2 | Proof of Theorem 2

By Proposition 1, there exist $g^* > 0$, $K = K(g) \in \mathbb{R}^{p \times p(N-p)}$ and $\ell^*(g, K) > 0$ such that, for all $g > g^*$ and $\ell > \ell^*(K, g)$, the choices (31) and (41) guarantee that the claim (9) of the proposition holds, and in particular that there exists a compact invariant attractor \mathcal{A} defined in (10). Pick now a solution (w, x, η) to $\Sigma_{cl}^{\mathcal{A}}$ and, define the signal η^* according to (18), which under Assumption 3 and 4 is well defined in view of Remark 7. With reference to the coordinates (27) and (35), consider the change of variables

$$\begin{aligned} \bar{e} &\mapsto \tilde{e} := \bar{e} - \eta_1^* \\ \eta &\mapsto \tilde{\eta} := \eta - \eta^* \\ \chi &\mapsto \tilde{\chi} := \chi - C_e^\top \eta_1^* \\ \bar{\zeta} &\mapsto \tilde{\zeta} := \bar{\zeta} + K C_e^\top \eta_1^* = \zeta - K \tilde{\chi}. \end{aligned}$$

Then

$$\dot{\tilde{e}} = C_e \chi - \eta_1^* = C_e \tilde{\chi},$$

and, in view of (28) and noting that $F C_e^\top = 0$ and $C_e C_e^\top = I_{n_e}$, we also have

$$\dot{\tilde{\chi}} = (F + C_e^\top G_1 C_e + H K) \tilde{\chi} + H \tilde{\zeta} + C_e^\top (\tilde{\eta}_2 - G_1 \tilde{\eta}_1). \quad (47)$$

With $\Delta(g) := \text{diag}(I, gI, \dots, g^{d-1}I)$, let

$$\tilde{\mu} := \Delta(g)^{-1}\tilde{\eta},$$

the same argument used in the proof of Proposition 1 in dealing with μ show that $\tilde{\mu}$ satisfies

$$\dot{\tilde{\mu}} = gM\tilde{\mu} + g^{1-d}E(\phi(\Delta(g)\tilde{\mu} + \eta^*) - \eta_1^{*(d)}) + gRC_e\tilde{\chi}$$

with M , E and R defined as in (32).

Then, as ϕ satisfies (6), for some $L_\phi > 0$, it holds that

$$\begin{aligned} |\phi(\Delta(g)\tilde{\mu} + \eta^*) - \eta_1^{*(d)}| &\leq |\phi(\Delta(g)\tilde{\mu} + \eta^*) - \phi(\eta^*)| + |\phi(\eta^*) - \eta_1^{*(d)}| \\ &\leq L_\phi g^{d-1}|\tilde{\mu}| + \bar{\delta}. \end{aligned}$$

where $\bar{\delta}$ is defined in (20).

Standard high-gain arguments can be thus used to prove that for large enough g the following estimate holds

$$|\tilde{\mu}(t)| \leq \beta_{\tilde{\mu}}(|\tilde{\mu}(0)|, t) + \frac{q_1}{g^d}\bar{\delta} + q_1|\tilde{e}|_{[0,t]}. \quad (48)$$

for some class- KL function $\beta_{\tilde{\mu}}$ and some $q_2 > 0$. We can thus assume without loss of generality that g is taken large enough to guarantee this.

Since in view of (31), $|G_1|\tilde{\eta}_1| + |\tilde{\eta}_2| \leq q_2g|\tilde{\mu}|$, for some $q_2 > 0$ independent from g , then in view of Lemma 1 and (47), we obtain that K can be chosen without loss of generality so that there exist $\beta_{\tilde{\chi}}$ and $\beta_{\tilde{e}}$ of class- KL , $q_3 > 0$ and $\epsilon < (2q_1g)^{-1}$ for which the following bounds hold

$$\begin{aligned} |\tilde{\chi}^e(t)| &\leq \beta_{\tilde{\chi}}(|\tilde{\chi}(0)|, t) + q_3\left(|\tilde{\zeta}^e|_{[0,t]} + g|\tilde{\mu}|_{[0,t]}\right) \\ |\tilde{e}(t)| &\leq \beta_{\tilde{e}}(|\tilde{e}(0)|, t) + \epsilon\left(|\tilde{\zeta}^e|_{[0,t]} + g|\tilde{\mu}|_{[0,t]}\right), \end{aligned} \quad (49)$$

in which $\tilde{\chi}^e$ and $\tilde{\zeta}^e$ are defined as in (4). Therefore, by small-gain arguments³⁷ (48) and (49) imply the following bound

$$\limsup_{t \rightarrow \infty} |(\tilde{\mu}(t), \tilde{\chi}^e(t))| \leq q_4 \limsup_{t \rightarrow \infty} |\tilde{\zeta}^e|_{[0,t]} + q_4g^{-d}\bar{\delta}, \quad (50)$$

for some $q_4 = q_4(g) > 0$.

In view of (41) and since

$$\xi = \chi - C_e^\top \eta_1 = \tilde{\chi} - C_e^\top \tilde{\eta}_1, \quad \zeta = \tilde{\zeta} + K\chi = \tilde{\zeta} + K\tilde{\chi},$$

we obtain

$$u = \mathcal{L}\left(\mathcal{K}_\xi \xi + \mathcal{K}_\zeta \zeta + \mathcal{K}_\eta \eta_1\right) = \mathcal{L}\left(\mathcal{K}_\xi(\tilde{\chi} - C_e^\top \tilde{\eta}_1) + \mathcal{K}_\zeta(\tilde{\zeta} + K\tilde{\chi}) + \mathcal{K}_\eta(\tilde{\eta}_1 + \eta_1^*(w, x))\right) = -\ell \mathcal{L}\tilde{\zeta} + \mathcal{L}\mathcal{K}_\eta \eta_1^*(w, x).$$

In view of the definition of η_1^* in (16), and of the block-diagonal structure of K in Lemma 1, by letting

$$Q := \begin{pmatrix} I_{n_e} \\ 0_{(n_y - n_e) \times n_e} \end{pmatrix},$$

we thus obtain

$$\begin{aligned} \dot{\tilde{\zeta}}^e &= \rho^e(\tilde{\chi}^e, \tilde{\zeta}^e, \tilde{\eta}) + q^e(w, x) + Q^\top B(w, x)\left(-\ell \mathcal{L}\tilde{\zeta} + \mathcal{L}\mathcal{K}_\eta \eta_1^*(w, x)\right) = \rho^e(\tilde{\chi}^e, \tilde{\zeta}^e, \tilde{\eta}) - \ell Q^\top B(w, x)\mathcal{L}Q\tilde{\zeta}^e \\ &= \rho^e(\tilde{\chi}^e, \tilde{\zeta}^e, \tilde{\eta}) - \ell D^{e,e}(w, x)\tilde{\zeta}^e \end{aligned}$$

with ρ^e a linear map that, in view of the lower-triangular structure of F and of the diagonal structure of H , only depends on $(\tilde{\chi}^e, \tilde{\zeta}^e, \tilde{\eta})$ and satisfies $\rho^e(\tilde{\chi}^e, \tilde{\zeta}^e, \tilde{\eta}) = Q^\top \rho(\tilde{\eta}, \tilde{\chi}, \tilde{\zeta})$, with ρ defined in (40). We now observe that, since ρ_{cl} in the claim of Proposition 1 does not depend on ℓ , then in view of Assumption (4) and on the fact that $(w(t), x(t)) \in \mathcal{O}$ implies the existence of $\underline{\lambda}', \bar{\lambda}' > 0$ such that $\underline{\lambda}'I \leq \mathcal{M}(w, x) \leq \bar{\lambda}'I$, arguments similar to those of Lemma 2 applied with $q = 0$ and $\tilde{X} = \{x \in \mathbb{R}^{n_x} : (w, x) \in \mathcal{O}\}$ to the Lyapunov candidate

$$V_e(w, x) := \sqrt{\tilde{\zeta}^{e\top} \mathcal{M}(w, x) \tilde{\zeta}^e},$$

can be used to show that for large enough ℓ , $\tilde{\zeta}^e$ satisfies

$$\limsup_{t \rightarrow \infty} |\tilde{\zeta}^e(t)| \leq \frac{q_5 g}{\ell} \limsup_{t \rightarrow \infty} |(\tilde{\chi}^e, \tilde{\mu})|_{[0,t]}$$

for some $q_5 > 0$ independent from ℓ and g . Hence, in view of (50), and since ℓ can be taken without loss of generality larger enough so that

$$\ell > 2q_5q_4g, \quad (51)$$

then we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} |(\tilde{\mu}(t), \tilde{\chi}^e(t))| &\leq \frac{q_4q_5g}{\ell} \limsup_{t \rightarrow \infty} |(\tilde{\mu}(t), \tilde{\chi}^e(t))| + q_4g^{-d}\bar{\delta} \\ \limsup_{t \rightarrow \infty} |\tilde{\zeta}^e(t)| &\leq \frac{q_4q_5g}{\ell} \limsup_{t \rightarrow \infty} |\tilde{\zeta}^e(t)| + \frac{q_4q_5g}{\ell} g^{-d}\bar{\delta}. \end{aligned}$$

As (51) implies both

$$\frac{1}{2} < 1 - \frac{q_4q_5g}{\ell}, \quad \frac{q_4q_5g}{\ell} < \frac{1}{2},$$

then we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} |(\tilde{\mu}(t), \tilde{\chi}^e(t))| &\leq 2 \left(1 - \frac{q_4q_5g}{\ell}\right) \limsup_{t \rightarrow \infty} |(\tilde{\mu}(t), \tilde{\chi}^e(t))| \leq 2q_4g^{-d}\bar{\delta} \\ \limsup_{t \rightarrow \infty} |\tilde{\zeta}^e(t)| &\leq 2 \left(1 - \frac{q_4q_5g}{\ell}\right) \limsup_{t \rightarrow \infty} |\tilde{\zeta}^e(t)| \leq g^{-d}\bar{\delta}. \end{aligned} \quad (52)$$

Now, pick a point $(\bar{w}, \bar{x}, \bar{\eta})$ in \mathcal{A} . Define arbitrarily a sequence $\{t_n\}_{n \in \mathbb{N}}$ of positive scalars t_n satisfying $t_n \rightarrow \infty$. As \mathcal{A} is invariant, it is backwards invariant. Hence, for each $n \in \mathbb{N}$, we can find a solution (w^n, x^n, η^n) to $\Sigma_{cl}^{\mathcal{A}}$ which satisfies

$$(w^n(t_n), x^n(t_n), \eta^n(t_n)) = (\bar{w}, \bar{x}, \bar{\eta}).$$

In view of (52), which holds for any solution of $\Sigma_{cl}^{\mathcal{A}}$ and thus in particular for each (w^n, x^n, η^n) , it follows that for each $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that the quantities $\tilde{\mu}^n := \Delta(g)^{-1}(\eta^n - \eta^*(w^n, x^n))$ and $\tilde{\chi}^{en} := \chi(w^n, x^n) - C_e^\top \eta_1^*(w^n, x^n)$ satisfy

$$|\tilde{\chi}^{en}(t_n)| + |\tilde{\mu}^n(t_n)| \leq 4q_4g^{-d}\bar{\delta} + \varepsilon$$

for all $n \geq N(\varepsilon)$. Let $\bar{e} := h_e(\bar{w}, \bar{x})$ denote the regulation error computed at $(\bar{w}, \bar{x}, \bar{\eta})$. Then for each $n \in \mathbb{N}$ it satisfies

$$|\bar{e}| = |C_e \tilde{\chi}^n(t_n) - C_e C_e^\top \tilde{\mu}_1^n(t_n)| \leq |\tilde{\chi}^{en}(t_n)| + |\tilde{\mu}^n(t_n)|,$$

so that, for all $\varepsilon > 0$, by taking $n \geq N(\varepsilon)$, we obtain

$$|\bar{e}| \leq 4q_4g^{-d}\bar{\delta} + \varepsilon.$$

For the arbitrariness of ε and $(\bar{w}, \bar{x}, \bar{\eta}) \in \mathcal{A}$ we thus conclude that

$$|e| = |h_e(w, x)| \leq 4q_4g^{-d}\bar{\delta}$$

holds for all $(w, x) \in \mathcal{O}$. Then the claim (21) of the theorem follows from the continuity of h_e and uniform attractiveness of \mathcal{A} from $W \times X \times H$ by assuming, without loss of generality, $g > \sqrt[4]{4q_4/\varepsilon}$. \square

6 | ON THE CONTROLLABILITY ASSUMPTIONS

In this section we complement Assumptions 2 and 4, by showing how they are implied by many state-of-art assumptions routinely used in the context of high-gain stabilization and regulation of multivariable systems, and thus also showing how the matrix \mathcal{L} can be constructed in the respective frameworks using only quantities that are known. In the following we assume that $B(w, x)$ is C^1 and, for ease of notation, we let $\mathbf{x} := (w, x)$.

6.1 | Strong Invertibility in the Sense of Wang and Isidori 2015 Implies Assumption 2

Here we prove that the assumption of invertibility used, for instance, in recent papers by Wang, Isidori and Su³¹ and by Wang, Isidori, Su and Marconi¹⁷ implies Assumption 2.

Lemma 3. Suppose that $n_u = n_y$ (i.e. $B(\mathbf{x})$ is square), $B(\mathbf{x})$ is bounded, $L_{g(\mathbf{x})u}^{(x)} B(\mathbf{x}) = 0$ for all $\mathbf{x} \in W \times \mathbb{R}^{n_x}$, and there exists $\varepsilon > 0$ such that all its principal minors $\Delta_i(\mathbf{x})$, $i = 1, \dots, r$ satisfy

$$|\Delta_i(\mathbf{x})| \geq \varepsilon, \quad (53)$$

for all $\mathbf{x} \in W \times \mathbb{R}^{n_x}$. Then Assumption 2 holds.

Proof. It can be shown³¹ that if (53) holds and $B(\mathbf{x})$ is bounded then $B(\mathbf{x})$ can be written as

$$B(\mathbf{x}) = EM(\mathbf{x})(I + U(\mathbf{x})),$$

with $M(\mathbf{x}) = M(\mathbf{x})^\top$ positive definite for all $\mathbf{x} \in \mathcal{W} \times \mathbb{R}^{n_x}$ and satisfying $a_1 I \leq M(\mathbf{x}) \leq a_2 I$ for all $\mathbf{x} \in \mathcal{W} \times \mathbb{R}^{n_x}$ and for some $a_1, a_2 > 0$, $U(\mathbf{x})$ a strictly upper triangular matrix and with E a diagonal matrix satisfying $EE = I$. Moreover, simple computations³¹ show that, if $B(\mathbf{x})$ is bounded, then there exists $c > 1$ such that, with $C := \text{diag}(c^{n_y-1}, c^{n_y-2}, \dots, c, 1)$, we have

$$(I + U(\mathbf{x}))C + C(I + U(\mathbf{x}))^\top \geq I.$$

Let

$$\mathcal{L} := EC, \quad \mathcal{P}(\mathbf{x}) := EM(\mathbf{x})^{-1}E.$$

Since $a_1 I \leq M(\mathbf{x}) \leq a_2 I$ holds at each $\mathbf{x} \in \mathcal{W} \times \mathbb{R}^{n_x}$, then the eigenvalues of $M(\mathbf{x})^{-1}$ are lower and upper-bounded by a_2^{-1} and a_1^{-1} respectively, and hence point a of Assumption 2 holds. Furthermore, we observe that $\mathcal{L} = \mathcal{L}^\top = EC = CE$. Hence, noting that

$$B(\mathbf{x})^\top EM(\mathbf{x})^{-1} = (I + U(\mathbf{x}))^\top M(\mathbf{x})^\top E^\top EM(\mathbf{x})^{-1} = (I + U(\mathbf{x}))^\top,$$

we then have

$$\mathcal{L}^\top B(\mathbf{x})^\top \mathcal{P}(\mathbf{x}) + \mathcal{P}(\mathbf{x}) B(\mathbf{x}) \mathcal{L} = EC B(\mathbf{x})^\top EM(\mathbf{x})^{-1} E + EM(\mathbf{x})^{-1} E B(\mathbf{x}) C E = E \left[C(I + U(\mathbf{x}))^\top + (I + U(\mathbf{x})) C \right] E \geq I,$$

for all $\mathbf{x} \in \mathcal{W} \times \mathbb{R}^{n_x}$. Thus, since $EE = I$, point c of Assumption 2 holds.

Finally, we observe that³¹ $L_{g(\mathbf{x})u}^{(x)} B(\mathbf{x}) = 0$ implies $L_{g(\mathbf{x})u}^{(x)} M(\mathbf{x}) = 0$. Since

$$L_{g(\mathbf{x})u}^{(x)} M(\mathbf{x})^{-1} = -M(\mathbf{x})^{-1} L_{g(\mathbf{x})u}^{(x)} M(\mathbf{x}) M(\mathbf{x})^{-1} = 0,$$

then also point b holds, hence the result. \square

6.2 | Strong invertibility in the sense of Wang and Isidori 2015 and 2017^{32,18} implies Assumption 2

Here we prove that the assumption of invertibility used, for instance, in the works of Wang, Isidori and Su^{32,18} implies Assumption 2.

Lemma 4. Suppose that $n_u = n_y$ (i.e. $B(\mathbf{x})$ is square) and that there exist a non singular matrix $M \in \mathbb{R}^{n_u \times n_u}$ and a constant $\delta_0 \in (0, 1)$ such that

$$\max_{\substack{\Lambda \in \mathbb{R}^{n_u \times n_u} \\ |\Lambda| \leq 1}} \left| (B(\mathbf{x}) - M) \Lambda M^{-1} \right| \leq \delta_0 \quad (54)$$

holds for all $\mathbf{x} \in \mathcal{W} \times \mathbb{R}^{n_x}$. Then Assumption 2 holds.

Proof. As (54) holds for all $\Lambda \in \mathbb{R}^{n_u \times n_u}$ satisfying $|\Lambda| \leq 1$, it holds in particular for $\Lambda = I$, thus yielding

$$|B(\mathbf{x})M^{-1} - I| \leq \delta_0.$$

Thus, for all $\mathbf{p} \in \mathbb{R}^{n_u}$ and $\mathbf{x} \in \mathcal{W} \times \mathbb{R}^{n_x}$, it holds that

$$2\mathbf{p}^\top \left(I - B(\mathbf{x})M^{-1} \right) \mathbf{p} \leq |2\mathbf{p}^\top (B(\mathbf{x})M^{-1} - I) \mathbf{p}| \leq 2|\mathbf{p}|^2 \cdot |I - B(\mathbf{x})M^{-1}| \leq 2\delta_0 |\mathbf{p}|^2 = \mathbf{p}^\top (2\delta_0 I) \mathbf{p}.$$

Therefore, we obtain

$$\mathbf{p}^\top \left(2I - M^{-T} B(\mathbf{x})^\top - B(\mathbf{x})M^{-1} \right) \mathbf{p} = 2\mathbf{p}^\top \mathbf{p} - 2\mathbf{p}^\top B(\mathbf{x})M^{-1} \mathbf{p} = 2\mathbf{p}^\top \left(I - B(\mathbf{x})M^{-1} \right) \mathbf{p} \leq \mathbf{p}^\top (2\delta_0 I) \mathbf{p}.$$

As it holds for all $\mathbf{p} \in \mathbb{R}^{n_u}$ and $\mathbf{x} \in \mathcal{W} \times \mathbb{R}^{n_x}$ then, necessarily

$$2I - M^{-T} B(\mathbf{x})^\top - B(\mathbf{x})M^{-1} \leq 2\delta_0 I,$$

and thus, letting $\mathcal{L} := M^{-1}$ and $\mathcal{P}(\mathbf{x}) := I/(2(1 - \delta_0))$ yields the item c of Assumption 2. Point a instead follows by noting that $\delta_0 \in (0, 1)$, while point b is straightforward as \mathcal{P} is constant in \mathbf{x} . \square

6.3 | Positivity and negativity in the sense of McGregor, Byrnes and Isidori 2006¹⁵, Back 2009³³ and Astolfi, Isidori, Marconi and Praly 2013¹⁶ imply Assumption 2

Finally, here we show that the “negativity” and “positivity” assumptions on B , made in some recent papers^{15,33,16}, all imply Assumption 2. The following lemma refers to the positivity assumption (Assumption 1) of Astolfi, Isidori, Marconi and Praly 2013¹⁶.

Lemma 5. Suppose that $n_u = n_y$ (i.e. $B(\mathbf{x})$ is square) and that there exists $K \in \mathbb{R}^{n_y \times n_y}$ such that the following positivity condition holds

$$B(\mathbf{x})K + K^\top B(\mathbf{x})^\top \geq I \quad (55)$$

for all $\mathbf{x} \in W \times \mathbb{R}^{n_x}$. Then Assumption 2 holds.

Proof. We first observe that equation (55) implies that K is invertible. In fact, if we suppose that K is singular, then there exists $\mathbf{p} \neq 0$ satisfying $K\mathbf{p} = 0$. But this yields $\mathbf{p}^\top(B(\mathbf{x})K + K^\top B(\mathbf{x}))\mathbf{p} = 0$, which contradicts (55). Then, Assumption 2 is satisfied by simply letting $\mathcal{L} = K$ and $\mathcal{P}(\mathbf{x}) = I$. \square

The following lemma, instead, refers to a slightly relaxed version of Assumption 4.4 of McGregor, Byrnes and Isidori 2006¹⁵, which is however implied by that assumption in each compact subset of $W \times \mathbb{R}^{n_x}$.

Lemma 6. Suppose that $n_u = n_y$ (i.e. $B(\mathbf{x})$ is square) and that there exists $M \in \mathbb{R}^{n_y \times n_y}$ and $\kappa > 0$ such that the following negativity condition holds

$$B(\mathbf{x})M + M^\top B(\mathbf{x})^\top < -\kappa I \quad (56)$$

for all $\mathbf{x} \in W \times \mathbb{R}^{n_x}$. Then Assumption 2 holds.

Proof. The proof follows by Lemma 5 by noticing that (56) implies (55) with $K = -M/\kappa$. \square

Finally, the following lemma concerns a slightly relaxed version of Assumption 3 of Back 2009³³, which is equivalent to that assumption in each compact set.

Lemma 7. Suppose that $n_u = n_y$ (i.e. $B(\mathbf{x})$ is square), and assume that there exist $\kappa > 0$, a non singular matrix K , $G^- := \text{diag}(g_1^-, \dots, g_m^-)$, and $G^+ := \text{diag}(g_1^+, \dots, g_m^+)$ such that $0 < G^- < G^+$ and that

$$\left(B(\mathbf{x})K\mathbf{p} - G^-\mathbf{p} \right)^\top \Pi^2 \left(B(\mathbf{x})K\mathbf{p} - G^+\mathbf{p} \right) \leq -\kappa \mathbf{p}^\top \mathbf{p}, \quad (57)$$

for all $\mathbf{p} \in \mathbb{R}^{n_u}$ and all $\mathbf{x} \in W \times \mathbb{R}^{n_x}$ and where $\Pi := 2(G^+ + G^-)^{-1}$. Then Assumption 2 holds.

Proof. Equation (57) implies $(G^- = (G^-)^\top)$

$$-K^\top B(\mathbf{x})^\top \Pi^2 G^+ - G^- \Pi^2 B(\mathbf{x})K + K^\top B(\mathbf{x})^\top \Pi^2 B(\mathbf{x})K + G^- \Pi^2 G^+ \leq -\kappa I,$$

that in turn implies

$$\mathcal{M}(\mathbf{x}) := K^\top B(\mathbf{x})^\top \Pi^2 G^+ + G^- \Pi^2 B(\mathbf{x})K > \kappa I$$

for all $\mathbf{x} \in W \times \mathbb{R}^{n_x}$. Noting that

$$\begin{aligned} \Pi^2 G^+ &= \Pi \cdot 2(G^+ + G^-)^{-1} G^+ = \Pi \cdot 2(G^+ + G^-)^{-1} (G^+ + G^- - G^-) = 2\Pi - \Pi^2 G^- \\ G^- \Pi^2 &= 2 \cdot G^- (G^+ + G^-)^{-1} \Pi = 2\Pi - G^+ \Pi^2 \end{aligned}$$

then

$$\mathcal{M}(\mathbf{x}) = 2 \left(K^\top B(\mathbf{x})^\top \Pi + \Pi B(\mathbf{x})K \right) - \mathcal{M}(\mathbf{x})^\top$$

Thus, $\mathcal{M}(\mathbf{x}) > \kappa I$ implies

$$K^\top B(\mathbf{x})^\top \Pi + \Pi B(\mathbf{x})K = \frac{1}{2} \left(\mathcal{M}(\mathbf{x}) + \mathcal{M}(\mathbf{x})^\top \right) > \kappa I,$$

and by letting $M = -K\Pi^{-1}$ we obtain

$$M^\top B(\mathbf{x})^\top + B(\mathbf{x})M < -\kappa \Pi^2,$$

and the claim follows from Lemma 6 by noting that Π^2 is positive definite. \square

6.4 | About Assumption 4

In this section we show that Assumption 4 holds in all the frameworks reported in the previous sections. Regarding the framework of Section 6.1, we observe that if $B(\mathbf{x})$ is bounded, $L_{g(\mathbf{x})u}^{(x)} B(\mathbf{x}) = 0$, and $|\Delta_i(\mathbf{x})| \geq \epsilon > 0$ for all $\mathbf{x} \in \mathcal{W} \times \mathbb{R}^{n_x}$ and all $i = 1, \dots, n_y$, then the same properties apply for the upper-left submatrix $B^{e,e}(\mathbf{x}) \in \mathbb{R}^{n_e \times n_e}$ defined by eliminating the last $n_y - n_e$ rows and columns of $B(\mathbf{x})$. Thus, according to Lemma 3, there exist \mathcal{L}^e and $\mathcal{P}^e(\mathbf{x})$ satisfying the same properties of Assumption 2 for the reduced matrix $B^{e,e}(\mathbf{x})$. Since by Lemma 3 the result for the full-size $B(\mathbf{x})$ can be obtained with a matrix \mathcal{L} of the form $\mathcal{L} = \text{diag}(c^{n_y - n_e} \mathcal{L}^e, \mathcal{L}^a)$, for some $c > 1$ and some $\mathcal{L}^a \in \mathbb{R}^{(n_y - n_e) \times (n_y - n_e)}$, then in view of (13), $D^{e,e}(\mathbf{x}) = c^{n_e - n_y} \mathcal{L}^e B^{e,e}(\mathbf{x})$, and it is readily seen that \mathcal{L} can be chosen so that Assumptions 2 and 4 hold simultaneously with $\mathcal{M}(\mathbf{x}) = \mathcal{P}^e(\mathbf{x})$.

Regarding all the other frameworks presented above, the fact that also Assumption 4 holds is a consequence of the following Lemma.

Lemma 8. Suppose that $n_u = n_y$ and Assumption 2 holds \mathcal{P} of the form

$$\mathcal{P}(\mathbf{x}) = \text{diag}(\mathcal{P}^e(\mathbf{x}), \mathcal{P}^a(\mathbf{x}))$$

for some $\mathcal{P}^e(\mathbf{x}) \in \mathbb{R}^{n_e \times n_e}$ and $\mathcal{P}^a(\mathbf{x}) \in \mathbb{R}^{(n_y - n_e) \times (n_y - n_e)}$. Then Assumption 4 holds.

Proof. The proof directly by noticing that point a and b of Assumption 2 imply the same properties on $\mathcal{P}^e(\mathbf{x})$, while point c, which by definition of $D(\mathbf{x})$ is equivalent to

$$D(\mathbf{x})^\top \mathcal{P}(\mathbf{x}) + \mathcal{P}(\mathbf{x}) D(\mathbf{x}) \geq I,$$

implies

$$\begin{pmatrix} D^{e,e}(\mathbf{x})^\top \mathcal{P}^e(\mathbf{x}) + \mathcal{P}^e(\mathbf{x}) D^{e,e}(\mathbf{x}) & M_1(\mathbf{x}) \\ M_1(\mathbf{x})^\top & M_2(\mathbf{x}) \end{pmatrix} \geq I \quad (58)$$

for some properly defined M_1 and M_2 . Pick $\mathbf{p} \in \mathbb{R}^{n_e}$ arbitrary. Then (58) implies

$$|\mathbf{p}|^2 \leq \begin{pmatrix} \mathbf{p} \\ 0 \end{pmatrix}^\top \begin{pmatrix} D^{e,e}(\mathbf{x})^\top \mathcal{P}^e(\mathbf{x}) + \mathcal{P}^e(\mathbf{x}) D^{e,e}(\mathbf{x}) & M_1(\mathbf{x}) \\ M_1(\mathbf{x})^\top & M_2(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ 0 \end{pmatrix} = \mathbf{p}^\top \left(D^{e,e}(\mathbf{x})^\top \mathcal{P}^e(\mathbf{x}) + \mathcal{P}^e(\mathbf{x}) D^{e,e}(\mathbf{x}) \right) \mathbf{p}$$

and thus Assumption 4 holds with $\mathcal{M}(\mathbf{x}) := \mathcal{P}^e(\mathbf{x})$. \square

7 | CONCLUSION

In this paper we have proposed a post-processing regulator for multivariable nonlinear systems possessing a partial normal form. Contrary to previous approaches, the proposed regulator can handle additional measurements that need not to vanish at the steady state but that can be useful for stabilization purposes or, for instance, to fit into the minimum-phase requirement. The proposed approach can also handle a control input dimension of arbitrarily large size, provided that a controllability assumption is fulfilled. About the controllability requirement, we have shown that it is implied by many state-of-art assumption in the literature of high-gain stabilization and regulation of normal forms. Among the drawbacks of the proposed approach, we underline how the stabilization phase is still strongly based on a ‘‘high-gain’’ perspective, thus allowing us to restrict to linear stabilizers, and how the proposed conditions to obtain asymptotic regulation may be not constructive in general.

Future research directions spread in many ways. In the first place, we aim to introduce adaptation in the control loop, so as to cope with model uncertainties and to weaken the chicken-egg dilemma that link the choice of the stabilizer gains and the internal model dynamics, as envisioned in⁴. A second research direction concerns the extension of the framework to include more general stabilizing laws, going beyond the linear high-gain technique and trying to relax the minimum-phase requirement.

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APPENDIX

A PROOF OF LEMMA 1

Pick $i \in \{1, \dots, r\}$ and, with $k_i > 0$, define the matrix $\Lambda_i(k_i) := \text{diag}(k_i^{N_i-2} I_{p_i}, k_i^{N_i-3} I_{p_i}, \dots, k_i I_{p_i}, I_{p_i}) \in \mathbb{R}^{p_i(N_i-1) \times p_i(N_i-1)}$ and change coordinates as

$$\chi^i \mapsto z^i := \Lambda_i(k_i) \chi^i.$$

With reference to the matrices defined in (3a)-(3c), noting that

$$\Lambda_i(k_i) F_{ii} \Lambda_i(k_i)^{-1} = k_i F_{ii}, \quad \Lambda_i(k_i) H_{ii} = H_{ii}, \quad \Lambda_i(k_i) C_i^\top G_1 C_i \Lambda_i(k_i)^{-1} = C_i^\top G_1 C_i, \quad \Lambda_i(k_i) C_i^\top = k_i^{N_i-2} C_i^\top$$

then z^i fulfills

$$\dot{z}^i = k_i F_{ii} z^i + H_{ii} \zeta_i + \Lambda_i(k_i) \sum_{j=1}^{i-1} (F_{ij} \Lambda_j(k_j)^{-1} z^j + H_{ij} \zeta_j) + C_i^\top G_1 C_i z^i + k_i^{N_i-2} C_i^\top (\eta_2 - G_1 \eta_1)$$

Let $(\alpha_1^i, \dots, \alpha_{N_i-1}^i)$ be such that the polynomial $\lambda^{N_i-1} + \alpha_{N_i-1}^i \lambda^{N_i-2} + \dots + \alpha_1^i$ has roots with negative real part and, with

$$D^i := \begin{pmatrix} -\alpha_1^i I_{p_i} & \dots & -\alpha_{N_i-1}^i I_{p_i} \end{pmatrix}$$

and $\bar{\zeta}^i \in \mathbb{R}^{p_i}$, let

$$\zeta^i = \bar{\zeta}^i + k_i D^i z^i.$$

Since $F_{ii} + H_{ii} D^i$ is Hurwitz, then there exists $k^* > 1$ such that, for any $k_i > k^*$, the following holds

$$|z^i(t)| \leq b_1 e^{-b_2 k_i t} |z^i(0)| + b_3 k_i^{N_i-3} \left(|\bar{\zeta}_{[1,i]}^i|_{[0,t]} + |v|_{[0,t]} \right) + b_4 k_i^{N_i-3} \sum_{j=1}^{i-1} k_j \int_0^t e^{-b_2 k_i (t-\tau)} |z^j(\tau)| d\tau \quad (\text{A1})$$

for some $b_1, b_2, b_3, b_4 > 0$, and in which we let for convenience $v := |G_1| |\eta_1| + |\eta_2|$, $\bar{\zeta}_{[1,i]}^i := (\bar{\zeta}_1, \dots, \bar{\zeta}_i)$. We can partition \bar{e} as $\bar{e} = \text{col}(\bar{e}^1, \dots, \bar{e}^e)$, with $\bar{e}^i := k_i^{2-N_i} C_i z^i$. Pick $\epsilon > 0$ and pick $i \in \{2, \dots, r\}$. Let for convenience $\bar{k}_i := \max_{1 \leq j < i} k_j$ and suppose that, for each $j = 1, \dots, i-1$, $z^j(t)$ fulfills

$$|z^j(t)| \leq h_1^i(\bar{k}_i) e^{-\frac{b_2}{2} k_i t} |z(0)| + h_2^i(\bar{k}_i) \left(|\bar{\zeta}_{[1,j]}^i|_{[0,t]} + |v|_{[0,t]} \right), \quad (\text{A2})$$

for some $h_1^i(\bar{k}_i), h_2^i(\bar{k}_i) \geq 0$. Then, if $k_i > \bar{k}_i > k_1$, for each $\tau \leq t$ we have

$$e^{-b_2 k_i (t-\tau)} \leq e^{-b_2 k_1 (t-\tau)} \leq e^{-\frac{b_2}{2} k_1 t} e^{b_2 k_1 \tau},$$

and equations (A1)-(A2) give

$$\begin{aligned} |z^i(t)| &\leq b_1 e^{-b_2 k_i t} |z(0)| + b_3 k_i^{N_i-3} \left(|\bar{\zeta}_{[1,i]}^i|_{[0,t]} + |v|_{[0,t]} \right) \\ &\quad + b_4 k_i^{N_i-3} (i-1) \bar{k}_i \left(e^{-b_2 k_i t} \int_0^t e^{\frac{b_2}{2} k_1 \tau} h_1^i(\bar{k}_i) |z(0)| d\tau + \int_0^t e^{-b_2 k_i (t-\tau)} h_2^i(\bar{k}_i) \left(|\bar{\zeta}_{[1,i]}^i|_{[0,t]} + |v|_{[0,t]} \right) d\tau \right) \\ &\leq \left(b_1 + \frac{2b_4 r \bar{k}_i h_1^i(\bar{k}_i) k_i^{N_i-3}}{b_2 k_1} \right) e^{-\frac{b_2}{2} k_1 t} |z(0)| + \left(b_3 + \frac{b_4 r \bar{k}_i h_2^i(\bar{k}_i)}{b_2 k_i} \right) k_i^{N_i-3} \left(|\bar{\zeta}_{[1,i]}^i|_{[0,t]} + |v|_{[0,t]} \right). \end{aligned} \quad (\text{A3})$$

Hence, if

$$k_i > \max \left\{ k^*, \bar{k}_i, \frac{r}{\epsilon} \left(b_3 + \frac{b_4 r \bar{k}_i h_2^i(\bar{k}_i)}{b_2} \right) \right\}, \quad (\text{A4})$$

then, in view of (A3), the fact that (A2) holds for $j = 1, \dots, i-1$ implies that the same bounds also hold for $j = 1, \dots, i$, with $\bar{k}_{i+1} = \max\{\bar{k}_i, k_i\} = k_i$ and

$$h_1^{i+1}(\bar{k}_{i+1}) := h_1^i(\bar{k}_{i+1}) + b_1 + \frac{2b_4 r h_1^i(\bar{k}_{i+1}) \bar{k}_{i+1}^{N_i-2}}{b_2 k_1}, \quad h_2^{i+1}(\bar{k}_{i+1}) := h_2^i(\bar{k}_{i+1}) + \frac{r \bar{k}_{i+1}^{N_i-2}}{\epsilon}.$$

Moreover, noting that $|\bar{e}^i(t)| \leq k_i^{2-N_i} |z^i(t)|$, in view of (A3), we have

$$|\bar{e}^i(t)| \leq q_1^i(\bar{k}_i, k_i) e^{-\frac{b_2}{2} k_1 t} |z(0)| + \frac{\epsilon}{r} \left(|\bar{\zeta}^e|_{[0,t]} + |v|_{[0,t]} \right) \quad (\text{A5})$$

with $q_1^i(\bar{k}_i, k_i) := h_1^{i+1}(\bar{k}_i) k_i^{2-N_i}$ and $\bar{\zeta}^e := \bar{\zeta}_{[1,r_e]}$. Fix k_1 so that

$$k_1 \geq \max \left\{ k^*, \frac{b_3 r}{\epsilon} \right\}.$$

Since $e^{-b_2 k_1 t} \leq e^{-b_2 k_1 t/2}$ for all $k_1, t > 0$ and $\bar{k}_2 = k_1$, then (A1) implies (A2) for $j < 2$ with $h_1^2(\bar{k}_2) = b_1$ and $h_2^2(\bar{k}_2) = b_3 \bar{k}_2^{N_1-3}$ and (A5) for $i = 1$, with $q_1^1 := b_1 k_1^{2-N_1}$, $q_2^1 := b_2 k_1$. Hence the sequence (A4) is well-defined and by induction we conclude that, for each $i = 1, \dots, r$, it holds that

$$|z^i(t)| \leq \bar{h}_1(\bar{k}_{i+1}) e^{-\frac{b_2}{2} k_1 t} |z(0)| + \bar{h}_2(\bar{k}_{i+1}) \left(|\bar{\zeta}_{[1,i]}|_{[0,t]} + |v|_{[0,t]} \right)$$

with $\bar{h}_1 := h_1^{r+1}$, $\bar{h}_2 := h_2^{r+1}$, and for each $i = 1, \dots, r_e$, (A5) holds true. Noting that $|\chi^i(t)| \leq |z^i(t)|$, $|z(0)| \leq k^{N_r-1} |\chi(0)|$ and $|\bar{e}(t)| \leq \sum_{i=1}^{r_e} |\bar{e}^i(t)|$, then we obtain (29)-(30), with

$$\beta_\chi(s, \tau) := k^{N_r-1} \bar{h}_1(\bar{k}_{r+1}) \exp\left(-\frac{b_2 b_3 r \tau}{2}\right) s, \quad a_1 := k^{N_r-1} \bar{h}_2(\bar{k}_{r+1}), \quad \beta_\bar{e}(s, t/\epsilon) := s \sum_{i=1, \dots, r_e} q_1^i(\bar{k}_i, k_i) \exp\left(-\frac{b_2 b_3 r \tau}{2}\right)$$

and the claim of the lemma follows with

$$K := \text{diag} \left(k_1 D^1 \Lambda_1(k_1), \dots, k_r D^r \Lambda_r(k_r) \right).$$

□

B PROOF OF LEMMA 2

With \mathcal{P} given by Assumption 2, define the function

$$V(w, x) = \sqrt{\bar{\zeta}^\top \mathcal{P}(w, x) \bar{\zeta}} \quad (\text{B6})$$

By point a of Assumption 2 V satisfies

$$\sqrt{\underline{\lambda}} |\bar{\zeta}| \leq V(w, x) \leq \sqrt{\bar{\lambda}} |\bar{\zeta}|$$

for all $(w, x) \in \mathcal{W} \times \mathbb{R}^{n_x}$. Taking the Dini derivative of V along the solutions of the closed-loop system yields

$$\begin{aligned} D^+ V(w, x) &= \frac{1}{2V(w, x)} \left(-\ell \bar{\zeta}^\top \left(\mathcal{L}^\top \mathbf{B}(w, x)^\top \mathcal{P}(w, x) + \mathcal{P}(w, x) \mathbf{B}(w, x) \mathcal{L} \right) \bar{\zeta} \right. \\ &\quad \left. + 2\bar{\zeta}^\top \mathcal{P}(w, x) (\rho(\eta, \chi, \bar{\zeta}) + q(w, x)) \right. \\ &\quad \left. + \bar{\zeta} \left(L_s^{(w)} \mathcal{P}(w, x) + L_f^{(x)} \mathcal{P}(w, x) + L_{b(w,x)u}^{(x)} \mathcal{P}(w, x) \right) \bar{\zeta} \right). \end{aligned}$$

Point c of Assumption 2 implies

$$\bar{\zeta}^\top \left(\mathcal{L}^\top \mathbf{B}(w, x)^\top \mathcal{P}(w, x) + \mathcal{P}(w, x) \mathbf{B}(w, x) \mathcal{L} \right) \bar{\zeta} \geq |\bar{\zeta}|^2$$

so as

$$-\ell \bar{\zeta}^\top \left(\mathcal{L}^\top \mathbf{B}(w, x)^\top \mathcal{P}(w, x) + \mathcal{P}(w, x) \mathbf{B}(w, x) \mathcal{L} \right) \bar{\zeta} \leq -\ell |\bar{\zeta}|^2.$$

As \mathcal{P} is continuous, ρ is Lipschitz and q is Lipschitz on $\mathcal{W} \times \bar{\mathcal{X}}$, then there exists $M_1 > 0$ such that

$$2\bar{\zeta}^\top \mathcal{P}(w, x) (\rho(\eta, \chi, \bar{\zeta}) + q(w, x)) \leq M_1 |\bar{\zeta}| \left(|\bar{\zeta}| + |(w, x, \chi, \eta)| \right),$$

as long as $(w, x) \in \mathcal{W} \times \bar{\mathcal{X}}$. Point b of Assumption 2 implies that $L_{b(w,x)u}^{(x)} \mathcal{P}(w, x) = 0$, so as by continuity of \mathcal{P} , as long as $(w, x) \in \mathcal{W} \times \bar{\mathcal{X}}$, we can write

$$\bar{\zeta} \left(L_s^{(w)} \mathcal{P}(w, x) + L_f^{(x)} \mathcal{P}(w, x) + L_{b(w,x)u}^{(x)} \mathcal{P}(w, x) \right) \bar{\zeta} \leq M_2 |\bar{\zeta}|^2$$

for some $M_2 > 0$. Since

$$\frac{1}{\sqrt{\lambda}} \leq \frac{|\bar{\xi}|}{V(w, x)} \leq \frac{1}{\sqrt{\lambda}},$$

then there exists $\alpha_2 > 0$ such that, as long as $(w, x) \in W \times \bar{X}$, we have

$$D^+V(w, x) \leq (\alpha_2 - \ell \alpha_1)V(w, x) + \alpha_2|(w, x, \eta, \chi)|,$$

with $\alpha_1 := 1/(2\sqrt{\lambda})$. The result thus follows with $\beta_{\bar{\xi}}(s, \tau) := (s\sqrt{\lambda}/\sqrt{\lambda}) \exp(-\alpha_1 \tau/2)$ and $a_4 := 2\alpha_2/(\alpha_1\sqrt{\lambda})$ by taking $\ell_0^* := 2\alpha_2/\alpha_1$ and noticing that $\beta_{\bar{\xi}}$ does not depend on \bar{X} . \square

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