Persistent manifolds of the special Euclidean group SE(3): A review

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Highlights

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- A systematic review of the concept of persistent manifolds and their application to mechanism design.
- Classification of three classes of persistent manifolds, namely, the Lie subgroups, the product-of-exponential (POE) manifolds, and the symmetric subspaces.
- Ample examples illustrating the application of the three classes of persistent manifolds in mechanism design.

Persistent Manifolds of the special Euclidean group SE(3): a Review

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Abstract

Mechanisms and robots often share the following fundamental property: the instantaneous twist space generated by the end-effector at a generic pose is a rigidly-displaced copy of the one generated at the home configuration, i.e., the tangent spaces at all points of its motion manifold (a manifold of the Lie group of rigid displacements SE(3)) are mutually congruent. A manifold of this kind, hereafter denoted as persistent, can be seen as the envelope of a persistent twist subspace rigidly moving in SE(3). In this paper, we shall summarize three important classes of persistent manifolds that have so far been discovered and systematically investigated in the literature, namely the Lie subgroups, the persistent product-of-exponential (POE) manifolds, and the symmetric subspaces. In each case, the persistence property arises from a distinct manifold structure, which dictates the ensuing classification and underlies the framework for the synthesis of mechanical devices that are capable of generating such manifolds. In this regard, we attempt to offer a guideline to classification and mechanism synthesis of persistent manifolds for a general audience.

Keywords: persistent manifold, Lie subgroup, product-of-exponential manifold, symmetric subspace, mechanism design.

1. Introduction

A *robot* or a *mechanism* in general is an ensemble of rigid bodies, called *links*, interconnected by *kinematic pairs* or *joints*. The *base* link is fixed to the ground, whereas the *end-effector* interacts with the environment by exerting motion and forces through a change of *configuration* of the mechanism. A pose of the end-effector is unambiguously specified by six parameters, which accounts for the six dimensions (three for orientation and three for position) of the *special Euclidean group* SE(3), namely the Lie group of all rigid *displacement*s in the 3-D Euclidean space. Many practical tasks require less than six degrees of freedom (DOFs), so that the corresponding *task spaces* can be identified with manifolds of SE(3). We shall henceforth refer to manifolds of SE(3) (containing the identity transformation $\mathbf{I}$) as *motion manifolds*.

In engineering applications, it is of equal importance to study the vector fields (or tangent spaces) of motion manifolds as local linear approximations of the latter, corresponding to instantaneous motions or *twists* of the end-effector. Moreover, the instantaneous motion is related, by the principle of virtual work, to the forces and moments (collectively referred to as *wrenches*) that the end-effector may exert. Since the instantaneous motion can be studied by the simple tools of linear algebra, there is a long-lasting aspiration of engineers to describe (and design) the finite end-effector motion through the instantaneous one. However, any linearization is necessarily local and configuration dependent. The extrapolation from local to general, from infinitesimal to finite, is a challenging task, which we wish to address in this contribution.

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From an engineering perspective, the study of motion manifolds can arguably be organized into the following three stages. *First*, the properties characterizing the motion manifolds in consideration are identified from task-specific functional requirements. *Second*, the collection of all motion manifolds that share the established properties are classified. *Third*, mechanisms that are able to generate the desired motion manifolds are synthesized. We shall henceforth speak of the *characterization*, *classification* and *synthesis* stage. Depending on the scope of characterization and classification, the task of mechanism synthesis may exhibit varying difficulties.

Mechanisms in practice often share the following fundamental property: the instantaneous twist subspace generated by the end-effector at a generic pose is a rigidly-displaced copy of the one generated at the home configuration, i.e., the tangent spaces at all points of its motion manifold are mutually congruent. A *manifold* of this kind, hereafter denoted as *persistent*, can be seen as the *envelope* of a *persistent twist subspace* rigidly moving in SE(3). In this paper, we shall summarize, in a chronological order, three important classes of persistent manifolds that have so far been discovered and systematically investigated in the literature, namely the *Lie subgroups* [1], the *persistent product-of-exponential* (POE) *manifolds* [2], and the *symmetric subspaces* [3]. In each case, the persistence property arises from a distinct manifold structure that dictates the ensuing classification and synthesis problem. In this regard, we attempt to offer a guideline to classification and mechanism synthesis of persistent manifolds for a general audience.

The paper is organized as follows. In Sect. 2, we give a brief review of the Lie group properties of the Euclidean group SE(3). Then we introduce in Sect. 3 the first class of persistent manifolds, namely the Lie subgroups of SE(3). In Sect. 4, we introduce, as a generalization to Lie subgroups, the persistent POE manifolds. Finally, in Sect. 5, we introduce the symmetric subspaces of SE(3). The paper concludes with a summary of the three classes of persistent motion manifolds.

2. Review of SE(3)

2.1. Lie group of rigid-body displacements SE(3)

As illustrated in Fig. 1(a), given a coordinate system a attached to the inertial or reference frame, and another coordinate system b attached to a freely movable rigid body, the rigid-body or Euclidean transformation $\mathbf{g}_{ab}$ of b with respect to a can be described by a 4×4 homogeneous matrix $\mathbf{g}_{ab}$:

$$\mathbf{g}_{ab} \triangleq \begin{pmatrix} \mathbf{R}_{ab} & \mathbf{p}_{ab} \\ \mathbf{0}^T & 1 \end{pmatrix} \in \mathbb{R}^{4\times 4} \tag{1}$$

where $\mathbf{R}_{ab}$ is a 3×3 proper orthonormal matrix and $\mathbf{p}_{ab}$ is a 3-D translation vector. For subsequent exposition, the subscript ab will simply be dropped. The set of all rigid-body transformations forms a 6-D Lie group called the *special Euclidean group* SE(3):

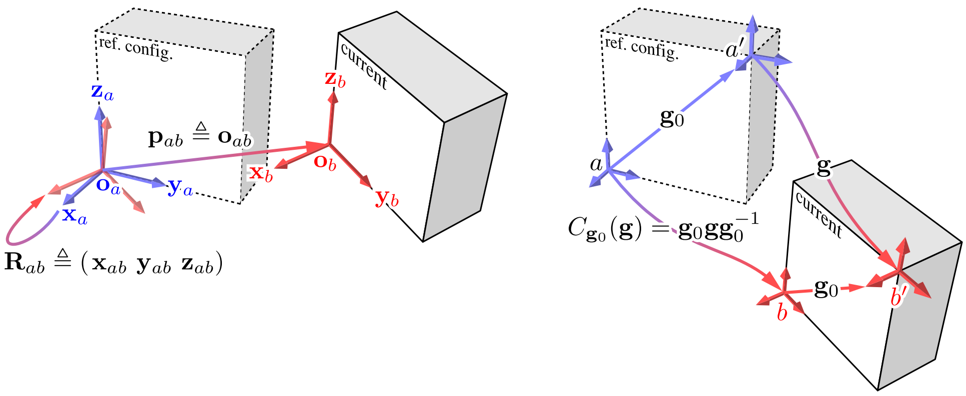
$$\mathrm{SE}(3) \triangleq \left\{ \mathbf{g} = \begin{pmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}^T & 1 \end{pmatrix} \in \mathbb{R}^{4\times 4} \;\middle|\; \mathbf{R} \in \mathrm{SO}(3), \mathbf{p} \in \mathbb{R}^3 \right\} \tag{2}$$

where SO(3) is the 3-D special orthogonal group. $\mathbf{g} \in \mathrm{SE}(3)$ transforms a point $\mathbf{q}_0 \in \mathbb{R}^3$ via its homogeneous coordinate $(\mathbf{q}_0^T, 1)^T$:

$$\mathbf{g}\begin{pmatrix} \mathbf{q}_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}^T & 1 \end{pmatrix}\begin{pmatrix} \mathbf{q}_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}\mathbf{q}_0 + \mathbf{p} \\ 1 \end{pmatrix} \tag{3}$$

Given $\mathbf{g}_0 \in \mathrm{SE}(3)$, we may define *left* and *right translation* on SE(3), denoted $L_{\mathbf{g}_0}$ and $R_{\mathbf{g}_0}$ respectively:

$$L_{\mathbf{g}_0}(\mathbf{g}) \triangleq \mathbf{g}_0\mathbf{g}, \quad R_{\mathbf{g}_0}(\mathbf{g}) \triangleq \mathbf{g}\mathbf{g}_0, \qquad \forall \mathbf{g} \in \mathrm{SE}(3) \tag{4}$$



(a) (b)

Figure 1: (a) Rigid-body transformation $\mathbf{g}_{ab}$ of body frame b with respect to reference frame a; (b) Conjugation of a rigid motion $\mathbf{g}$ due to change of reference frame and body frame by $\mathbf{g}_0$.

They correspond to a change of reference and body coordinate frame, respectively (Fig. 1). Similarly, we define the *conjugation* map $C_{\mathbf{g}_0}$ associated to a given $\mathbf{g}_0 \in \mathrm{SE}(3)$ as:

$$C_{\mathbf{g}_0}(\mathbf{g}) \triangleq \mathbf{g}_0\mathbf{g}\mathbf{g}_0^{-1} = L_{\mathbf{g}_0} \circ R_{\mathbf{g}_0^{-1}}(\mathbf{g}), \qquad \forall \mathbf{g} \in \mathrm{SE}(3) \tag{5}$$

We define a *rigid-body motion* (with respect to an unspoken reference frame) to be a trajectory $\mathbf{g}(t) \in \mathrm{SE}(3), t \in \mathbb{R}$, such that $\mathbf{g}(0) = \mathbf{I}$, i.e., the body frame coincides with the reference frame at the initial configuration. In general, we can nullify the initial coordinate transformation $\mathbf{g}(0) \neq \mathbf{I}$ by considering the right-translated trajectory $\mathbf{g}(t)\mathbf{g}(0)^{-1}$. As illustrated in Fig. 1(b), a motion $\mathbf{g}(t)$ as observed in a reference frame a' becomes $C_{\mathbf{g}_0}(\mathbf{g}(t))$ when observed in a reference frame a with $\mathbf{g}_{aa'} = \mathbf{g}_0$; $C_{\mathbf{g}_0}(\mathbf{g}(t))$ may as well be considered a rigidly displaced version of $\mathbf{g}(t)$ observed from the same reference frame.

2.2. Lie algebra $\mathfrak{se}(3)$ *of* SE(3)

The *Lie algebra* of SE(3), denoted $\mathfrak{se}(3)$, is its tangent space at the identity element $\mathbf{I}$:

$$\mathfrak{se}(3) \triangleq T_{\mathbf{I}}(\mathrm{SE}(3)) = \left\{ \boldsymbol{\xi} \triangleq \begin{pmatrix} \widehat{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0}^T & 0 \end{pmatrix} \in \mathbb{R}^{4\times4} \;\middle|\; \boldsymbol{\omega}, \mathbf{v} \in \mathbb{R}^3 \right\} \tag{6}$$

where $\widehat{\boldsymbol{\omega}}$ is a 3×3 skew-symmetric matrix such that $\widehat{\boldsymbol{\omega}}\boldsymbol{\omega}' = \boldsymbol{\omega}\times\boldsymbol{\omega}', \forall\boldsymbol{\omega}' \in \mathbb{R}^3$. $\mathfrak{se}(3)$ is equipped with the following skew-symmetric bilinear *Lie bracket* operator:

$$[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] \triangleq \boldsymbol{\xi}_1\boldsymbol{\xi}_2 - \boldsymbol{\xi}_2\boldsymbol{\xi}_1, \qquad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathfrak{se}(3) \tag{7}$$

For any $\mathbf{g}_0 \in \mathrm{SE}(3)$, we define an *Adjoint transformation* $\mathrm{Ad}_{\mathbf{g}_0}$ to be the following linear transformation on $\mathfrak{se}(3)$:

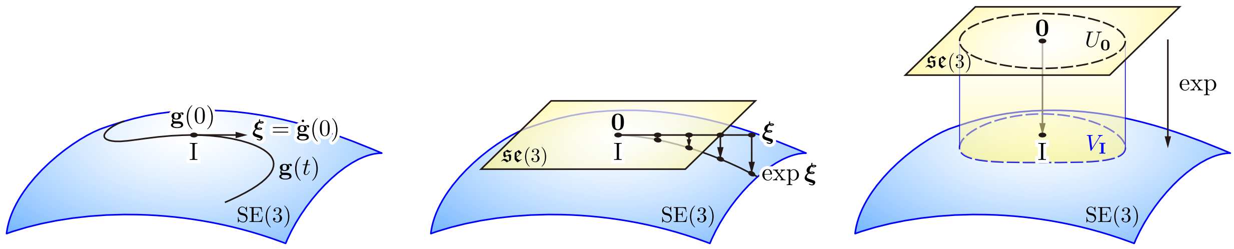
$$\mathrm{Ad}_{\mathbf{g}_0}(\boldsymbol{\xi}) \triangleq \mathbf{g}_0\boldsymbol{\xi}\mathbf{g}_0^{-1} = L_{\mathbf{g}_0} \circ R_{\mathbf{g}_0^{-1}}(\boldsymbol{\xi}), \qquad \forall \boldsymbol{\xi} \in \mathfrak{se}(3) \tag{8}$$

$\mathrm{Ad}_{\mathbf{g}_0}(\boldsymbol{\xi})$ corresponds to the observation of a twist $\boldsymbol{\xi}$ under changing reference frame or the observation of a displaced copy of $\boldsymbol{\xi}$ from the same frame. The tangent space of SE(3) at a generic point $\mathbf{g}$ is related to $\mathfrak{se}(3)$ via either left or right translation:

$$T_{\mathbf{g}}(\mathrm{SE}(3)) = \mathbf{g}\mathfrak{se}(3) = \mathfrak{se}(3)\mathbf{g} \tag{9}$$

which leads to the following *congruence-invariance property* of $\mathfrak{se}(3)$:

$$\mathrm{Ad}_{\mathbf{g}}(\mathfrak{se}(3)) = \mathbf{g}\mathfrak{se}(3)\mathbf{g}^{-1} = \mathfrak{se}(3), \qquad \forall \mathbf{g} \in \mathrm{SE}(3) \tag{10}$$



(a) (b) (c)

Figure 2: (a) A twist $\boldsymbol{\xi}$ as a tangent vector of a motion $\mathbf{g}(t)$ at the identity $\mathbf{I}$; (b) the exponential map; (c) the exponential map maps an open neighborhood $U_{\mathbf{0}}$ of $\mathbf{0} \in \mathfrak{se}(3)$ smoothly onto an open neighborhood $V_{\mathbf{I}}$ of $\mathbf{I} \in \mathrm{SE}(3)$.

Physically, $\boldsymbol{\xi} \in \mathfrak{se}(3)$ corresponds to the instantaneous velocity $\dot{\mathbf{g}}$ of a rigid motion $\mathbf{g}(t), t \in \mathbb{R}$:

$$\boldsymbol{\xi} = \dot{\mathbf{g}}(t)\mathbf{g}(t)^{-1} \tag{11}$$

and it is often called a *twist* (see Fig. 2(a)). The velocity $\dot{\mathbf{q}}(t)$ of a point $\mathbf{q}(t)$ on the moving body is given by:

$$\begin{aligned} \begin{pmatrix} \mathbf{q}(t) \\ 1 \end{pmatrix} &= \mathbf{g}(t) \begin{pmatrix} \mathbf{q}_0 \\ 1 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} \dot{\mathbf{q}}(t) \\ 0 \end{pmatrix} &= \dot{\mathbf{g}}(t) \begin{pmatrix} \mathbf{q}_0 \\ 1 \end{pmatrix} = \dot{\mathbf{g}}(t)\mathbf{g}(t)^{-1}\mathbf{g}(t) \begin{pmatrix} \mathbf{q}_0 \\ 1 \end{pmatrix} = \boldsymbol{\xi} \begin{pmatrix} \mathbf{q}(t) \\ 1 \end{pmatrix} \end{aligned} \tag{12}$$

where $\mathbf{q}_0 = \mathbf{q}(0)$. A vector subspace $\mathfrak{T}$ of $\mathfrak{se}(3)$ is often called a *twist subspace*.

$\mathfrak{se}(3)$ can be integrated back into SE(3) via the *exponential map* $\exp : \mathfrak{se}(3) \to \mathrm{SE}(3)$ (see Fig. 2(b)):

$$\exp(\boldsymbol{\xi}) \triangleq e^{\boldsymbol{\xi}} = \sum_{k=0}^{\infty} \frac{\boldsymbol{\xi}^k}{k!} = \begin{cases} \begin{pmatrix} e^{\widehat{\boldsymbol{\omega}}} & (\mathbf{I} - e^{\widehat{\boldsymbol{\omega}}})\frac{1}{\|\boldsymbol{\omega}\|^2}\widehat{\boldsymbol{\omega}}\mathbf{v} + \frac{\boldsymbol{\omega}^T\mathbf{v}}{\|\boldsymbol{\omega}\|^2}\boldsymbol{\omega} \\ \mathbf{0}^T & 1 \end{pmatrix} & \|\boldsymbol{\omega}\| \neq 0 \\ \begin{pmatrix} \mathbf{I}_{3\times3} & \mathbf{v} \\ \mathbf{0}^T & 1 \end{pmatrix} & \|\boldsymbol{\omega}\| = 0 \end{cases} \tag{13}$$

where

$$e^{\widehat{\boldsymbol{\omega}}} = \mathbf{I}_{3\times3} + \tfrac{\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\widehat{\boldsymbol{\omega}} + \tfrac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}\widehat{\boldsymbol{\omega}}^2, \qquad \|\boldsymbol{\omega}\| \neq 0 \tag{14}$$

The exponential map maps an open neighborhood $U_{\mathbf{0}}$ of the zero matrix $\mathbf{0} \in \mathfrak{se}(3)$ onto an open neighborhood $V_{\mathbf{I}}$ of $\mathbf{I} \in \mathrm{SE}(3)$ (see Fig. 2(c)). Given a basis $(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_6)$ of $\mathfrak{se}(3)$, the exponential map leads to the following two parameterizations of $V_{\mathbf{I}}$ [4]:

$$\exp_1 : (\theta_1, \ldots, \theta_6) \mapsto \exp(\theta_1\boldsymbol{\xi}_1 + \cdots + \theta_6\boldsymbol{\xi}_6) \tag{15}$$

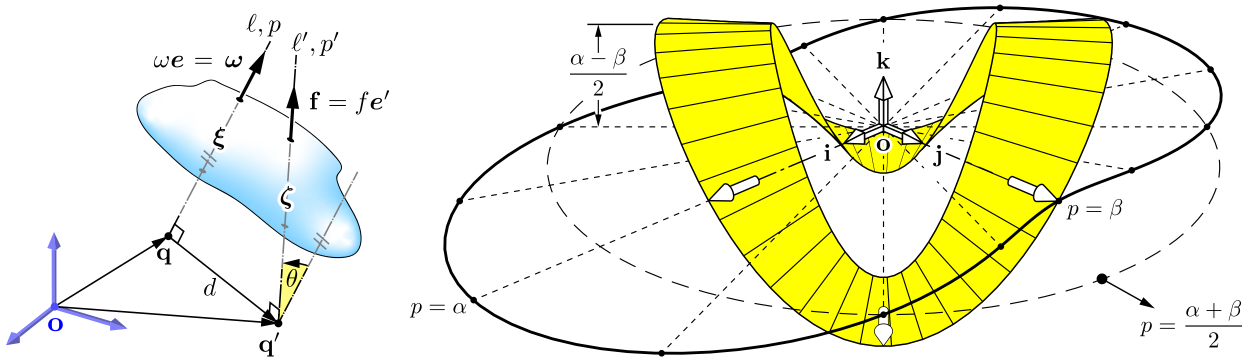
$$\exp_2 : (\theta_1, \ldots, \theta_6) \mapsto \exp(\theta_1\boldsymbol{\xi}_1) \cdots \exp(\theta_6\boldsymbol{\xi}_6) \tag{16}$$

The coordinates $\boldsymbol{\theta} \triangleq (\theta_1, \ldots, \theta_6)$ in Eq. (15) and Eq. (16) are referred to as *canonical coordinates* of the *first* and *second kind* respectively [4]. Eq. (16) is often referred to as the *product-of-exponential* (POE) formula [5].

For any $\boldsymbol{\xi} \in \mathfrak{se}(3)$, we define an *adjoint map* $\mathrm{ad}_{\boldsymbol{\xi}}$ to be the following linear map on $\mathfrak{se}(3)$:

$$\mathrm{ad}_{\boldsymbol{\xi}}(\boldsymbol{\xi}') \triangleq \frac{d}{dt}\left(\mathrm{Ad}_{e^{t\boldsymbol{\xi}}}(\boldsymbol{\xi}')\right)\Big|_{t=0} = [\boldsymbol{\xi}, \boldsymbol{\xi}'], \qquad \forall \boldsymbol{\xi}' \in \mathfrak{se}(3) \tag{17}$$

which gives the rate of the Adjoint transformation of $\boldsymbol{\xi}'$ along $\boldsymbol{\xi}$.



(a) (b)

Figure 3: (a) A wrench $\boldsymbol{\zeta}$ acting on a body moving with twist $\boldsymbol{\xi}$; (b) geometry of the cylindroid.

2.3. Screw coordinates of twists and wrenches

In mechanism analysis and synthesis, a twist $\boldsymbol{\xi} \in \mathfrak{se}(3)$ as shown in Eq. (6) is often identified with a pair of 3-D vectors $(\boldsymbol{\omega}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$, and interpreted as the instantaneous velocity of a twisting or screwing motion about an axis ℓ with pitch p and magnitude ω. The Plücker-coordinate representation of $\boldsymbol{\xi}$ is accordingly:

$$\boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = \omega \mathbf{S} \triangleq \omega \begin{pmatrix} \boldsymbol{e} \\ \mathbf{q} \times \boldsymbol{e} + p\boldsymbol{e} \end{pmatrix} \quad \Leftrightarrow \quad \begin{cases} \omega = \|\boldsymbol{\omega}\|, & p = \frac{\boldsymbol{\omega}^T \mathbf{v}}{\|\boldsymbol{\omega}\|^2} \\ \boldsymbol{e} = \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}, & \mathbf{q} = \frac{1}{\|\boldsymbol{\omega}\|^2} \widehat{\boldsymbol{\omega}} \mathbf{v} \end{cases} \tag{18}$$

where $\boldsymbol{\omega}$ is the angular velocity about ℓ, $\mathbf{q} \in \mathbb{R}^3$ is a point on ℓ, and $\mathbf{v}$ is the linear velocity of a reference point. A twist of the form $(\mathbf{0}, \mathbf{v})$ is said to have an infinite-pitch; it has a direction, but not an axis. An *n-system of screws* is a projective space underlying a twist subspace of dimension n. When there is no danger of ambiguity, the term can refer to the twist subspace itself. A screw $\mathbf{S}$ may also be used to characterize a generalized force called *wrench* and denoted $\boldsymbol{\zeta}$, which is simply a vector in the dual space $\mathfrak{se}(3)^*$ of $\mathfrak{se}(3)$:

$$\boldsymbol{\zeta} = \begin{pmatrix} \mathbf{f} \\ \mathbf{m} \end{pmatrix} = f\mathbf{S} \triangleq f \begin{pmatrix} \boldsymbol{e}' \\ \mathbf{q} \times \boldsymbol{e}' + p\boldsymbol{e}' \end{pmatrix} \tag{19}$$

where $\mathbf{f}$ and $\mathbf{m}$ are respectively its force and moment vector along the screw axis of $\mathbf{S}$. Unlike $\mathfrak{se}(3)$, $\mathfrak{se}(3)^*$ is not equipped with a Lie algebra structure. A vector subspace of $\mathfrak{se}(3)^*$ is often called a *wrench subspace*.

The power developed by a wrench $\boldsymbol{\zeta}$ acting on a body moving with twist $\boldsymbol{\xi}$ is given by the *reciprocal product* between $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}$:

$$\boldsymbol{\xi} \odot \boldsymbol{\zeta} \triangleq \omega f \mathbf{S} \odot \mathbf{S}' \triangleq \mathbf{v}^T \mathbf{f} + \boldsymbol{\omega}^T \mathbf{m} = \omega f ((p + p') \cos\theta - d \sin\theta) \tag{20}$$

where θ is the signed angle formed by $\boldsymbol{e}$ and $\boldsymbol{e}'$, and d is the signed distance between the two screw axes (see Fig. 3(a)). Two screws are said to be *reciprocal* if their reciprocal product is zero. Given a twist subspace denoted $\mathfrak{T}$, we define its reciprocal wrench subspace denoted $\mathfrak{W}$ to be:

$$\mathfrak{W} \triangleq \mathfrak{T}^{\perp} = \{\boldsymbol{\zeta} \in \mathfrak{se}(3)^* \mid \boldsymbol{\xi} \odot \boldsymbol{\zeta} = 0, \forall \boldsymbol{\xi} \in \mathfrak{T}\} \tag{21}$$

2.4. Screw systems

A classification, up to an Adjoint transformation, of all screw systems of $\mathfrak{se}(3)$ was first obtained by Hunt [6] (and later refined in [7, 8]). We first introduce the following basis twists:

$$\begin{aligned} \boldsymbol{r}_{\mathrm{x}} &\triangleq \begin{pmatrix} \widehat{\mathbf{x}} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix}_{4\times 4}, & \boldsymbol{r}_{\mathrm{y}} &\triangleq \begin{pmatrix} \widehat{\mathbf{y}} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix}_{4\times 4}, & \boldsymbol{r}_{\mathrm{z}} &\triangleq \begin{pmatrix} \widehat{\mathbf{z}} & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix}_{4\times 4}, \\ \boldsymbol{t}_{\mathrm{x}} &\triangleq \begin{pmatrix} \widehat{\mathbf{0}} & \mathbf{x} \\ \mathbf{0}^T & 0 \end{pmatrix}_{4\times 4}, & \boldsymbol{t}_{\mathrm{y}} &\triangleq \begin{pmatrix} \widehat{\mathbf{0}} & \mathbf{y} \\ \mathbf{0}^T & 0 \end{pmatrix}_{4\times 4}, & \boldsymbol{t}_{\mathrm{z}} &\triangleq \begin{pmatrix} \widehat{\mathbf{0}} & \mathbf{z} \\ \mathbf{0}^T & 0 \end{pmatrix}_{4\times 4} \end{aligned} \tag{22}$$

where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are a canonical basis of $\mathbb{R}^3$. $\boldsymbol{r}_\mathrm{x}, \boldsymbol{r}_\mathrm{y}, \boldsymbol{r}_\mathrm{z}$ represent unit rotation twists along $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ axes; $\boldsymbol{t}_\mathrm{x}, \boldsymbol{t}_\mathrm{y}, \boldsymbol{t}_\mathrm{z}$ represent unit translation twists along $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ directions. We also define p-pitch twists along the coordinate axes:

$$\mathbf{i}_p \triangleq \boldsymbol{r}_\mathrm{x} + p\boldsymbol{t}_\mathrm{x}, \qquad \mathbf{j}_p \triangleq \boldsymbol{r}_\mathrm{y} + p\boldsymbol{t}_\mathrm{y}, \qquad \mathbf{k}_p \triangleq \boldsymbol{r}_\mathrm{z} + p\boldsymbol{t}_\mathrm{z} \tag{23}$$

A *general* one-system, denoted $S_{1,g}$, is spanned by a single screw $\mathbf{S} \in \mathfrak{se}(3)$, denoted $S_{1,g} = \{\mathbf{S}\}_\mathrm{sp}$. When $\mathbf{S}$ has infinite pitch, we have the first special one-system $S_{1,1}$.

A *general* two-system, denoted $S_{2,g}$, has its screw axes lying on a cylindroid, thus all intersecting a single axis at right-angle [6]. The intersection point between the screw and the cylindroid axis, and the screw pitch, are prescribed by the circle diagram, from which two screws that perpendicularly intersect at the mid-way of the cylindroid are shown to have the maximal and minimal pitch, denoted α and β respectively (see Fig. 3(b)). By a suitable Adjoint transformation, the basis screws may be brought to align with the $\mathbf{x}$ and $\mathbf{y}$-axis respectively:

$$S_{2,g} \triangleq \{\mathbf{i}_\alpha, \mathbf{j}_\beta\}_\mathrm{sp} \tag{24}$$

For any $p \in (\beta, \alpha)$, there exists exactly two screws in the system having pitch value p. Aside from a general two-system, Hunt summarized a total of five special two-systems [6], which we denote by $S_{2,i}, i = 1, \ldots, 5$, and list in Tab. 1.

Similarly, a *general* three-system, denoted $S_{3,g}$ is spanned by three mutually orthogonal and intersecting basis screws with pitch value $\alpha > \beta > \gamma$. By a suitable Adjoint transformation, the basis screws may be brought to align with the $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ axes, respectively:

$$S_{3,g} \triangleq \{\mathbf{i}_\alpha, \mathbf{j}_\beta, \mathbf{k}_\gamma\}_\mathrm{sp} \tag{25}$$

For any $p \in (\gamma, \beta) \cup (\beta, \alpha)$, there exists a one-parameter family of screws in the system having pitch value p; this family forms one regulus on a hyperboloid of one sheet. There is exactly one screw, namely the basis screw $\mathbf{i}_\alpha$ (or $\mathbf{k}_\gamma$), that has a pitch value α (or γ). The screws with pitch value β lie on a degenerate regulus on two planes. Aside from a general two-system, Hunt summarized a total of ten special three-systems [6], $S_{3,i}, i = 1, \ldots, 10$; see Tab. 1.

The geometry of four-systems and five-systems are determined, respectively, by their reciprocal two-systems and one-systems; their classifications follow that of two-systems and one-systems [6]. There are a total of six classes of four-systems, namely the general four-system $S_{4,g}$ and the five special four-systems $S_{4,i}, i = 1, \ldots, 5$, and two classes of five-systems; see Tab. 1.

3. Lie subgroups of SE(3)

A manifold G of SE(3) is called a *Lie subgroup* if it is closed under group multiplication and inverse, i.e., for any $\mathbf{h}, \mathbf{h}' \in \mathrm{G}$,

$$\mathbf{h}\mathbf{h}' \in \mathrm{G}, \qquad \mathbf{h}^{-1} \in \mathrm{G} \tag{26}$$

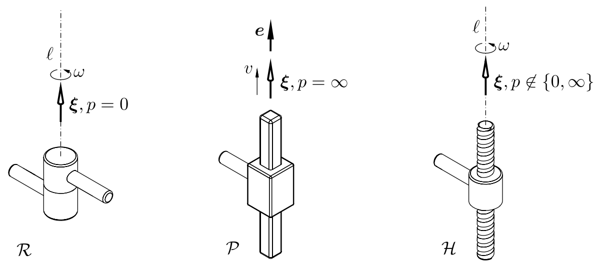
The Lie algebra $\mathfrak{g} \triangleq T_\mathbf{I}(\mathrm{G})$ of G becomes a twist subspace in $\mathfrak{se}(3)$ that is closed under the Lie bracket:

$$[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] \in \mathfrak{g}, \qquad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathfrak{g} \tag{27}$$

and is called a *Lie subalgebra* of $\mathfrak{se}(3)$.

Table 1: A complete classification of screw systems of $\mathfrak{se}(3)$.

dim	screw system	reciprocal sys.
1	$S_{1,g} \triangleq \{\mathbf{i}_\alpha\}_{\mathrm{sp}}$	$S_{5,g}$
	$S_{1,1} \triangleq \{\boldsymbol{t}_{\mathrm{x}}\}_{\mathrm{sp}}$	$S_{5,1}$
2	$S_{2,g} \triangleq \{\mathbf{i}_\alpha, \mathbf{j}_\beta\}_{\mathrm{sp}}$	$S_{4,g}$
	$S_{2,1} \triangleq \{\mathbf{i}_\alpha, \mathbf{j}_\alpha\}_{\mathrm{sp}}$	$S_{4,1}$
	$S_{2,2} \triangleq \{\mathbf{i}_\alpha, \boldsymbol{t}_{\mathrm{y}}\}_{\mathrm{sp}}$	$S_{4,2}$
	$S_{2,3} \triangleq \{\boldsymbol{t}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{y}}\}_{\mathrm{sp}}$	$S_{4,3}$
	$S_{2,4} \triangleq \{\mathbf{i}_\alpha, \boldsymbol{t}_{\mathrm{x}} \cos\zeta + \boldsymbol{t}_{\mathrm{y}} \sin\zeta\}_{\mathrm{sp}}, \zeta \in (0, \pi/2)$	$S_{4,4}$
	$S_{2,5} \triangleq \{\boldsymbol{r}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{x}}\}_{\mathrm{sp}}$	$S_{4,5}$
3	$S_{3,g} \triangleq \{\mathbf{i}_\alpha, \mathbf{j}_\beta, \mathbf{k}_\gamma\}_{\mathrm{sp}}$	$S_{3,g}$
	$S_{3,1} \triangleq \{\mathbf{i}_\alpha, \mathbf{j}_\alpha, \mathbf{k}_\gamma\}_{\mathrm{sp}}$	$S_{3,1}$
	$S_{3,2} \triangleq \{\mathbf{i}_\alpha, \mathbf{j}_\alpha, \mathbf{k}_\alpha\}_{\mathrm{sp}}$	$S_{3,2}$
	$S_{3,3} \triangleq \{\mathbf{i}_\alpha, \mathbf{j}_\beta, \boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}}$	$S_{3,3}$
	$S_{3,4} \triangleq \{\mathbf{i}_\alpha, \mathbf{j}_\alpha, \boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}}$	$S_{3,4}$
	$S_{3,5} \triangleq \{\mathbf{i}_\alpha, \boldsymbol{t}_{\mathrm{y}}, \boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}}$	$S_{3,5}$
	$S_{3,6} \triangleq \{\boldsymbol{t}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{y}}, \boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}}$	$S_{3,6}$
	$S_{3,7} \triangleq \{\mathbf{i}_\alpha, \mathbf{j}_\beta, \boldsymbol{t}_{\mathrm{x}} \cos\zeta + \boldsymbol{t}_{\mathrm{z}} \sin\zeta\}_{\mathrm{sp}}, \zeta \in (0, \frac{\pi}{2})$	$S_{3,7}$
	$S_{3,8} \triangleq \{\boldsymbol{r}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{x}}, \mathbf{j}_\beta\}_{\mathrm{sp}}$	$S_{3,8}$
	$S_{3,9} \triangleq \{\boldsymbol{t}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{y}}, \mathbf{i}_\alpha \cos\zeta + \mathbf{k}_\alpha \sin\zeta\}_{\mathrm{sp}}, \zeta \in (0, \frac{\pi}{2})$	$S_{3,9}$
	$S_{3,10} \triangleq \{\boldsymbol{r}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{y}}\}_{\mathrm{sp}}$	$S_{3,10}$
4	$S_{4,g} \triangleq \{\mathbf{i}_\alpha, \mathbf{j}_\beta, \boldsymbol{r}_{\mathrm{z}}, \boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}}$	$S_{2,g}$
	$S_{4,1} \triangleq \{\mathbf{i}_\alpha, \mathbf{j}_\alpha, \boldsymbol{r}_{\mathrm{z}}, \boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}}$	$S_{2,1}$
	$S_{4,2} \triangleq \{\mathbf{i}_\alpha, \boldsymbol{t}_{\mathrm{y}}, \boldsymbol{r}_{\mathrm{z}}, \boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}}$	$S_{2,2}$
	$S_{4,3} \triangleq \{\boldsymbol{t}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{y}}, \boldsymbol{r}_{\mathrm{z}}, \boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}}$	$S_{2,3}$
	$S_{4,4} \triangleq \{\mathbf{i}_\alpha, \boldsymbol{t}_{\mathrm{x}} \cos\zeta + \boldsymbol{t}_{\mathrm{y}} \sin\zeta, \boldsymbol{r}_{\mathrm{z}}, \boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}}, \zeta \in (0, \frac{\pi}{2})$	$S_{2,4}$
	$S_{4,5} \triangleq \{\boldsymbol{r}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{x}}, \boldsymbol{r}_{\mathrm{z}}, \boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}}$	$S_{2,5}$
5	$S_{5,g} \triangleq \{\mathbf{i}_\alpha, \boldsymbol{r}_{\mathrm{y}}, \boldsymbol{t}_{\mathrm{y}}, \boldsymbol{r}_{\mathrm{z}}, \boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}}$	$S_{1,g}$
	$S_{5,1} \triangleq \{\boldsymbol{t}_{\mathrm{x}}, \boldsymbol{r}_{\mathrm{y}}, \boldsymbol{t}_{\mathrm{y}}, \boldsymbol{r}_{\mathrm{z}}, \boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}}$	$S_{1,1}$



(a) (b) (c)

Figure 4: Illustration of 1-D Lie subgroup generators: (a) the $\mathcal{R}$ joint; (b) the $\mathcal{P}$ joint; (c) the $\mathcal{H}$ joint.

3.1. Classification of Lie subgroups of SE(3)

Lie's third theorem [4] states that there is a one-to-one correspondence, via the exponential map, between Lie subalgebras $\mathfrak{g}$ of $\mathfrak{se}(3)$ (up to an Adjoint transformation) and connected Lie subgroups G of SE(3) (up to conjugation). Moreover, each Lie subgroup G shares the same Lie group and Lie algebra properties as those of SE(3). The systematic classification of a total of ten classes of connected Lie subgroups of SE(3) was (probably) first studied in [9] based on a classification of Lie subalgebras of $\mathfrak{se}(3)$; see also [1]. The complete classification of Lie subgroups of SE(3) is now presented for quick reference.

1-D Lie subgroups of SE(3) are of the form $\exp(\{\boldsymbol{\xi}\}_{\mathrm{sp}})$ for some twist $\boldsymbol{\xi} \in \mathfrak{se}(3)$. After bringing $\boldsymbol{\xi}$ to lie on the $\mathbf{z}$-axis by a suitable Adjoint transformation, we have the following three conjugacy classes of 1-D Lie subgroups:

i) $\mathcal{R} \triangleq \exp(\{\boldsymbol{r}_{\mathrm{z}}\}_{\mathrm{sp}})$, the 1-D rotation group;

ii) $\mathcal{P} \triangleq \exp(\{\boldsymbol{t}_{\mathrm{z}}\}_{\mathrm{sp}})$, the 1-D translation group;

iii) $\mathcal{H}_p \triangleq \exp(\{\mathbf{k}_p\}_{\text{sp}})$, the 1-D helical group with pitch p.

Here, with a slight abuse of notation, we denote these 1-D Lie subgroups by their corresponding lower kinematic pair, namely, the revolute ($\mathcal{R}$), prismatic ($\mathcal{P}$) and helical ($\mathcal{H}$) joints, respectively (Fig. 4).

Following the Lie bracket relationships of the basis twists of $\mathfrak{se}(3)$:

$$\begin{array}{lllll} [\boldsymbol{t}_{\text{y}},\boldsymbol{t}_{\text{z}}]=\mathbf{0}, & [\boldsymbol{t}_{\text{x}},\boldsymbol{t}_{\text{y}}]=\mathbf{0}, & [\boldsymbol{t}_{\text{z}},\boldsymbol{t}_{\text{x}}]=\mathbf{0}, & [\boldsymbol{r}_{\text{z}},\boldsymbol{r}_{\text{x}}]=\boldsymbol{r}_{\text{y}}, & [\boldsymbol{r}_{\text{x}},\boldsymbol{r}_{\text{y}}]=\boldsymbol{r}_{\text{z}}, \\ [\boldsymbol{r}_{\text{y}},\boldsymbol{r}_{\text{z}}]=\boldsymbol{r}_{\text{x}}, & [\boldsymbol{r}_{\text{y}},\boldsymbol{t}_{\text{x}}]=-\boldsymbol{t}_{\text{z}}, & [\boldsymbol{r}_{\text{x}},\boldsymbol{t}_{\text{x}}]=\mathbf{0}, & [\boldsymbol{r}_{\text{z}},\boldsymbol{t}_{\text{x}}]=-\boldsymbol{t}_{\text{y}}, & [\boldsymbol{r}_{\text{x}},\boldsymbol{t}_{\text{y}}]=\boldsymbol{t}_{\text{z}}, \\ [\boldsymbol{r}_{\text{y}},\boldsymbol{t}_{\text{y}}]=\mathbf{0}, & [\boldsymbol{r}_{\text{z}},\boldsymbol{t}_{\text{y}}]=-\boldsymbol{t}_{\text{x}}, & [\boldsymbol{r}_{\text{x}},\boldsymbol{t}_{\text{z}}]=-\boldsymbol{t}_{\text{y}}, & [\boldsymbol{r}_{\text{y}},\boldsymbol{t}_{\text{z}}]=\boldsymbol{t}_{\text{x}}, & [\boldsymbol{r}_{\text{z}},\boldsymbol{t}_{\text{z}}]=\mathbf{0} \end{array} \tag{28}$$

and the fact that Lie bracket is a skew-symmetric bi-linear operator, one can directly verify that there are altogether two conjugacy classes of 2-D Lie subalgebras of $\mathfrak{se}(3)$, i.e., 2-D vector subspaces that are closed under the Lie bracket. Consequently, there are two classes of 2-D Lie subgroups of SE(3):

i) $\mathcal{T}_2 \triangleq \exp(\{\boldsymbol{t}_{\text{x}}, \boldsymbol{t}_{\text{y}}\}_{\text{sp}})$, the planar translation group.

ii) $\mathcal{C} \triangleq \exp(\{\boldsymbol{r}_{\text{z}}, \boldsymbol{t}_{\text{z}}\}_{\text{sp}})$, the cylindrical group, the motion manifold of a cylindrical joint.

It is also clear that $\exp(\{\boldsymbol{t}_{\text{x}}, \boldsymbol{t}_{\text{y}}\}_{\text{sp}})$ contains any 1-D translation group with axis lying in the $\mathbf{xy}$-plane, and that $\exp(\{\boldsymbol{r}_{\text{z}}, \boldsymbol{t}_{\text{z}}\}_{\text{sp}})$ contains any 1-D subgroup with axis being the $\mathbf{z}$-axis (Fig. 5 (a) and (b)).

Following a similar approach, we can verify that there are four classes of 3-D Lie subgroups of SE(3) (illustrated in Fig. 5 (c) to (f)):

i) $\mathcal{T}_3 \triangleq \exp(\{\boldsymbol{t}_{\text{x}}, \boldsymbol{t}_{\text{y}}, \boldsymbol{t}_{\text{z}}\}_{\text{sp}})$, the spatial translation algebra, which is the motion manifold of a 3-DOF gantry robot and the 3-DOF DELTA parallel robot;

ii) $\mathcal{E} \triangleq \exp(\{\boldsymbol{r}_{\text{z}}, \boldsymbol{t}_{\text{x}}, \boldsymbol{t}_{\text{y}}\}_{\text{sp}})$, the planar Euclidean group;

iii) $\mathcal{Y}_p \triangleq \exp(\{\mathbf{k}_p, \boldsymbol{t}_{\text{x}}, \boldsymbol{t}_{\text{y}}\}_{\text{sp}})$, the planar helical group with characteristic pitch p;

iv) $\mathcal{S} \triangleq \exp(\{\boldsymbol{r}_{\text{x}}, \boldsymbol{r}_{\text{y}}, \boldsymbol{r}_{\text{z}}\}_{\text{sp}})$, the spatial rotation group, which is the motion manifold of a spherical joint, or a satellite in space;

and there is only one class of 4-D Lie subgroup of SE(3), namely:

x) $\mathcal{X} \triangleq \exp(\{\boldsymbol{r}_{\text{z}}, \boldsymbol{t}_{\text{x}}, \boldsymbol{t}_{\text{y}}, \boldsymbol{t}_{\text{z}}\}_{\text{sp}})$

which is often referred to as the Schönflies group, and is the motion manifold of a 4-DOF pick-and-place or palletizer robot.

Finally, there is no 5-D Lie subalgebras of $\mathfrak{se}(3)$ and consequently no 5-D Lie subgroups of SE(3). A total of ten classes of connected Lie subgroups of SE(3) and their corresponding Lie subalgebras are listed in Tab. 2.

3.2. Mechanism synthesis for Lie subgroups of SE(3)

The screw representation of twists leads to the following mechanical interpretation of the POE formula in Eq. (16) for SE(3). The canonical coordinates $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_6)$ in Eq. (16) represent the joint variables of a kinematic chain or serial mechanism with six 1-DOF joints with joint screw axes $\mathbf{S}_i, i = 1, \ldots, 6$ (denoted $(\mathbf{S}_1, \ldots, \mathbf{S}_6)$); the POE formula prescribes the generation of SE(3) by the end-effector of the serial mechanism $(\mathbf{S}_1, \ldots, \mathbf{S}_6)$.

The above elaboration immediately leads to a synthesis procedure for serial mechanisms that generate a Lie subgroup G of SE(3). Given the desired k-D Lie subgroup $\mathrm{G} \subset \mathrm{SE}(3)$,

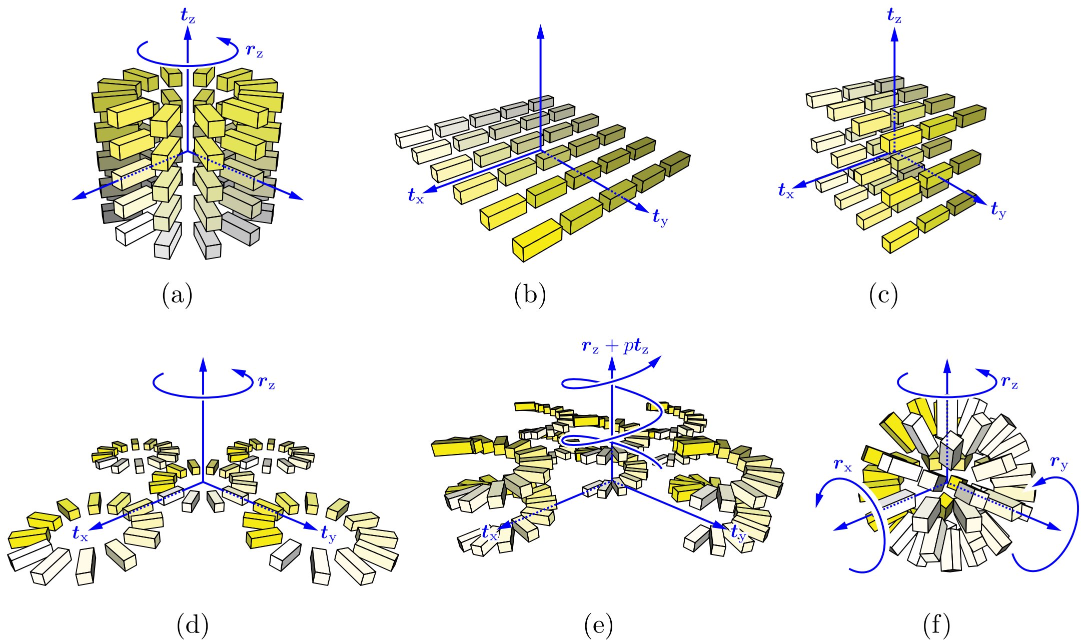


Figure 5: Traces of a rigid body undergoing motions of the: (a) cylindrical group $\mathcal{C}(\mathrm{z}) = \exp\{\boldsymbol{t}_\mathrm{z}, \boldsymbol{r}_\mathrm{z}\}_\mathrm{sp}$; (b) planar translation group $\mathcal{T}_2 = \exp(\{\boldsymbol{t}_\mathrm{x}, \boldsymbol{t}_\mathrm{y}\}_\mathrm{sp})$. (c) spatial translation group $\mathcal{T}_3 = \exp\{\boldsymbol{t}_\mathrm{x}, \boldsymbol{t}_\mathrm{y}, \boldsymbol{t}_\mathrm{z}\}_\mathrm{sp}$; (d) planar Euclidean group $\mathcal{E}(\mathrm{z}) = \exp(\{\boldsymbol{t}_\mathrm{x}, \boldsymbol{t}_\mathrm{y}, \boldsymbol{r}_\mathrm{z}\}_\mathrm{sp})$;(e) planar helical group $\mathcal{Y} = \exp(\{\boldsymbol{t}_\mathrm{x}, \boldsymbol{t}_\mathrm{y}, \mathbf{k}_p\}_\mathrm{sp})$; (f) spatial rotation group $\mathcal{S} = \exp(\{\boldsymbol{r}_\mathrm{x}, \boldsymbol{r}_\mathrm{y}, \boldsymbol{r}_\mathrm{z}\}_\mathrm{sp})$.

Table 2: A complete list of connected Lie subgroups of SE(3).

dim	Lie subalgebra	screw system	Lie subgroup	conjugate form	characteristic feature
1	$\mathfrak{r}_p \triangleq \{\mathbf{k}_p\}_\mathrm{sp}$	$S_{1,g}$	$\mathcal{H}_p \triangleq \exp(\mathfrak{r}_p)$	$\mathcal{H}_p(\ell)$	axis ℓ, pitch p
	$\mathfrak{r} \triangleq \{\boldsymbol{r}_\mathrm{z}\}_\mathrm{sp}$	$S_{1,g}$	$\mathcal{R} \triangleq \exp(\mathfrak{r})$	$\mathcal{R}(\ell)$	axis ℓ
	$\mathfrak{t}_1 \triangleq \{\boldsymbol{t}_\mathrm{z}\}_\mathrm{sp}$	$S_{1,1}$	$\mathcal{P} \triangleq \exp(\mathfrak{t}_1)$	$\mathcal{P}(\mathbf{v})$	direction $\mathbf{v}$
2	$\mathfrak{t}_2 \triangleq \{\boldsymbol{t}_\mathrm{x}, \boldsymbol{t}_\mathrm{y}\}_\mathrm{sp}$	$S_{2,3}$	$\mathcal{T}_2 \triangleq \exp(\mathfrak{t}_2)$	$\mathcal{T}_2(\mathbf{n})$	normal $\mathbf{n}$
	$\mathfrak{c} \triangleq \{\boldsymbol{r}_\mathrm{z}, \boldsymbol{t}_\mathrm{z}\}_\mathrm{sp}$	$S_{2,5}$	$\mathcal{C} \triangleq \exp(\mathfrak{c})$	$\mathcal{C}(\ell)$	axis ℓ
3	$\mathfrak{s} \triangleq \{\boldsymbol{r}_\mathrm{x}, \boldsymbol{r}_\mathrm{y}, \boldsymbol{r}_\mathrm{z}\}_\mathrm{sp}$	$S_{3,2}$	$\mathcal{S} \triangleq \exp(\mathfrak{s})$	$\mathcal{S}(\mathbf{p})$	center $\mathbf{p}$
	$\mathfrak{t}_3 \triangleq \{\boldsymbol{t}_\mathrm{x}, \boldsymbol{t}_\mathrm{y}, \boldsymbol{t}_\mathrm{z}\}_\mathrm{sp}$	$S_{3,6}$	$\mathcal{T}_3 \triangleq \exp(\mathfrak{t}_3)$	$\mathcal{T}_3$	—
	$\mathfrak{e} \triangleq \{\boldsymbol{t}_\mathrm{x}, \boldsymbol{t}_\mathrm{y}, \boldsymbol{r}_\mathrm{z}\}_\mathrm{sp}$	$S_{3,5}$	$\mathcal{E} \triangleq \exp(\mathfrak{e})$	$\mathcal{E}(\mathbf{n})$	normal $\mathbf{n}$,
	$\mathfrak{y}_p \triangleq \{\boldsymbol{t}_\mathrm{x}, \boldsymbol{t}_\mathrm{y}, \mathbf{k}_p\}_\mathrm{sp}$	$S_{3,5}$	$\mathcal{Y}_p \triangleq \exp(\mathfrak{y}_p)$	$\mathcal{Y}_p(\mathbf{n})$	normal $\mathbf{n}$, pitch p
4	$\mathfrak{x} \triangleq \{\boldsymbol{t}_\mathrm{x}, \boldsymbol{t}_\mathrm{y}, \boldsymbol{t}_\mathrm{z}, \boldsymbol{r}_\mathrm{z}\}_\mathrm{sp}$	$S_{4,3}$	$\mathcal{X} \triangleq \exp(\mathfrak{x})$	$\mathcal{X}(\mathbf{v})$	direction $\mathbf{v}$

we assign to its k-D Lie algebra $\mathfrak{g} \subset \mathfrak{se}(3)$ a basis $\{\mathbf{S}_i\}_{i=1}^k$. The resulting serial mechanism $(\mathbf{S}_1, \ldots, \mathbf{S}_k)$ generates G (locally) via the local diffeomorphism (Eq. (16) adapted to G):

$$\exp_2 : (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k \mapsto e^{\theta_1 \mathbf{S}_1} \cdots e^{\theta_k \mathbf{S}_k} \in \mathrm{G} \tag{29}$$

More generally, given a redundant generator $\{\mathbf{S}_i\}_{i=1}^r, r > k$ of $\mathfrak{g}$,

$$\exp_2 : (\theta_1, \ldots, \theta_r) \in \mathbb{R}^r \mapsto e^{\theta_1 \mathbf{S}_1} \cdots e^{\theta_r \mathbf{S}_r} \in \mathrm{G} \tag{30}$$

is a local submersion into G. By implicit function theorem [10], $\exp_2^{-1}(\mathbf{I})$, namely the configuration space of a closed-loop linkage formed by connecting the base and the end-effector of $(\mathbf{S}_1, \ldots, \mathbf{S}_r)$, is a manifold of $\mathbb{R}^r$ with dimension $r - k$. This serves as a generalization of the Grübler mobility formula; see the trivial linkages in [1].

In reference to Eq. (29), a Lie subgroup generator is exactly the same as a Lie subalgebra generator. Indeed, if we define

$$\mathbf{S}'_j \triangleq \mathrm{Ad}_{\exp(\theta_1 \mathbf{S}_1) \cdots \exp(\theta_{j-1} \mathbf{S}_{j-1})}(\mathbf{S}_j), j = 2, \ldots, k \tag{31}$$

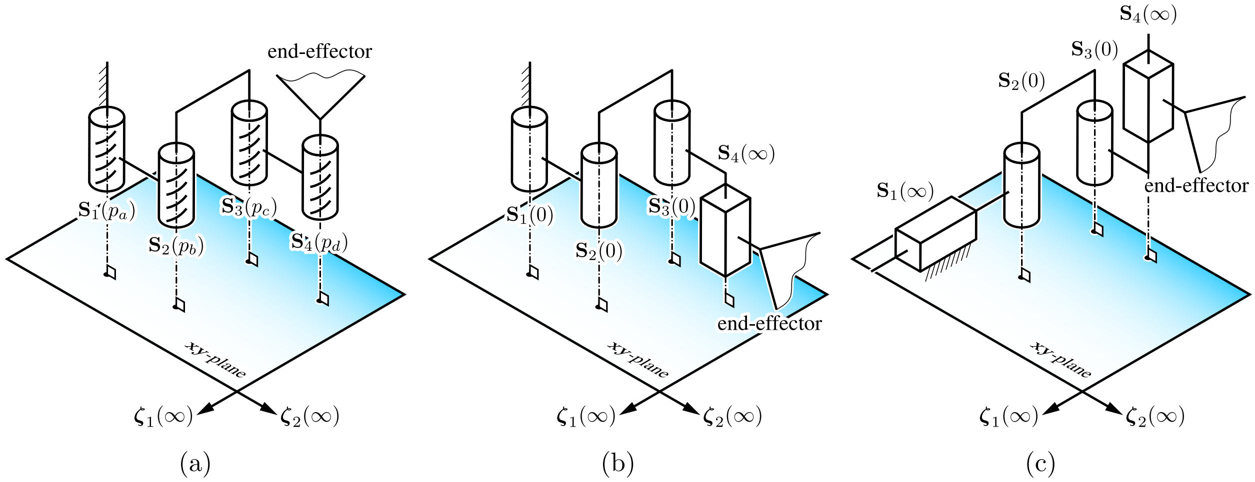


Figure 6: Several instances of serial mechanical generators for $\mathcal{X}$. (a) An $\mathcal{HHHH}$ chain; (b) An $\mathcal{RRRP}$ chain; (c) A $\mathcal{PRRP}$ chain.

for any non-singular configuration $(\theta_1, \ldots, \theta_k)$ of the serial mechanism $(\mathbf{S}_1, \ldots, \mathbf{S}_k)$, then

$$\{\mathbf{S}_1, \mathbf{S}_2', \ldots, \mathbf{S}_k'\}_{\text{sp}} = \{\mathbf{S}_1, \ldots, \mathbf{S}_k\}_{\text{sp}} = \mathfrak{g} \tag{32}$$

thanks to the invariance property $\text{Ad}_{\mathbf{g}}(\mathfrak{g}) = \mathfrak{g}, \forall \mathbf{g} \in \text{G}$, with $\mathfrak{se}(3)$ and SE(3) in Eq. (10) replaced by $\mathfrak{g}$ and G, respectively. Conversely, if Eq. (10) is satisfied for any $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$ in a neighborhood of $\mathbf{0} \in \mathbb{R}^k$, we have:

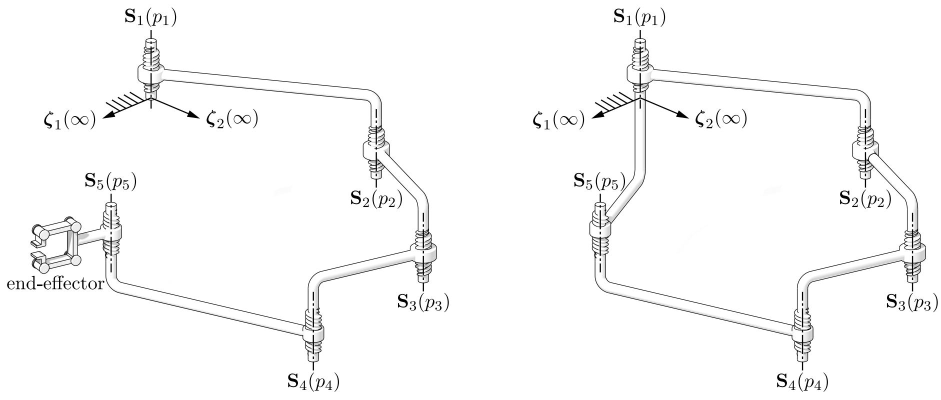
$$[\mathbf{S}_i, \mathbf{S}_j] = \frac{d}{d\theta_i} \left(\text{Ad}_{e^{\theta_i \mathbf{S}_i}}(\mathbf{S}_j)\right)\bigg|_{\theta_i = 0} \in \mathfrak{g} \tag{33}$$

where we assume $i < j$ and that $\theta_l = 0, l = 2, \ldots, i-1, i+1, \ldots, j-1$ in Eq. (31). This shows that $\mathfrak{g}$ and G are necessarily a Lie subalgebra and its corresponding Lie subgroup. Lie subalgebra generators were known to Hunt [6] as kinematic chains that admit full-cycle mobility. Hervé was probably the first to point out their equivalence to Lie subgroup generators [11]. Due to Eq. (32), Lie subalgebra generators are also said to generate *invariant screw systems* [2], since the end-effector twist space (and thus the corresponding screw system) does not vary as the chain moves.

Example 1 (Serial mechanism for the Schönflies group $\mathcal{X}$)**.** Recall that the Schönflies group $\mathcal{X}$ is a 4-D Lie subgroup of SE(3), with Lie algebra $\mathfrak{x} = \{\boldsymbol{t}_\text{x}, \boldsymbol{t}_\text{y}, \boldsymbol{t}_\text{z}, \boldsymbol{r}_\text{z}\}_{\text{sp}}$. To synthesize a kinematic chain for $\mathcal{X}$, we specify a basis $(\mathbf{S}_1, \ldots, \mathbf{S}_4)$ of $\mathfrak{x}$ as the joint screws:

$$\begin{pmatrix}\mathbf{S}_1 & \mathbf{S}_2 & \mathbf{S}_3 & \mathbf{S}_4\end{pmatrix} = \begin{pmatrix}\boldsymbol{t}_\text{x} & \boldsymbol{t}_\text{y} & \boldsymbol{t}_\text{z} & \boldsymbol{r}_\text{z}\end{pmatrix} \cdot \mathbf{A} \tag{34}$$

where $\mathbf{A}$ is a non-singular 4×4 co-efficient matrix. It can be verified that we need at least one screw $\mathbf{S}_i$ with non-infinite pitch parallel to the $\mathbf{z}$-axis. Several possible choices are illustrated in Fig. 6. In Fig. 6(a), the four screws $\mathbf{S}_1, \ldots, \mathbf{S}_4$ are all parallel to the $\mathbf{z}$-axis, with finite pitch p_a, p_b, p_c and p_d not all equal to the same value; the four screw axes should not lie in a common plane. In Fig. 6(b), the basis comprises three zero-pitch screws $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$, and one infinite-pitch screw $\mathbf{S}_4$ parallel to the $\mathbf{z}$-axis; the three zero-pitch screws should not lie in a common plane. In Fig. 6(c), the basis comprises an infinite-pitch screw $\mathbf{S}_1$ lying in the $\mathbf{xy}$-plane, two zero-pitch screws $\mathbf{S}_2$ and $\mathbf{S}_3$, and one infinite-pitch screw $\mathbf{S}_4$ parallel to the $\mathbf{z}$-axis; $\mathbf{S}_1$ should not be perpendicular to the plane defined by the axes of $\mathbf{S}_2$ and $\mathbf{S}_3$. The corresponding serial mechanisms are illustrated in Fig. 6. ♢



(a) (b)

Figure 7: (a) A redundant $\mathcal{H}\mathcal{H}\mathcal{H}\mathcal{H}\mathcal{H}$ chain generating the Schönflies group $\mathcal{X}$; (b) A 1-DOF linkage resulting from closing the loop of the $\mathcal{H}\mathcal{H}\mathcal{H}\mathcal{H}\mathcal{H}$ chain.

Example 2 (Redundant chain for Schönflies motion)**.** Continuing on Example 1, consider a chain $(\mathbf{S}_1, \ldots, \mathbf{S}_5)$ with $\mathbf{S}_i \in \mathfrak{x}, i = 1, \ldots, 5$ and satisfying

$$\{\mathbf{S}_1, \ldots, \mathbf{S}_5\}_{\text{sp}} = \mathfrak{x} \tag{35}$$

One such kinematic chain is shown in Fig. 7(a), where $\mathbf{S}_i, i = 1, \ldots, 5$ are all parallel to the $\mathbf{z}$-axis and have distinct pitches $p_1, \ldots, p_5$. In light of the discussion after Eq. (30), $(\mathbf{S}_1, \ldots, \mathbf{S}_5)$ is a 5-DOF kinematic chain that redundantly generates the 4-D Schönflies group $\mathcal{X}$; by locking its end-effector to the base, as shown in Fig. 7(b), we obtain a movable closed chain with mobility $5 - 4 = 1$ [12]. $\diamondsuit$

Example 3 (Parallel mechanism for the spatial translation group $\mathcal{T}_3$)**.** Once we have synthesized a Schönflies chain $(\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4)$ as in the previous case, we can use it to synthesize a spatial translational parallel mechanism with motion manifold $\mathcal{T}_3$ as follows [1]. We first make l ($l \geq 3$) rigidly displaced copies of $(\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4)$

$$\mathcal{M}_i \triangleq (\mathbf{S}_{i1}, \ldots, \mathbf{S}_{i4}) = (\text{Ad}_{\mathbf{g}_i}(\mathbf{S}_1), \ldots, \text{Ad}_{\mathbf{g}_i}(\mathbf{S}_4)), \qquad i = 1, \ldots, l \tag{36}$$

for some $\mathbf{g}_i \in \text{SE}(3), i = 1, \ldots, l$. $(\mathbf{S}_{i1}, \ldots, \mathbf{S}_{i4})$ is then a chain of the conjugate subgroup $\mathcal{X}(\mathbf{v}_i), i = 1, \ldots, l$. Finally, we connect the l chains to the same base and end-effector. By realizing that the intersection of a collection of subgroups of SE(3) is another subgroup, and the fact that $\mathcal{T}_3$ is a subgroup of $\mathcal{X}(\mathbf{v}_i)$ for any $\mathbf{v}_i \in \mathbb{R}^3$, we see that:

$$\bigcap_{i=1}^{l} \mathcal{X}(\mathbf{v}_i) = \mathcal{T}_3 \tag{37}$$

if $\mathbf{v}_1, \ldots, \mathbf{v}_l$ span $\mathbb{R}^3$. This suggests that we can construct a $\mathcal{T}_3$ parallel mechanism from a collection of (at least three) Schönflies chains that generate $\mathcal{X}(\mathbf{v}_i), i = 1, \ldots, l$. A $\mathcal{T}_3$ parallel mechanism with three chains is illustrated in Fig. 8. For future reference, we denote the aforementioned parallel mechanism by $\mathcal{M}_1 \| \cdots \| \mathcal{M}_l$. Hervé was the first to use the group intersection argument to generalize the famous DELTA robot [13] to a class of $\mathcal{T}_3$ parallel robots [14]. $\diamondsuit$

Example 4 (Wrench analysis of $\mathcal{T}_3$ parallel mechanism)**.** Continuing on Example 3, a dual interpretation of fundamental engineering value emerges by looking at wrench spaces. Denote the twist subspaces of all chains by $\mathfrak{T}_i$ and their corresponding reciprocal wrench subspaces by

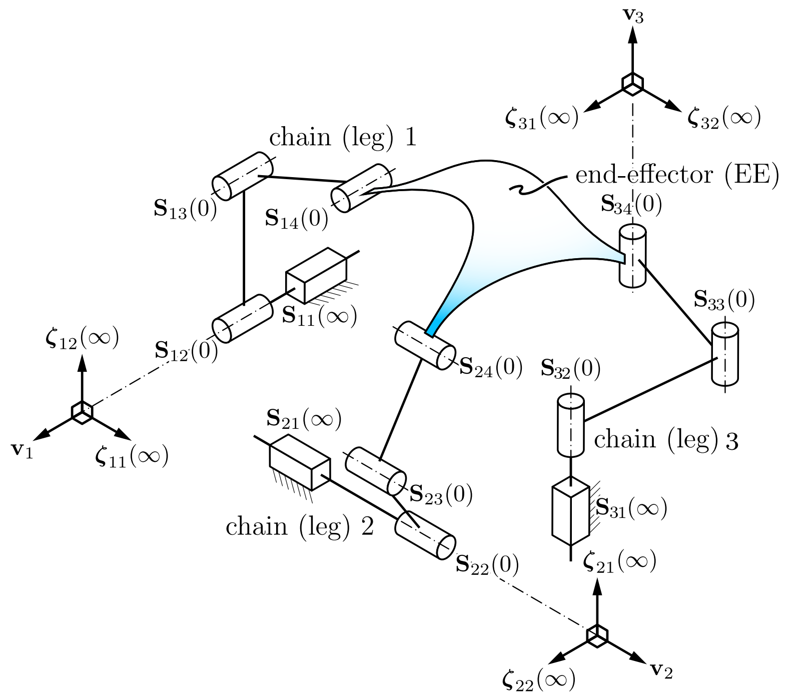


Figure 8: Illustration of a $\mathcal{T}_3$ parallel mechanism with l $\mathcal{X}(\mathbf{v}_i)$ chains ($\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ must span $\mathbb{R}^3$, namely they must not be perpendicular to the same direction).

$\mathfrak{W}_i$ for $i = 1, \ldots, l$. If all $\mathfrak{T}_i$'s are invariant, so are their wrench subspaces $\mathfrak{W}_i$'s. This makes it possible to conceive the constraints that generate the desired end-effector motion in a single configuration, with the guarantee that they hold for finite motions away from it.

In the above case, we see that each chain $\mathcal{M}_i$ has an invariant wrench subspace spanned by two infinite-pitch basis wrenches $\boldsymbol{\zeta}_{i1}$, $\boldsymbol{\zeta}_{i2}$ that are perpendicular to the unit direction vector $\mathbf{v}_i$. It is not difficult to see that the parallel mechanism $\mathcal{M}_1 \| \cdots \| \mathcal{M}_l$ is a $\mathcal{T}_3$ generator if the span of all chain basis wrenches equals the wrench subspace of $\mathcal{T}_3$:

$$\begin{aligned} \sum_{i=1}^{l} \mathfrak{W}_i = \sum_{i=1}^{l} \mathfrak{T}_i^{\perp} &= \{\boldsymbol{\zeta}_{11}, \boldsymbol{\zeta}_{12}, \ldots, \boldsymbol{\zeta}_{l1}, \boldsymbol{\zeta}_{l2}\}_{\mathrm{sp}} \\ &= \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{x} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{z} \end{pmatrix} \right\}_{\mathrm{sp}} = (\mathfrak{t}_3)^{\perp} \end{aligned} \tag{38}$$

which is consistent with Hervé's group intersection argument. As we shall see later, the dual interpretation can be applied to more general chain types. $\diamondsuit$

4. Persistent POE manifolds of SE(3)

Though mechanical generators of Lie subgroups have a fundamental importance, the world of mechanical devices goes beyond them. There are plenty of ingenious mechanisms with very useful properties that generate persistent manifolds that are not subgroups. In this cases, the "enveloping" twist subspace of the manifold is not invariant, but does move in space. Persistent manifolds that are not subgroups have not been systematically studied until recently, when persistent Product-of-Exponential (POE) manifolds and symmetric subspaces of SE(3) have been revealed and investigated.

4.1. POE manifolds of SE(3)

Recall the POE formula for SE(3):

$$\exp_2 : (\theta_1, \ldots, \theta_6) \in W_{\mathbf{0}} \subset \mathbb{R}^6 \mapsto e^{\theta_1 \mathbf{S}_1} \cdots e^{\theta_6 \mathbf{S}_6} \in V_{\mathbf{I}} \subset \mathrm{SE}(3) \tag{39}$$

which is a local diffeomorphism of a neighborhood $W_{\mathbf{0}}$ of $\mathbf{0} \in \mathbb{R}^6$ onto a neighborhood $V_{\mathbf{I}}$ of $\mathbf{I} \in \mathrm{SE}(3)$. If one or more joint variables, say $\theta_{m+1}, \ldots, \theta_6$, are constantly set to zero, the image of the map:

$$\exp_2 : (\theta_1, \ldots, \theta_m, 0, \ldots, 0) \mapsto e^{\theta_1 \mathbf{S}_1} \cdots e^{\theta_m \mathbf{S}_m} \tag{40}$$

is in general a manifold of SE(3), which we shall refer to as a *POE manifold*, and denote by $\prod_{i=1}^{m} \exp(\{\mathbf{S}_i\}_{\mathrm{sp}})$ or $\exp(\{\mathbf{S}_1\}_{\mathrm{sp}}) \cdots \exp(\{\mathbf{S}_m\}_{\mathrm{sp}})$. POE manifolds naturally describe the motion of all serial chains. Indeed, any serial chain can be considered as the serial connection of 1-D Lie subgroup generators, so that the resulting end-effector motion is the product of the corresponding 1-D Lie subgroups (each one of which is the exponential of a 1-D Lie subalgebra). In reference to the canonical parameterization Eq. (16) for Lie subgroups, POE manifolds include Lie subgroups as special cases.

4.2. Persistence of POE manifolds

If the twist space $\mathfrak{T}(\mathbf{g})$ of a manifold M at a generic point $\mathbf{g} \in$ M is a congruent copy of its twist subspace $\mathfrak{T}(\mathbf{I})$ at the identity $\mathbf{I} \in$ M, M is referred to as a *persistent manifold*: the shape of its twist subspace persists through a change of configuration. A POE manifold $\prod_{i=1}^{m} \exp(\{\mathbf{S}_i\}_{\mathrm{sp}})$ is persistent if for any generic configuration $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$:

$$\{\mathbf{S}_1, \mathbf{S}'_2, \ldots, \mathbf{S}'_m\}_{\mathrm{sp}} = \mathrm{Ad}_{\mathbf{g}(\boldsymbol{\theta})}(\{\mathbf{S}_1, \ldots, \mathbf{S}_m\}_{\mathrm{sp}}) \tag{41}$$

for some $\mathbf{g}(\boldsymbol{\theta}) \in \mathrm{SE}(3)$ parameterized by $\boldsymbol{\theta}$, where $\mathbf{S}'_j, j = 2, \ldots, m$, is defined in Eq. (31). The invariance property in Eq. (32) is a particular case of the congruence-invariance property in Eq. (41), occurring when $\mathbf{g}(\boldsymbol{\theta}) = \mathbf{I}$ for any configuration $\boldsymbol{\theta}$.

Since $\mathrm{Ad}_{\exp(\theta_1 \mathbf{S}_1)}(\mathbf{S}_1) = \mathbf{S}_1$, we have:

$$\begin{aligned} \{\mathbf{S}_1, \mathrm{Ad}_{\exp(\theta_1 \mathbf{S}_1)}(\mathbf{S}_2), \ldots\}_{\mathrm{sp}} &= \{\mathrm{Ad}_{\exp(\theta_1 \mathbf{S}_1)}(\mathbf{S}_1), \mathrm{Ad}_{\exp(\theta_1 \mathbf{S}_1)}(\mathbf{S}_2), \ldots\}_{\mathrm{sp}} \\ &= \mathrm{Ad}_{\exp(\theta_1 \mathbf{S}_1)}(\{\mathbf{S}_1, \mathbf{S}_2, \ldots\}_{\mathrm{sp}}) \end{aligned} \tag{42}$$

which turns Eq. (41) into:

$$\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}'_3, \ldots, \mathbf{S}'_m\}_{\mathrm{sp}} = \mathrm{Ad}_{\mathbf{g}'(\boldsymbol{\theta})}(\{\mathbf{S}_1, \ldots, \mathbf{S}_m\}_{\mathrm{sp}}) \tag{43}$$

where $\mathbf{g}'(\boldsymbol{\theta}) = \exp(-\theta_1 \mathbf{S}_1)\mathbf{g}(\boldsymbol{\theta})$. We may therefore, without a loss of generality, set $\theta_1 = 0$ in Eq. (41). From a similar argument, we may also set $\theta_m = 0$ in Eq. (41).

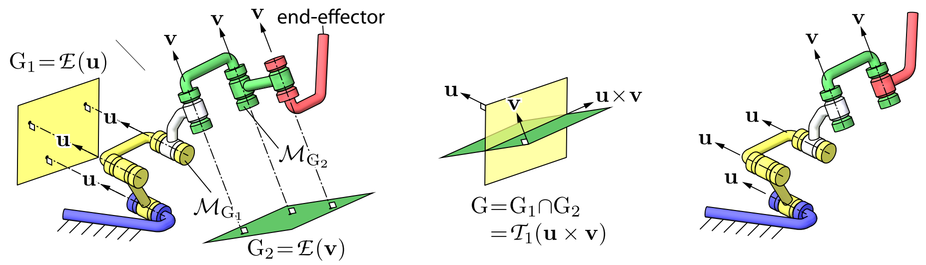
In general, a POE manifold is not persistent, unless its kinematic generator meets special requirements. It is worth pointing out that the order by which $\mathbf{S}_1, \ldots, \mathbf{S}_m$ are concatenated is important: for example, even if the manifold $\exp(\{\mathbf{S}_1\}_{\mathrm{sp}}) \exp(\{\mathbf{S}_2\}_{\mathrm{sp}}) \exp(\{\mathbf{S}_3\}_{\mathrm{sp}})$ is persistent, the manifold $\exp(\{\mathbf{S}_1\}_{\mathrm{sp}}) \exp(\{\mathbf{S}_3\}_{\mathrm{sp}}) \exp(\{\mathbf{S}_2\}_{\mathrm{sp}})$ may not be so.

A guaranteed source of persistent POE manifolds comes in the form of a point-wise product of two Lie subgroups (or a *binary product*) G_1 and G_2 of SE(3), namely the set $\mathrm{G}_1\mathrm{G}_2$ defined by:

$$\mathrm{G}_1\mathrm{G}_2 \triangleq \{\mathbf{g}_1\mathbf{g}_2 \in \mathrm{SE}(3) \mid \mathbf{g}_1 \in \mathrm{G}_1, \mathbf{g}_2 \in \mathrm{G}_2\} \tag{44}$$

which is a manifold of dimension $\dim \mathrm{G}_1 + \dim \mathrm{G}_2 - \dim(\mathrm{G}_1 \cap \mathrm{G}_2)$ [1]. We denote the Lie algebra of G_1 and G_2 by $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. To see that the binary product $\mathrm{G}_1\mathrm{G}_2$ is indeed a persistent manifold, note that for any motion $\mathbf{h}_1(t)\mathbf{h}_2(t) \in \mathrm{G}_1\mathrm{G}_2$, where $\mathbf{h}_1 \in \mathrm{G}_1, \mathbf{h}_2 \in \mathrm{G}_2$, any twist $\boldsymbol{\xi}$ in the twist space $\mathfrak{T}(\mathbf{h}_1\mathbf{h}_2)$ is given by:

$$\boldsymbol{\xi} = \frac{d(\mathbf{h}_1\mathbf{h}_2)}{dt}(\mathbf{h}_1\mathbf{h}_2)^{-1} = \dot{\mathbf{h}}_1\mathbf{h}_2\mathbf{h}_2^{-1}\mathbf{h}_1^{-1} + \mathbf{h}_1\dot{\mathbf{h}}_2\mathbf{h}_2^{-1}\mathbf{h}_1^{-1} = \dot{\mathbf{h}}_1\mathbf{h}_1^{-1} + \mathrm{Ad}_{\mathbf{h}_1}(\dot{\mathbf{h}}_2\mathbf{h}_2^{-1}) \in \mathfrak{g}_1 + \mathrm{Ad}_{\mathbf{h}_1}(\mathfrak{g}_2) \tag{45}$$



(a) (b) (c)

Figure 9: Serial mechanism of a binary product $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$. (a) A redundant serial generator; (b) Redundant subgroup $\mathcal{P}(\mathbf{u}\times\mathbf{v})$; (c) A non-redundant serial generator.

which, together with the fact that $\mathrm{Ad}_{\mathbf{h}_1}(\mathfrak{g}_1)=\mathfrak{g}_1, \forall \mathbf{h}_1 \in G_1$, leads to the following congruence-invariance property:

$$\mathfrak{T}(\mathbf{h}_1\mathbf{h}_2)=\mathrm{Ad}_{\mathbf{h}_1}(\mathfrak{g}_1+\mathfrak{g}_2)=\mathrm{Ad}_{\mathbf{h}_1}(\mathfrak{T}(\mathbf{I})), \qquad \forall \mathbf{h}_1\mathbf{h}_2\in G_1G_2 \tag{46}$$

By concatenating a G_1 chain with another G_2 chain, we have thus created a persistent G_1G_2 chain.

Example 5 (Serial mechanism of $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$)**.** Consider the binary product formed by the product of two planar Euclidean groups $G_1=\mathcal{E}(\mathbf{u})$ and $G_2=\mathcal{E}(\mathbf{v})$. We may construct a kinematic chain for $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$ by first constructing an $\mathcal{R}\mathcal{R}\mathcal{R}$ chain for $\mathcal{E}(\mathbf{u})$ and $\mathcal{E}(\mathbf{v})$ respectively, say:

$$\mathcal{M}_{\mathcal{E}(\mathbf{u})}=(\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3), \quad \mathcal{M}_{\mathcal{E}(\mathbf{v})}=(\mathbf{S}_4,\mathbf{S}_5,\mathbf{S}_6) \tag{47}$$

with

$$\{\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3\}_{\mathrm{sp}}=\mathfrak{e}(\mathbf{u})=\mathrm{Ad}_{\mathbf{g}(\mathbf{z},\mathbf{u})}(\{\boldsymbol{t}_{\mathrm{x}},\boldsymbol{t}_{\mathrm{y}},\boldsymbol{r}_{\mathrm{z}}\}_{\mathrm{sp}}),\ \{\mathbf{S}_4,\mathbf{S}_5,\mathbf{S}_6\}_{\mathrm{sp}}=\mathfrak{e}(\mathbf{v})=\mathrm{Ad}_{\mathbf{g}(\mathbf{z},\mathbf{v})}(\{\boldsymbol{t}_{\mathrm{x}},\boldsymbol{t}_{\mathrm{y}},\boldsymbol{r}_{\mathrm{z}}\}_{\mathrm{sp}}) \tag{48}$$

where $\mathbf{g}(\mathbf{z},\mathbf{u}),\mathbf{g}(\mathbf{z},\mathbf{v})\in SE(3)$ denote transformations that take $\mathbf{z}$ to $\mathbf{u}$ and $\mathbf{v}$, respectively. The concatenated 6-$\mathcal{R}$ chain $(\mathbf{S}_1,\ldots,\mathbf{S}_6)$ is then a serial mechanism for $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$, as illustrated in Fig. 9(a). This observation suggests that $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$ can be expressed into the following POE form:

$$\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})=\prod_{i=1}^{6}\exp(\{\mathbf{S}_i\}_{\mathrm{sp}}) \tag{49}$$

Since $\mathcal{E}(\mathbf{u})$ and $\mathcal{E}(\mathbf{v})$ have a non-trivial intersection, namely the 1-D translation group $\mathcal{P}(\mathbf{u}\times\mathbf{v})$ (see Fig. 9(b)), $(\mathbf{S}_1,\ldots,\mathbf{S}_6)$ is redundant by a degree of one. A non-redundant chain for generating $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$, say $(\mathbf{S}_1,\ldots,\mathbf{S}_5)$, is formed by removing a redundant screw $\mathbf{S}_6$ from $\mathcal{M}_{\mathcal{E}(\mathbf{v})}$ while keeping the remaining freedoms intact:

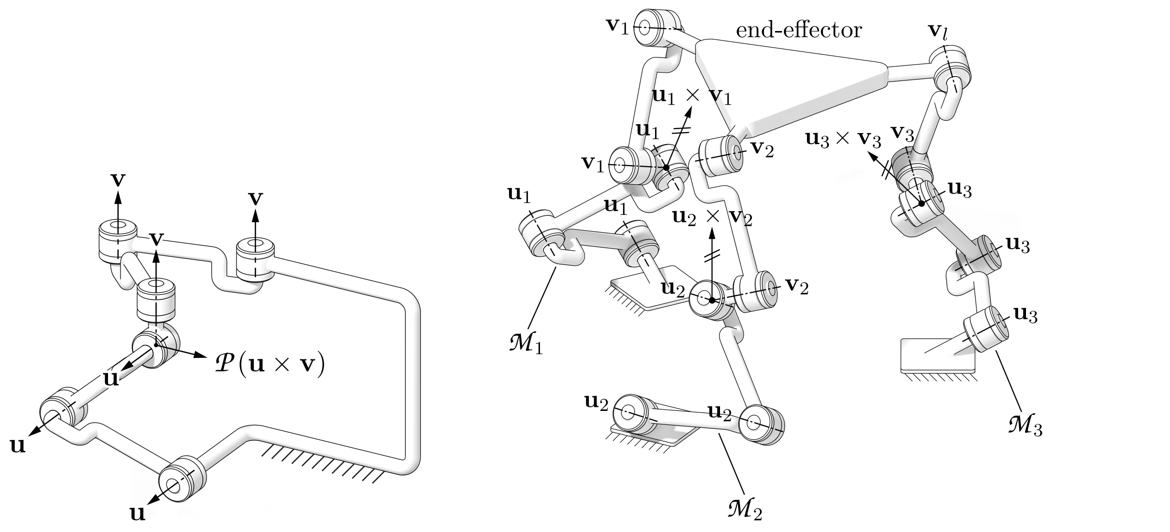
$$\left\{\mathbf{S}_4,\mathbf{S}_5,\begin{pmatrix}\mathbf{0}\\ \mathbf{u}\times\mathbf{v}\end{pmatrix}\right\}_{\mathrm{sp}}=\mathfrak{e}(\mathbf{v}) \tag{50}$$

We arrive at a non-redundant POE representation for $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$:

$$\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})=\prod_{i=1}^{5}\exp(\{\mathbf{S}_i\}_{\mathrm{sp}}) \tag{51}$$

and its corresponding 5-$\mathcal{R}$ chain is shown in Fig. 9(c).

Notice that $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$ contains the subgroup $\mathcal{T}_3$ of spatial translations, as well as all subgroups of SE(3) that comprise rotational or helicoidal motions around lines that are perpendicular to $\mathbf{u}\times\mathbf{v}$, though no spherical subgroups. $\diamondsuit$



(a) (b)

Figure 10: (a) A 1-DOF linkage resulting from closing the loop of a 6-DOF redundant $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$ chain; (b) A purely translational parallel mechanism comprising 3 or more $\mathcal{E}\mathcal{E}$ chains.

Notice that, similarly to Lie subalgebra generators, if a k-D persistent POE manifold is generated by a redundant chain $\{\mathbf{S}_i\}_{i=1}^r$, $r > k$, as in Fig. 9, the degree of redundancy $r - k$ is constant (throughout the motion of the chain and out of singularities, vectors $\mathbf{S}_1, \ldots, \mathbf{S}_r$ remain linearly dependent and form a twist space of dimension k). Accordingly, the configuration space of a closed-loop linkage formed by connecting the base and end-effector of $(\mathbf{S}_1, \ldots, \mathbf{S}_r)$ is a manifold of $\mathbb{R}^r$ with dimension $r - k$. This serves as a further generalization of the Grübler mobility formula; see the ordinary linkages in [2].

Example 6 (Sarrus mechanism)**.** Continuing on Example 5, consider closing the loop of the redundant $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$ chain $(\mathbf{S}_1, \ldots, \mathbf{S}_6)$ shown in Fig. 9(a). Since $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$ is a 5-D persistent manifold, the closed chain, as shown in Fig. 10(a), becomes a movable mechanism with mobility $6 - 5 = 1$, following an argument similar to that in Example 2. $\diamondsuit$

4.3. Classification of persistent POE manifolds of SE(3)

A systematic classification of persistent POE manifolds resulting from *binary products* can be accomplished based on classification of Lie subgroups of SE(3). The results were known to Hervé and researchers in the mechanism synthesis field [15, 16]; a relatively recent summary can be found in [16]. Carricato *et al.* [17, 2] were the first to recognize the persistence of tangent spaces of products of two Lie subgroups. Moreover, they investigated the persistence of general k-D POE manifolds by resorting to screw-system geometry, by conducting systematic studies that revealed a large number of persistent POE manifolds [18, 19, 20, 2, 21].

While all POE manifolds of dimension one, two and six are trivially persistent, the identification and classification of persistent POE manifolds of dimension three, four and five is not trivial. The exhaustive classification of persistent POE manifolds of dimension three [18] and four [20, 19] was recently completed, whereas the classification of 5-D persistent POE manifolds is an open issue (though a number of relevant examples, such as Example 5, are available in the literature [2]).

Since a 2-D POE manifold is a binary product of two 1-D Lie subgroups, any basis $(\mathbf{S}_1, \mathbf{S}_2)$ of any type of two-systems is automatically congruence-invariant, and thus the corresponding POE manifold is persistent.

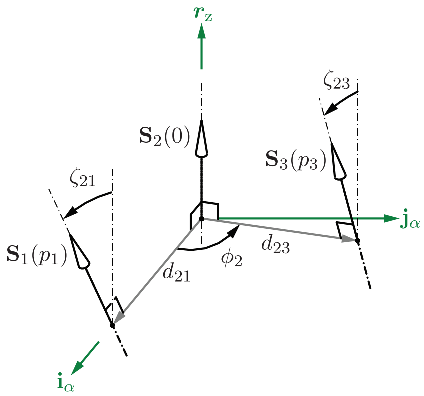


Figure 11: Persistent $S_{3,1}$ system ($\mathcal{HRH}$ type).

4.4. 3-D persistent POE manifolds

3-D persistent POE manifolds were systematically identified in [18], with an addendum in [22].

- $S_{3,2}$, $S_{3,5}$ and $S_{3,6}$ correspond, respectively, to the spherical algebra, the planar helical algebra and the spatial translation algebra; any basis $(\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3)$ of these two three-systems will automatically form a persistent generator. The corresponding persistent POE manifolds are equivalent to the spherical group, the planar helical group and the spatial translation group, respectively.

- Among the three-systems that include no infinite-pitch screws and are not Lie subalgebras, namely $S_{3,g}$ and $S_{3,1}$, only $S_{3,1} = \{\mathbf{i}_\alpha, \mathbf{j}_\alpha, \boldsymbol{r}_z\}_{\text{sp}}$ admits a persistent generator $(\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3)$, which via a suitable Adjoint transformation and change of configuration, can be brought into the following form (Fig. 11):

$$\begin{aligned}&\mathbf{S}_1 = \mathrm{Ad}_{\exp(\zeta_{21}\boldsymbol{r}_x + d_{21}\boldsymbol{t}_x)}(\mathbf{k}_{p_1}),\ \mathbf{S}_2 = \boldsymbol{r}_z,\ \mathbf{S}_3 = \mathrm{Ad}_{\exp(\phi_2\mathbf{S}_2)\exp(\zeta_{23}\boldsymbol{r}_x + d_{23}\boldsymbol{t}_x)}(\mathbf{k}_{p_3})\\ &(p_i = \alpha\sin^2\zeta_{2i}, d_{2i} = \alpha\sin\zeta_{2i}\cos\zeta_{2i}, i = 1, 3)\end{aligned} \tag{52}$$

  where ϕ_2 must not be a multiple of π for $\mathbf{S}_1, \mathbf{S}_2$ and $\mathbf{S}_3$ to be linearly independent. The corresponding persistent manifold $\exp(\{\mathbf{S}_1\}_{\text{sp}})\exp(\{\mathbf{S}_2\}_{\text{sp}})\exp(\{\mathbf{S}_3\}_{\text{sp}})$ represents the motion manifold of an $\mathcal{HRH}$ chain. Notice that when $\alpha = 0$ or $\zeta_{21} = \zeta_{23} = 0$, $S_{3,1}$ becomes a Lie subalgebra ($S_{3,1}$ or $S_{3,5}$, respectively) and is automatically persistent.

- Among the three-systems that include exactly one infinite-pitch screw, only $S_{3,7} = \{\mathbf{i}_\alpha, \mathbf{j}_\beta, \cos\zeta\boldsymbol{t}_x + \sin\zeta\boldsymbol{t}_z, 0 \leq \zeta \leq \pi/2\}_{\text{sp}}$ admits a persistent generator of type $\mathcal{HPH}$ in the form (Fig. 12(a)):

$$\mathbf{S}_1 = \mathbf{j}_\beta, \mathbf{S}_2 = \cos\zeta\boldsymbol{t}_x + \sin\zeta\boldsymbol{t}_z, \mathbf{S}_3 = \mathrm{Ad}_{\exp(\zeta_{13}\boldsymbol{r}_z + d_{13}\boldsymbol{t}_z)}(\mathbf{j}_{p_3}) \tag{53}$$

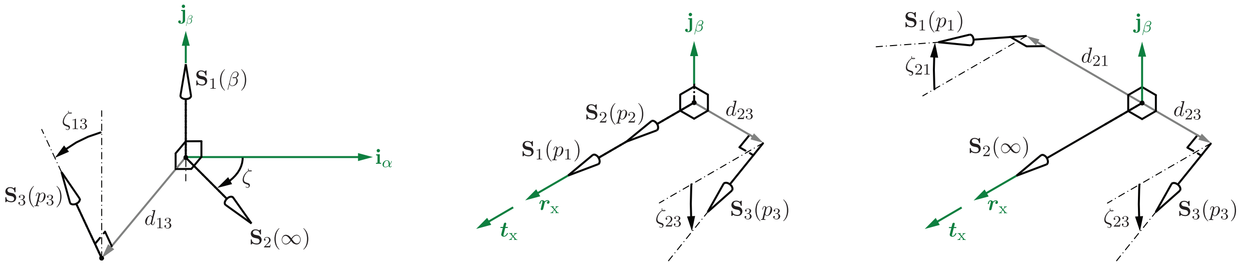
- Since $S_{3,8} = \{\boldsymbol{r}_x, \boldsymbol{t}_x, \mathbf{j}_\beta\}_{\text{sp}}$ contains the cylindrical algebra, the following basis (Fig. 12(b)):

$$\mathbf{S}_1{=}\mathbf{i}_{p_1},\ \mathbf{S}_2{=}\mathbf{i}_{p_2},\ \mathbf{S}_3{=}\mathrm{Ad}_{\exp(\zeta_{23}\boldsymbol{r}_z + d_{23}\boldsymbol{t}_z)}(\mathbf{i}_{p_3})\ (p_1{\neq}p_2, p_3{=}\beta{-}d_{23}\cot\zeta_{23}, \zeta_{23}{\neq}0) \tag{54}$$

  results in a persistent binary product $\exp(\{\boldsymbol{r}_x, \boldsymbol{t}_x\}_{\text{sp}})\exp(\{\mathbf{S}_3\}_{\text{sp}})$ of type $\mathcal{CH}$. The other persistent generator of $S_{3,8}$ is given by (Fig. 12(c)):

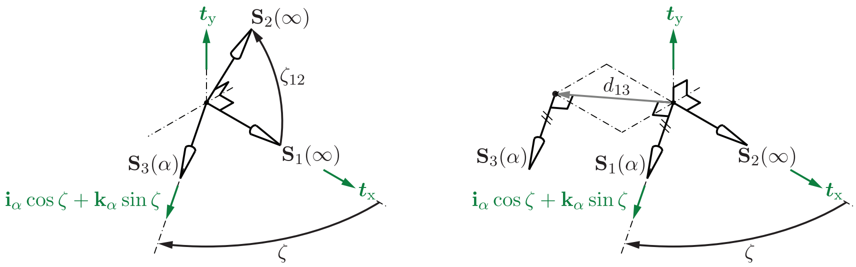
$$\mathbf{S}_1{=}\mathrm{Ad}_{\exp(\zeta_{21}\boldsymbol{r}_z + d_{21}\boldsymbol{t}_z)}(\mathbf{i}_{p_1}),\ \mathbf{S}_2{=}\boldsymbol{t}_x,\ \mathbf{S}_3{=}\mathrm{Ad}_{\exp(\zeta_{23}\boldsymbol{r}_z + d_{23}\boldsymbol{t}_z)}(\mathbf{i}_{p_3})\ (p_i{=}\beta{-}d_{2i}\cot\zeta_{2i}, i{=}1,3) \tag{55}$$

  which leads to a ternary product of type $\mathcal{HPH}$.



(a) $\mathcal{HPH}$ type (b) $\mathcal{CH}$ type (c) $\mathcal{HPH}$ type

Figure 12: Persistent $S_{3,7}$ (a) and $S_{3,8}$ (b, c) systems.



(a) $\mathcal{T}_2\mathcal{H}$ type (b) $\mathcal{HPH}$ type

Figure 13: Persistent $S_{3,9}$ system.

– Since $S_{3,9} = \{\boldsymbol{t}_\mathrm{x}, \boldsymbol{t}_\mathrm{y}, \mathbf{i}_\alpha \cos\zeta + \mathbf{k}_\alpha \sin\zeta\}_\mathrm{sp}, \zeta \notin \{0, \frac{\pi}{2}\}$ contains the planar translation algebra, the following basis (Fig. 13(a)):

$$\mathbf{S}_1 = \boldsymbol{t}_\mathrm{x},\ \mathbf{S}_2 = \mathrm{Ad}_{\exp(\zeta_{12}\boldsymbol{r}_\mathrm{z})}(\mathbf{S}_1),\ \mathbf{S}_3 = \mathbf{i}_\alpha \cos\zeta + \mathbf{k}_\alpha \sin\zeta\ (\zeta_{12} \neq 0) \tag{56}$$

results in the 3-D manifold $\exp(\{\mathbf{S}_1, \mathbf{S}_2\}_\mathrm{sp}) \exp(\{\mathbf{S}_3\}_\mathrm{sp})$, which is a binary product of the planar translation group $\mathcal{T}_2$ and the 1-D helical group $\mathcal{H}$, and is automatically persistent. The corresponding motion manifold is that of a $\mathcal{T}_2\mathcal{H}$ chain. The other persistent generator of $S_{3,9}$ is given by (Fig. 13(b)):

$$\mathbf{S}_1 = \mathbf{i}_\alpha \cos\zeta + \mathbf{k}_\alpha \sin\zeta,\ \mathbf{S}_2 = \boldsymbol{t}_\mathrm{x},\ \mathbf{S}_3 = \mathbf{S}_1 + d_{13}\boldsymbol{t}_\mathrm{y}\ (d_{13} \neq 0) \tag{57}$$

The corresponding motion manifold is that of a $\mathcal{H}(\mathbf{S}_1)\mathcal{P}(\mathbf{S}_2)\mathcal{H}(\mathbf{S}_3)$ chain.

– $S_{3,10} = \{\boldsymbol{r}_\mathrm{x}, \boldsymbol{t}_\mathrm{x}, \boldsymbol{t}_\mathrm{y}\}_\mathrm{sp}$ admits the following three persistent generators (Fig. 14):

$$\mathbf{S}_1 = \mathbf{i}_{p_1},\ \mathbf{S}_2 = \mathbf{i}_{p_2},\ \mathbf{S}_3 = \mathrm{Ad}_{\exp(\zeta_{23}\boldsymbol{r}_\mathrm{z})}(\boldsymbol{t}_\mathrm{x})\ (p_1 \neq p_2, \zeta_{23} \neq 0) \tag{58}$$

which corresponds to a binary product of type $\mathcal{CP}$;

$$\mathbf{S}_1 = \mathbf{i}_{p_1},\ \mathbf{S}_2 = \mathbf{i}_{p_2},\ \mathbf{S}_3 = \mathrm{Ad}_{\exp(d_{23}\boldsymbol{t}_\mathrm{z})}(\mathbf{i}_{p_3})\ (p_1 \neq p_2, d_{23} \neq 0) \tag{59}$$

which corresponds to a binary product of type $\mathcal{CH}$;

$$\mathbf{S}_1 = \mathrm{Ad}_{\exp(\zeta_{31}\boldsymbol{r}_\mathrm{z})}(\boldsymbol{t}_\mathrm{x}),\ \mathbf{S}_2 = \mathrm{Ad}_{\exp(\zeta_{32}\boldsymbol{r}_\mathrm{z})}(\boldsymbol{t}_\mathrm{x}),\ \mathbf{S}_3 = \mathbf{i}_{p_3}\ (\zeta_{31} \neq \zeta_{32}) \tag{60}$$

which corresponds to a binary product of type $\mathcal{T}_2\mathcal{H}$.

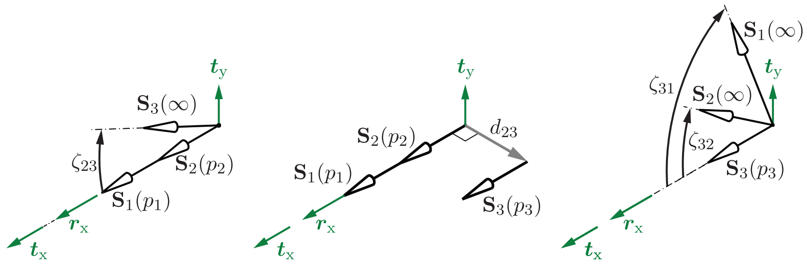
Aside from the above non-redundant persistent generators, the binary product $\mathcal{CH}$ resulting from Eq. (59) has the following redundant generator:

$$\mathbf{S}_1{=}\mathbf{i}_{p_1},\ \mathbf{S}_2{=}\mathbf{i}_{p_2},\ \mathbf{S}_3{=}\mathrm{Ad}_{\exp(d_{23}\boldsymbol{t}_\mathrm{z})}(\mathbf{i}_{p_3}),\ \mathbf{S}_4{=}\mathrm{Ad}_{\exp(d_{23}\boldsymbol{t}_\mathrm{z})}(\mathbf{i}_{p_4})\ (p_1{\neq}p_2, p_3{\neq}p_4, d_{23}{\neq}0) \tag{61}$$

which leads to a binary product of type $\mathcal{CC}$. Similary, the binary product $\mathcal{T}_2\mathcal{H}$ resulting from Eq. (60) has the following redundant generator:

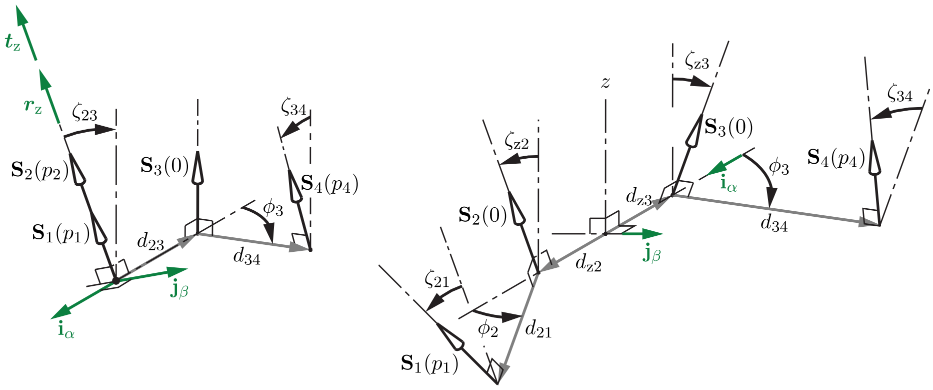
$$\mathbf{S}_1 = \mathrm{Ad}_{\exp(\zeta_{31}\boldsymbol{r}_\mathrm{z})}(\boldsymbol{t}_\mathrm{x}),\ \mathbf{S}_2 = \mathrm{Ad}_{\exp(\zeta_{32}\boldsymbol{r}_\mathrm{z})}(\boldsymbol{t}_\mathrm{x}),\ \mathbf{S}_3 = \mathbf{i}_{p_3},\ \mathbf{S}_4 = \mathbf{i}_{p_4}\ (\zeta_{31} \neq \zeta_{32}, p_3 \neq p_4) \tag{62}$$

which leads to a binary product of type $\mathcal{T}_2\mathcal{C}$.



(a) $\mathcal{CP}$ type (b) $\mathcal{CH}$ type (c) $\mathcal{T}_2\mathcal{H}$ type

Figure 14: Persistent $S_{3,10}$ system.



(a) $\mathcal{CRH}$ type (b) $\mathcal{HRRH}$ type

Figure 15: Persistent $S_{4,g}$ systems.

4.5. 4-D persistent POE manifolds

In [19, 20], a systematic identification of 4-D persistent POE manifolds has been carried out, and the results are summarized below.

– $S_{4,g} = \{\mathbf{i}_\alpha, \mathbf{j}_\beta, \boldsymbol{r}_\mathrm{z}, \boldsymbol{t}_\mathrm{z}\}_\mathrm{sp}$ admits two persistent generators, with the first being [20] (Fig. 15(a)):

$$\begin{gathered}\mathbf{S}_1{=}\mathbf{k}_{p_1},\ \mathbf{S}_2{=}\mathbf{k}_{p_2},\ \mathbf{S}_3{=}\mathrm{Ad}_{e^{\zeta_{23}\boldsymbol{r}_\mathrm{x}+d_{23}\boldsymbol{t}_\mathrm{x}}}(\boldsymbol{r}_\mathrm{z}),\ \mathbf{S}_4{=}\mathrm{Ad}_{e^{\phi_3\mathbf{S}_3}e^{\zeta_{23}\boldsymbol{r}_\mathrm{x}+d_{23}\boldsymbol{t}_\mathrm{x}}e^{\zeta_{34}\boldsymbol{r}_\mathrm{x}+d_{34}\boldsymbol{t}_\mathrm{x}}}(\mathbf{k}_{p_4})\\ (p_1{\neq}p_2, p_4{=}\alpha\sin^2\zeta_{34}, d_{34}{=}\alpha\sin\zeta_{34}\cos\zeta_{34}, d_{23}{=}\alpha\cos\zeta_{23}\sin\zeta_{23}, \beta{=}\alpha\cos^2\zeta_{23}, \alpha{\geq}\beta{\geq}0)\end{gathered} \tag{63}$$

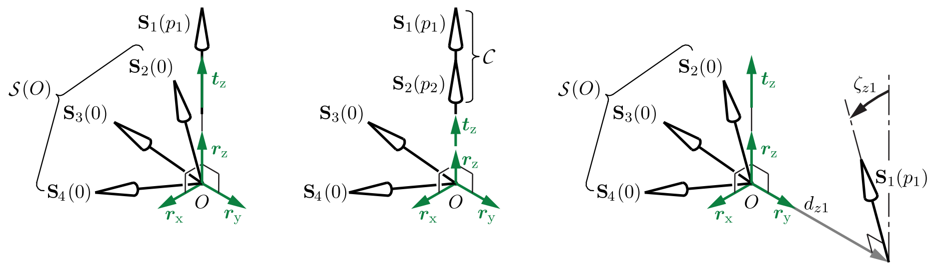
where ϕ_3 is chosen so that $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ and $\mathbf{S}_4$ are linearly independent. The corresponding persistent manifold $\exp(\{\mathbf{S}_1, \mathbf{S}_2\}_\mathrm{sp})\exp(\{\mathbf{S}_3\}_\mathrm{sp})\exp(\{\mathbf{S}_4\}_\mathrm{sp})$ is a ternary product of the type $\mathcal{CRH}$.

The second persistent generator of $S_{4,g}$ is given by (Fig. 15(b)):

$$\begin{gathered}\mathbf{S}_1{=}\mathrm{Ad}_{\exp(\phi_2\mathbf{S}_2)\exp(\zeta_{z2}\boldsymbol{r}_\mathrm{x}+d_{z2}\boldsymbol{t}_\mathrm{x})\exp(\zeta_{21}\boldsymbol{r}_\mathrm{x}+d_{21}\boldsymbol{t}_\mathrm{x})}(\mathbf{k}_{p_1}),\ \mathbf{S}_2{=}\mathrm{Ad}_{\exp(\zeta_{z2}\boldsymbol{r}_\mathrm{x}+d_{z2}\boldsymbol{t}_\mathrm{x})}(\boldsymbol{r}_\mathrm{z}),\\ \mathbf{S}_3{=}\mathrm{Ad}_{\exp(\zeta_{z3}\boldsymbol{r}_\mathrm{x}+d_{z3}\boldsymbol{t}_\mathrm{x})}(\boldsymbol{r}_\mathrm{z}),\ \mathbf{S}_4{=}\mathrm{Ad}_{\exp(\phi_3\mathbf{S}_3)\exp(\zeta_{z3}\boldsymbol{r}_\mathrm{x}+d_{z3}\boldsymbol{t}_\mathrm{x})\exp(\zeta_{34}\boldsymbol{r}_\mathrm{x}+d_{34}\boldsymbol{t}_\mathrm{x})}(\mathbf{k}_{p_4})\\ (0{\neq}\zeta_{z2}{=}\zeta_{z3}{\neq}\tfrac{\pi}{2}, \beta{=}\alpha\cos^2\zeta_{z2}, d_{z2}{=}d_{z3}{=}\alpha\sin\zeta_{z2}\cos\zeta_{z2}, p_1{=}\alpha\sin^2\zeta_{21},\\ d_{21}{=}\alpha\sin\zeta_{21}\cos\zeta_{21}, p_4{=}\alpha\sin^2\zeta_{34}, d_{34}{=}\alpha\sin\zeta_{34}\cos\zeta_{34})\end{gathered} \tag{64}$$

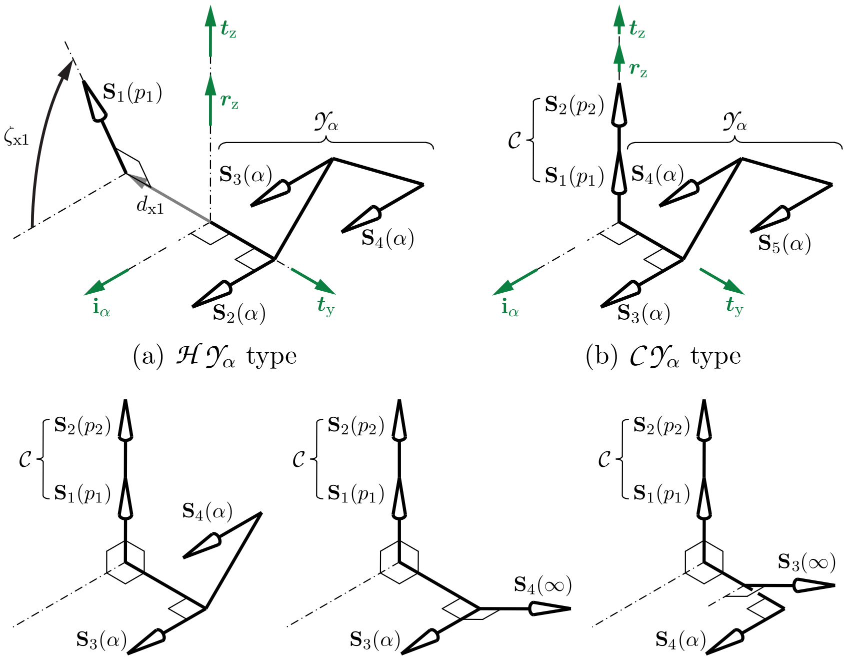
where ϕ_2 and ϕ_3 are chosen so that $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ and $\mathbf{S}_4$ are linearly independent. The corresponding persistent manifold $\exp(\{\mathbf{S}_1\}_\mathrm{sp})\exp(\{\mathbf{S}_2\}_\mathrm{sp})\exp(\{\mathbf{S}_3\}_\mathrm{sp})\exp(\{\mathbf{S}_4\}_\mathrm{sp})$ is a quaternary product of type $\mathcal{HRRH}$.

– $S_{4,1} = \{\mathbf{i}_\alpha, \mathbf{j}_\alpha, \boldsymbol{r}_\mathrm{z}, \boldsymbol{t}_\mathrm{z}\}_\mathrm{sp}$ admits persistent bases only when the principal pitch α in Tab. 1 is zero, namely $S_{4,1} = \{\boldsymbol{r}_\mathrm{x}, \boldsymbol{r}_\mathrm{y}, \boldsymbol{r}_\mathrm{z}, \boldsymbol{t}_\mathrm{z}\}_\mathrm{sp}$, which leads to the following persistent manifolds.



(a) $\mathcal{PS}$ or $\mathcal{HS}$ type (b) $\mathcal{CRR}$ type (c) $\mathcal{HS}$ or $\mathcal{RS}$ type

Figure 16: Persistent $S_{4,1}$ systems.



(a) $\mathcal{HY}_\alpha$ type (b) $\mathcal{CY}_\alpha$ type

(c) $\mathcal{CHH}$ type (d) $\mathcal{CHP}$ type (e) $\mathcal{CPH}$ type

Figure 17: Persistent $S_{4,2}$ systems.

The first one, as illustrated in Fig. 16(a),

$$\exp(\{\mathbf{k}_{p_1}\}_\mathrm{sp})\exp(\{\boldsymbol{r}_\mathrm{x},\boldsymbol{r}_\mathrm{y},\boldsymbol{r}_\mathrm{z}\}_\mathrm{sp})\ (p_1\neq 0) \tag{65}$$

is a binary product of type $\mathcal{PS}$ ($p_1=\infty$) or $\mathcal{HS}$ ($p_1\notin\{0,\infty\}$).

In particular, when $\mathbf{S}_2(0)$ is aligned with the **z**-axis, it may be replaced by a screw of arbitrary pitch p_2, thereby resulting in a second persistent manifold (Fig. 16(b))

$$\exp(\{\mathbf{S}_1(p_1),\mathbf{S}_2(p_2)\}_\mathrm{sp})\exp(\{\mathbf{S}_3(0)\}_\mathrm{sp})\exp(\{\mathbf{S}_4(0)\}_\mathrm{sp})\ (p_1\neq p_2) \tag{66}$$

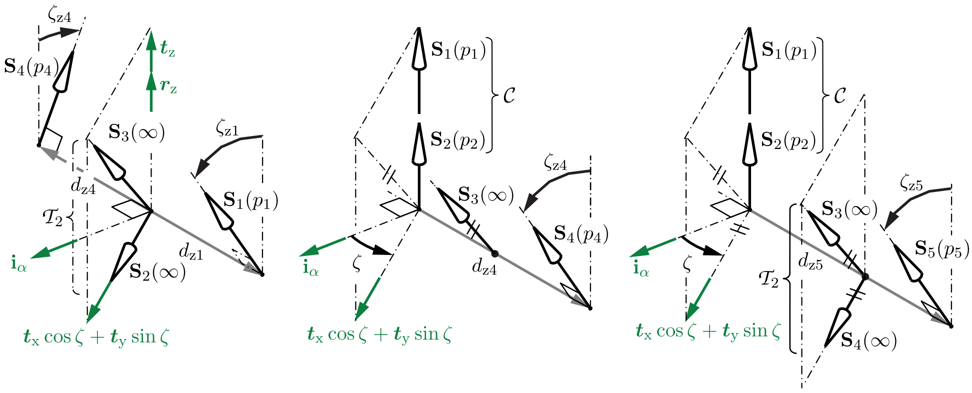
of type $\mathcal{CRR}$.

The third persistent manifold, as illustrated in Fig. 16(c),

$$\exp(\{\mathrm{Ad}_{\exp(\zeta_{z1}\boldsymbol{r}_\mathrm{y}+d_{z1}\boldsymbol{t}_\mathrm{y})}(\mathbf{k}_{p_1})\}_\mathrm{sp})\exp(\{\boldsymbol{r}_\mathrm{x},\boldsymbol{r}_\mathrm{y},\boldsymbol{r}_\mathrm{z}\}_\mathrm{sp})\ (d_{z1}=-p_1\tan\zeta_{z1}) \tag{67}$$

is a binary product of type $\mathcal{RS}$ ($\zeta_{z1}=\frac{\pi}{2}$) or $\mathcal{HS}$ ($\zeta_{z1}\notin\{0,\frac{\pi}{2}\}$).

Finally, the binary product shown in Eq. (66) can be shown to be equivalent to the binary product $\exp(\{\mathbf{S}_1(p_1),\mathbf{S}_2(p_2)\}_\mathrm{sp})\exp(\{\boldsymbol{r}_\mathrm{x},\boldsymbol{r}_\mathrm{y},\boldsymbol{r}_\mathrm{z}\}_\mathrm{sp})$ of type $\mathcal{CS}$.



(a) $\mathcal{HT}_2\mathcal{H}$ type (b) $\mathcal{CPH}$ type (c) $\mathcal{CT}_2\mathcal{H}$ type

Figure 18: Persistent $S_{4,4}$ systems.

– $S_{4,2} = \{\mathbf{i}_\alpha, \boldsymbol{t}_y, \boldsymbol{r}_z, \boldsymbol{t}_z\}_{sp}$ leads to the following persistent manifolds.

The first one, as illustrated in Fig. 17(a):

$$\exp(\{\mathbf{S}_1(p_1)\}_{sp}) \exp(\{\mathbf{S}_2(\alpha), \mathbf{S}_3(\alpha), \mathbf{S}_4(\alpha)\}_{sp}) \ (p_1 - \alpha = d_{x1} \tan\zeta_{x1}, \zeta_{x1} \neq 0) \tag{68}$$

is a binary product of type $\mathcal{HY}_\alpha$.

Besides, the fact that $S_{4,2}$ contains both the cylindrical algebra $\{\boldsymbol{r}_z, \boldsymbol{t}_z\}_{sp}$ and the planar helical algebra $\{\mathbf{i}_\alpha, \boldsymbol{t}_y, \boldsymbol{t}_z\}_{sp}$ leads to the following binary product (Fig. 17(b)):

$$\exp(\{\mathbf{S}_1(p_1), \mathbf{S}_2(p_2)\}_{sp}) \exp(\{\mathbf{S}_3(\alpha), \mathbf{S}_4(\alpha), \mathbf{S}_5(\alpha)\}_{sp}) \ (p_1 \neq p_2) \tag{69}$$

of type $\mathcal{CY}_\alpha$. Three more ternary products arise from removing the redundancy in Eq. (69):

$$\exp(\{\mathbf{S}_1(p_1), \mathbf{S}_2(p_2)\}_{sp}) \exp(\{\mathbf{S}_3(\alpha)\}_{sp}) \exp(\{\mathbf{S}_4(\alpha)\}_{sp} \qquad \text{(Fig. 17(c))} \tag{70}$$

$$\exp(\{\mathbf{S}_1(p_1), \mathbf{S}_2(p_2)\}_{sp}) \exp(\{\mathbf{S}_3(\alpha)\}_{sp}) \exp(\{\mathbf{S}_4(\infty)\}_{sp} \qquad \text{(Fig. 17(d))} \tag{71}$$

$$\exp(\{\mathbf{S}_1(p_1), \mathbf{S}_2(p_2)\}_{sp}) \exp(\{\mathbf{S}_3(\infty)\}_{sp}) \exp(\{\mathbf{S}_4(\alpha)\}_{sp} \qquad \text{(Fig. 17(e))} \tag{72}$$

– Since $S_{4,3} = \{\boldsymbol{t}_x, \boldsymbol{t}_y, \boldsymbol{r}_z, \boldsymbol{t}_z\}_{sp}$ is the Schönflies algebra, any basis of $S_{4,3}$ is automatically persistent; the corresponding persistent POE manifold is equivalent to the Schönflies group.

– $S_{4,4} = \{\mathbf{i}_\alpha, \boldsymbol{t}_x \cos\zeta + \boldsymbol{t}_y \sin\zeta, \boldsymbol{r}_z, \boldsymbol{t}_z\}_{sp}$ admits the following persistent manifolds.

The first one, as illustrated in Fig. 18(a):

$$\exp(\{\mathbf{S}_1(p_1)\}_{sp})\exp(\{\mathbf{S}_2(\infty),\mathbf{S}_3(\infty)\}_{sp})\exp(\{\mathbf{S}_4(p_4)\}_{sp}) \ (\zeta_{z4}{\neq}\zeta_{z1}, p_i{=}\alpha{+}d_{zi}\cot\zeta_{zi}, i{=}1,4) \tag{73}$$

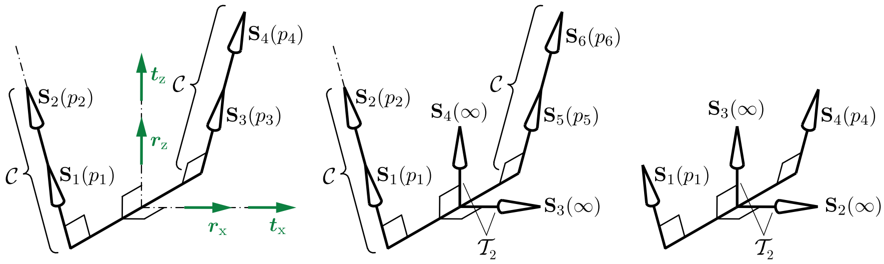
is a ternary product of type $\mathcal{HT}_2\mathcal{H}$.

The second one, as illustrated in Fig. 18(b):

$$\exp(\{\mathbf{S}_1(p_1),\mathbf{S}_2(p_2)\}_{sp})\exp(\{\mathbf{S}_3(\infty)\}_{sp})\exp(\{\mathbf{S}_4(p_4)\}_{sp}) \ (p_1{\neq}p_2, p_4{=}\alpha{+}d_{z4}\cot\zeta_{z4}) \tag{74}$$

is a ternary product of type $\mathcal{CPH}$. The latter can also be turned into a ternary $\mathcal{CT}_2\mathcal{H}$-type product (Fig. 18(c)):

$$\exp(\{\mathbf{S}_1(p_1),\mathbf{S}_2(p_2)\}_{sp})\exp(\{\mathbf{S}_3(\infty),\mathbf{S}_4(\infty)\}_{sp})\exp(\{\mathbf{S}_5(p_5)\}_{sp}) \ (p_1{\neq}p_2, p_5{=}\alpha{+}d_{z5}\cot\zeta_{z5}) \tag{75}$$



(a) $\mathcal{C}\mathcal{C}$ type (b) $\mathcal{C}\mathcal{T}_2\mathcal{C}$ type (c) $\mathcal{H}\mathcal{T}_2\mathcal{H}$ type

Figure 19: Persistent $S_{4,5}$ systems.

Table 3: A complete list of 3-D persistent POE manifolds of SE(3).

dim	screw sys.	persistent basis	POE manifold	subalg.
3	$S_{3,1}, \gamma = 0$	Eq. (52)	$\mathcal{H}\mathcal{R}\mathcal{H}$	
	$S_{3,2}, \alpha = 0$	any basis	$\mathcal{S}$	$\mathfrak{s}$
	$S_{3,5}$	any basis	$\mathcal{Y}_\alpha$	$\mathfrak{y}_\alpha$
	$S_{3,6}$	any basis	$\mathcal{T}_3$	$\mathfrak{t}_3$
	$S_{3,7}$	Eq. (53)	$\mathcal{H}\mathcal{P}\mathcal{H}$	
	$S_{3,8}$	Eq. (54), Eq. (55)	$\mathcal{C}\mathcal{H}$, $\mathcal{H}\mathcal{P}\mathcal{H}$	
	$S_{3,9}$	Eq. (56), Eq. (57)	$\mathcal{T}_2\mathcal{H}$, $\mathcal{H}\mathcal{P}\mathcal{H}$	
	$S_{3,10}$	Eq. (58) – (62)	$\mathcal{C}\mathcal{P}$, $\mathcal{C}\mathcal{H}$, $\mathcal{C}\mathcal{C}$, $\mathcal{T}_2\mathcal{H}$, $\mathcal{T}_2\mathcal{C}$	

– $S_{4,5} = \{\boldsymbol{r}_x, \boldsymbol{t}_x, \boldsymbol{r}_z, \boldsymbol{t}_z\}_{\rm sp}$ admits the following persistent manifolds.

The first one, as illustrated in Fig. 19(a):

$$\exp(\{\mathbf{S}_1(p_1), \mathbf{S}_2(p_2)\}_{\rm sp}) \exp(\{\mathbf{S}_3(p_3), \mathbf{S}_4(p_4)\}_{\rm sp}) \tag{76}$$

is a binary product of type $\mathcal{C}\mathcal{C}$.

Eq. (76) can be turned into a ternary product of type $\mathcal{C}\mathcal{T}_2\mathcal{C}$, as shown in Fig. 19(b):

$$\exp(\{\mathbf{S}_1(p_1),\mathbf{S}_2(p_2)\}_{\rm sp})\exp(\{\mathbf{S}_3(\infty),\mathbf{S}_4(\infty)\}_{\rm sp})\exp(\{\mathbf{S}_5(p_5),\mathbf{S}_6(p_6)\}_{\rm sp}) \ (p_1{\neq}p_2,p_5{\neq}p_6) \tag{77}$$

which is equivalent to a ternary product of type $\mathcal{H}\mathcal{T}_2\mathcal{H}$ (Fig. 19(c)):

$$\exp(\{\mathbf{S}_1(p_1)\}_{\rm sp}) \exp(\{\mathbf{S}_2(\infty), \mathbf{S}_3(\infty)\}_{\rm sp}) \exp(\{\mathbf{S}_4(p_4)\}_{\rm sp}) \tag{78}$$

Obvious ternary products of type $\mathcal{C}\mathcal{T}_2\mathcal{H}$, $\mathcal{C}\mathcal{P}\mathcal{C}$ and $\mathcal{C}\mathcal{P}\mathcal{H}$ can also be obtained.

The aforementioned persistent POE manifolds are summarized in Tab. 3 and Tab. 4.

4.6. Mechanism synthesis with persistent POE manifolds

A m-D persistent POE manifold $\mathrm{M} \triangleq \prod_{i=1}^{m} \exp(\{\boldsymbol{\xi}_i\}_{\rm sp})$ can be readily generated by the corresponding serial kinematic chain $(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m)$. Persistent POE manifolds can then be used to construct more parallel mechanisms. Hunt was probably the first to use binary product-equivalent kinematic chains in the synthesis of parallel mechanisms [23].

Example 7 (Translational parallel mechanism with persistent $\mathcal{E}\mathcal{E}$ chains)**.** We have already seen in Example 5 that the spatial translation group $\mathcal{T}_3$ is contained in the binary product $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$, which suggests the synthesis of a $\mathcal{T}_3$ parallel mechanism by using multiple $\mathcal{E}\mathcal{E}$ chains. Consider l ($l \geq 3$) rigidly displaced copies of a $\mathcal{E}(\mathbf{u})\mathcal{E}(\mathbf{v})$ chain, say $(\mathbf{S}_1, \ldots, \mathbf{S}_5)$ as shown in Fig. 9(c):

$$\mathcal{M}_i \triangleq (\mathbf{S}_{i1}, \ldots, \mathbf{S}_{i5}) = (\mathrm{Ad}_{\mathbf{g}_i}(\mathbf{S}_1), \ldots \mathrm{Ad}_{\mathbf{g}_i}(\mathbf{S}_5)), \qquad i = 1, \ldots, l \tag{79}$$

Table 4: A complete list of 4-D persistent POE manifolds of SE(3).

dim	screw sys.	persistent basis	POE manifold	subalg.
4	$S_{4,g}$	Eq. (63), Eq. (64)	$\mathcal{CRH}$, $\mathcal{HRRH}$	
	$S_{4,1}, \alpha = 0$	Eq. (65) – (67)	$\mathcal{PS}$, $\mathcal{HS}$, $\mathcal{CRR}$ ($\mathcal{CS}$), $\mathcal{RS}$	
	$S_{4,2}$	Eq. (68) – (72)	$\mathcal{HY}$, $\mathcal{CHH}$, $\mathcal{CHP}$, $\mathcal{CPH}$, $\mathcal{YC}$	
	$S_{4,3}$	any basis	$\mathcal{X}$	$\mathfrak{x}$
	$S_{4,4}$	Eq. (73) – (75)	$\mathcal{HT}_2\mathcal{H}$, $\mathcal{CPH}$, $\mathcal{CT}_2\mathcal{H}$	
	$S_{4,5}$	Eq. (76) – (78)	$\mathcal{CC}$, $\mathcal{HT}_2\mathcal{H}$, $\mathcal{CT}_2\mathcal{C}$, $\mathcal{CT}_2\mathcal{H}$, $\mathcal{CPC}$, $\mathcal{CPH}$	

for some $\mathbf{g}_i \in \mathrm{SE}(3), i = 1, \ldots, l$. $(\mathbf{S}_{i1}, \ldots, \mathbf{S}_{i5})$ is then a chain of the conjugate manifold $\mathcal{E}(\mathbf{u}_i)\mathcal{E}(\mathbf{v}_i), \mathbf{u}_i = \mathbf{g}_i\mathbf{u}, \mathbf{v}_i = \mathbf{g}_i\mathbf{v}, i = 1, \ldots, l$.

Since the reciprocal wrench subspace $\mathfrak{W}_i(\mathbf{g})$ of each chain manifold $\mathcal{E}(\mathbf{u}_i)\mathcal{E}(\mathbf{v}_i)$ is persistently generated by an infinite-pitch wrench $\boldsymbol{\zeta}_i(\infty)$:

$$\boldsymbol{\zeta}_i(\infty) = \begin{pmatrix} \mathbf{0} \\ \mathbf{u}_i \times \mathbf{v}_i \end{pmatrix} \qquad i = 1, \ldots, l \tag{80}$$

the constraint wrench subspace of the parallel mechanism $\mathcal{M}_1 \| \cdots \| \mathcal{M}_l$ shown in Fig. 10(b) is given by:

$$\sum_{i=1}^{l} \mathfrak{W}_i = \{\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_l\}_{\mathrm{sp}} = \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{x} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{z} \end{pmatrix} \right\}_{\mathrm{sp}} = (\mathfrak{t}_3)^{\perp} \tag{81}$$

so that the end-effector of $\mathcal{M}_1 \| \cdots \| \mathcal{M}_l$ cannot rotate so long as $\mathbf{u}_i \times \mathbf{v}_i, i = 1, \ldots, l$, span $\mathbb{R}^3$. This validates $\mathcal{M}_1 \| \cdots \| \mathcal{M}_l$ as a $\mathcal{T}_3$ parallel mechanism. ♢

We have seen from Example 6 and 7 that persistent POE manifolds inherit a number of properties that hold for their subgroup factors, such as the trivial identification of serial mechanical generators, and the full-cycle linear dependence of joint twists that allows the design of closed-chain mechanisms with guaranteed mobility. Since the wrench space exerted by a persistent POE generator preserves its dimension and "shape" for finite motions, also the synthesis of complex parallel mechanisms is highly benefited: plenty of examples can be found in the literature [24].

5. Symmetric subspaces of SE(3)

So far, we have introduced two classes of persistent manifolds, namely the Lie subgroups and the persistent POE manifolds. The latter may be considered a generalization of the former, since both can be represented by the POE formula. However, there are mechanisms that defy a description in terms of Lie subgroups or POE manifolds. For example, there are parallel mechanisms whose end-effectors have a persistent twist subspace and whose chains are not persistent POE manifold generators, but chains with peculiar symmetric properties.

The most relevant example is provided by homokinetic couplings (also known as constant-velocity joints), which are parallel mechanisms used for connecting shafts with intersecting axes (with or without plunging) [25, 23]. They have a special plane (called homokinetic plane) with respect to which all chains are symmetric and whose motion embodies the rigid movement of a persistent twist subspace. For example, Fig. 20(a) shows a homokinetic kinematic chain with five $\mathcal{R}$ joints with joint axes being mirror-symmetric about plane Π. More precisely, the joint axes $\ell_1, \ell_2, \ell_3, \ell_2'$ and ℓ_1' are pairwise symmetric about Π, with ℓ_3 lying in Π. The corresponding twist space $\mathfrak{T}$ and wrench space $\mathfrak{W}$ are both underlied by the fourth special three-system $S_{3,4}$ for all configurations, as long as mirror symmetry is preserved, with zero-pitch screws lying in Π

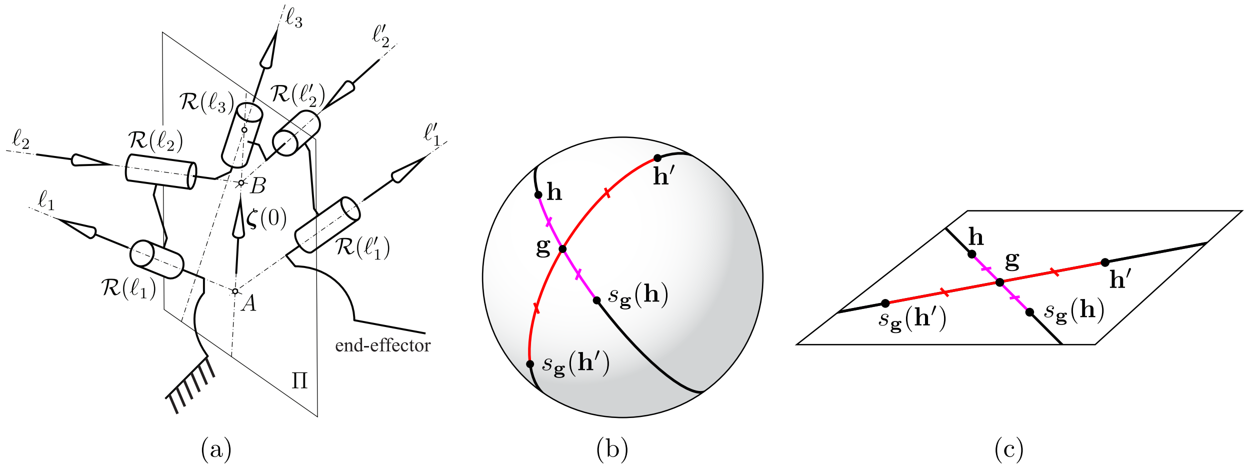


Figure 20: (a) Example of a 5-$\mathcal{R}$ homokinetic chain mirror-symmetric about the *homokinetic plane* Π; (b) and (c) Two examples of symmetric space: (b) the unit 2-D sphere; (c) the Euclidean space $\mathbb{R}^m$ ($m = 2$ here).

and an infinite-pitch screw perpendicular to Π; the homokinetic chain is inherently redundant and maintain the homokinetic (mirror-symmetry) condition only within a closed-loop (e.g., parallel) mechanism. As we shall shortly see, the congruence-invariance of the corresponding 3-D motion manifold arises from yet another class of persistent manifolds, called symmetric subspaces.

5.1. Symmetric space preliminaries

Aside from being a 6-D Lie group, SE(3) also admits the structure of a symmetric space [26]. A *symmetric space*, again denoted M, is a manifold that can be isometrically point-reflected onto itself about any point on the manifold. More precisely, we associate to each $\mathbf{g} \in \mathrm{M}$ a diffeomorphism called an *inversion symmetry* denoted $s_{\mathbf{g}} : \mathrm{M} \to \mathrm{M}$ such that:

1. $s_{\mathbf{g}}$ is an *involution* map, i.e., $s_{\mathbf{g}} \circ s_{\mathbf{g}}$ is the identity map on M for any $\mathbf{g} \in \mathrm{M}$;
2. the only fixed point of $s_{\mathbf{g}}$ in a neighborhood of $\mathbf{g}$ is $\mathbf{g}$ itself;
3. $s_{\mathbf{g}}$ is an isometry, i.e., it reverses every geodesic passing through $\mathbf{g} \in \mathrm{M}$.

Typical examples of symmetric spaces include: the unit 2-D sphere S^2 (Fig. 20(b)), where $s_{\mathbf{g}}, \mathbf{g} \in S^2$ is given by $s_{\mathbf{g}}(\mathbf{h}) \triangleq (\mathbf{I} + 2\widehat{\mathbf{g}}^2)\mathbf{h}, \forall \mathbf{h} \in S^2$; the Euclidean space $\mathbb{R}^m$ (Fig. 20(c)), where $s_{\mathbf{g}}, \mathbf{g} \in \mathbb{R}^m$, is given by $s_{\mathbf{g}}(\mathbf{h}) \triangleq 2\mathbf{g} - \mathbf{h}, \forall \mathbf{h} \in \mathbb{R}^m$. It is less obvious that a Lie group such as SE(3) is also a symmetric space, with $s_{\mathbf{g}}, \mathbf{g} \in \mathrm{SE}(3)$ defined by $s_{\mathbf{g}}(\mathbf{h}) \triangleq \mathbf{g}\mathbf{h}^{-1}\mathbf{g}, \forall \mathbf{h} \in \mathrm{SE}(3)$.

A manifold N of a symmetric space M is called a *symmetric subspace* of M, if it is closed under inversion symmetry:

$$s_{\mathbf{g}}(\mathbf{h}) = \mathbf{g}\mathbf{h}^{-1}\mathbf{g} \in \mathrm{N}, \qquad \forall \mathbf{g}, \mathbf{h} \in \mathrm{N} \tag{82}$$

For example, $\mathbb{R}^n$, with $n < m$, is a symmetric subspace of $\mathbb{R}^m$.

5.2. Classification of symmetric subspaces of SE(3)

In [3], we have shown that a symmetric subspace M of SE(3) is always given by the exponential image of a special type of twist subspace $\mathfrak{m}$ called a *Lie triple subsystem* (LTS), which is defined to be closed under double Lie brackets:

$$[[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2], \boldsymbol{\xi}_3] \subset \mathfrak{m}, \qquad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3 \in \mathfrak{m} \tag{83}$$

All Lie subalgebras of $\mathfrak{se}(3)$ are (trivial) LTSs, and accordingly all subgroups of SE(3) are (trivial) symmetric subspaces of SE(3). In general, however, $\mathfrak{m}$ need not be a Lie subalgebra of $\mathfrak{se}(3)$. All symmetric subspaces $\mathrm{M} = \exp(\mathfrak{m})$ of SE(3) share the following properties:

Table 5: A complete list of the connected (non-trivial) symmetric subspaces of SE(3).

dim	LTS $\mathfrak{m}$	symmetric subspace	completion group	characteristics
2	$\mathfrak{m}_{2A} \triangleq \{t_z, r_x\}_{\text{sp}}$	$\text{M}_{2A} \triangleq \exp(\mathfrak{m}_{2A})$	$\text{G}_{2A} = \mathcal{E}(\mathbf{x})$	$\mathbf{xy}$-plane, $\mathbf{x}$-direction
	$\mathfrak{m}^p_{2A} \triangleq \{t_z, r_x + p t_x\}_{\text{sp}}$	$\text{M}^p_{2A} \triangleq \exp(\mathfrak{m}^p_{2A})$	$\text{G}^p_{2A} = \mathcal{Y}_p(\mathbf{x})$	$\mathbf{xy}$-plane, $\mathbf{x}$-direction
	$\mathfrak{m}_{2B} \triangleq \{r_x, r_y\}_{\text{sp}}$	$\text{M}_{2B} \triangleq \exp(\mathfrak{m}_{2B})$	$\text{G}_{2B} = \mathcal{S}$	$\mathbf{xy}$-plane, center $\mathbf{o}$
3	$\mathfrak{m}_{3A} \triangleq \{t_x, t_z, r_x\}_{\text{sp}}$	$\text{M}_{3A} \triangleq \exp(\mathfrak{m}_{3A})$	$\text{G}_{3A} = \mathcal{X}(\mathbf{x})$	$\mathbf{xy}$-plane, $\mathbf{x}$-direction
	$\mathfrak{m}_{3B} \triangleq \{r_x, r_y, t_z\}_{\text{sp}}$	$\text{M}_{3B} \triangleq \exp(\mathfrak{m}_{3B})$	$\text{G}_{3B} = \text{SE}(3)$	$\mathbf{xy}$-plane
4	$\mathfrak{m}_4 \triangleq \{t_x, t_y, r_x, r_y\}_{\text{sp}}$	$\text{M}_4 \triangleq \exp(\mathfrak{m}_4)$	$\text{G}_4 = \text{SE}(3)$	$\mathbf{z}$-axis
5	$\mathfrak{m}_5 \triangleq \{t_x, t_y, t_z, r_x, r_y\}_{\text{sp}}$	$\text{M}_5 \triangleq \exp(\mathfrak{m}_5)$	$\text{G}_5 = \text{SE}(3)$	$\mathbf{z}$-direction

1) For any LTS $\mathfrak{m}$, $\mathfrak{h}_\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$ is a lie subalgebra of $\mathfrak{se}(3)$ and is called *derived algebra* of $\mathfrak{m}$.

2) The sum of $\mathfrak{m}$ and $\mathfrak{h}_\mathfrak{m}$, denoted $\mathfrak{g}_\mathfrak{m}$:

$$\mathfrak{g}_\mathfrak{m} \triangleq \mathfrak{m} + \mathfrak{h}_\mathfrak{m} \tag{84}$$

is the *completion algebra* of $\mathfrak{m}$ in $\mathfrak{se}(3)$, i.e., the minimal Lie subalgebra that contains $\mathfrak{m}$; the corresponding Lie subgroup $\text{G}_\text{M} \triangleq \exp(\mathfrak{g}_\mathfrak{m})$ is the *completion group* of M in SE(3), i.e., the minimal Lie subgroup that contains M.

3) For any $\mathbf{g}$ in the Lie subgroup $\text{H}_\text{M} \triangleq \exp(\mathfrak{h}_\mathfrak{m})$ generated by $\mathfrak{h}_\mathfrak{m}$, i.e., $\mathbf{g} = e^{\boldsymbol{\eta}}$ for some $\boldsymbol{\eta} \in \mathfrak{h}_\mathfrak{m}$, we have:

$$C_\mathbf{g}(\text{M}) = \text{M}, \qquad \forall \mathbf{g} \in \text{H}_\text{M} \tag{85}$$

and correspondingly

$$\text{Ad}_\mathbf{g}(\mathfrak{m}) = \mathfrak{m} \tag{86}$$

This property is very important for the synthesis of parallel mechanisms for symmetric subspaces.

4) The twist subspace $\mathfrak{T}(\mathbf{g})$ of M at $\mathbf{g} = e^{\boldsymbol{\xi}} \in \text{M}$ is given by:

$$\mathfrak{T}(\mathbf{g}) = \text{Ad}_{\exp(\boldsymbol{\xi}/2)}(\mathfrak{T}(\mathbf{I})) = \text{Ad}_{\exp(\boldsymbol{\xi}/2)}(\mathfrak{m}) \tag{87}$$

The last property, known as the *half-angle property* in [3], is a special type of congruence-invariance. However, non-trivial symmetric subspaces are fundamentally different from POE manifolds (including Lie subgroups). A m-D POE manifold $\text{M} = \prod_{i=1}^m \exp(\{\boldsymbol{\xi}_i\}_{\text{sp}})$ always admits the second canonical parameterization in Eq. (16) (i.e., the POE formula):

$$(\theta_1, \ldots, \theta_m) \mapsto e^{\theta_1 \boldsymbol{\xi}_1} \cdots e^{\theta_m \boldsymbol{\xi}_m} \tag{88}$$

where $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m$ is a basis of $\mathfrak{T}(\mathbf{I}) = T_\mathbf{I}(\text{M})$. When M is a Lie subgroup, the second canonical parameterization is equivalent to the first canonical parameterization Eq. (15):

$$(\theta_1, \ldots, \theta_m) \mapsto \exp(\theta_1 \boldsymbol{\xi}_1 + \cdots + \theta_m \boldsymbol{\xi}_m) \tag{89}$$

However, in general, the two parameterizations are not equivalent (see [27] for a proof). While a POE manifold comes with a natural POE parameterization, the exponential form of a symmetric subspace $\text{M} \triangleq \exp(\mathfrak{m})$ naturally lends itself to the first canonical parameterization (c.f. Eq. (15))

$$(\theta_1, \ldots, \theta_m) \mapsto \exp(\theta_1 \boldsymbol{\xi}_1 + \cdots + \theta_m \boldsymbol{\xi}_m), \qquad \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m\}_{\text{sp}} = \mathfrak{m} \tag{90}$$

Just as for the classification of Lie subgroups of SE(3), the classification of conjugacy classes of symmetric subspaces of SE(3) can be accomplished by a classification of conjugacy classes

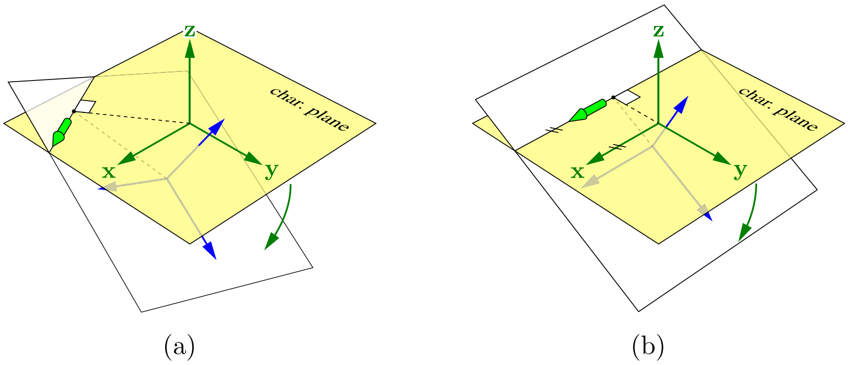


Figure 21: Physical interpretation of M_{3B} (a) and M_{2A} (b) motions.

of their corresponding LTSs in $\mathfrak{se}(3)$ [28], leading to the discovery of a total of seven classes of non-trivial symmetric subspaces of SE(3), which are listed in Tab. 5 for quick reference. We shall follow [3] and denote a m-D LTS with finite-pitch screws parallel to a single direction by $\mathfrak{m}_{mA}$, and a m-D LTS with finite-pitch screws parallel to a single plane by $\mathfrak{m}_{mB}$; the corresponding symmetric subspace, derived algebra, completion algebra and completion group shall be indicated by the same subscripts. When there is only one LTS for the specified dimension, the subscript $()_{mA}$ or $()_{mB}$ will be simply written as $()_m$.

There are three classes of 2-D symmetric subspaces of SE(3):

i) $\mathrm{M}_{2A} \triangleq \exp(\{\boldsymbol{t}_{\mathrm{z}}, \boldsymbol{r}_{\mathrm{x}}\}_{\mathrm{sp}})$, with $\mathfrak{h}_{2A} = \{\boldsymbol{t}_{\mathrm{y}}\}_{\mathrm{sp}}$ and $\mathfrak{g}_{2A} = \{\boldsymbol{t}_{\mathrm{y}}, \boldsymbol{t}_{\mathrm{z}}, \boldsymbol{r}_{\mathrm{x}}\}_{\mathrm{sp}} = \mathfrak{e}(\mathbf{x})$; we have $\mathrm{M}_{2A} \cdot \mathcal{P}(\mathbf{y}) = \mathcal{E}(\mathbf{x})$.

ii) $\mathrm{M}_{2A}^{p} \triangleq \exp(\{\boldsymbol{t}_{\mathrm{z}}, \mathbf{i}_p\}_{\mathrm{sp}})$, with $\mathfrak{h}_{2A} = \{\boldsymbol{t}_{\mathrm{y}}\}_{\mathrm{sp}}$ and $\mathfrak{g}_{2A} = \{\boldsymbol{t}_{\mathrm{y}}, \boldsymbol{t}_{\mathrm{z}}, \mathbf{i}_p\}_{\mathrm{sp}}$; if $p = 0$, $\mathrm{M}_{2A}^{p} = \mathrm{M}_{2A}$. We have $\mathrm{M}_{2A}^{p} \cdot \mathcal{P}(\mathbf{y}) = \mathcal{Y}_p(\mathbf{x})$.

iii) $\mathrm{M}_{2B} \triangleq \exp(\{\boldsymbol{r}_{\mathrm{x}}, \boldsymbol{r}_{\mathrm{y}}\}_{\mathrm{sp}})$, with $\mathfrak{h}_{2B} = \{\boldsymbol{r}_{\mathrm{z}}\}_{\mathrm{sp}} = \mathfrak{r}(\mathbf{z})$ and $\mathfrak{g}_{2B} = \{\boldsymbol{r}_{\mathrm{x}}, \boldsymbol{r}_{\mathrm{y}}, \boldsymbol{r}_{\mathrm{z}}\}_{\mathrm{sp}} = \mathfrak{s}(\mathbf{o})$; we have $\mathrm{M}_{2B} \cdot \mathcal{R}(\mathbf{z}) = \mathcal{S}(\mathbf{o})$.

There are three classes of 3-D symmetric subspaces of SE(3):

i) $\mathrm{M}_{3A} \triangleq \exp(\{\boldsymbol{t}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{z}}, \boldsymbol{r}_{\mathrm{x}}\}_{\mathrm{sp}})$, with $\mathfrak{h}_{3A} = \{\boldsymbol{t}_{\mathrm{y}}\}_{\mathrm{sp}} = \{\boldsymbol{t}_{\mathrm{y}}\}_{\mathrm{sp}}$ and $\mathfrak{g}_{3A} = \{\boldsymbol{t}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{y}}, \boldsymbol{t}_{\mathrm{z}}, \boldsymbol{r}_{\mathrm{x}}\}_{\mathrm{sp}} = \mathfrak{x}(\mathbf{x})$; we have $\mathrm{M}_{3A} \cdot \mathcal{P}(\mathbf{y}) = \mathcal{X}(\mathbf{x})$. It is obvious that M_{3A} contains M_{2A} and M_{2A}^{p} as manifolds.

ii) $\mathrm{M}_{3B} \triangleq \exp(\{\boldsymbol{t}_{\mathrm{z}}, \boldsymbol{r}_{\mathrm{x}}, \boldsymbol{r}_{\mathrm{y}}\}_{\mathrm{sp}})$, with $\mathfrak{h}_{3B} = \{\boldsymbol{t}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{y}}, \boldsymbol{r}_{\mathrm{z}}\}_{\mathrm{sp}} = \mathfrak{e}(\mathbf{z})$, and $\mathfrak{g}_{3B} = \mathfrak{se}(3)$; we have $\mathrm{M}_{3B} \cdot \mathcal{E}(\mathbf{z}) = \mathrm{SE}(3)$. It is obvious that M_{3B} contains M_{2B} as a manifold.

There is exactly one class of 4-D symmetric subspace of SE(3):

i) $\mathrm{M}_{4} \triangleq \exp(\{\boldsymbol{t}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{y}}, \boldsymbol{r}_{\mathrm{x}}, \boldsymbol{r}_{\mathrm{y}}\}_{\mathrm{sp}})$, with $\mathfrak{h}_{4} = \{\boldsymbol{t}_{\mathrm{z}}, \boldsymbol{r}_{\mathrm{z}}\}_{\mathrm{sp}} = \mathfrak{c}(\mathbf{z})$, and $\mathfrak{g}_{4} = \mathfrak{se}(3)$; we have $\mathrm{M}_{4} \cdot \mathcal{C}(\mathrm{z}) = \mathrm{SE}(3)$. M_4 contains all lower dimensional symmetric subspaces, except M_{3B}.

There is exactly one class of 5-D symmetric subspace of SE(3):

i) $\mathrm{M}_{5} \triangleq \exp(\{\boldsymbol{t}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{y}}, \boldsymbol{t}_{\mathrm{z}}, \boldsymbol{r}_{\mathrm{x}}, \boldsymbol{r}_{\mathrm{y}}\}_{\mathrm{sp}})$, with $\mathfrak{h}_{5} = \{\boldsymbol{t}_{\mathrm{x}}, \boldsymbol{t}_{\mathrm{y}}, \boldsymbol{t}_{\mathrm{z}}, \boldsymbol{r}_{\mathrm{z}}\}_{\mathrm{sp}} = \mathfrak{x}(\mathbf{z})$, and $\mathfrak{g}_{5} = \mathfrak{se}(3)$; we have $\mathrm{M}_{5} \cdot \mathcal{R}(\mathbf{z}) = \mathrm{SE}(3)$. M_5 contains all lower dimensional symmetric subspaces.

In [27], we gave a physical explanation of the type of motion represented by symmetric subspaces of SE(3). For example, as shown in Fig. 21, M_{3B} comprises non-redundant motions of the xy-plane (called the *characteristic plane*) to a new location via a single rotation about the intersection line of the two plane locations without incurring a redundant self-motion of

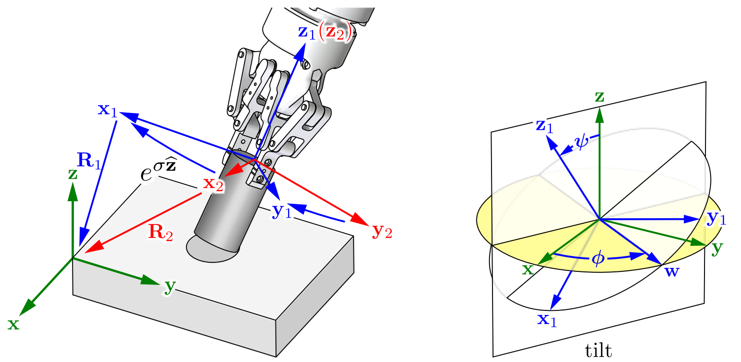
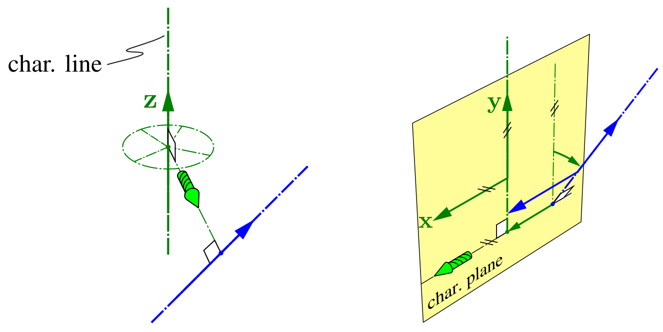


Figure 22: A motion in M_{2B} corresponds to the tilt of the symmetry-axis (the $\mathbf{z}$-axis) via a rotation about an axis in the $\mathbf{xy}$-plane that is perpendicular to the initial and final location of the $\mathbf{z}$-axis.



(a) (b)

Figure 23: Physical interpretation of M_4 (a) and M_{3A} (b).

the plane, represented by $H_{3B} = \mathcal{E}(\mathbf{z})$. M_{2A} may be considered a submanifold of M_{3B} that maintains the perpendicularity of the plane normal to the $\mathbf{x}$-axis.

Similarly, M_{2B} characterizes the rotation of an object that is axis symmetric about the $\mathbf{z}$-axis (called the *characteristic direction*) without incurring a redundant self-spin of the object, represented by $H_{2B} = \mathcal{R}(\mathbf{z})$ (see Fig. 22). If we augment M_{2B} with all translational freedoms, we arrive at the 5-D symmetric subspace M_5.

Finally, M_4 characterizes the motion of the $\mathbf{z}$-axis (called the *characteristic line*) to a target location by twisting along the common perpendicular of the two lines without incurring redundant self-spin and sliding, which is represented by $H_4 = \mathcal{C}(\mathbf{z})$ (see Fig. 23). M_{3A} may be considered a submanifold of M_4 that maintains the perpendicularity of the line to the $\mathbf{x}$-axis.

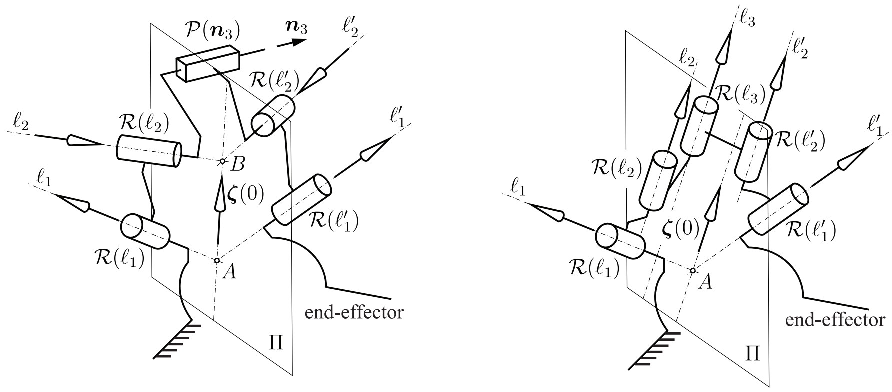
5.3. Mechanism synthesis of symmetric subspaces of SE(3)

The key to synthesizing mechanisms for the seven symmetric subspaces lies in the inversion symmetry property in Eq. (82) [3, 27].

Example 8 (M_{3B} parallel mechanism)**.** Consider the 3-D symmetric subspace $M_{3B} = \exp(\{\mathfrak{m}_{3B}\}_{\text{sp}})$, with the corresponding LTS given by $\mathfrak{m}_{3B} = \{\boldsymbol{r}_{\text{x}}, \boldsymbol{r}_{\text{y}}, \boldsymbol{t}_{\text{z}}\}_{\text{sp}}$. Due to the incompatibility of the two canonical parameterizations in Eq. (15) and Eq. (16) for symmetric subspaces, we cannot synthesize M_{3B} by assigning an ordered basis $(\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3)$ to its LTS $\mathfrak{m}_{3B}$ and then generate M_{3B} by the corresponding serial mechanism: indeed, the resulting POE manifold $\exp(\{\mathbf{S}_1\}_{\text{sp}})\exp(\{\mathbf{S}_2\}_{\text{sp}})\exp(\{\mathbf{S}_3\}_{\text{sp}})$ will in general induce residual planar motion in the decomposition $\mathrm{SE}(3) = M_{3B}\mathcal{E}(\mathbf{z})$.

On the other hand, by repeatedly applying the inversion symmetry property of M_{3B}:

$$\mathbf{g}\mathbf{h}^{-1}\mathbf{g} \in M_{3B}, \qquad \forall \mathbf{g}, \mathbf{h} \in M_{3B} \tag{91}$$



(a) (b)

Figure 24: Two examples of symmetric chains for M_{3B}.

we see that:

$$(\theta_1, \theta_2, \theta_3) \mapsto e^{\theta_1 \mathbf{S}_1} e^{\theta_2 \mathbf{S}_2} e^{2\theta_3 \mathbf{S}_3} e^{\theta_2 \mathbf{S}_2} e^{\theta_1 \mathbf{S}_1} \in \mathrm{M}_{3B} \tag{92}$$

gives a valid parameterization of M_{3B}.

Therefore, a *symmetric chain* of the form $(\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_2, \mathbf{S}_1)$ will generate M_{3B} and maintain the congruence-invariance of its twist subspace if each pair of joints with identical screw axes move with equal joint angles. In [3, 27], we showed that the symmetric chain can be generalized to have different but mirror-symmetric (about the characteristic plane) joint screws and still generate the desired symmetric subspace M_{3B}. More precisely, a general symmetric chain has the following form:

$$(\mathbf{S}_1^+, \mathbf{S}_2^+, \mathbf{S}_3, \mathbf{S}_2^-, \mathbf{S}_1^-) \tag{93}$$

where:

$$\begin{cases} \mathbf{S}_i^+ = \boldsymbol{\xi}_i + \boldsymbol{\eta}_i \\ \mathbf{S}_i^- = \boldsymbol{\xi}_i - \boldsymbol{\eta}_i \end{cases} \qquad \boldsymbol{\xi}_i \in \mathfrak{m}_{3B}, \boldsymbol{\eta}_i \in \mathfrak{e}(\mathbf{z}) \tag{94}$$

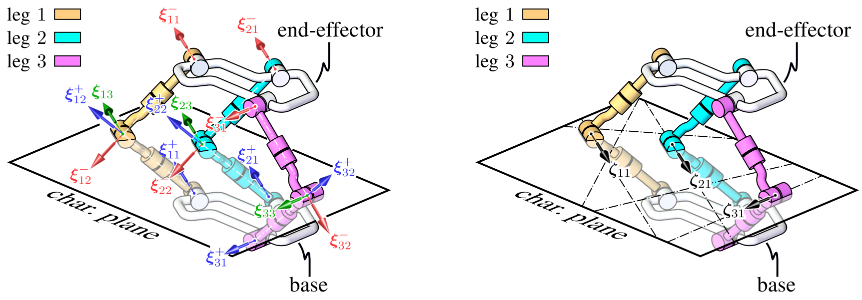
for $i = 1, 2$ and so that $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \mathbf{S}_3\}_{\mathrm{sp}} = \mathfrak{m}_{3B}$. The 5-$\mathcal{R}$ chain we have seen in Fig. 20(a) is one such example. Two more examples of symmetric chains for M_{3B} are given in Fig. 24

The symmetric chain will not move with symmetric joint angles on its own. To ensure symmetric joint movements, we can impose additional constraint by forming a parallel mechanism with multiple symmetric chains $\mathcal{M}_i = (\mathbf{S}_{i1}^+, \mathbf{S}_{i2}^+, \mathbf{S}_{i3}, \mathbf{S}_{i2}^-, \mathbf{S}_{i1}^-), i = 1, \ldots, l$. To ensure that the chain motion manifolds $\mathrm{M}_i = \exp(\{\mathbf{S}_{i1}^+\}_{\mathrm{sp}}) \exp(\{\mathbf{S}_{i2}^+\}_{\mathrm{sp}}) \exp(\{\mathbf{S}_3\}_{\mathrm{sp}}) \exp(\{\mathbf{S}_{i2}^-\}_{\mathrm{sp}}) \cdots \exp(\{\mathbf{S}_{i1}^-\}_{\mathrm{sp}})$ do contain the desired manifold $\mathrm{M} = \mathrm{M}_{3B}$ (step 2), it suffices to assemble the chains so that they are all mirror-symmetric about the same characteristic plane of M_{3B}, namely the **xy**-plane, at the home configuration, as shown in Fig. 25(a). Since each chain $\mathcal{M}_i$ has 5-DOFs, it contributes to one basis wrench $\boldsymbol{\zeta}_i$. The wrench condition (step 3) is met if:

$$\{\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_l\}_{\mathrm{sp}} = \mathfrak{W}(\mathbf{I}) = \mathfrak{m}_{3B}^{\perp} = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{y} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{z} \end{pmatrix} \right\}_{\mathrm{sp}} \tag{95}$$

In the case of 5-$\mathcal{R}$ symmetric chains (as shown in Fig. 25(a)), each $\boldsymbol{\zeta}_i$ is a zero-pitch wrench lying in the **xy**-plane. Consequently, a total of three chains is needed to form a valid parallel mechanism for M_{3B}, as shown in Fig. 25(b).

Hunt was the first to analyze the persistent mirror symmetry of M_{3B} parallel mechanisms when working on his general theory of parallel constant velocity couplings [23]. Carricato [25] made further analysis of the synthesis condition for such parallel mechanisms. The theory was finally completed in our recent publications [3, 27], where we also showed that the synthesis procedure demonstrated here can be generalized to general symmetric subspaces in a similar manner. $\diamondsuit$



(a) (b)

Figure 25: Example of a M_{3B} parallel mechanism: (a) chain geometry; (b) constraint wrenches.

6. Conclusion

In this paper, we provided a unified treatment of three important classes of motion manifolds of SE(3) that admit a persistent twist subspace over the entire motion manifold (around the identity):

1. a Lie subgroup $\mathrm{G} = \exp(\mathfrak{g})$ has, for any configuration $\mathbf{g} \in \mathrm{G}$, an invariant twist subspace $\mathfrak{T}(\mathbf{g})$ equal to its Lie algebra $\mathfrak{g}$ for any $\mathbf{g} \in \mathrm{G}$.

2. a persistent POE manifold $\mathrm{M} \triangleq \prod_{i=1}^{m} \exp(\{\boldsymbol{\xi}_i\}_{\mathrm{sp}})$ has, for any configuration $\mathbf{g} \in \mathrm{M}$, a congruence-invariant twist subspace $\mathfrak{T}(\mathbf{g})$ equal to $\mathrm{Ad}_{\mathbf{g}'}(\mathfrak{T}(\mathbf{I})) = \mathrm{Ad}_{\mathbf{g}'}(\{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m\}_{\mathrm{sp}})$ for a certain $\mathbf{g}' \in \mathrm{SE}(3)$, which is a conjugacy copy of its identity tangent space $\{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m\}_{\mathrm{sp}}$.

3. a symmetric subspace $\mathrm{M} = \exp(\mathfrak{m})$ has, for any configuration $e^{\boldsymbol{\xi}} \in \mathrm{M}$, a congruence-invariant twist subspace $\mathfrak{T}(e^{\boldsymbol{\xi}})$ equal to $\mathrm{Ad}_{\exp(\boldsymbol{\xi}/2)}(\mathfrak{T}(\mathbf{I})) = \mathrm{Ad}_{\exp(\boldsymbol{\xi}/2)}(\mathfrak{m})$, which is a conjugacy copy of its LTS $\mathfrak{m}$.

The persistence properties of twist subspaces arise from the persistence property of their underlying manifold structure. Indeed, persistent motion manifolds are the maximal integral manifolds of certain persistent distributions, which are spanned by persistent vector fields, such as the left-invariant and right-invariant vector fields for Lie subgroups and binary products, and the derivations (a mixture of left and right-invariant vector fields) for symmetric subspaces.

Due to space limit, we have intentionally left out further detailed treatment on mechanism synthesis. However, we hope that this both introductory and summarizing treatment of persistent motion manifolds has demonstrated the combined power of Lie theory of SE(3) and screw theory of $\mathfrak{se}(3)$ in the field of mechanism synthesis.

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