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# Adaptive Output Regulation for Multivariable Linear Systems via Slow Identifiers

Alessandro Melis, Michelangelo Bin and Lorenzo Marconi

**Abstract**—This paper deals with the problem of adaptive output regulation for linear multivariable systems. The proposed solution employs a continuous-time identifier that adapts the parameters of the internal model to match the (unknown) exosystem frequencies. Boundedness of the closed-loop trajectories is established and, under a persistence of excitation condition, asymptotic regulation is shown.

## I. INTRODUCTION

The output regulation problem for linear systems was firstly addressed by Francis, Wonham and Davison in the 70s (see e.g. [1], [2] and [3]), where the so-called internal model principle was introduced. The linear regulator boasts an exceptional robustness property with respect to the plant's parameters, i.e., the regulation error is ensured to vanish even in the presence of large perturbations, as long as linearity and closed-loop stability are preserved. Nevertheless, asymptotic regulation is inexorably lost whenever the exosystem is not perfectly known, namely, the linear regulator is not robust with respect to any, although arbitrarily small, perturbation of the exosystem.

The general problem of designing a regulator for a linear system ensuring asymptotic regulation in the presence of uncertainties in the exosystem is still open, even though in the last decades many papers have been written on the topic. In [4] and [5] adaptive observers have been used to asymptotically estimate the internal model's parameters in the single-input single-output (SISO) case. In both papers, perfect knowledge of the plant is assumed, sacrificing robustness with respect to plant's perturbations for robustness to uncertainties in the exosystem. Multivariable linear systems have been considered in [6], under a strong minimum-phase assumption, and in [7], where only state-feedback tracking is addressed. Further approaches can be found in the context of nonlinear SISO minimum-phase normal forms. In [8] an estimation law based on Lyapunov-like arguments is proposed to deal with linear uncertain exosystems. Instead of adaptation, immersion arguments have been used in [9], [10], [11], [12] and [13] for linear and some classes of nonlinear exosystems. More recently, a different approach based on identification techniques has been proposed in [14] for SISO normal forms, while in [15] a hybrid adaptive observer is designed for SISO stable plants, and an adaptive design for multivariable linear systems, based on discrete-time identification schemes, has been proposed in [16].

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In this paper we consider the output regulation problem for general multivariable linear systems, with the reference signals and the disturbances that are generated by an unknown exosystem. On the heels of [16], we augment a canonical linear regulator with an identification unit (referred to as the “identifier”) that adapts the internal model on the basis of the measurable data. Differently from [16], the identifier is continuous-time, and the asymptotic properties of the regulator are obtained thanks to a time-scale separation of the two units. The identifier is designed to solve a least-squares optimization problem defined by the available measurements. Linearity and persistency of excitation ensure the existence of a unique global solution matching the parameters of the exosystem, despite possibly large deviations of the plant's state from the ideal error-zeroing steady state. The design of the stabilization and the adaptation laws turns out to be decoupled, making thus possible to handle general linear non-minimum phase systems.

Examples of application of the proposed work are reported in [17] and [18], where the problem of an aircraft landing on an oscillating platform, and active control of mechanic suspensions subject to unknown disturbances, generated by neutrally stable linear exosystems, are respectively studied.

The paper is organized as follows. In Section 2 the system and the main assumptions are presented. In Section 3 the different subsystems composing the regulator are detailed. In Section 4 we state the main result of this paper, whose proof is reported in Section 5. In Section 6 we provide an example where the proposed regulator is applied.

*Notation:*  $\mathbb{R}$  and  $\mathbb{N}$  denote the sets of real and natural numbers respectively, and we let  $\mathbb{R}_{\geq 0} := [0, \infty)$  and  $\mathbb{R}_{> 0} := (0, \infty)$ . We denote by  $\|\cdot\|$  any vector or matrix norm whenever the underlying normed space is clear. With  $x : \mathbb{R} \rightarrow \mathbb{R}^n$  and  $t \in \mathbb{R}$ , we let  $\|x\|_t := \sup_{\tau \in [0, t]} |x(\tau)|$  denote the uniform norm of  $x(t)$ . With  $\mathcal{X}$  an Euclidean space,  $X \subset \mathcal{X}$  and  $z \in \mathcal{X}$ , we denote by  $|z|_X := \inf_{x \in X} |z - x|$  the distance from  $z$  to the set  $X$  and by  $p_X(z) := \{x : |z - x| = |z|_X\}$  the projection of  $z$  onto  $X$ . For a square matrix  $A$ , we denote by  $\sigma(A)$  the set of its eigenvalues and by  $\varphi_A(s)$  its characteristic polynomial. The symbol  $\otimes$  indicates the Kronecker product of matrices. For  $x \in \mathbb{R}^n$ ,  $n \in \mathbb{N}$ , and  $i < j$ , we let  $x_{[i, j]} := \text{col}(x_i, \dots, x_j)$ .

## II. PROBLEM FORMULATION

We consider linear systems of the form

$$\dot{w} = Sw \quad (1)$$

$$\dot{x} = Ax + Bu + Pw \quad (2)$$

$$y_m = C_m x + Q_m w \quad (3)$$

$$e = C_e x + Q_e w, \quad (4)$$

with  $w \in \mathbb{R}^{n_w}$  an exogenous input,  $x \in \mathbb{R}^{n_x}$  the state,  $u \in \mathbb{R}^{n_u}$  the control input,  $e \in \mathbb{R}^{n_e}$  the regulation error,  $y_m \in \mathbb{R}^{n_m}$  additional measurements and  $n_w, n_x, n_u, n_e, n_m \in \mathbb{N}$  such that  $n_u \geq n_e$ . The signal  $w(t)$  models exogenous disturbances and reference signals acting on the system, whose modes are defined by the matrix  $S$ , that we do not assume to be known but we suppose to be neutrally stable.

In this paper we consider the problem of output regulation for the system (1)-(4), that is, we aim to design an output feedback regulator of the form

$$\dot{\mu} = f_c(\mu, y) \quad (5)$$

$$u = \gamma(\mu, y), \quad (6)$$

where  $y := \text{col}(y_m, e)$  and  $\mu \in \Xi$ , with  $\Xi$  an Euclidean space, such that the trajectories solution of (1)-(5) are bounded and

$$\lim_{t \rightarrow \infty} e(t) = 0. \quad (7)$$

In the rest of the paper we make the following standing assumptions<sup>1</sup>

*Assumption 1:*  $(A, B)$  is stabilizable,  $(C, A)$ , with  $C := [C_m \ C_e]$ , is detectable and  $\text{rank } B \geq \text{rank } C_e$ .

*Assumption 2:*  $S$  is neutrally stable and the initial conditions of (1) range in a compact invariant set  $W \subset \mathbb{R}^{n_w}$ .

## III. THE REGULATOR STRUCTURE

The regulator is composed of three different subsystems: the *internal model unit*, the *identifier* and the *stabiliser*. The internal model unit, based on the design proposed in [3], is an error-driven dynamical system that, ideally, incorporates the modes of the exosystem. The identifier is a continuous-time system whose objective is to adapt the internal model unit to asymptotically match the actual exosystem's parameters. If the exosystem were known, the internal model could be designed as a linear system in normal form with the same characteristic polynomial of  $S$  [3]. As in this paper we do not assume to know  $S$ , we still retain a similar structure, with the parameters defining the internal model's dynamics that are decided at runtime by the identifier. The stabiliser is a subsystem that, for each fixed value of the identifier, stabilizes the cascade interconnection of the plant and internal model unit. We detail the three subsystems in the rest of the section.

<sup>1</sup>We observe that Assumption 1 is also necessary for the solvability of the problem at hand.

### A. The Internal Model

The internal model unit is designed as the following dynamical system

$$\dot{\eta} = \Phi(\eta, z) + Ge, \quad (8)$$

with  $\eta \in \mathbb{R}^{n_e(n_w+1)}$ ,  $z \in \mathcal{Z}$ , with  $\mathcal{Z}$  an Euclidean space that will be defined in the next subsection, the state of the identifier that will adapt the internal model and

$$\Phi(\eta, z) := \begin{pmatrix} \eta_{[2, n_w+1]} \\ \Psi(\eta, z) \end{pmatrix}, \quad G := \begin{pmatrix} 0_{n_e n_w \times n_e} \\ I_{n_e} \end{pmatrix}, \quad (9)$$

with  $\Psi : \mathbb{R}^{n_e(n_w+1)} \times \mathbb{R}^{n_w} \times \mathcal{Z} \rightarrow \mathbb{R}^{n_e}$  to be fixed. If the closed-loop system is stable, it reaches a steady-state in which all the variables oscillate with the same modes of the exosystem. As the dimension of  $\eta$  is  $n_w + 1$ , the Cayley-Hamilton Theorem implies that, at such steady state,  $\eta$  must satisfy a regression of the kind

$$\eta_{n_w+1} = (\theta^{\circ T} \otimes I_{n_e}) \eta_{[1, n_w]}, \quad (10)$$

with  $\theta^{\circ}$  matching the coefficients of the characteristic polynomial of  $S$  (modulo a change of sign). The intuition behind the proposed approach is to look at (10) as a *prediction error model*, asymptotically relating the state  $\eta$  of the internal model (measured) with the sought unknown characteristic polynomial of  $S$ .

We postpone the design of the identifier  $z$  to find  $\theta^{\circ}$  in (10) to the next section, while for the moment we assume that we have a guess of  $\theta^{\circ}$  given by

$$\theta = \omega(z),$$

with  $\omega : \mathcal{Z} \rightarrow \mathbb{R}^{n_w}$ , to be defined later.

The design of the internal model is completed by letting

$$\Psi(\eta, z) := (p_{\mathcal{E}}(\omega(z))^T \otimes I_{n_e}) \eta_{[2, n_w+1]} + \bar{\rho}(\eta, z), \quad (11)$$

with  $p_{\mathcal{E}} : \mathbb{R}^{n_w} \rightarrow \mathcal{E}$  the projection operator onto a compact convex set  $\mathcal{E} \subset \mathbb{R}^{n_w}$  to be fixed, and with  $\bar{\rho} : \mathbb{R}^{n_e(n_w+1)} \times \mathcal{Z} \rightarrow \mathbb{R}^{n_e}$  a bounded function to be chosen later according to the identifier's structure.

With the choice (11), and for a suitable choice of  $\bar{\rho}$  such that  $\bar{\rho}(\eta, z)$  vanishes whenever  $\theta$  equals its "ideal" value  $\theta^{\circ}$ , the system (8) with  $\theta = \theta^{\circ}$  is able to reproduce all the modes of the exosystem, and it thus candidates as a proper internal model.

To fix the set  $\mathcal{E}$  in (11), we first define the set

$$\mathcal{Q} := \left\{ \theta \in \mathbb{R}^{n_w} : \text{rank} \begin{pmatrix} A - \mu I & B \\ C_e & 0 \end{pmatrix} < n_x + n_e, \right. \\ \left. \mu \in \sigma \left( \begin{pmatrix} 0_{n_e n_w \times n_e} & I_{n_e n_w} \\ 0_{n_e \times n_e} & \theta^T \otimes I_{n_e} \end{pmatrix} \right) \right\}$$

which represents the set of  $\theta \in \mathbb{R}^{n_w}$  for which the *non-resonance* condition is not satisfied, i.e. for which the cascade (2), (8) is not stabilizable. We thus fix the set  $\mathcal{E}$  as any (arbitrarily large) compact convex such that  $\mathcal{E} \cap \mathcal{Q} = \emptyset$ . The existence of such a set, when the transfer function of the plant has a finite number of zeros, has been proved in [16].

## B. The Identifier

We approach the design of the identifier  $z$  by looking at (10) as a linear regression relating the steady state values of the state  $\eta$ , and by casting the estimation problem of the ideal parameters  $\theta^\circ$  as a recursive least-squares problem in the variables  $\eta_{n_w+1}$  and  $\eta_{[1, n_w]}$ . More precisely, we define the “prediction error”

$$\varepsilon(t, \theta) := \eta_{n_w+1}(t) - (\theta^T \otimes I_{n_e})\eta_{[1, n_w]}(t), \quad (12)$$

and we associate to each signal  $\eta(t)$  the “cost functional”

$$(\mathcal{J}_\eta(\theta))(t) = \lambda \int_0^t e^{-\lambda(t-s)} |\varepsilon(s, \theta)|^2 ds, \quad (13)$$

with  $\lambda > 0$ . The parameter  $\theta$  is decided by the identifier so as to minimize (13).

We define the identifier as a continuous-time system defined on the state-space  $\mathcal{Z} := \mathbb{R}^{n_w \times n_w} \times \mathbb{R}^{n_w}$  and with state partitioned as  $z := (R, v)$ , with  $R \in \mathbb{R}^{n_w \times n_w}$  and  $v \in \mathbb{R}^{n_w}$ , whose evolution is described by the following equations

$$\begin{aligned} \dot{R} &= -\lambda R + \lambda \gamma(\eta_{[1, n_w]}) \gamma(\eta_{[1, n_w]})^T \\ \dot{v} &= -\lambda v + \lambda \gamma(\eta_{[1, n_w]}) \eta_{n_w+1} \\ \theta &= R^\dagger v, \end{aligned} \quad (14)$$

where  $\lambda$  is the same as in (13),  $\cdot^\dagger$  denotes the Moore-Penrose pseudoinverse and  $\gamma : \mathbb{R}^{n_e n_w} \rightarrow \mathbb{R}^{n_w \times n_w}$  is defined as

$$\gamma(\eta_{[1, n_w]}) := \text{col}(\eta_1^T \ \eta_2^T \ \cdots \ \eta_{n_w}^T).$$

We equip  $\mathcal{Z}$  with the norm  $|z| = |(R, v)| := \sqrt{|R|^2 + |v|^2}$ . In order to obtain the differentiability of the map  $t \mapsto R(t)^\dagger v(t)$ , and in order to have uniqueness of solutions to the minimization problem (13), we define the following *persistence of excitation* (PE) property for  $\eta$ .

*Definition 1:* With  $\epsilon, T > 0$ , the signal  $\eta$  is said to have the  $(\epsilon, T)$  – *persistence of excitation* property if for all  $t \geq T$

$$\det \int_0^t \gamma(\eta_{[1, n_w]}(s)) \gamma(\eta_{[1, n_w]}(s))^T ds \geq \epsilon. \quad (15)$$

We observe that, by continuity, the PE condition (15) can be checked online simply by looking at  $\det(R(t))$ . As a matter of fact it is easy to see that, for any initial condition  $R(0)$ , the difference between  $R(t)$  and the matrix appearing in (15) vanishes asymptotically.

For future readability we rewrite (14) in compact form

$$\begin{aligned} \dot{z} &= \lambda l(z, \eta) \\ \theta &= \omega(z), \end{aligned} \quad (16)$$

with  $l : \mathcal{Z} \times \mathbb{R}^{n_e(n_w+1)} \rightarrow \mathcal{Z}$  and  $\omega : \mathcal{Z} \rightarrow \mathbb{R}^{n_w}$  defined as

$$\begin{aligned} l(z, \eta) &:= \begin{pmatrix} -R + \gamma(\eta_{[1, n_w]}) \gamma(\eta_{[1, n_w]})^T \\ -v + \gamma(\eta_{[1, n_w]}) \eta_{n_w+1} \end{pmatrix} \\ \omega(z) &:= R^\dagger v. \end{aligned}$$

We conclude the design of the internal model unit by letting  $\bar{\rho}(\eta, z)$  in (11) be defined as

$$\bar{\rho}(\eta, z) := \text{sat} \left( \frac{\partial \Delta(\eta_{[1, n_w]}, z)}{\partial z} \lambda l(z, \eta) \right), \quad (17)$$

with  $\text{sat}(\cdot)$  any properly defined smooth saturation function and

$$\Delta(\eta_{[1, n_w]}, z) := (\omega(z)^T \otimes I_{n_e}) \eta_{[1, n_w]}.$$

## C. The Stabilizer

The stabilizer is a linear output feedback controller parametrized by  $\theta$ , and continuous in  $p_\mathcal{E}(\theta)$ , of the form:

$$\begin{aligned} \dot{\chi} &= H_\chi(p_\mathcal{E}(\theta)) \chi + H_y(p_\mathcal{E}(\theta)) y + H_\eta(p_\mathcal{E}(\theta)) \eta \\ u &= K_\chi(p_\mathcal{E}(\theta)) \chi + K_y(p_\mathcal{E}(\theta)) y + K_\eta(p_\mathcal{E}(\theta)) \eta, \end{aligned} \quad (18)$$

and it is designed in order to make the matrix

$$F(\theta) := \begin{pmatrix} A + BK_y(p_\mathcal{E}(\theta))C & BK_\eta(p_\mathcal{E}(\theta)) & BK_\chi(p_\mathcal{E}(\theta)) \\ G_\eta C_e & \Phi_\eta(\theta) & 0 \\ H_y(p_\mathcal{E}(\theta))C & H_\eta(p_\mathcal{E}(\theta)) & H_\chi(p_\mathcal{E}(\theta)) \end{pmatrix} \quad (19)$$

Hurwitz for all  $\theta \in \mathbb{R}^{n_w}$ , where:

$$\Phi_\eta(\theta) = \begin{pmatrix} 0_{n_e n_w \times n_e} & I_{n_e n_w} \\ 0_{n_e \times n_e} & p_\mathcal{E}(\theta)^T \otimes I_{n_e} \end{pmatrix}, \quad G_\eta = \begin{pmatrix} 0_{n_e n_w \times n_e} \\ I_{n_e} \end{pmatrix} \quad (20)$$

Under Assumption 1, and by construction of  $\mathcal{E}$ , a stabilizer of the form (18) making  $F(\theta)$  Hurwitz always exists.

## IV. MAIN RESULT

The closed-loop system reads as

$$\dot{z} = \lambda l(z, \xi) \quad (21)$$

$$\dot{w} = Sw \quad (22)$$

$$\dot{\xi} = F(\theta)\xi + P_\xi w + \lambda \rho_\xi(\eta, z), \quad (23)$$

where  $\xi := (x, \eta, \chi)$ ,  $F(\theta)$  is as in (19),  $P_\xi := \text{col}(P, 0_{(n_w+1+n_\chi) \times n_w})$  and

$$\rho_\xi(\eta, z) := \text{col} \left( 0_{n_x \times 1}, \frac{1}{\lambda} \bar{\rho}(\eta, z), 0_{n_\chi \times 1} \right),$$

The following proposition characterizes the asymptotic properties of the regulator.

*Proposition 1:* Suppose Assumptions 1 and 2 are satisfied, then the trajectories of the closed loop (21)-(23) are bounded. If in addition there exists  $\theta^\circ \in \mathcal{E}$  such that

$$-\sum_{i=1}^{n_w} \theta_i^\circ s^{i-1} + s^{n_w} = \varphi_S(s),$$

then for any  $\epsilon > 0$  there exists  $\lambda^* > 0$  such that, if  $\lambda \leq \lambda^*$ , any solution of the closed-loop system (21)-(23) such that, for some  $T > 0$ ,  $\eta$  has the  $(\epsilon, T)$  – *persistence of excitation* property, also satisfies

$$\lim_{t \rightarrow \infty} e(t) = 0.$$

□

Proposition 1 states that, in order to obtain asymptotic regulation along the persistently exciting solutions, the dynamics of the identifier have to be slow enough compared to the rest of the control system. In this respect it is worth comparing this result with the approach of [16], where the time-separation of the adaptation dynamics is obtained by means of a discrete-time identifier working on time instants that must be separated, on average, by a sufficiently large amount of time.

We also observe that the convergence of  $e$  to zero is uniform only inside the set of the solutions for which the signals  $\eta(t)$  satisfy Definition 1 with the same  $\epsilon$  and  $T$ . In this respect we also observe that if Definition 1 is satisfied with  $\epsilon_1$  and  $T_1$ , then it is satisfied with  $\epsilon_2$  and  $T_2$  for any  $\epsilon_2 < \epsilon_1$  and  $T_2 > T_1$ .

## V. PROOF OF PROPOSITION 1

Boundedness of the trajectories of (21), (22) and (23) is readily shown by noticing that, due to the design of the stabilizer, (23) is an asymptotically stable system driven by two bounded inputs, namely  $P_\xi w$  and  $\lambda \rho_\xi(z, \xi)$ . The input to the identifier is thus bounded, and, by the structure of (14), boundedness is proved.

Consider the closed loop system (21), (22) and (23) under the change of coordinates  $\xi \mapsto \tilde{\xi} = \xi - \Pi(\theta)w$ , with  $\Pi(\theta)$  the unique (smooth in  $p_\mathcal{E}(\theta)$ ) solution to the Sylvester equation  $S\Pi(\theta) - F(\theta)\Pi(\theta) = P_\xi$ . In the new coordinates the closed-loop system reads as

$$\begin{aligned} \dot{z} &= \lambda l(z, \eta) \\ \dot{w} &= Sw \\ \dot{\tilde{\xi}} &= F(\theta)\tilde{\xi} + \lambda \rho_\xi(\tilde{\eta}, w, \theta, z), \end{aligned} \quad (24)$$

where:

$$\begin{aligned} \rho_\xi(\tilde{\eta}, w, \theta, z) &= \\ &= \frac{1}{\lambda} \bar{\rho}(\tilde{\eta} + \Pi_\eta(\theta)w, z) - \frac{\partial \Pi(\theta)}{\partial \theta} w \frac{\partial \omega(z)}{\partial z} l(z, \tilde{\eta} + \Pi_\eta(\theta)w). \end{aligned} \quad (25)$$

We will first analyze the stability properties of the identifier  $z$ .

To this end, consider  $\eta^*(t) = \Pi_\eta(\theta)w(t)$  and define  $z^*(t) \in \mathcal{Z}$  as  $z^* = (R^*, v^*)$ , where

$$\begin{aligned} R^*(t) &:= \lambda \int_0^t e^{-\lambda(t-s)} \gamma(\eta_{[1, n_w]}^*(s)) \gamma(\eta_{[1, n_w]}^*(s))^T ds \\ v^*(t) &:= \lambda \int_0^t e^{-\lambda(t-s)} \gamma(\eta_{[1, n_w]}^*(s)) \eta_{n_w+1}^*(s) ds. \end{aligned}$$

Then  $z^*(t)$  is solution of (14) for  $z(0) = 0$  and  $\eta = \eta^*$ .

Define  $\tilde{\eta} := \eta - \eta^*$ . It is possible to show, by Lipschitz continuity of  $\bar{u}(\eta) := (\gamma(\eta_{[1, n_w]}) \gamma(\eta_{[1, n_w]})^T, \gamma(\eta_{[1, n_w]}) \eta_{n_w+1})$ , that

$$|\tilde{z}(t)| \leq \beta_{\tilde{z}} (|\tilde{z}(0)|, t) + \alpha_{\tilde{z}} \|\tilde{\eta}\|_t, \quad (26)$$

where  $\tilde{z} := z - z^*$ ,

$$\beta_{\tilde{z}} (|\tilde{z}^*(0)|, t) := e^{-\lambda t} |\tilde{z}(0)|, \quad (27)$$

and  $\alpha_{\tilde{z}} > 0$  is a constant chosen independently of  $\eta$  and  $\eta^*$  since  $\bar{u}(\eta)$  is bounded.

It follows from (26) that  $\tilde{z}$  is Input-to-State Stable (ISS) with respect to  $\tilde{\eta}$  with respect to the origin.

By differentiating (13) with  $\eta = \eta^*$  with respect to  $\theta$ , we get

$$\begin{aligned} \nabla_\theta(\mathcal{J}_{\eta^*}(\theta))(t) &= \\ &= -2\lambda \int_0^t e^{-\lambda(t-s)} (\gamma(\eta_{[1, d]}^*(s)) \varepsilon(s, \theta)) ds = \\ &= 2(R^*(t)\theta - v^*(t)). \end{aligned}$$

Since the set of minimizers of (13) for  $\eta = \eta^*$  is given by  $\{\theta \in \mathbb{R}^{n_\theta} : \nabla_\theta(\mathcal{J}_{\eta^*}(\theta))(t) = 0\}$ , it follows that

$$\theta^*(t) = (R^*(t))^\dagger v^*(t).$$

minimizes (13).

From  $\eta^* = \Pi_\eta(\theta)w$ , the definition of  $\Pi(\theta)$  gives

$$\begin{aligned} \Pi_{\eta_i}(\theta)S &= \Pi_{\eta_{i+1}}(\theta), \quad i = 1, \dots, n_w \\ \Pi_{\eta_i}(\theta) &= \Pi_{\eta_1}(\theta)S^{i-1} \quad i = 1, \dots, n_w + 1, \end{aligned} \quad (28)$$

By letting  $c_i$ ,  $i = 0, \dots, n_w - 1$ , be the coefficients of the characteristic polynomial of  $S$  and by the Cayley-Hamilton Theorem, we have

$$\begin{aligned} \Pi_{\eta_{n_w+1}}(\theta) &= \Pi_{\eta_1}(\theta)S^{n_w} = -\Pi_{\eta_1}(\theta) \sum_{i=0}^{n_w-1} c_i S^i = \\ &= - \sum_{i=0}^{n_w-1} c_i \Pi_{\eta_1}(\theta)S^i = - \sum_{i=0}^{n_w-1} c_i \Pi_{\eta_{i+1}}(\theta), \end{aligned}$$

The prediction error (12) would read in this case as

$$\begin{aligned} \varepsilon(t, \theta) &:= \\ &= \sum_{i=0}^{n_w-1} c_i \Pi_{\eta_{i+1}}(\theta)w(t) - (\theta^T \otimes I_{n_e}) \Pi_{\eta_{[1, n_w]}}(\theta)w(t). \end{aligned} \quad (29)$$

It follows that (13) has a global solution given by

$$\theta^\circ = -\text{col}(c_0, \dots, c_{n_w-1}).$$

Under persistency of excitation,  $\theta^\circ$  is also the unique solution to (13).

We show now that the term  $\rho_\xi$  can be linearly bounded by  $|\tilde{z}|$ .

In view of (25), consider, with abuse of notation, the term

$$\frac{\partial \Pi(\theta)}{\partial \theta} w = \frac{\partial \Pi(p_\mathcal{E}(\theta))}{\partial p_\mathcal{E}(\theta)} \frac{\partial p_\mathcal{E}(\theta)}{\partial \theta} w. \quad (30)$$

The first term of the right-hand side of (30) is bounded, since  $p_\mathcal{E}(\theta) \in \mathcal{E}$ ,  $\Pi(p_\mathcal{E}(\theta))$  is continuous in  $p_\mathcal{E}(\theta)$  and  $w$  is bounded. We thus claim the existence of a  $\bar{b} > 0$  such that

$$\left| \frac{\partial \Pi(p_\mathcal{E}(\theta))}{\partial p_\mathcal{E}(\theta)} w \right| \leq \bar{b}.$$

Regarding the term  $\partial p_{\mathcal{E}}(\theta)/\partial\theta$ , the map  $p_{\mathcal{E}}(\theta)$  is in general non-differentiable. By convexity of  $\mathcal{E}$ , however, the function  $p_{\mathcal{E}}$  is Lipschitz continuous, and thus differentiable almost everywhere. As a consequence, since  $|\partial p_{\mathcal{E}}(\theta)/\partial\theta| \leq 1$  almost everywhere, we obtain

$$\left| \frac{\partial \Pi(\theta)}{\partial \theta} w \right| \leq \bar{b}$$

almost everywhere.

By definition,  $\omega(z) = R^\dagger v$ . As the elements of  $R^\dagger$  are rational functions of the elements of  $R$ ,  $\omega$  is locally Lipschitz. Moreover, for each  $\epsilon > 0$  there exists  $\kappa > 0$  such that, along the solutions of the closed-loop system (21)-(23), for which  $\eta$  has the  $(\epsilon, T)$ -PE property, for some  $T > 0$ , we can write

$$|\omega(z)| \leq \kappa |z|.$$

Since  $\omega(z)$  is locally Lipschitz and linearly bounded, and the trajectory of  $z$  is bounded as well, the derivative  $\partial\omega(z)/\partial z$  is bounded. From continuity of  $\partial\omega(z)/\partial z$ , and noticing that for  $z = z^*$ ,  $\partial\omega(z)/\partial z = 0$ , there exists a  $\bar{c} > 0$  such that:

$$\left| \frac{\partial\omega(z)}{\partial z} \right| \leq \bar{c} |\tilde{z}|, \quad (31)$$

along the solutions for which  $\eta$  has the  $(\epsilon, T)$ -PE property. Moreover, we have

$$|l(z, \xi)| \leq \bar{d} |\tilde{z}| + \bar{d} |z^*| + \bar{e} |\tilde{\eta}| + \bar{e} |\eta^*|.$$

Finally we notice that in (17) we have

$$\frac{\partial \Delta(\eta_{[1, n_w]}, z)}{\partial z} = \frac{\partial \Delta(\eta_{[1, n_w]}, \omega(z))}{\partial \omega(z)} \frac{\partial \omega(z)}{\partial z}.$$

The matrix  $\partial \Delta(\eta_{[1, n_w]}, \omega(z))/\partial \omega(z)$  is a matrix dependent only on  $\eta_{[1, n_w]}$ . As a consequence there exists  $\bar{a} > 0$  such that we can write

$$\left| \frac{\partial \Delta(\eta_{[1, n_w]}, z)}{\partial z} \right| \leq \bar{a} |\tilde{\eta}| + \bar{a} |\eta^*|.$$

For some constant  $\bar{l}$ , we can thus write

$$\begin{aligned} \left| \rho_{\tilde{\xi}}(\tilde{\xi}, w, \theta, z) \right| &\leq \\ &\leq [\bar{a}(|\tilde{\eta}| + |\eta^*|) + \bar{b}] \bar{c} |\tilde{z}| [\bar{d}(|\tilde{z}| + |z^*|) + \bar{e}(|\tilde{\eta}| + |\eta^*|)] \\ &\leq \bar{l} |\tilde{z}|, \end{aligned} \quad (32)$$

By (26), (32) and the fact that  $F(\theta)$  is Hurwitz for every  $\theta \in \mathbb{R}^{n_w}$ , we can write the interconnection of the systems (21) and (23) as

$$\begin{aligned} |\tilde{\xi}(t)| &\leq \beta_{\tilde{\xi}}(|\tilde{\xi}(0)|, t) + \alpha_{\tilde{\xi}} \|\tilde{z}\|_{\infty} \\ |\tilde{z}(t)| &\leq \beta_{\tilde{z}}(|\tilde{z}(0)|, t) + \alpha_{\tilde{z}} \|\tilde{\xi}\|_{\infty}, \end{aligned}$$

where

$$\begin{aligned} \beta_{\tilde{\xi}}(|\tilde{\xi}(0)|, t) &:= e^{F(\theta)t} |\tilde{\xi}(0)| \\ \alpha_{\tilde{\xi}} &:= \lambda \bar{l} \int_0^{\infty} \left| e^{F(\theta)s} \right| ds \end{aligned}$$

Thus, standard small gain arguments (see e.g. [19]) can be used to show that there exists  $\lambda^* > 0$  such that, whenever  $\lambda < \lambda^*$ ,  $\tilde{\xi}, \tilde{z} \rightarrow 0$ .

Suppose now that  $\theta^\circ \in \mathcal{E}$ , then by the definition of  $\Pi(\theta)$ , the structure of the identifier, and by using (28), we obtain that, for  $\theta = \theta^\circ$ , the quantity  $\Pi_e(\theta) := C_e \Pi_x(\theta) + Q_e$  fulfils

$$\begin{aligned} \Pi_e(\theta^\circ) &= \Pi_{\eta_{n_w+1}}(\theta^\circ) S - \sum_{i=1}^{n_w} \theta_i^\circ \Pi_{\eta_{i+1}}(\theta^\circ) = \\ &= - \sum_{i=1}^{n_w} (c_{i-1} + \theta_i^\circ) \Pi_{\eta_i}(\theta^\circ) S = 0, \end{aligned}$$

hence  $e \rightarrow 0$ , proving the claim of Proposition 1.

## VI. EXAMPLE

As an example of application we will consider a linear system of the form (2) defined by the following matrices

$$\begin{aligned} A &= \begin{pmatrix} 0 & -1 & 1 \\ 0 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 \\ 0 & 0 \\ 2 & 1 \end{pmatrix}, \\ C_e &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \\ Q_e &= \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \end{aligned}$$

The exosystem matrix  $S$  is defined as  $S = \text{blkdiag}(S_1, S_2, S_3)$ , with:

$$S_1 = \gamma_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S_2 = \gamma_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S_3 = 0,$$

and where  $\gamma_1, \gamma_2 > 0$  are unknown parameters. The errors to be regulated are thus defined as

$$e_1 := x_1 - w_5, \quad e_2 := x_3 - w_3.$$

We observe that the system considered here is not minimum-phase with respect to the input  $u$  and the output  $e$ , and relative to the ideal error-zeroing steady state given by the graph of  $\Pi$ , where  $\Pi$  is such that, for some  $\Gamma \in \mathbb{R}^{m \times n_w}$ ,  $(\Pi, \Gamma)$  is the unique solution of the regulator equations

$$\Pi S = A \Pi + B \Gamma + P, \quad C_e \Pi + Q_e = 0.$$

As a matter of fact, changing coordinates as  $x \mapsto \tilde{x} = x - \Pi w$ , and letting  $e = 0$ , yields

$$\dot{\tilde{x}}_2 = 3\tilde{x}_2.$$

For simplicity, we will assume  $C_m := \text{col}(0, 1, 0)^T$  and  $Q_m := 0_{1 \times 5}$ , and define the output of the considered system as  $y := \text{col}(e_1, x_2, e_2)$ . The set  $\mathcal{E}$  was chosen, after experimental tests on the non-resonance of the extended system (2)-(8), as  $\mathcal{E} = [-3, 7] \times [-12, -3] \times [-6, 3] \times [-20, -7] \times [-7, 2]$ . The stabilizer can thus be designed as the static feedback regulator  $u = K(\theta) \text{col}(y, \eta)$ , with  $K(\theta) := \text{col}(K_y(\theta), K_\eta(\theta))$  a gain scheduling controller. For simplicity, in this example,  $K(\theta) = \bar{K}$ , with  $\bar{K}$  constant, and  $\theta(0) \in \mathcal{E}$ . Figure 1 shows the results of the simulation of the control system obtained with  $\lambda = 0.01$ ,  $\gamma_1 = 3$ ,  $\gamma_2 = 1$ ,  $w(0) = \text{col}(1, -1, 0, -1, 7)$  and  $x(0) = (10, -2, 3)$ .

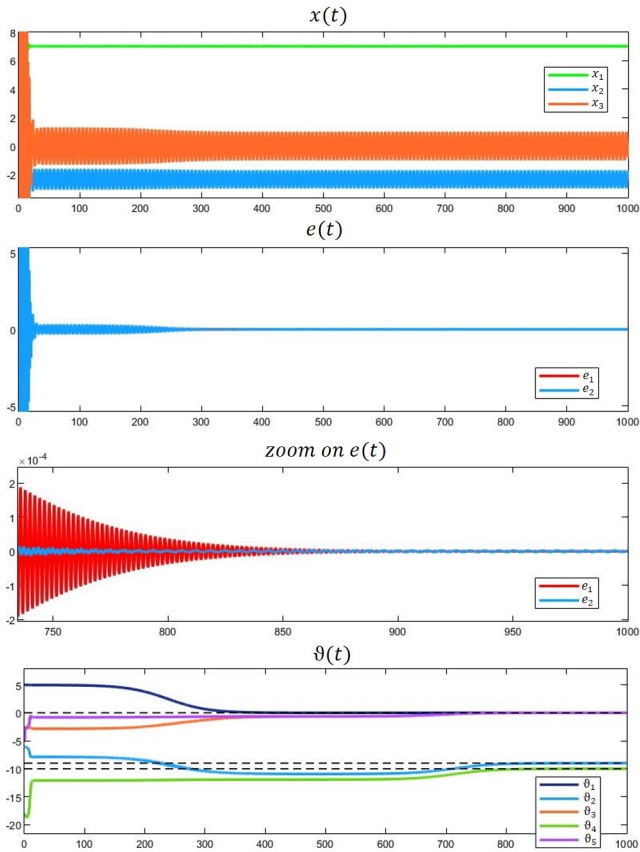


Fig. 1: Plots of the trajectories of  $x(t)$ ,  $e(t)$ ,  $\theta(t)$  resulting from the simulation. In the fourth plot, the black dashed lines represent the values of the coefficients of  $\varphi_S(s)$ .

## VII. CONCLUSIONS

In this paper we proposed a regulator that exploits a continuous-time identifier to tune the internal model unit, in order to solve the output regulation problem without assuming knowledge of the exosystem. With the proposed approach, we showed that if the identifier is designed to be slow enough with respect to the control system dynamics, asymptotic regulation is achieved under a persistency of excitation condition. Moreover, it is worth noticing that no minimum-phase assumptions were made, indeed non-minimum phase plants can be treated without any additional effort, as shown in the example.

## REFERENCES

- [1] B. A. Francis and W. M. Wonham, "The internal model principle for multivariable regulators," *Applied Mathematics and Optimization*, vol. 2, no. 2, pp. 170–196, 1975.
- [2] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica*, vol. 12, pp. 457–465, 1976.
- [3] E. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," *IEEE Transactions on Automatic Control*, vol. 21, pp. 25–34, feb 1976.
- [4] R. Marino and P. Tomei, "Output regulation for linear systems via adaptive internal model," *IEEE Transactions on Automatic Control*, vol. 48, pp. 2199–2202, dec 2003.
- [5] R. Marino and G. L. Santosuosso, "Regulation of Linear Systems With Unknown Exosystems of Uncertain Order," *IEEE Transactions on Automatic Control*, vol. 52, pp. 352–359, feb 2007.
- [6] I. Mizumoto and Z. Iwai, "Adaptive output regulation of unknown MIMO linear systems with unknown exosystems," in *2007 European Control Conference (ECC)*, pp. 804–809, IEEE, jul 2007.
- [7] M. Bando and A. Ichikawa, "Adaptive Output Regulation for Linear Systems," in *2006 SICE-ICASE International Joint Conference*, pp. 3217–3222, IEEE, 2006.
- [8] A. Serrani, A. Isidori, and L. Marconi, "Semi-global nonlinear output regulation with adaptive internal model," *IEEE Transactions on Automatic Control*, vol. 46, no. 8, pp. 1178–1194, 2001.
- [9] Z. Ding, "Global stabilization and disturbance suppression of a class of nonlinear systems with uncertain internal model," *Automatica*, vol. 39, pp. 471–479, 2003.
- [10] R. Marino and P. Tomei, "Adaptive nonlinear regulation for uncertain minimum phase systems with unknown exosystem," in *2008 47th IEEE Conference on Decision and Control*, pp. 2569–2574, IEEE, 2008.
- [11] A. Isidori, L. Marconi, and A. Serrani, "Robust design of nonlinear internal models without adaptation," *Automatica*, vol. 48, pp. 2409–2419, 2012.
- [12] F. Forte, L. Marconi, and A. Isidori, "Robust design of internal models by nonlinear regression," *IFAC, 9th Symposium on Nonlinear Control Systems*, pp. 283–288, 2013.
- [13] M. Bin, D. Astolfi, and L. Marconi, "Robust internal model design by nonlinear regression via low-power high-gain observers," in *2016 IEEE 55th Conference on Decision and Control (CDC)*, pp. 4740–4745, IEEE, dec 2016.
- [14] F. Forte, L. Marconi, and A. R. Teel, "Robust Nonlinear Regulation: Continuous-Time Internal Models and Hybrid Identifiers," *IEEE Transactions on Automatic Control*, vol. 62, pp. 3136–3151, jul 2017.
- [15] R. Marino and P. Tomei, "Hybrid Adaptive Multi Sinusoidal Disturbance Cancellation," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 4023–4030, 2017.
- [16] M. Bin, L. Marconi, and A. R. Teel, "Adaptive Output Regulation for Linear Systems via Discrete-Time Identifiers," *to be published in Automatica*, 2019.
- [17] L. Marconi, A. Isidori, and A. Serrani, "Autonomous vertical landing on an oscillating platform: an internal-model approach," *Automatica*, vol. 38, no. 1, pp. 21–32, 2002.
- [18] I. D. Landau, A. Constantinescu, and D. Rey, "Adaptive narrow band disturbance rejection applied to an active suspensionan internal model principle approach," *Automatica*, vol. 41, no. 4, pp. 563–574, 2005.
- [19] H. K. Khalil, *Nonlinear Systems*. 3 ed., 2002.