Interpolation in Extensions of First-Order Logic

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Interpolation in extensions of first-order logic

Abstract. We prove a generalization of Maehara's lemma to show that the extensions of classical and intuitionistic first-order logic with a special type of geometric axioms, called singular geometric axioms, have Craig's interpolation property. As a corollary, we obtain a direct proof of interpolation for (classical and intuitionistic) first-order logic with identity, as well as interpolation for several mathematical theories, including the theory of equivalence relations, (strict) partial and linear orders, and various intuitionistic order theories such as apartness and positive partial and linear orders.

Keywords: Craig's interpolation theorem, Maehara's lemma, sequent calculi, first-order theories, singular geometric rules.

1. Introduction

Craig's interpolation theorem [4] is a central result in first-order logic. It asserts that for any theorem $A \to B$ there exists a formula C, called *interpolant*, such that $A \to C$ and $C \to B$ are also theorems and C only contains non-logical symbols that are contained in both A and B (and if A and B have no non-logical symbols in common, then either $\neg A$ is a theorem or B is). The aim of this paper is to extend interpolation beyond first-order logic. In particular, we show how to prove interpolation in extensions of intuitionistic and classical sequent calculi with *singular geometric rules*, a special case of geometric rules investigated in [14]. Interpolation for singular geometric rules will be obtained by generalizing a standard result, reportedly due to Maehara in [20] and known as "Maehara's lemma" [12].[1]

The proof of Maehara's lemma for intuitionistic and classical first-order logic requires cut elimination. For systems extending first-order logic with axioms it is not all straightforward to prove Maehara's lemma, since such systems are not generally cut-free (cf. [21, §4.5] and [16, §6.3] for different approaches to non-logical axioms). For example, in the calculus $\mathsf{LK_e}$, an extension of Gentzen's LK for first-order logic with identity, cuts on identi-

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[1] In this work we shall not consider semantic methods to prove interpolation. These have been applied extensively to non-classical logics in [7]; there are also proofs of interpolation for non-classical logics that are more similar to our approach, especially [1, 6, 11].

ties $s = t$ are not eliminable (cf. Theorem 6 in [20], where these cuts are called "inessential"). Fortunately, interpolation can still be proved for first-order logic with identity. The drawback of the existing proofs, however, is that they are indirect, in the sense that the interpolant is not built using exclusively the rules of the calculus. In [21], for example, a translation is used to reduce interpolation for first-order logic with identity to the case of pure first-order logic.[2] A different route is taken in [8], using the method of "axioms in the context", where interpolation is again not proved directly in $\mathsf{LK_e}$, but in a variant of LK, equivalent to $\mathsf{LK_e}$, in which all derivable sequents have the axioms governing the identity predicate in the context.[3] On the other hand, in this paper interpolation is proved via a generalization of Maehara's lemma to a class of extensions of first-order logic (which include first-order logic with identity as a particular case) and using no other means than the rules of the calculus (Lemma 13).

Our generalization of Maehara's lemma is based on previous work by Negri and von Plato who have shown (in a series of papers starting from [15]) how to recover cut elimination (as well as the admissibility of other structural rules) for extensions of the calculi $\mathsf{G3c}$ and $\mathsf{m\text{-}G3i}$ for classical and intuitionistic first-order logic. Of particular interest for the present work are the extensions with geometric rules, investigated in [14].[4] Once cut elimination is recovered in this way, we impose a singularity condition on geometric rules to isolate those containing at most one non-logical predicate (identity will be counted as logical). Our main result is to show that Maehara's lemma holds when $\mathsf{G3c}$ and $\mathsf{G3i}$ are extended with singular geometric rules (Lemma 13). Then interpolation follows easily from the generalized Maehara's lemma (Theorem 14). Finally, we consider applications of Theorem 14 and we show that singular geometric rules include many interesting extensions of intuitionistic and classical first-order logic, especially (classical and intuitionistic) first-order logic with identity, the theory of equivalence relations, (strict) partial and linear orders, the theory of apartness and the theory of positive partial and linear orders.

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[2] For other proofs of interpolation via translation see [19] and [2].

[3] Thanks to a referee for bringing this to our attention.

[4] We depart from Negri's approach in taking the intuitionistic single-succedent calculus $\mathsf{G3i}$ instead of the multi-succedent $\mathsf{m\text{-}G3i}$ of [14]; in Theorem 8 we will also prove, along the way, that cut elimination holds for geometric extensions of $\mathsf{G3i}$.

2. Classical and intuitionistic sequent calculi

The language $\mathcal{L}$ is a first-order language with individual constants and no functional symbols. Terms $(s, t, u, \dots)$ are either variables $(x, y, z, \dots)$ or individual constants $(a, b, c \dots)$. $\mathcal{L}$ contains also denumerably many k-ary predicates $P^k, Q^k, R^k, \dots$ for each $k \geq 0$. $\mathcal{L}$ may also contain the identity. All predicates, except identity, are non-logical. Moreover, it is convenient to have two propositional constants $\bot$ (falsity) and $\top$ (truth). Formulas are built up from atoms $P^k(t_1, \dots, t_k)$, the constants $\bot$ and $\top$ using logical operators $\wedge$, $\vee$, $\to$, $\exists$ and $\forall$ as usual. We use $P, Q, R, \dots$ for atoms, $A, B, C, \dots$ for formulas and $\Gamma, \Delta, \Pi, \dots$ for (possibly empty) finite multisets of formulas. The negation $\neg A$ of a formula A is defined as $A \to \bot$. Moreover, let Γ, Δ be an abbreviation for $\Gamma \cup \Delta$ (where $\cup$ is the multiset union) and $\bigwedge \Gamma$ ($\bigvee \Gamma$) stand for the conjunction (disjunction, respectively) of all formulas in Γ. Moreover, if Γ is empty, then $\bigwedge \Gamma \equiv \top$ and $\bigvee \Gamma \equiv \bot$, where $\equiv$ indicates syntactic identity (up to α-congruence) between expressions of the object-language.

The substitution of a variable x with a term t in a term s (in a formula A, in a multiset Γ) will be indicated as $s[\frac{t}{x}]$ ($A[\frac{t}{x}]$ and $\Gamma[\frac{t}{x}]$, respectively) and defined as usual. To indicate the simultaneous substitution of the list of variables $x_1, \dots, x_n$ (abbreviated in $\bar{x}$) with the list of terms $t_1, \dots, t_n$ (abbreviated in $\bar{t}$), we use $[\frac{\bar{t}}{\bar{x}}]$ in place of $[\frac{t_1 \dots t_n}{x_1 \dots x_n}]$. Later on, we shall also need a more general notion of substitution of terms for terms (not just variables) which will be proved to preserve derivability (Lemma 6).

Finally, let $\mathsf{FV}(A)$ be the set of free variables of a formula A and let $\mathsf{Con}(A)$ be the set of its individual constants. Let the set of terms $\mathsf{Ter}(A)$ be A is $\mathsf{FV}(A) \cup \mathsf{Con}(A)$. Moreover, if $\mathsf{Rel}(A)$ is the set of non-logical predicates of A then we define the language $\mathcal{L}(A)$ of A as $\mathsf{Ter}(A) \cup \mathsf{Rel}(A)$. Notice that $= \notin \mathcal{L}(A)$, for all A. Such notions are immediately extended to multisets of formulas Γ, by letting $\mathsf{FV}(\Gamma)$ to be defined as $\bigcup_{A \in \Gamma} \mathsf{FV}(A)$, and analogously for $\mathsf{Con}(\Gamma)$, $\mathsf{Ter}(\Gamma)$, $\mathsf{Rel}(\Gamma)$ and $\mathcal{L}(\Gamma)$.

The calculus Gc (Gi) is a variant of LK (LI) for classical (intuitionistic, respectively) first-order logic, originally introduced by Gentzen in [9]. In the literature, especially in [21] and [16], Gc and Gi are commonly referred to as $\mathsf{G3c}$ and $\mathsf{G3i}$ but we will omit '3' in the interest of readability. Moreover, we will write G to refer to either Gc or Gi. A sequent in Gc is a pair $\langle \Gamma, \Delta \rangle$ of multisets, indicated as $\Gamma \Rightarrow \Delta$. The calculus Gc consists of the following initial sequents and logical rules (where y is an *eigenvariable* in $R\forall$ and $L\exists$, i.e. y must not occur free in the conclusion of these rules):

The calculus Gc

$$P, \Gamma \Rightarrow \Delta, P$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\bot} \qquad \frac{}{\Gamma \Rightarrow \Delta, \top}\,{\scriptstyle R\top}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\wedge} \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\,{\scriptstyle R\wedge}$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\vee} \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\,{\scriptstyle R\vee}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\to} \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\,{\scriptstyle R\to}$$

$$\frac{A[\genfrac{}{}{0pt}{}{t}{x}], \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\forall} \qquad \frac{\Gamma \Rightarrow \Delta, A[\genfrac{}{}{0pt}{}{y}{x}]}{\Gamma \Rightarrow \Delta, \forall x A}\,{\scriptstyle R\forall}$$

$$\frac{A[\genfrac{}{}{0pt}{}{y}{x}], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\exists} \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A[\genfrac{}{}{0pt}{}{t}{x}]}{\Gamma \Rightarrow \Delta, \exists x A}\,{\scriptstyle R\exists}$$

Sequents in Gi are defined as in Gc, except that Δ must contain exactly one formula. The calculus Gi has the following initial sequents and logical rules (again, y is an *eigenvariable* in $R\forall$ and $L\exists$).

The calculus Gi

$$P, \Gamma \Rightarrow P$$

$$\frac{}{\bot, \Gamma \Rightarrow C}\,{\scriptstyle L\bot} \qquad \frac{}{\Gamma \Rightarrow \top}\,{\scriptstyle R\top}$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\,{\scriptstyle L\wedge} \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,{\scriptstyle R\wedge}$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\,{\scriptstyle L\vee} \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\,{\scriptstyle R\vee_1} \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\,{\scriptstyle R\vee_2}$$

$$\frac{A \to B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C}\,{\scriptstyle L\to} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\,{\scriptstyle R\to}$$

$$\frac{A[\genfrac{}{}{0pt}{}{t}{x}], \forall x A, \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\,{\scriptstyle L\forall} \qquad \frac{\Gamma \Rightarrow A[\genfrac{}{}{0pt}{}{y}{x}]}{\Gamma \Rightarrow \forall x A}\,{\scriptstyle R\forall}$$

$$\frac{A[\genfrac{}{}{0pt}{}{y}{x}], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\,{\scriptstyle L\exists} \qquad \frac{\Gamma \Rightarrow A[\genfrac{}{}{0pt}{}{t}{x}]}{\Gamma \Rightarrow \exists x A}\,{\scriptstyle R\exists}$$

A derivation in G is a tree of sequents which grows according to the rules of G and whose leaves are initial sequents or conclusions of a 0-premise rule. A derivation of a sequent is a derivation concluding that sequent and a sequent is derivable when there is a derivation of it. As usual, we consider only *pure-variable derivations*: bound and free variables are kept distinct, and no two rule instances have the same variable as *eigenvariable*, see [21, p. 62]. Moreover, α-congruent formulas are identified and we permit renaming of bound variables in order to always keep bound and free variables disjoint, see [21, p. 67]. The height h of a derivation is defined inductively as follows: the derivation height of an initial sequent or of a conclusion of a 0-premise rule is 0, the derivation height of a derivation of a conclusion of a one-premise rule is the derivation height of its premise plus 1, and the derivation height of a derivation of a conclusion of a n-premise rule ($n \geq 2$) is the maximum of the derivation heights of its premises plus 1. A sequent is h-derivable if it is derivable with a derivation of height less than or equal to h. A rule is admissible if the conclusion is derivable whenever the premises are derivable; a rule is height-preserving admissible if the conclusion is h-derivable whenever the premises are h-derivable. Derivations will be denoted by $\mathcal{D}, \mathcal{D}_1, \mathcal{D}_2, \ldots$. We agree to use $\mathcal{D} \vdash \Gamma \Rightarrow \Delta$ to indicate that $\mathcal{D}$ is a derivation in G of $\Gamma \Rightarrow \Delta$ and $\vdash \Gamma \Rightarrow \Delta$ to indicate that $\Gamma \Rightarrow \Delta$ is derivable; finally, $\vdash^h \Gamma \Rightarrow \Delta$ indicates that $\Gamma \Rightarrow \Delta$ is h-derivable. We will use a double-line rule of the form

$$\frac{\Pi \Rightarrow \Sigma}{\overline{\Gamma \Rightarrow \Delta}}\ R$$

to indicate that $\Gamma \Rightarrow \Delta$ is derivable from $\Pi \Rightarrow \Sigma$ by a (possibly empty) sequence of instances of the rule R. It is easy to see that initial sequents with $A, \Gamma \Rightarrow \Delta, A$, for an arbitrary A, are derivable in G (where Δ is empty for Gi).

The following structural rules for Gc (weakening, contraction and cut) are valid in the standard semantics of Gc.

Structural rules of Gc

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathit{Wkn} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ \mathit{Wkn}$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathit{Ctr} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \mathit{Ctr}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \mathit{Cut}$$

However, we can safely leave them out without impairing the completeness of Gc, since they are all admissible in it. In fact, weakening and contraction are also height-preserving admissible. Regarding Gi, the structural rules are:

Structural rules of Gi

$$\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\;{\scriptstyle Wkn} \qquad \frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\;{\scriptstyle Ctr}$$

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\;{\scriptstyle Cut}$$

These rules are also valid in the model-theoretic semantics for intuitionistic logic, but just like in the classical case, they are all admissible in Gi (again, weakening and contracting are height-preserving admissible) and there is no need to take any of them as primitive. The proof of the admissibility of the structural rules in any of the two calculi requires some preparatory results. First, the height-preserving admissibility of substitution in G.

Lemma 1. *In* G*, if* $\vdash^h \Gamma \Rightarrow \Delta$ *and* t *is free for* x *in* Γ, Δ *then* $\vdash^h \Gamma[\frac{t}{x}] \Rightarrow \Delta[\frac{t}{x}]$.

Second, the so-called inversion lemma. Intuitively, a rule is invertible when it can be applied backwards, from the conclusion to its premises, and it is height-preserving invertible when it is invertible with the preservation of the derivation height (for a precise definition of height-preserving invertible rule see [21, p. 76-77]).

Lemma 2. *In* Gc *all rules are height-preserving invertible. In* Gi *all rules, except* $R\vee$*,* $L\rightarrow$ *and* $R\exists$*, are height-preserving invertible. However,* $L\rightarrow$ *is height-preserving invertible with respect to its right premise.*

With height-preserving admissibility of substitution and inversion lemma it is possible to prove the admissibility of the structural rules.

Theorem 3. *In* G *weakening and contraction are height-preserving admissible. Moreover, cut is admissible.*

The proof of Lemma 1, Lemma 2, and Theorem 3 are standard and the interested reader is referred to [21] and [16].

2.1. From axioms to rules

Extensions of G are not, in general, cut free; this means that Theorem 3 does not necessarily hold in the presence of new initial sequents or rules.

For example, a natural way to extend Gc to cover first-order logic with identity is to allow derivations to start with initial sequents of the form $\Rightarrow s = s$ and $s = t, P[^s_x] \Rightarrow P[^t_x]$, corresponding to the reflexivity of identity and Leibniz's principle of indescernibility of identicals, respectively (we call these sequents S_1 and S_2). Notice that S_2 is in fact a scheme which becomes $s = t, s = s \Rightarrow t = s$, when P is $x = s$. From this, via cut on $\Rightarrow s = s$, one derives $s = t \Rightarrow t = s$, namely the symmetry of identity. However, such a sequent has no derivation without cut. Therefore, cut is not admissible in $\mathsf{Gc} + \{S_1, S_2\}$, though it is admissible in the underlying system Gc.

In [15] Negri and von Plato have shown how to recover cut elimination for (classical) first-order logic with identity by transforming S_1 and S_2 into an equivalent pair of rules of the form:

$$\frac{s = s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Ref_=} \qquad \frac{P[^t_x], s = t, P[^s_x], \Gamma \Rightarrow \Delta}{s = t, P[^s_x], \Gamma \Rightarrow \Delta}\,{\scriptstyle Repl_=}$$

If one replaces S_1 and S_2 with the corresponding rules, it is easy to derive $s = t \Rightarrow t = s$ without any application of cut. More generally, cut elimination holds in $\mathsf{Gc} + \{\mathit{Ref}, \mathit{Repl}\}$ (cf. Theorem 4.2 in [15] and [16, §6.5]). This result can be, and has been, extended in different directions. Here we are particularly interested in the fact, established by [14], that cut elimination holds in extensions of Gc with geometric rules (of which the rules of identity are special cases). The result will be reviewed briefly below, while for a more thorough discussion on this topic the reader is referred to [14] or the monograph [17].

In [14] Negri also showed that cut elimination holds for geometric theories formulated as extensions of the multi-succedent calculus m-G3i for intuitionistic logic, introduced in [5]. For our purposes, however, it is better to work in Gi as the underlying logical calculus for intuitionistic logic. In this way we can rely on the proof of Maehara's lemma for Gi already available in the literature (whereas to our knowledge no attempt has been made to obtain a similar result for m-G3i). In fact, it is not entirely obvious how to prove Maehara's lemma for m-G3i. Working in Gi is thus more advantageous as far as Maehara's lemma is concerned, but one needs first to make sure that cut elimination holds in the presence of geometric rules. Thus, after introducing geometric rules, we will show that the standard cut-elimination procedures goes through with minor adjustment in geometric extensions of Gi (Theorem 8).

2.2. Geometric theories

A *geometric axiom* is a formula following the *geometric axiom scheme* below:

$$\forall \bar{x}(P_1 \wedge \cdots \wedge P_n \to \exists \bar{y}_1 M_1 \vee \cdots \vee \exists \bar{y}_m M_m)$$

where each P_j is an atom and each M_i is a conjunction of a list of atoms $Q_{i_1}, \ldots, Q_{i_\ell}$ and none of the variables in any $\bar{y}_i$ are free in the P_js. We shall conveniently abbreviate $Q_{i_1}, \ldots, Q_{i_\ell}$ in $\mathbf{Q}_i$. In a geometric axiom, if $m = 0$ then the consequent of $\to$ becomes $\bot$, whereas if $n = 0$ the antecedent of $\to$ becomes $\top$. A *geometric theory* is a theory containing only geometric axioms. An m-premise *geometric rule*, for $m \geq 0$, is a rule following the *geometric rule scheme* below:

$$\frac{\mathbf{Q}_1^*, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta \quad \cdots \quad \mathbf{Q}_m^*, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_n, \Gamma \Rightarrow \Delta} \; R$$

where each $\mathbf{Q}_i^*$ is obtained from $\mathbf{Q}_i$ by replacing every variable in $\bar{y}_i$ with a variable which does not occur free in the conclusion. Such variables will be called the *eigenvariables* of R. Without loss of generality, we assume that each $\bar{y}_i$ consists of a single variable. In sequent calculus a geometric theory can be formulated by adding on top of G finitely many geometric rules (recall that Δ contains exactly one formula in Gi).

Moreover, geometric rules are assumed to satisfy a natural closure property for contraction (see [16, 6.1.7]).

Definition 4 (Closure condition). If a geometric extension G' of G contains a rule where a substitution instance of the principal formulas produces a rule with repetition of the form:

$$\frac{\mathbf{Q}_1^*, P_1, \ldots, P_{n-2}, P, P, \Gamma \Rightarrow \Delta \quad \cdots \quad \mathbf{Q}_m^*, P_1, \ldots, P_{n-2}, P, P, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_{n-2}, P, P, \Gamma \Rightarrow \Delta} \; R$$

then G' contains or is closed under the following contracted instance of the rule:

$$\frac{\mathbf{Q}_1^*, P_1, \ldots, P_{n-2}, P, \Gamma \Rightarrow \Delta \quad \cdots \quad \mathbf{Q}_m^*, P_1, \ldots, P_{n-2}, P, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_{n-2}, P, \Gamma \Rightarrow \Delta} \; R^c$$

As an illustration, we consider the rule $Trans_{\leqslant}$ in the theory PO (see § 5.3):

$$\frac{s \leqslant u, s \leqslant t, t \leqslant u, \Gamma \Rightarrow \Delta}{s \leqslant t, t \leqslant u, \Gamma \Rightarrow \Delta}\ \scriptstyle{\mathit{Trans}_{\leqslant}}$$

Clearly, as an instance of such a rule we have:

$$\frac{s \leqslant s, s \leqslant s, s \leqslant s, \Gamma \Rightarrow \Delta}{s \leqslant s, s \leqslant s, \Gamma \Rightarrow \Delta}\ \scriptstyle{\mathit{Trans}_{\leqslant}}$$

Hence PO has to be closed under the following contracted instance

$$\frac{s \leqslant s, s \leqslant s, \Gamma \Rightarrow \Delta}{s \leqslant s, \Gamma \Rightarrow \Delta}\ \scriptstyle{\mathit{Trans}^{c}_{\leqslant}}$$

For PO we don't need to add the contracted rule $\mathit{Trans}^{c}_{\leqslant}$, because it is admissible thanks to rule $\mathit{Ref}_{\leqslant}$. In general, however, this is not the case.

Let G^{g} be any extension of G with finitely many geometric rules satisfying the closure condition (from now on, we will tacitly assume that the closure condition is always met). We now show that cut elimination and the admissibility of the structural rules hold in G^{g}. Although we will heavily rely on [14], we start by introducing a more general notion of substitution that allows an arbitrary term u (possibly a constant) to be replaced by a term t. In the presence of such general substitutions, special care is needed in order to maintain the height-preserving admissibility of substitutions. In particular, general substitutions are height-preserving admissible, provided that the replaced term u does not occur essentially in the calculus. Intuitively, a term u occurs essentially in a rule R when u cannot be replaced (by an arbitrary term), namely when u is a constant and u already occurs in the axiom from which R is obtained. More precisely,

Definition 5. A constant u occurs essentially in a geometric axiom A if and only if, for some $t \not\equiv u$, $A[\,{}^{t}_{u}\,]$ is not an instance of the axiom A.

Moreover, we say that a term u occurs essentially in a geometric rule R when it does so in the corresponding axiom. For example, in the geometric axiom $\neg 1 \leqslant 0$ of non-degenerate partial orders (see [17, p. 116]) both 1 and 0 occur essentially; hence they also occur essentially in the corresponding geometric rule *Non-deg*:

$$\frac{}{1 \leqslant 0, \Gamma \Rightarrow \Delta}\ \scriptstyle{\mathit{Non\text{-}deg}}$$

Now we show that the general substitution $[\,{}^{t}_{u}\,]$ is height-preserving admissible in G^{g}, provided that u occurs essentially in none of its geometric rule.

Lemma 6. *In* G^{g}*, if* $\vdash^n \Gamma \Rightarrow \Delta$*,* t *is free for* u *in* Γ, Δ*, and* u *does not occur essentially in any rule of* G^{g}*, then* $\vdash^n \Gamma[{}^t_u] \Rightarrow \Delta[{}^t_u]$.

Proof. If u is a variable, the claim holds by extending Lemma 1 to G^{g}. Otherwise, let u be an individual constant. We can think of the derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$ as

$$\frac{\Gamma' \Rightarrow \Delta'}{\Gamma'[{}^u_z] \Rightarrow \Delta'[{}^u_z]}\ [{}^u_z]$$

where $\Gamma' \Rightarrow \Delta'$ is like $\Gamma \Rightarrow \Delta$ save that it has a fresh variable z in place of u. Note that this is always feasible for purely logical derivations, and it is feasible for derivations involving geometric rules as long as these rules do not involve essentially the constant u. We transform $\mathcal{D}$ into

$$\frac{\Gamma' \Rightarrow \Delta'}{\Gamma'[{}^t_z] \Rightarrow \Delta'[{}^t_z]}\ [{}^t_z]$$

where t is free for z since we assumed it is free for u in $\Gamma \Rightarrow \Delta$. We have thus found a derivation $(\mathcal{D}[{}^t_u])$ of $\Gamma[{}^t_u] \Rightarrow \Delta[{}^t_u]$ that has the same height as the derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. ■

We can now show that Lemma 2 and Theorem 3 still hold in G^{g}. In fact, for Gc^{g} a proof has already been given in [14].

Theorem 7 (Negri). *In* Gc^{g} *all geometric and logical rules are height-preserving invertible. Weakening and contraction are also height-preserving admissible and cut is admissible.*

At this point we need to show that the same holds for Gi. A similar result has been proved by Negri in [13] for a subclass of geometric rules, called universal rules. In fact, Negri only considers specific instances of universal rules expressing the axioms of the constructive theory of apartness and excess, see §5.5 and §5.6. Moreover, in [13] only the quantifier-free version of Gi is considered. Here we extend Negri's result and show the admissibility of the structural rules for the full calculus Gi extended by arbitrary geometric rules. Then,

Theorem 8. *In* Gi^{g} *all the geometric rules and all logical rules, except* $R\vee$*,* $L\rightarrow$ *and* $R\exists$*, are height-preserving invertible. However,* $L\rightarrow$ *is height-preserving invertible with respect to its right premise. Moreover, weakening and contraction are height-preserving admissible and cut is admissible.*

Proof. The proof of height-preserving invertibility of the geometric and logical rules for Gi^{g} does not differ substantially from that for Gi and is left to the reader. We take a closer look at the admissibility of the structural rules.

Weakening. To show that weakening is height-preserving admissible in Gi^{g}, we need to extend the proof for Gi with the cases arising from geometric rules R. These cases can be dealt with as for geometric rules over $\mathsf{m\text{-}Gi}$ and Gc [14, Thm. 2]. In particular, if R is an m-premises ($m \geq 1$) geometric rule with a variable condition on y, we replace y with a fresh variable not occurring in the weakening formula, then we apply the inductive hypothesis and, finally, we apply R. If R is an m-premises ($m \geq 1$) geometric rule without variable condition, we can apply directly the inductive hypothesis and then R. Finally, if R is a 0-premise geometric rule, the conclusion of weakening is obtained directly by R.

Contraction. Once again, the new cases arising by the addition of geometric rules to Gi are similar to the cases in which these rules are added to $\mathsf{m\text{-}Gi}$ or to Gc [14, Thm. 4]. This means we have three cases: of the occurrences of the contraction formula either (i) none, or (ii) exactly one, or (iii) both are principal in the final step of the derivation of the premise. The first and the second case can be dealt with by induction and the third by the closure condition.

Cut. To show that cut is admissible we need to prove that if $\vdash \Gamma \Rightarrow A$ and $\vdash A, \Delta \Rightarrow C$ then $\vdash \Gamma, \Delta \Rightarrow C$. The proof is by induction on the weight of the cut formula A with a sub-induction on the sum of heights of derivation of the two premises (cut-height, for short). As for the proof of the admissibility of Cut over $\mathsf{m\text{-}Gi}^{\mathsf{g}}$ [14, Thm. 5], we consider only the new cases arising from the geometric rules R.

1. The left premise of *Cut* is by a 0-premise geometric rule R. Hence also the conclusion of *Cut* is a conclusion of an instance of R.
2. The right premise is by a 0-premise geometric rule R and the cut formula is not principal in it. We proceed as in case 1.
3. The right premise is by an instance of a 0-premise geometric rule R and the cut formula is principal in it. In this case we know that A is atomic (or $\top$ or $\bot$) and we consider the last step of the derivation of the left premise. If it is by a 0-premise (logic or geometric) rule or it is an initial sequent, we proceed as in case.[5] If the left premise is inferred by an m-premises ($m \geq 1$) logical or geometric rule, then the cut formula is not

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[5] Observe that, unlike the cases of $\mathsf{m\text{-}Gi}^{\mathsf{g}}$ and Gc^{g}, the cut formula A must be principal

principal in it and we can permute the *cut* upwards in the left premise (if the last rule applied in the left premise has *eigenvariables*, we rename them before permuting the cut to avoid clashes).

4. If the cut formula is not principal either in the left or in the right premise and this premise is inferred by an m-premises (for $m \geq 1$) geometric rule R, then, after having renamed any *eigenvariable* of R to avoid clashes, we permute the cut upwards with respect to this premise.

5. Finally, if the cut formula is principal in both premises, neither premise has been derived by a geometric rule and we proceed as for Gi.

■

3. Singular geometric theories

To prove interpolation in extensions of first-order logic, the class of geometric rules seems too large. Thus, we restrict our attention to a proper sub-class of it and we introduce the class of singular geometric theories. In the next section we will prove (Lemma 13) that Maehara's lemma holds for singular geometric extensions of first-order logic.

A *singular geometric axiom* is a geometric axiom with at most one non-logical predicate and no constant occurring essentially. A *singular geometric theory* is a theory containing only singular geometric axioms. In sequent calculus a singular geometric theory can be formulated by extending G with finitely many geometric rules of form:

$$\frac{\mathbf{Q}_1^*, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta \quad \cdots \quad \mathbf{Q}_m^*, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}\ R$$

where no constant occurs essentially and that satisfy the following singularity condition:

$$|\mathsf{Rel}(\mathbf{Q}_1^*, \ldots, \mathbf{Q}_m^*, P_1, \ldots, P_n)| \leq 1 \qquad (\star)$$

Singular geometric axioms are ubiquitous in mathematics. Here, for example, is an incomplete list of singular geometric axioms for a binary relation R (the list is partly taken from [3, p. 48-50]).

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in the left premise when this premise is an initial sequent.

R is reflexive	$\forall x(\top \to xRx)$
R is irreflexive	$\forall x(xRx \to \bot)$
R is transitive	$\forall x\forall y\forall z(xRy \land yRz \to xRz)$
R is intransitive	$\forall x\forall y\forall z(xRy \land yRz \land xRz \to \bot)$
R is co-transitive	$\forall x\forall y\forall z(xRy \to xRz \lor zRy)$
R is splitting	$\forall x\forall y\forall z(xRy \to xRz \lor yRz)$
R is symmetric	$\forall x\forall y(xRy \to yRx)$
R is asymmetric	$\forall x\forall y(xRy \land yRx \to \bot)$
R is anti-symmetric	$\forall x\forall y(xRy \land yRx \to x = y)$
R is trichotomous	$\forall x\forall y(\top \to x = y \lor xRy \lor yRx)$
R is linear	$\forall x\forall y(\top \to xRy \lor yRx)$
R is Euclidean	$\forall x\forall y\forall z(xRz \land yRz \to xRy)$
R is left-unique	$\forall x\forall y\forall z(xRz \land yRz \to x = y)$
R is right-unique	$\forall x\forall y\forall z(zRx \land zRy \to x = y)$
R is connected	$\forall x\forall y\forall z(xRy \land xRz \to yRz \lor zRy)$
R is nilpotent	$\forall x\forall y\forall z(xRz \land zRy \to \bot)$
R is a left ideal	$\forall x\forall y\forall z(xRy \to xRz)$
R is a right ideal	$\forall x\forall y\forall z(xRy \to zRy)$
R is rectangular	$\forall x\forall y\forall z\forall v(xRz \land vRy \to xRy)$
R is dense	$\forall x\forall y(xRy \to \exists z(xRz \land zRy))$
R is total	$\forall x\exists y(\top \to xRy)$
R is confluent	$\forall x\forall y\forall z(xRy \land xRz \to \exists u(yRu \land zRu))$
R is left-oriented	$\forall x\forall y(\top \to \exists z(zRx \land zRy))$
R is right-oriented	$\forall x\forall y(\top \to \exists z(xRz \land yRz))$

It is evident that a number of important classical and intuitionistic mathematical theories are singular geometric. Regarding the classical ones, the theory of partial orders (R is reflexive, transitive and anti-symmetric), the theory of linear orders (R is a linear partial order), as well as the theories of strict partial orders (R is irreflexive and transitive) and strict linear orders (R is a trichotomous strict partial order) are singular geometric. Constructive singular geometric theories, on the other hand, include von Plato's theories of positive partial orders [18] (R is irreflexive and co-transitive) and positive linear orders (R is an asymmetric positive partial order), as well as the theory of apartness (R is irreflexive and splitting). Also the theory of equivalence relations (R is reflexive, transitive and symmetric) falls within the class of singular geometric. Finally, the fact that a relation R is functional (total and right-unique) can be axiomatized using singular geometric axioms. Singular geometric axioms are important in logic, too. Specifically, the axioms of identity are singular geometric.

$=$ is reflexive	$\forall x(x = x)$
$=$ satisfies the indescernibility of identicals	$\forall x \forall y(x = y \wedge P[{}^{x}_{z}] \to P[{}^{y}_{z}])$

Notice that the indiscernibility of identicals satisfies the singularity condition ($\star$) because identity is a logical predicate. Hence, first-order logic with identity is a singular geometric theory.

Cut elimination for singular geometric rules clearly follows from cut elimination for geometric rules. More precisely, let G^{s} be any extension of G with singular geometric rules. Then:

Corollary 9. *All derivability properties expressed in Lemma 6, Theorem 7 and Theorem 8 hold for* G^{s}.

Proof. Straightforward, since all singular geometric rules are geometric. ■

4. Interpolation with singular geometric rules

The standard proof of interpolation in sequent calculi rests on a result due to Maehara which appeared (in Japanese) in [12] and was later made available to international readership by Takeuti in his [20]. While interpolation is a result about logic, regardless the formal system (sequent calculus, natural deduction, axiom system, etc), Maehara's lemma is a "sequent-calculus version" of interpolation. Although originally Maehara proved his lemma for LK, it is easy to adapt the proof so that it holds also in G (cf. [21, §4.4]). We recall from [21] some basic definitions.

Definition 10 (partition, split-interpolant). A *partition of a sequent* $\Gamma \Rightarrow \Delta$ is an expression $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$, where $\Gamma = \Gamma_1, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$ (for $=$ the multiset-identity). A *split-interpolant* of a partition $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$ is a formula C such that:

I $\vdash \Gamma_1 \Rightarrow \Delta_1, C$

II $\vdash C, \Gamma_2 \Rightarrow \Delta_2$

III $\mathcal{L}(C) \subseteq \mathcal{L}(\Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2)$

We use $\Gamma_1 \ ; \ \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2$ to indicate that C is a split-interpolant for $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$.

Moreover, we say that a C satisfying conditions (I) and (II) satisfies the derivability conditions for being a split-interpolant for the partition $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$, whereas if C satisfies (III) we say that it satisfies the language condition for being a split-interpolant for the same partition.

Lemma 11 (Maehara). *In* Gc *every partition* $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$ *of a derivable sequent* $\Gamma \Rightarrow \Delta$ *has a split-interpolant. In* Gi *every partition* $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \ ; \ A$ *of a derivable sequent* $\Gamma \Rightarrow A$ *has a split-interpolant.*

The proof is by induction on the height h of the derivation. If $h = 0$ then $\Gamma \Rightarrow \Delta$ is an initial sequent or a conclusion of a 0-premise rule and the proof is as in [21].[6] If $h = n + 1$ one uses as induction hypothesis the fact that any partition of the premises of a rule R has a split-interpolant. For a detailed proof the reader is again referred to [21].

From Maehara's lemma it is immediate to prove Craig's interpolation theorem.

Theorem 12 (Craig). *If* $A \Rightarrow B$ *is derivable in* G *then there exists a* C *such that* $\vdash A \Rightarrow C$ *and* $\vdash C \Rightarrow B$ *and* $\mathcal{L}(C) \subseteq \mathcal{L}(A) \cap \mathcal{L}(B)$.

Proof. Let $A \Rightarrow B$ be derivable in G and consider the partition $A \ ; \ \varnothing \Rightarrow \varnothing \ ; \ B$ of $A \Rightarrow B$. By Lemma 11, this partition has a split-interpolant, namely there exists a C such that $A \ ; \ \varnothing \overset{C}{\Rightarrow} \varnothing \ ; \ B$. Hence $\vdash A \Rightarrow C$ and $\vdash C \Rightarrow B$ and $\mathcal{L}(C) \subseteq \mathcal{L}(A) \cap \mathcal{L}(B)$ by Definition 10. ∎

If a calculus satisfies Theorem 12, we say that it has the interpolation property. Now we extend Lemma 11 to extensions of G with singular geometric rules.

In the proof of Lemma 13 we shall only consider singular geometric rules where each $\mathbf{Q}_i^*$ is a single atom Q_i^*. More precisely, we consider singular geometric rules of the form

$$\frac{Q_1^*, P_1, \dots, P_n, \Gamma \Rightarrow \Delta \quad \cdots \quad Q_m^*, P_1, \dots, P_n, \Gamma \Rightarrow \Delta}{P_1, \dots, P_n, \Gamma \Rightarrow \Delta}\, R$$

where Δ consists of exactly one formula in Gi. This allows some notation simplification and will significantly improve the readability of the proof. It does not impair the generality of the result.

Lemma 13. *In* Gc^{s} *every partition* $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$ *of a derivable sequent* $\Gamma \Rightarrow \Delta$ *has a split-interpolant. In* Gi^{s} *every partition* $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \ ; \ A$ *of a derivable sequent* $\Gamma \Rightarrow A$ *has a split-interpolant.*

Proof. The proof extends that of Lemma 11. Let R be a singular geometric rule with m premises and let $\Pi, \Gamma \Rightarrow \Delta$ be its conclusion, where Π is the

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[6] Notice, however, that the proof given in [21] contains a misprint and the split-interpolant for the partition of the initial sequent $\Gamma_1, P \ ; \ \Gamma_2 \Rightarrow \Delta_1, P \ ; \ \Delta_2$ (their notation adjusted to ours) is $\bot$, and not $\bot \to \bot$ as stated in [21, p.117].

multiset $P_1, \dots, P_n$ of the atomic principal formulas of R, if any. We consider the following generic partition of the conclusion:

$$\Pi_1, \Gamma_1 \ ; \ \Pi_2, \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$$

where $\Pi_1, \Pi_2 = \Pi$ and $\Gamma_1, \Gamma_2 = \Gamma$ and $\Delta_1, \Delta_2 = \Delta$, and where $\Delta_1 = \varnothing$ and $\Delta_2 = A$ for Gi^{s}. Moreover, let Θ be the multiset $Q_1^*, \dots, Q_m^*$ of active formulas of R, if any. We organize the proof in three exhaustive cases:

1. $\mathsf{Rel}(\Theta, \Pi) \subseteq \mathsf{Rel}(\Pi_1, \Gamma_1, \Delta_1)$;
2. $\mathsf{Rel}(\Theta, \Pi) \subseteq \mathsf{Rel}(\Pi_2, \Gamma_2, \Delta_2)$;
3. $\mathsf{Rel}(\Theta, \Pi) \not\subseteq \mathsf{Rel}(\Pi, \Gamma, \Delta)$.

Observe that these three cases are exhaustive since singular geometric rules have at most one non-logical predicate in their principal and active formulas and, therefore, when Case 3 does not hold at least one of Cases 1 and 2 holds. We give a proof of the three cases for Gc, and then we show the modifications needed for Gi.

Case 1 for Gc^{s}. If R is an m-premise(s) rule for $m \geq 1$, then by the inductive hypothesis (IH) every partition of each of the m premises of R has a split-interpolant. In particular, for each $k \in \{1, \dots, m\}$, there is a C_k such that:

(I_k) $\vdash Q_k^*, \Pi_1, \Pi_2, \Gamma_1 \Rightarrow \Delta_1, C_k$

(II_k) $\vdash C_k, \Gamma_2 \Rightarrow \Delta_2$

(III_k) $\mathcal{L}(C_k) \subseteq \mathcal{L}(Q_k^*, \Pi_1, \Pi_2, \Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2)$

If, instead, R is a 0-premise rule then (I_1), (II_1), and (III_1) hold trivially if we impose that $C_1 \equiv \bot$ and we delete the two instances of Q_1^*.

We start by assuming that Π_2 is the non-empty multiset $P_{i_{j+1}}, \dots, P_{i_n}$, and then we show the modifications needed when $\Pi_2 = \varnothing$. Consider now the following derivation $\mathcal{D}_1$, where the topmost sequents are derivable by (I_1) - (I_m):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q_1^*, \Pi_2, \Pi_1, \Gamma_1 \Rightarrow \Delta_1, C_1}{Q_1^*, \Pi_2, \Pi_1, \Gamma_1 \Rightarrow \Delta_1, C_1, \dots, C_m}\,{\scriptstyle Wkn}}{Q_1^*, \Pi_2, \Pi_1, \Gamma_1 \Rightarrow \Delta_1, \bigvee_{i=1}^m C_i}\,{\scriptstyle R\vee} \quad \dots \quad \dfrac{\dfrac{Q_m^*, \Pi_2, \Pi_1, \Gamma_1 \Rightarrow \Delta_1, C_m}{Q_m^*, \Pi_2, \Pi_1, \Gamma_1 \Rightarrow \Delta_1, C_1, \dots, C_m}\,{\scriptstyle Wkn}}{Q_m^*, \Pi_2, \Pi_1, \Gamma_1 \Rightarrow \Delta_1, \bigvee_{i=1}^m C_i}\,{\scriptstyle R\vee}}{\Pi_2, \Pi_1, \Gamma_1 \Rightarrow \Delta_1, \bigvee_{i=1}^m C_i}\,{\scriptstyle R}}{\bigwedge \Pi_2, \Pi_1, \Gamma_1 \Rightarrow \Delta_1, \bigvee_{i=1}^m C_i}\,{\scriptstyle L\wedge}}{\Pi_1, \Gamma_1 \Rightarrow \Delta_1, \bigwedge \Pi_2 \to \bigvee_{i=1}^m C_i}\,{\scriptstyle R\to} \tag{1}$$

Notice that the application of R is legitimate because by assumption R is applicable to $Q_i^*, \Pi, \Gamma \Rightarrow \Delta$ and none of the *eigenvariables* of the Q_i^*'s can occur free in some C_k, since $\mathcal{L}(C_k) \subseteq \mathcal{L}(\Gamma_2, \Delta_2)$. Notice also that in some particular case the double-line stands for the empty sequence of instances, e.g., the steps by $R\vee$ when R is a 0- or 1-premise rule.

Consider a second derivation $\mathcal{D}_2$, where the left-topmost sequents are initial sequents since $\Pi_2 = P_{i_{j+1}}, \dots, P_{i_n}$ and the right-topmost ones are derivable by (II_1)–(II_m):

$$\begin{array}{c}
\begin{array}{c} \Pi_2, \Gamma_2 \Rightarrow \Delta_2, P_{i_{j+1}} \quad \cdots \quad \Pi_2, \Gamma_2 \Rightarrow \Delta_2, P_{i_n} \\ \hline\hline \Pi_2, \Gamma_2 \Rightarrow \Delta_2, \bigwedge \Pi_2 \end{array} {\scriptstyle R\wedge} \qquad
\begin{array}{c} \begin{array}{c} C_1, \Gamma_2 \Rightarrow \Delta_2 \quad \cdots \quad C_m, \Gamma_2 \Rightarrow \Delta_2 \\ \hline\hline \bigvee_{i=1}^m C_i, \Gamma_2 \Rightarrow \Delta_2 \end{array} {\scriptstyle L\vee} \\ \hline\hline \bigvee_{i=1}^m C_i, \Pi_2, \Gamma_2 \Rightarrow \Delta_2 \end{array} {\scriptstyle Wkn} \\ \hline
\bigwedge \Pi_2 \to \bigvee_{i=1}^m C_i, \Pi_2, \Gamma_2 \Rightarrow \Delta_2
\end{array} {\scriptstyle L\to} \tag{2}$$

When $\Pi_2 = \varnothing$ we modify $\mathcal{D}_1$ by using left weakening instead of $L\wedge$ to add $\bigwedge \Pi_2$ –i.e., $\top$ – to the antecedent, and we modify $\mathcal{D}_2$ by deriving the conclusion of $R\wedge$ by an instance of $R\top$ instead of by instances of $R\wedge$.

Let $t_1, \dots, t_\ell$ be all terms such that $t_1, \dots, t_\ell \in \mathsf{Ter}(\bigwedge \Pi_2 \to \bigvee_{i=1}^m C_i)$ and $(\bullet)$ $t_1, \dots, t_\ell \notin \mathsf{Ter}(\Pi_1, \Gamma_1, \Delta_1) \cap \mathsf{Ter}(\Pi_2, \Gamma_2, \Delta_2)$. We use $\bar{t}$ to denote $t_1, \dots, t_\ell$. We show that

$$(\ddagger) \qquad t_1, \dots, t_\ell \notin \mathsf{Ter}(\Pi_1, \Gamma_1, \Delta_1)$$

For each $k \leq m$, (III_k) entails that $\mathsf{Ter}(C_k) \subseteq \mathsf{Ter}(\Gamma_2, \Delta_2)$. Hence, $\mathsf{Ter}(\bigwedge \Pi_2 \to \bigvee_{i=1}^m C_i) \subseteq \mathsf{Ter}(\Pi_2, \Gamma_2, \Delta_2)$. By this and $(\bullet)$ we immediately get that $(\ddagger)$ holds.

Let now $\bar{z}$ be variables $z_1, \dots, z_\ell$ not occurring in $\mathcal{D}_1$ and $\mathcal{D}_2$. Lemma 6 applied to $\mathcal{D}_1$ shows that:

$$\vdash \Pi_1, \Gamma_1 \Rightarrow \Delta_1, (\bigwedge \Pi_2 \to \bigvee_{i=1}^m C_i)[\tfrac{\bar{z}}{\bar{t}}]$$

Here $(\ddagger)$ ensures that the substitution $[\frac{\bar{z}}{\bar{t}}]$ has no effect on $\Pi_1, \Gamma_1, \Delta_1$. By ℓ applications of $R\forall$ to the derivable sequent above we obtain:

$$(\mathrm{I}_C) \qquad \vdash \Pi_1, \Gamma_1 \Rightarrow \Delta_1, \forall \bar{z}((\bigwedge \Pi_2 \to \bigvee_{i=1}^m C_i)[\tfrac{\bar{z}}{\bar{t}}])$$

Moreover, by applying ℓ instances of left weakening and then ℓ instances of $L\forall$ to the conclusion of $\mathcal{D}_2$ we obtain:

$$(\mathrm{II}_C) \qquad \vdash \forall\bar{z}((\bigwedge \Pi_2 \to \bigvee_{i=1}^{m} C_i)[\tfrac{\bar{z}}{\bar{t}}]), \Pi_2, \Gamma_2 \Rightarrow \Delta_2$$

Let C be $\forall\bar{z}((\bigwedge \Pi_2 \to \bigvee_{i=1}^{m} C_i)[\frac{\bar{z}}{\bar{t}}])$. By (I_C) and (II_C), we have established that C satisfies the derivability conditions for being a split-interpolant of the given partition. We now show that it also satisfies the language condition, namely:

$$(\mathrm{III}_C) \quad \mathcal{L}(C) \subseteq \mathcal{L}(\Pi_1, \Gamma_1, \Delta_1) \cap \mathcal{L}(\Pi_2, \Gamma_2, \Delta_2)$$

First, if s is a term in $\mathsf{Ter}(C)$, it is a term occurring in $\bigwedge \Pi_2 \to \bigvee_{i=1}^{m} C_i$ that is not in the list $\bar{t}$. By $(\bullet)$, we have:

$$(\mathrm{III.1}_C) \quad s \in \mathsf{Ter}(\Pi_1, \Gamma_1, \Delta_1) \cap \mathsf{Ter}(\Pi_2, \Gamma_2, \Delta_2)$$

Next, we show that:

$$(\mathrm{III.2}_C) \qquad \mathsf{Rel}(C) \subseteq \mathsf{Rel}(\Pi_1, \Gamma_1, \Delta_1) \cap \mathsf{Rel}(\Pi_2, \Gamma_2, \Delta_2)$$

By assumption, we are in Case 1, i.e., $\mathsf{Rel}(\Theta, \Pi) \subseteq \mathsf{Rel}(\Pi_1, \Gamma_1, \Delta_1)$. The following set-theoretic reasoning shows that $(\mathrm{III.2}_C)$ holds:

$$\begin{array}{ll}
\mathsf{Rel}(C) & \overset{\mathrm{III}_k}{\subseteq} \\
\mathsf{Rel}(\Pi_2) \cup (\mathsf{Rel}(\Theta, \Pi_1, \Pi_2, \Gamma_1, \Delta_1) \cap \mathsf{Rel}(\Gamma_2, \Delta_2)) & \overset{\text{distrib.}}{=} \\
\mathsf{Rel}(\Theta, \Pi_1, \Pi_2, \Gamma_1, \Delta_1) \cap \mathsf{Rel}(\Pi_2, \Gamma_2, \Delta_2) & \overset{\text{Case 1}}{=} \\
\mathsf{Rel}(\Pi_1, \Gamma_1, \Delta_1) \cap \mathsf{Rel}(\Pi_2, \Gamma_2, \Delta_2) &
\end{array}$$

We conclude that:

$$\boxed{\dfrac{Q_1^*, \Pi_1, \Pi_2, \Gamma_1 \ ; \ \Gamma_2 \xRightarrow{C_1} \Delta_1 \ ; \ \Delta_2 \quad \cdots \quad Q_m^*, \Pi_1, \Pi_2, \Gamma_1 \ ; \ \Gamma_2 \xRightarrow{C_m} \Delta_1 \ ; \ \Delta_2}{\Pi_1, \Gamma_1 \ ; \ \Pi_2, \Gamma_2 \xRightarrow{\forall\bar{z}((\bigwedge \Pi_2 \to \bigvee_{i=1}^{m} C_i)[\frac{\bar{z}}{\bar{t}}])} \Delta_1 \ ; \ \Delta_2}}$$

Observe that when $\Pi_2 = \varnothing$ the split-interpolant of the conclusion can be simplified as follows:

$$\boxed{\forall\bar{z}((\bigvee_{i=1}^{m} C_i)[\tfrac{\bar{z}}{\bar{t}}])}$$

Case 2 for $\mathsf{Gc^s}$. The proof differs substantially from that of Case 1 only as far as the derivability conditions are concerned. Thus, we give a detailed analysis of these and leave to the reader the task to check that also the language condition is satisfied. By IH every partition of each premise of an m-premises ($m \geq 1$) rule R has a split-interpolant. In particular, for all $k \in \{1, \ldots, m\}$, there is a C_k such that:

(I_k) $\vdash \Gamma_1 \Rightarrow \Delta_1, C_k$

(II_k) $\vdash C_k, Q^*_k, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta_2$

(III_k) $\mathcal{L}(C_k) \subseteq \mathcal{L}(\Gamma_1, \Delta_1) \cap \mathcal{L}(Q^*_k, \Pi_1, \Pi_2, \Gamma_2, \Delta_2)$

In case R is a 0-premise rule, (I_1), (II_1), and (III_1) hold by imposing $C_1 \equiv \top$ (and deleting the two instances of Q^*_1).

Let $\mathcal{D}_1$ be the following derivation, where the topmost sequents are derivable by (II_1) - (II_m):

$$\begin{prooftree}
\AxiomC{$C_1, Q^*_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta_2$}
\RightLabel{$\scriptstyle Wkn$}
\doubleLine
\UnaryInfC{$C_1, \dots, C_m, Q^*_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta_2$}
\AxiomC{$\cdots$}
\AxiomC{$C_m, Q^*_m, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta_2$}
\RightLabel{$\scriptstyle Wkn$}
\doubleLine
\UnaryInfC{$C_1, \dots, C_m, Q^*_m, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta_2$}
\RightLabel{$\scriptstyle R$}
\TrinaryInfC{$C_1, \dots, C_m, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta_2$}
\RightLabel{$\scriptstyle L\wedge$}
\doubleLine
\UnaryInfC{$\bigwedge_{i=1}^m C_i \wedge \bigwedge \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta_2$}
\end{prooftree} \tag{3}$$

Consider now another derivation $\mathcal{D}_2$ where the left topmost sequents are derivable by (I_1) - (I_m) and the right ones are initial sequents (we take $P_{i_1}, \dots, P_{i_j} = \Pi_1$ if $\Pi_1 \neq \varnothing$, else, as we did in (2), we derive the conclusion of the right top-most instance(s) of $R\wedge$ by $R\top$):

$$\begin{prooftree}
\AxiomC{$\Gamma_1 \Rightarrow \Delta_1, C_1 \quad \cdots \quad \Gamma_1 \Rightarrow \Delta_1, C_m$}
\RightLabel{$\scriptstyle R\wedge$}
\doubleLine
\UnaryInfC{$\Gamma_1 \Rightarrow \Delta_1, \bigwedge_{i=1}^m C_i$}
\RightLabel{$\scriptstyle Wkn$}
\doubleLine
\UnaryInfC{$\Pi_1, \Gamma_1 \Rightarrow \Delta_1, \bigwedge_{i=1}^m C_i$}
\AxiomC{$\Pi_1, \Gamma_1 \Rightarrow \Delta_1, P_{i_1} \quad \cdots \quad \Pi_1, \Gamma_1 \Rightarrow \Delta_1, P_{i_j}$}
\RightLabel{$\scriptstyle R\wedge$}
\doubleLine
\UnaryInfC{$\Pi_1, \Gamma_1 \Rightarrow \Delta_1, \bigwedge \Pi_1$}
\RightLabel{$\scriptstyle R\wedge$}
\BinaryInfC{$\Pi_1, \Gamma_1 \Rightarrow \Delta_1, \bigwedge_{i=1}^m C_i \wedge \bigwedge \Pi_1$}
\end{prooftree} \tag{4}$$

Let $\bar{t}$ be all terms $t_1, \dots, t_\ell$ such that $t_1, \dots, t_\ell \in \mathsf{Ter}(\bigwedge_{i=1}^m C_i \wedge \bigwedge \Pi_1)$ and $t_1, \dots, t_\ell \notin \mathsf{Ter}(\Pi_1, \Gamma_1, \Delta_1) \cap \mathsf{Ter}(\Pi_2, \Gamma_2, \Delta_2)$. As in the previous case, it is easy to show that:

$$(\ddagger) \qquad t_1, \dots, t_\ell \notin \mathsf{Ter}(\Pi_2, \Gamma_2, \Delta_2)$$

Moreover let $\bar{z}$ be variables $z_1, \dots, z_\ell$ new to $\mathcal{D}_1$ and $\mathcal{D}_2$. We reason analogously to the previous case to obtain:

$$(\mathrm{I}_C) \qquad \vdash \Pi_1, \Gamma_1 \Rightarrow \Delta_1, \exists \bar{z}((\bigwedge_{i=1}^{m} C_i \wedge \bigwedge \Pi_1)[\tfrac{\bar{z}}{\bar{t}}])$$

As above, thanks to $(\ddagger)$, we also obtain:

$$(\mathrm{II}_C) \qquad \vdash \exists \bar{z}((\bigwedge_{i=1}^{m} C_i \wedge \bigwedge \Pi_1)[\tfrac{\bar{z}}{\bar{t}}]), \Pi_2, \Gamma_2 \Rightarrow \Delta_2$$

Let C be $\exists\bar{z}((\bigwedge_{i=1}^{m} C_i \wedge \bigwedge \Pi_1)[\frac{\bar{z}}{\bar{t}}])$. Given that $\mathsf{Rel}(\Theta,\Pi) \subseteq \mathsf{Rel}(\Pi_2,\Gamma_2,\Delta_2)$, and given that we have quantified away all terms in $\bar{t}$, we have:

$$(\mathrm{III}_C) \qquad \mathcal{L}(C) \subseteq \mathcal{L}(\Pi_1,\Gamma_1,\Delta_1) \cap \mathcal{L}(\Pi_2,\Gamma_2,\Delta_2)$$

We conclude that C is a split-interpolant of the given partition.

$$\boxed{\dfrac{\Gamma_1\ ;\ Q_1^*,\Pi_1,\Pi_2,\Gamma_2 \overset{C_1}{\Longrightarrow} \Delta_1\ ;\ \Delta_2 \quad \ldots \quad \Gamma_1\ ;\ Q_m^*,\Pi_1,\Pi_2,\Gamma_2 \overset{C_m}{\Longrightarrow} \Delta_1\ ;\ \Delta_2}{\Pi_1,\Gamma_1\ ;\ \Pi_2,\Gamma_2 \xRightarrow{\exists\bar{z}((\bigwedge_{i=1}^{m} C_i\wedge\bigwedge \Pi_1)[\frac{\bar{z}}{\bar{t}}])} \Delta_1\ ;\ \Delta_2}}$$

As for the previous case, when $\Pi_1 = \varnothing$ we have a simpler split-interpolant of the conclusion:

$$\boxed{\exists\bar{z}((\bigwedge_{i=1}^{m} C_i)[\frac{\bar{z}}{\bar{t}}])}$$

Case 3 for Gc^{s}. We can proceed either as in Case 1 or as in Case 2. If we proceed as in Case 1, we obtain the following split-interpolant:

$$\boxed{\dfrac{Q_1^*,\Pi_1,\Pi_2,\Gamma_1\ ;\ \Gamma_2 \overset{C_1}{\Longrightarrow} \Delta_1\ ;\ \Delta_2 \quad \cdots \quad Q_m^*,\Pi_1,\Pi_2,\Gamma_1\ ;\ \Gamma_2 \overset{C_m}{\Longrightarrow} \Delta_1\ ;\ \Delta_2}{\Pi_1,\Gamma_1\ ;\ \Pi_2,\Gamma_2 \xRightarrow{\forall\bar{z}((\bigwedge \Pi_2\rightarrow\bigvee_{i=1}^{m} C_i)[\frac{\bar{z}}{\bar{t}}])} \Delta_1\ ;\ \Delta_2}}$$

The proof that the formula C presented above is the split-interpolant of the conclusion is exactly as for Case 1, save for the relational part $(\mathrm{III}.2_C)$ of the language condition. In this case we are assuming that $\mathsf{Rel}(\Theta,\Pi) \not\subseteq \mathsf{Rel}(\Pi,\Gamma,\Delta)$. By the singularity condition (i.e. $|\mathsf{Rel}(\Theta,\Pi)| \leq 1$), this implies

$$(+) \quad \mathsf{Rel}(\Pi_1,\Pi_2) = \varnothing$$

and

$$(++) \quad \mathsf{Rel}(\Theta) \cap \mathsf{Rel}(\Pi_2,\Gamma_2,\Delta_2) = \varnothing$$

Hence, we can show that $(\mathrm{III}.2_C)$ holds via the following set-theoretic reasoning

$$\begin{aligned}
&\mathsf{Rel}(C) && \overset{\mathrm{III}_k}{\subseteq} \\
&\mathsf{Rel}(\Pi_2) \cup (\mathsf{Rel}(\Theta,\Pi_1,\Pi_2,\Gamma_1,\Delta_1) \cap \mathsf{Rel}(\Gamma_2,\Delta_2)) && \overset{\text{distrib.}}{=} \\
&\mathsf{Rel}(\Theta,\Pi_1,\Pi_2,\Gamma_1,\Delta_1) \cap \mathsf{Rel}(\Pi_2,\Gamma_2,\Delta_2) && \overset{(+),(++)}{=} \\
&\mathsf{Rel}(\Gamma_1,\Delta_1) \cap \mathsf{Rel}(\Gamma_2,\Delta_2)
\end{aligned}$$

Case 1 for Gi^{s}. The proof is the same as for Case 1 in Gc^{s} (with $\Delta_1 = \varnothing$ and $\Delta_2 = A$) save for the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ presented in (1) and (2) that are not Gi^{s}-derivations. It is immediate to see that we can obtain a Gi^{s}-derivation from the derivation in (1) by simply omitting the instances of weakening and applying directly instances of $R\vee$ to the leaves. On the other hand, the derivation $\mathcal{D}_2$ presented in (2) becomes a Gi^{s}-derivation by simply dropping the singleton multiset Δ_2 from the left top-most sequents and then adding an instance of weakening on the left premise of $L\to$.

Case 2 for Gi^{s}. The proof is the same as for Case 2 in Gc^{s}, since the derivations presented in (3) and (4) are Gi^{s}-derivation when $\Delta_1 = \varnothing$ and $\Delta_2 = A$.

Case 3 for Gi^{s}. We may proceed as for Case 1 for Gi^{s} save for the relational part (III.2_C) of the language condition where we reason as in Case 3 for Gc^{s}. ■

From Lemma 13 it is immediate to conclude that singular geometric extensions of classical and intuitionistic logic satisfy the interpolation theorem, namely:

THEOREM 14. G^{s} *has the interpolation property.*

5. Applications

We now consider some corollaries of Theorem 14 in which the strategy for building interpolants provided in Lemma 13 is applied. Notice that in the theories considered in this section all contracted instances are admissible and, hence, we can ignore them, see the discussion after Definition 4.

5.1. First-order logic with identity

We start with first-order logic with identity. Recall that a cut-free calculus for classical first-order logic with identity has been presented in [15] by adding on top of Gc the rules $Ref_=$ and $Repl_=$ corresponding to the reflexivity of $=$ and Leibniz's principle of indescernibility of identicals, respectively. In intuitionistic theories, on the other hand, identity is often treated differently and we will provide a constructively more acceptable treatment of identity later in dealing with apartness. In general, however, nothing prevents us from building intuitionistic first-order logic with identity in a parallel fashion to the classical case. This is, for example, the route taken in [21] and we will follow suit. More specifically, let $\mathsf{G}^{=}$ be $\mathsf{G} + \{\mathit{Ref}_{=}, \mathit{Repl}_{=}\}$. Notice that, since $\mathit{Ref}_{=}$ and $\mathit{Repl}_{=}$ are geometric rules, cut elimination holds in $\mathsf{Gi}^{=}$ in virtue of Theorem 8. Moreover, since they are also singular geometric, it follows from our Theorem 14 that in $\mathsf{G}^{=}$ the interpolation property holds, i.e.

Corollary 15. $\mathsf{G}^{=}$ *has the interpolation property.*

Proof. We determine the split-interpolants as applications of the procedures given in the proof of Lemma 13. The rule $\mathit{Ref}_{=}$ can be treated as an instance of Case 1 with $\Pi_2 = \varnothing$ (obviously, it could also have been treated as an instance of Case 2). Depending on whether both $s \in \mathsf{Ter}(C)$ and $s \notin \mathsf{Ter}(\Gamma_1, \Delta_1)$ or not, we have then, respectively:

$$\boxed{\dfrac{s = s, \Gamma_1 \,;\, \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \,;\, \Delta_2}{\Gamma_1 \,;\, \Gamma_2 \xRightarrow{\forall z (C[\genfrac{}{}{0pt}{}{z}{s}])} \Delta_1 \,;\, \Delta_2} \qquad \dfrac{s = s, \Gamma_1 \,;\, \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \,;\, \Delta_2}{\Gamma_1 \,;\, \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \,;\, \Delta_2}}$$

For $\mathit{Repl}_{=}$, there are four possible partitions of the conclusion:

- $s = t, P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$
- $\Gamma_1 \,;\, s = t, P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$
- $P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_1 \,;\, s = t, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$
- $s = t, \Gamma_1 \,;\, P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$

Accordingly, we need to consider four sub-cases. As in Case 1 of Lemma 13, when $\Pi_2 = \varnothing$, the interpolant for the first partition is as follows:

$$\boxed{\dfrac{P[\genfrac{}{}{0pt}{}{t}{x}], s = t, P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_1 \,;\, \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \,;\, \Delta_2}{s = t, P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_1 \,;\, \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \,;\, \Delta_2}}$$

The interpolant for the second partition is obtained by reasoning as in Case 2 with $\Pi_1 = \varnothing$ of Lemma 13:

$$\boxed{\dfrac{\Gamma_1 \,;\, P[\genfrac{}{}{0pt}{}{t}{x}], s = t, P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \,;\, \Delta_2}{\Gamma_1 \,;\, s = t, P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \,;\, \Delta_2}}$$

The interpolant for the third partition is found as in Case 1 of Lemma 13, depending on whether $t \in \mathsf{Ter}(P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_1, \Delta_1)$ (left derivation in the box below) or not (right derivation in the box below).

$$\boxed{\dfrac{P[\genfrac{}{}{0pt}{}{t}{x}], s = t, P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_1 \,;\, \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \,;\, \Delta_2}{P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_1 \,;\, s = t, \Gamma_2 \xRightarrow{s = t \to C} \Delta_1 \,;\, \Delta_2} \quad \dfrac{P[\genfrac{}{}{0pt}{}{t}{x}], s = t, P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_1 \,;\, \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \,;\, \Delta_2}{P[\genfrac{}{}{0pt}{}{s}{x}], \Gamma_1 \,;\, s = t, \Gamma_2 \xRightarrow{\forall z (s = z \to C[\genfrac{}{}{0pt}{}{z}{t}])} \Delta_1 \,;\, \Delta_2}}$$

Lastly, the interpolant for the fourth partition is found as in Case 2 of Lemma 13, depending on whether $t \in \mathsf{Ter}(P[^{s}_{x}], \Gamma_2, \Delta_2)$ or not:

$$\boxed{\dfrac{\Gamma_1 \ ; \ P[^{t}_{x}], s = t, P[^{s}_{x}], \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2}{s = t, \Gamma_1 \ ; \ P[^{s}_{x}], \Gamma_2 \xRightarrow{s=t\wedge C} \Delta_1 \ ; \ \Delta_2} \qquad \dfrac{\Gamma_1 \ ; \ P[^{t}_{x}], s = t, P[^{s}_{x}], \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2}{s = t, \Gamma_1 \ ; \ P[^{s}_{x}], \Gamma_2 \xRightarrow{\exists z(s=z\wedge C[^{z}_{t}])} \Delta_1 \ ; \ \Delta_2}}$$

■

5.2. Equivalence relations

In a perfectly parallel fashion, we obtain the theory of equivalence relations by adding to G the rules corresponding to the reflexivity, transitivity and symmetry of a binary relation $\sim$. Thus, $\mathsf{EQ} = \mathsf{G} + \{Ref_{\sim}\ , \ Trans_{\sim}\ , \ Sym_{\sim}\}$.

$$\dfrac{s \sim s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Ref_{\sim} \qquad \dfrac{s \sim u, s \sim t, t \sim u, \Gamma \Rightarrow \Delta}{s \sim t, t \sim u, \Gamma \Rightarrow \Delta}\, Trans_{\sim}$$

$$\dfrac{t \sim s, s \sim t, \Gamma \Rightarrow \Delta}{s \sim t, \Gamma \Rightarrow \Delta}\, Sym_{\sim}$$

From the fact that these rules are singular geometric, it follows that:

Corollary 16. EQ *has the interpolation property.*

Proof. The case of $Ref_{\sim}$ is like that for $Ref_{=}$ in $\mathsf{G}^{=}$, the only difference being that, when $\sim$ is not in $\mathsf{Rel}(\Gamma, \Delta)$, the rule $Ref_{\sim}$ becomes an instance of Case 3.[7] We consider in detail the cases of $Trans_{\sim}$ and $Sym_{\sim}$.

Regarding $Trans_{\sim}$, there are four possible partitions of the conclusion:

- $s \sim t, t \sim u, \Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$
- $\Gamma_1 \ ; \ s \sim t, t \sim u, \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$
- $s \sim t, \Gamma_1 \ ; \ t \sim u, \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$
- $t \sim u, \Gamma_1 \ ; \ s \sim t, \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$

For the first two partitions, we find the split-interpolant by reasoning as in Case 1 with $\Pi_2 = \varnothing$ and Case 2 with $\Pi_1 = \varnothing$, respectively. Hence, a split-interpolant for the first and second partitions is:

---

[7] Otherwise, it is an instance of Case 1 or of Case 2, and then the split-interpolant of the conclusion can be determined as we have shown for $Ref_{=}$, except for the use of the existential quantifier when we have an instance of Case 2 only and we must quantify away s.

$$\boxed{\dfrac{s \sim u, s \sim t, t \sim u, \Gamma_1 \ ; \ \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2}{s \sim t, t \sim u, \Gamma_1 \ ; \ \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2} \qquad \dfrac{\Gamma_1 \ ; \ s \sim u, s \sim t, t \sim u, \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2}{\Gamma_1 \ ; \ s \sim t, t \sim u, \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2}}$$

For the last two partitions we can proceed as in Case 1 or as in Case 2. By proceeding as in Case 1 we find the following split-interpolants, assuming, respectively, $u \notin \mathsf{Ter}(s \sim t, \Gamma_1, \Delta_1)$ and $s \notin \mathsf{Ter}(t \sim u, \Gamma_1, \Delta_1)$:

$$\boxed{\dfrac{s \sim u, s \sim t, t \sim u, \Gamma_1 \ ; \ \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2}{s \sim t, \Gamma_1 \ ; \ t \sim u, \Gamma_2 \xrightarrow{\forall z (t \sim z \to C[{}^{z}_{u}])} \Delta_1 \ ; \ \Delta_2} \quad \dfrac{s \sim u, s \sim t, t \sim u, \Gamma_1 \ ; \ \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2}{t \sim u, \Gamma_1 \ ; \ s \sim t, \Gamma_2 \xrightarrow{\forall z (z \sim t \to C[{}^{z}_{s}])} \Delta_1 \ ; \ \Delta_2}}$$

If, instead, $u \in \mathsf{Ter}(s \sim t, \Gamma_1, \Delta_1)$ or $s \in \mathsf{Ter}(t \sim u, \Gamma_1, \Delta_1)$, then we do not quantify them away and we have, respectively:

$$\boxed{\dfrac{s \sim u, s \sim t, t \sim u, \Gamma_1 \ ; \ \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2}{s \sim t, \Gamma_1 \ ; \ t \sim u, \Gamma_2 \xrightarrow{t \sim u \to C} \Delta_1 \ ; \ \Delta_2} \quad \dfrac{s \sim u, s \sim t, t \sim u, \Gamma_1 \ ; \ \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2}{t \sim u, \Gamma_1 \ ; \ s \sim t, \Gamma_2 \xrightarrow{s \sim t \to C} \Delta_1 \ ; \ \Delta_2}}$$

Regarding $Sym_{\sim}$, there are two possible partitions of the conclusion:

- $s \sim t, \Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$
- $\Gamma_1 \ ; \ s \sim t, \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$

We find the split-interpolant by reasoning as in Case 1 with $\Pi_2 = \varnothing$ and Case 2 with $\Pi_1 = \varnothing$, respectively. Hence we have:

$$\boxed{\dfrac{t \sim s, s \sim t, \Gamma_1 \ ; \ \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2}{s \sim t, \Gamma_1 \ ; \ \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2} \qquad \dfrac{\Gamma_1 \ ; \ t \sim s, s \sim t, \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2}{\Gamma_1 \ ; \ s \sim t, \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2}}$$

■

5.3. Partial and linear orders

Now we consider some well-known order theories. We start with partial orders. In sequent calculus, the theory of partial orders is obtained by extending $\mathsf{Gc}^{=}$ with the following rules corresponding to the axioms of reflexivity, transitivity and anti-symmetry of a binary relation $\leqslant$. Thus, let $\mathsf{PO} = \mathsf{Gc}^{=} + \{Ref_{\leqslant}\,,\ Trans_{\leqslant}\,,\ Anti\text{-}sym_{\leqslant}\}$:

$$\dfrac{s \leqslant s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle Ref_{\leqslant}} \qquad \dfrac{s \leqslant u, s \leqslant t, t \leqslant u, \Gamma \Rightarrow \Delta}{s \leqslant t, t \leqslant u, \Gamma \Rightarrow \Delta}\ {\scriptstyle Trans_{\leqslant}}$$

$$\dfrac{s = t, s \leqslant t, t \leqslant s, \Gamma \Rightarrow \Delta}{s \leqslant t, t \leqslant s, \Gamma \Rightarrow \Delta}\ {\scriptstyle Anti\text{-}sym_{\leqslant}}$$

Linear orders are obtained by assuming that the partial order $\leqslant$ is also linear, i.e $\mathsf{LO} = \mathsf{PO} + \{\mathit{Lin}_{\leqslant}\}$.

$$\frac{s \leqslant t, \Gamma \Rightarrow \Delta \quad t \leqslant s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Lin}_{\leqslant}$$

Both PO and LO are singular geometric theories, hence:

Corollary 17. LO *(hence,* PO*) has the interpolation property.*

Proof. The procedure for building the interpolants for *Ref*$_{\leqslant}$ and *Trans*$_{\leqslant}$ are the same as those for *Ref*$_{\sim}$ and *Trans*$_{\sim}$, respectively, in EQ; that for *Anti-sym*$_{\leqslant}$ is like that for *Trans*$_{\sim}$, save that here there is no need to quantify away any term occurring in the split-interpolant.

For *Lin*$_{\leqslant}$, only one partition of the conclusion has to be considered, namely $\Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$. Its interpolant can be found as in Case 3 of Lemma 13 with $\Pi_2 = \varnothing$, provided that $\leqslant$ is not in $\mathsf{Rel}(\Gamma, \Delta)$. Assuming that both s and t are in $\mathsf{Ter}(C_1, C_2)$ but not in $\mathsf{Ter}(\Gamma_1, \Delta_1)$:

$$\boxed{\frac{s \leqslant t, \Gamma_1 \,;\, \Gamma_2 \overset{C_1}{\Longrightarrow} \Delta_1 \,;\, \Delta_2 \quad t \leqslant s, \Gamma_1 \,;\, \Gamma_2 \overset{C_2}{\Longrightarrow} \Delta_1 \,;\, \Delta_2}{\Gamma_1 \,;\, \Gamma_2 \xRightarrow{\forall z_1 \forall z_2 ((C_1 \vee C_2)[^{z_1}_{s}\,^{z_2}_{t}])} \Delta_1 \,;\, \Delta_2}}$$

If, instead, s or t is in $\mathsf{Ter}(\Gamma_1, \Delta_1)$, or if it is not in $\mathsf{Ter}(C_1, C_2)$, then it is not quantified away. Else, if $\leqslant$ is in $\mathsf{Rel}(\Gamma, \Delta)$, we proceed as in Case 1 or 2 of Lemma 13 as for rule *Ref*$_{\sim}$, cf. footnote 7. ■

Unlike $\mathsf{G}^{=}$ and EQ, the underlying logical calculus of both PO and LO is the classical one. The reason is that linearity is intuitionistically contentious and normally it requires a different, more constructively acceptable, axiomatization that will be considered in Section 5.6.

5.4. Strict partial and linear orders

The theory of strict partial orders consists of the axioms of first-order logic with identity plus the irreflexivity and transitivity of $<$. As we did for PO and LO, we consider this theory to be based on classical logic, i.e. by adding on top of $\mathsf{Gc}^{=}$ the following rules:

$$\frac{}{s < s, \Gamma \Rightarrow \Delta}\ \mathit{Irref}_{<} \qquad \frac{s < u, s < t, t < u, \Gamma \Rightarrow \Delta}{s < t, t < u, \Gamma \Rightarrow \Delta}\ \mathit{Trans}_{<}$$

Let SPO be $\mathsf{Gc}^{=} + \{\mathit{Irref}_{<}, \mathit{Trans}_{<}\}$. Total strict partial orders are then obtained assuming that $<$ is also trichotomous, i.e. $\mathsf{SLO} = \mathsf{SPO} + \{\mathit{Trich}_{<}\}$:

$$\frac{s = t, \Gamma \Rightarrow \Delta \quad s < t, \Gamma \Rightarrow \Delta \quad t < s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Trich}_{<}$$

Corollary 18. SLO *(hence,* SPO*) has the interpolation property.*

Proof. We show how to find the interpolants for $\mathit{Irref}_{<}$ and $\mathit{Trich}_{<}$, while $\mathit{Trans}_{<}$ is identical to $\mathit{Trans}_{\sim}$. We start with $\mathit{Irref}_{<}$. There are two possible partitions of its conclusion, namely

- $s < s, \Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$
- $\Gamma_1 \ ; \ s < s, \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$

As in Case 1 with $\Pi_2 = \varnothing$ (and $m = 0$) and as in Case 2 with $\Pi_1 = \varnothing$ (and $m = 0$) of Lemma 13, we find the split-interpolant for each partition as follows:

$$\boxed{\frac{}{s < s, \Gamma_1 \ ; \ \Gamma_2 \xRightarrow{\bot} \Delta_1 \ ; \ \Delta_2} \qquad \frac{}{\Gamma_1 \ ; \ s < s, \Gamma_2 \xRightarrow{\top} \Delta_1 \ ; \ \Delta_2}}$$

Regarding $\mathit{Trich}_{<}$, we need to consider only one partition of the conclusion, namely $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$, whose interpolant can be found as in Case 3 of Lemma 13 with $\Pi_2 = \varnothing$ when $<$ is not in $\mathsf{Rel}(\Gamma, \Delta)$. Assuming that both s and t are in $\mathsf{Ter}(C_1, C_2, C_3)$ but not in $\mathsf{Ter}(\Gamma_2, \Delta_2)$:

$$\boxed{\frac{s = t, \Gamma_1 \ ; \ \Gamma_2 \xRightarrow{C_1} \Delta_1 \ ; \ \Delta_2 \quad s < t, \Gamma_1 \ ; \ \Gamma_2 \xRightarrow{C_2} \Delta_1 \ ; \ \Delta_2 \quad t < s, \Gamma_1 \ ; \ \Gamma_2 \xRightarrow{C_3} \Delta_1 \ ; \ \Delta_2}{\Gamma_1 \ ; \ \Gamma_2 \xRightarrow{\forall z_1 \forall z_2 ((C_1 \vee C_2 \vee C_3)[^{z_1}_{s}\,^{z_2}_{t}])} \Delta_1 \ ; \ \Delta_2}}$$

If s or t is in $\mathsf{Ter}(\Gamma_2, \Delta_2)$, or if it is not in $\mathsf{Ter}(C_1, C_2, C_3)$, then it is not quantified away. Else, if $<$ is in $\mathsf{Rel}(\Gamma, \Delta)$, we proceed as in Case 1 or 2 of Lemma 13 as for rule $\mathit{Ref}_{\sim}$, cf. footnote 7.

■

5.5. Apartness

We noticed earlier that in intuitionistic theories the identity relation is not always treated as in classical logic. In particular, identity is defined in terms of the more constructively acceptable relation of apartness. Apartness was originally introduced by Brouwer (and later axiomatized by Heyting in [10]) to express inequality between real numbers in the constructive analysis of the continuum: whereas saying that two real numbers a and b are unequal only means that the assumption $a = b$ is contradictory, to say that a and b are apart expresses the constructively stronger requirement that their distance on the real line can be effectively measured, i.e. that $|\, a - b \,| > 0$ has a constructive proof. Classically, inequality and apartness coincide, but intuitionistically two real numbers can be unequal without being apart. The theory of apartness consists of intuitionistic first-order logic plus the irreflexivity and splitting of $\neq$. Following [13], the theory of apartness is formulated by adding on top of Gi the following rules:[8]

$$\frac{}{s \neq s, \Gamma \Rightarrow A}\ \mathit{Irref}_{\neq} \qquad \frac{s \neq u, s \neq t, \Gamma \Rightarrow A \quad t \neq u, s \neq t, \Gamma \Rightarrow A}{s \neq t, \Gamma \Rightarrow A}\ \mathit{Split}_{\neq}$$

Let $\mathsf{AP} = \mathsf{Gi} + \{\mathit{Irref}_{\neq}\,, \mathit{Split}_{\neq}\}$. Given that these two rules are singular geometric rules, it follows that:

Corollary 19. AP *has the interpolation property.*

Proof. As above, we show how to find the interpolants for $\mathit{Irref}_{\neq}$ and $\mathit{Split}_{\neq}$. The former is identical to that of $\mathit{Irref}_{<}$ in SPO.

In the case of $\mathit{Split}_{\neq}$, there are two possible partitions of the conclusion:

- $s \neq t, \Gamma_1 \;;\; \Gamma_2 \Rightarrow \;;\; A$
- $\Gamma_1 \;;\; s \neq t, \Gamma_2 \Rightarrow \;;\; A$

For the first partition, we use Case 1 of Lemma 13 with $\Pi_2 = \varnothing$. Thus, if $u \notin \mathsf{Ter}(s \neq t, \Gamma_1)$ and $u \in \mathsf{Ter}(C_1, C_2)$, a split-interpolant for the first partition is:

$$\boxed{\frac{s \neq u, s \neq t, \Gamma_1 \;;\; \Gamma_2 \overset{C_1}{\Longrightarrow} \;;\; A \qquad t \neq u, s \neq t, \Gamma_1 \;;\; \Gamma_2 \overset{C_2}{\Longrightarrow} \;;\; A}{s \neq t, \Gamma_1 \;;\; \Gamma_2 \xRightarrow{\forall z (C_1[\frac{z}{u}] \vee C_2[\frac{z}{u}])} \;;\; A}}$$

For the second partition, we use Case 2 of Lemma 13 with $\Pi_1 = \varnothing$. Thus, if $u \notin \mathsf{Ter}(s \neq t, \Gamma_2, A)$ and $u \in \mathsf{Ter}(C_1, C_2)$, a split-interpolant for the second partition is:

---

[8] Notice that Negri's underlying calculus is a quantifier-free version of Gi.

$$\boxed{\dfrac{\Gamma_1\ ;\ s \neq u, s \neq t, \Gamma_2 \overset{C_1}{\Longrightarrow}\ ;\ A \qquad \Gamma_1\ ;\ t \neq u, s \neq t, \Gamma_2 \overset{C_2}{\Longrightarrow}\ ;\ A}{\Gamma_1\ ;\ s \neq t, \Gamma_2 \xRightarrow{\exists z(C_1[\frac{z}{u}] \wedge C_2[\frac{z}{u}])}\ ;\ A}}$$

When u is, respectively, in $\mathsf{Ter}(s \neq t, \Gamma_1)$ or in $\mathsf{Ter}(s \neq t, \Gamma_2, A)$, as well as when it is not in $\mathsf{Ter}(C_1, C_2)$, we do not quantify it away . ■

5.6. Positive partial and linear orders

Just like apartness is a positive version of inequality, so excess $\nleqslant$ is a positive version of the negation of a partial order $\leqslant$. Excess relation was introduced by von Plato in [18] and has been further investigated by Negri in [13]. The theory of positive partial orders consists of intuitionistic first-order logic plus the irreflexivity and co-transitivity of $\nleqslant$.[9] Let $\mathsf{PPO} = \mathsf{Gi} + \{\mathit{Irref}_{\nleqslant}, \mathit{Co\text{-}trans}_{\nleqslant}\}$

$$\frac{}{s \nleqslant s, \Gamma \Rightarrow A}\ \mathit{Irref}_{\nleqslant} \qquad \frac{s \nleqslant u, s \nleqslant t, \Gamma \Rightarrow A \quad u \nleqslant t, s \nleqslant t, \Gamma \Rightarrow A}{s \nleqslant t, \Gamma \Rightarrow A}\ \mathit{Co\text{-}trans}_{\nleqslant}$$

The theory of positive linear orders extends the theory of positive partial orders with the asymmetry of $\nleqslant$. Specifically, let $\mathsf{PLO} = \mathsf{PPO} + \{\mathit{Asym}_{\nleqslant}\}$:

$$\frac{}{s \nleqslant t, t \nleqslant s, \Gamma \Rightarrow A}\ \mathit{Asym}_{\nleqslant}$$

Given that all these rules are singular geometric, from Theorem 14 it follows that

Corollary 20. PPO *and in* PLO *have the interpolation property.*

Proof. The cases of $\mathit{Irref}_{\nleqslant}$ and of $\mathit{Co\text{-}Trans}_{\nleqslant}$ are like the analogous cases for rules $\mathit{Irref}_{\neq}$ and $\mathit{Split}_{\neq}$ and the split-interpolants can be obtained by those in the proof of Corollary 19. For rule $\mathit{Asym}_{\nleqslant}$ we have four possible partitions of the conclusion

- $s \nleqslant t, t \nleqslant s, \Gamma_1\ ;\ \Gamma_2 \Rightarrow\ ;\ A$
- $\Gamma_1\ ;\ s \nleqslant t, t \nleqslant s, \Gamma_2 \Rightarrow\ ;\ A$

---

[9] Co-transitivity and splitting should not be confused. In particular, splitting (along with irreflexivity) gives symmetry, whereas co-transitivity does not. This is what distinguishes apartness (which is symmetric) from excess (which in general is not).

- $s \nleqslant t, \Gamma_1 \,;\, t \nleqslant s, \Gamma_2 \Rightarrow \,;\, A$
- $t \nleqslant s, \Gamma_1 \,;\, s \nleqslant t, \Gamma_2 \Rightarrow \,;\, A$

Their split-interpolants are like those for rule *Anti-sym*$_{\leqslant}$, except that here we have a 0-premise rule. For the first and second partitions we have, respectively:

$$\boxed{\dfrac{}{s \nleqslant t, t \nleqslant s, \Gamma_1 \,;\, \Gamma_2 \overset{\bot}{\Rightarrow} \,;\, A} \qquad \dfrac{}{\Gamma_1 \,;\, s \nleqslant t, t \nleqslant s, \Gamma_2 \overset{\top}{\Rightarrow} \,;\, A}}$$

Finally, for the last two partitions we have, respectively:

$$\boxed{\dfrac{}{s \nleqslant t, \Gamma_1 \,;\, t \nleqslant s, \Gamma_2 \xRightarrow{t \nleqslant s \to \bot} \,;\, A} \quad \dfrac{}{t \nleqslant s, \Gamma_1 \,;\, s \nleqslant t, \Gamma_2 \xRightarrow{s \nleqslant t \to \bot} \,;\, A}}$$

■

To conclude, we have shown (Lemma 13) how to extend Maehara's lemma to extensions of classical and intuitionistic sequent calculi with singular geometric rules and provided a number of interesting examples of singular geometric rules that are important both in logic and mathematics, especially in order theories. In particular, we have shown that Lemma 13 covers first-order logic with identity and its extension with the theory of (strict) partial and linear orders. We have also proved that the same holds for the intuitionistic theories of apartness, as well as for positive partial and linear order. Along the way, we have also provided a cut-elimination theorem for geometric extensions Gi^{g} of the intuitionistic single-succedent calculus Gi.

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