# Additive Cellular Automata Over Finite Abelian Groups: Topological and Measure Theoretic Properties

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## Abstract

We study the dynamical behavior of $D$-dimensional ($D \geq 1$) additive cellular automata where the alphabet is any finite abelian group. This class of discrete time dynamical systems is a generalization of the systems extensively studied by many authors among which one may list [36, 43, 44, 40, 12, 11]. Our main contribution is the proof that topologically transitive additive cellular automata are ergodic. This result represents a solid bridge between the world of measure theory and that of topology theory and greatly extends previous results obtained in [12, 43] for linear CA over $\mathbb{Z}_m$ i.e. additive CA in which the alphabet is the cyclic group $\mathbb{Z}_m$ and the local rules are linear combinations with coefficients in $\mathbb{Z}_m$. In our scenario, the alphabet is any finite abelian group and the global rule is any additive map. This class of CA strictly contains the class of linear CA over $\mathbb{Z}_m^n$, i.e., with the local rule defined by $n \times n$ matrices with elements in $\mathbb{Z}_m$ which, in turn, strictly contains the class of linear CA over $\mathbb{Z}_m$. In order to further emphasize that finite abelian groups are more expressive than $\mathbb{Z}_m$ we prove that, contrary to what happens in $\mathbb{Z}_m$, there exist additive CA over suitable finite abelian groups which are roots (with arbitrarily large indices) of the shift map.

As a consequence of our results, we have that, for additive CA, ergodic mixing, weak ergodic mixing, ergodicity, topological mixing, weak topological mixing, topological total transitivity and topological transitivity are all equivalent properties. As a corollary, we have that invertible transitive additive CA are isomorphic to Bernoulli shifts. Finally, we provide a first characterization of strong transitivity for additive CA which we suspect it might be true also for the general case.

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# 1 Introduction

*Cellular automata (CA)* are widely known formal models for studying and simulating complex systems [9]. They are used in many disciplines ranging from physics to biology, stepping through sociology and, of course, computer science (for recent results and an up-to date bibliography on CA, see [25, 28, 17, 16, 3], while for other models of natural computing see for instance [21, 19, 26, 18]). Their extensive use is due to the huge variety of dynamical behaviors. In computer science, applications can be found in many different contexts such as cryptography, pseudo-random number generation, image processing, data compression, *etc.*

More formally, a CA can be defined as an infinite set of finite automata arranged on the regular lattice $\mathbb{Z}^D$, where $D \in \mathbb{N}$ is the *dimension* of the CA. Each finite automaton has a *state* which is chosen from a finite set $G$, called the *set of states* or the *alphabet*. All finite automata update synchronously according to a *local rule* which takes into account the current state of the automaton and the states of a *neighborhood* $\mathcal{N}$ of neighboring automata. The local rule $f$ induces a *global map* $F\colon G^{\mathbb{Z}^D} \to G^{\mathbb{Z}^D}$ which describes the overall evolution of the CA at each time tick.

Despite of their apparent simplicity, CA may exhibit extremely complex dynamical behaviors. Indeed, in most cases the problem of deciding if a given CA has a certain dynamical property or not is undecidable [5, 30, 37] and some Rice-like theorems have been proved [34, 39]. The complex dynamics of CA has been described through a great number of properties (see Section 2 for definitions) involving both the measure theoretical and the topological point of views. Figure 1 illustrates the relations between those that are studied in this paper.

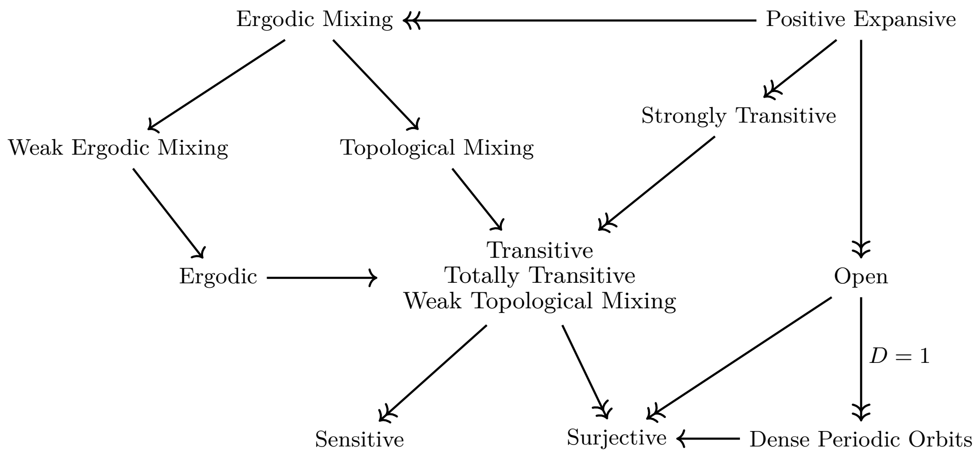


**Figure 1** Known relations between dynamical properties of CA. An arrow with single tip indicates that the converse relation is unknown, an arrow with double tip means that the converse relation is false. Labels on arrows indicate that implications have been proved only for specific dimensions. Note that there are no expansive CA in dimension $D > 1$.

Imposing some additional constraints to the global update map allows a complete and efficient description of the dynamical behavior. These properties can take the form of a conservation law [33, 29, 32, 6, 49] or superposition principles induced by an algebraic structure imposed on the alphabet [36, 43, 44, 40, 12, 11] (in both cases the literature is really huge, only a small excerpt is cited here).

Similarly, in this paper, it is required that the alphabet $G$ of the CA is a finite abelian group and its global update map is additive, i.e., an endomorphism of $G^{\mathbb{Z}^D}$. This is a pretty broad requirement which characterizes a class of CA generalizing those with linear local rule defined by $n \times n$ matrices (see the previous citations for the case $n = 1$ and [40, 8] for a generic $n$). Indeed, the local rule of an additive CA over a group $\langle G, + \rangle$ can be written as

$$f(x_1, \ldots, x_r) = \sum_{i=1}^{r} f_i(x_i)$$

where the functions $f_i$ are endomorphisms of $G$ and $\{x_1, \ldots, x_r\}$ is the neighborhood $\mathcal{N}$.

The fundamental theorem of finite abelian groups states that every finite abelian group is isomorphic to $\bigoplus_{i=1}^{h} \mathbb{Z}_{k_i}$ where the numbers $k_1, \ldots, k_h$ are powers of (not necessarily distinct) primes and $\oplus$ is the direct sum operation. Hence, the global rule $F$ of an additive CA over $G$ splits into the direct sum of a suitable number $h'$ of additive CA over subgroups $G_1, \ldots, G_{h'}$ with $h' \leq h$ and such that $\gcd(|G_i|, |G_j|) = 1$ for each pair of distinct $i, j \in \{1, \ldots, h'\}$. Each of them can be studied separately and then the analysis of the dynamical behavior of $F$ can be carried out by combining together the results obtained for each component.

In order to make things clearer, consider the following example. If $F$ is an additive CA over $G \cong \mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$ then $F$ splits into the direct sum of 3 additive CA over $\mathbb{Z}_4 \times \mathbb{Z}_8$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_{25}$, respectively. Therefore, $F$ will be topologically transitive iff each component is topologically transitive while $F$ will be sensitive to initial conditions iff at least one component is sensitive to initial conditions (see Section 2.1 for the precise definitions of these properties).

The above considerations lead us to three distinct scenarios:

- **$G \cong \mathbb{Z}_{p^k}$.** Then, $G$ is cyclic and we can define each $f_i$ simply assigning the value of $f_i$ applied to the unique generator of $G$. Moreover, every pair $f_i, f_j$ commutes, i.e. $f_i \circ f_j = f_j \circ f_i$, and this makes it possible a detailed analysis of the global behavior of $F$. For additive cellular automata over $\mathbb{Z}_{p^k}$ almost all dynamical properties are well understood and characterized [43].
- **$G \cong \mathbb{Z}_{p^k}^n$.** In this case, $G$ is not cyclic anymore and has $n$ generators. We can define each $f_i$ assigning the value of $f_i$ for each generator of $G$. This gives rise to the class of linear CA over $\mathbb{Z}_{p^k}^n$ that have been studied in [20, 40, 8]. Now, $f_i$ and $f_j$ do not commute in general and this makes the analysis of the dynamical behavior much harder. As pointed out in [20], we also recall that linear CA over $\mathbb{Z}_{p^k}^n$ allow the investigation of some classes of non-uniform CA over $\mathbb{Z}_{p^k}$ (for these latter see [22, 10, 24, 23]).
- **$G \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{p^{k_i}}$.** In this case ($\mathbb{Z}_4 \times \mathbb{Z}_8$ in the example), $G$ is again not cyclic and $F$ turns out to be a subsystem (in the sense of topological dynamics) of a suitable linear CA. In this last case the analysis of the dynamical behavior of $F$ is even more complex than in the previous case. We do not even know easy checkable characterizations of basic properties like surjectivity or injectivity.

Even if the superposition principle still allows us to prove deep and interesting results on the asymptotic behavior of additive CA over finite abelian groups, their dynamics is definitely more interesting and expressive than that of linear CA over $\mathbb{Z}_m$ and exhibits much more complex features. As a first example, consider the set $A$ of graphs with $n$ nodes represented by their adjacency matrices. $A$ can be equipped with a binary operation "+" that makes it a finite (or finitely generated) abelian group $G$ (isomorphic to the group of all $n \times n$ matrices over $\mathbb{Z}_2$ with the "+" operation). The group operation can be defined in many different ways,

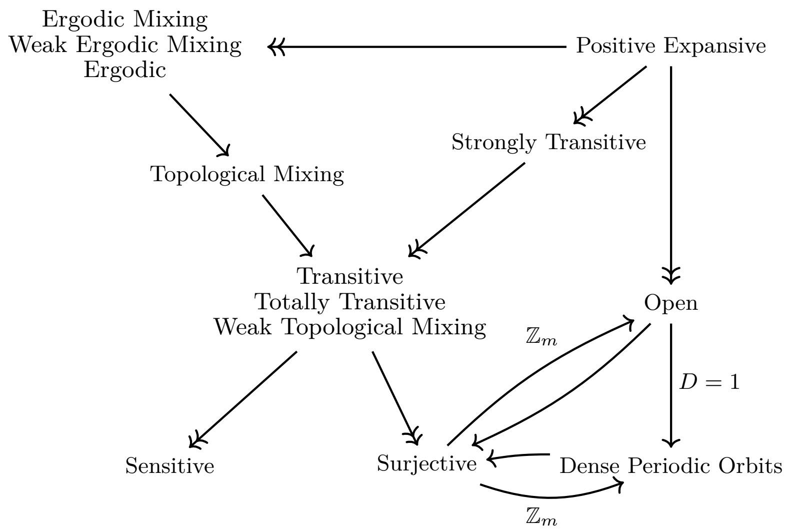


**Figure 2** Known relations among dynamical properties of additive CA before the present paper. An arrow with single tip indicates that the converse relation is unknown, an arrow with double tip means that the converse relation is false. Labels on arrows indicate that implications have been proved only for specific alphabets or dimensions. Note that there are no expansive CA in dimension $D > 1$.

e.g. sum of matrices or, for undirected graphs, product of (symmetric) matrices. The local rule of the CA can be any map preserving the group structure. It is easy to verify that the dynamics of this kind of automata cannot be simulated by any linear CA over $\mathbb{Z}_m$.

This richness in terms of expressive power is further stressed by Theorem 9 which shows how the group structure of $\mathbb{Z}_{p^k}$ constraints the dynamics. However, there exist additive one-dimensional CA over suitable abelian groups that are arbitrarily "small" roots of the shift map, as illustrated in Example 10. This means that the group structure helps out in constructing CA which are able of transmitting the information (encoded in the initial configuration) at arbitrary slow speed. In particular, this allows the construction of ergodic maps with arbitrary low Lyapunov exponents. Theorem 8 tells that the same cannot be done by one-dimensional CA over alphabets of prime cardinality.

Figure 2 illustrates the known relations among dynamical properties of linear CA before the present paper. Figure 3 illustrates the impact of the results of the present paper. As a matter of fact, the overall picture have been greatly simplified and the dynamics much better understood.

The paper is structured as follows. Section 2 introduces all the necessary background and formal definitions. Section 3 states the main contributions of the paper and the next one provides all the proofs. In the last section we draw our conclusions and provide some perspectives.

## 2 Background

We begin by reviewing some general notions and introducing notations we will use throughout the paper.

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and natural numbers, respectively. For any $v \in \mathbb{Z}^D$ we denote $||v|| = ||(v_1, \ldots, v_D)|| = \max\{|v_1|, \ldots, |v_D|\}$.

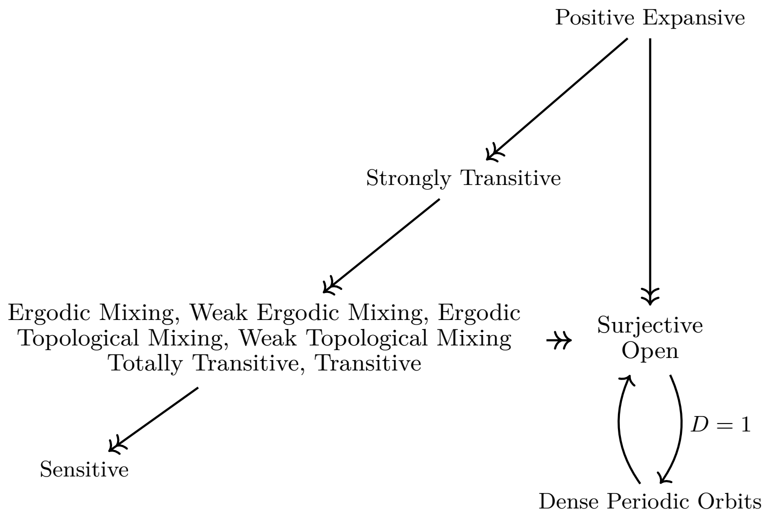


**Figure 3** Relations among dynamical properties of additive CA taking into account the results of the present paper. An arrow with single tip indicates that the converse relation is unknown, an arrow with double tip means that the converse relation is false. Labels on arrows indicate that implications have been proved only for specific dimensions. Note that there are no expansive CA in dimension $D > 1$.

Let $G$ be a finite alphabet. A **configuration** over $G$ is a map from $\mathbb{Z}^D$ to $G$. For any configuration $c \in G^{\mathbb{Z}^D}$ and any vector $v \in \mathbb{Z}^D$, $c(v)$ is the value of $c$ in position $v$. The configuration space $G^{\mathbb{Z}^D}$ is equipped with the distance $d$ defined as follows

$$\forall c, c' \in G^{\mathbb{Z}^D}, \quad d(c,c') = \begin{cases} 0, & \text{if} \quad c = c', \\ 2^{-\min\{||\boldsymbol{v}|| \,:\, c(v) \neq c'(v)\}}, & \text{otherwise} \end{cases} .$$

In this way, the set $G^{\mathbb{Z}^D}$ equipped with the topology induced by $d$ turns out to be a compact, perfect, and totally disconnected topological space (i.e., a Cantor space). In the sequel, the configuration space $G^{\mathbb{Z}^D}$ will be sometimes denoted by $X$.

A **pattern** $P$ is a function from $\{-\ell, \ldots, \ell\}^D$ to $G$, for some $\ell \in \mathbb{N}$. For any $\ell \in \mathbb{N}$, denote by $\mathcal{P}_\ell$ the set of all patterns with domain $\{-\ell, \ldots, \ell\}^D$. For any $P \in \mathcal{P}_\ell$, the **cylinder** individuated by the pattern $P$ is the set $[P] = \{c \in G^{\mathbb{Z}^D} | \forall v \in \{-\ell, \ldots, \ell\}^D, c(v) = P(v)\}$. Cylinders are clopen sets and the set $\{[P] : \ell \in \mathbb{N}, P \in \mathcal{P}_\ell\}$ of all cylinders is a basis for the topology induced by $d$.

For some fixed integer $s \geq 1$, let $f$ (named, $s$-sized *local rule*) and $\mathcal{N}$ ($s$-sized *neighborhood frame*) be any map from $G^s$ to $G$ and an ordered set of distinct vectors $u_1, \ldots, u_s$, respectively. A ***D*-dimensional CA** over $G$ is a pair $(G^{\mathbb{Z}^D}, F)$, where $F\colon G^{\mathbb{Z}^D} \to G^{\mathbb{Z}^D}$ is the function (named, *global transition map*) defined on the basis of $f$ and $\mathcal{N}$ as follows

$$\forall c \in G^{\mathbb{Z}^D}, \forall v \in \mathbb{Z}^D, \quad F(c)(v) = f\left(c(v+u_1), \ldots, c(v+u_s)\right) . \tag{1}$$

Recall that $F$ is a uniformly continuous map w.r.t. the metric $d$ and any function $F : G^{\mathbb{Z}^D} \to G^{\mathbb{Z}^D}$ is the global transition map of a $D$-dimensional CA iff it is uniformly continuous and shift-commuting (Hedlund's theorem from [35]), i.e., $F \circ \sigma^u = \sigma^u \circ F$ for any $u \in \mathbb{Z}^D$, where $\sigma^u : G^{\mathbb{Z}^D} \to G^{\mathbb{Z}^D}$ is the $D$-dimensional **shift map** defined by $\forall c \in G^{\mathbb{Z}^D}, \forall v \in \mathbb{Z}^D, \sigma^u(c)(v) = c(v+u)$. From now on, for the sake of simplicity, we identify a CA with its global map. Moreover, we will denote $\sigma^1$ simply by $\sigma$.

In the sequel, the alphabet $G$ will be an **abelian group**, with group operation $+$, neutral element 0, and inverse operation $-$. In this way, both the configuration space $G^{\mathbb{Z}^D}$ and the set $\mathcal{P}_\ell$ turn out to be abelian groups, too, where the group operation of $G^{\mathbb{Z}^D}$ and $\mathcal{P}_\ell$ is the component-wise extension of $+$ to $G^{\mathbb{Z}^D}$ and $\mathcal{P}_\ell$. With an abuse of notation, we denote by the same symbols $+$, 0, and $-$ the group operation, the neutral element, and the inverse operation, respectively, both of $\mathbb{Z}^D$, $G$, $G^{\mathbb{Z}^D}$, and $\mathcal{P}_\ell$. Observe that $+$ and $-$ are continuous functions in the topology induced by cylinders. Hence, $G^{\mathbb{Z}^D}$ is a compact topological group.

A configuration $c \in G^{\mathbb{Z}^D}$ is said to be finite if the number of positions $v \in \mathbb{Z}^D$ with $c(v) \neq 0$ is finite. A CA $(G^{\mathbb{Z}^D}, F)$ over $G$ is **additive** if

$$\forall c, c' \in G^{\mathbb{Z}^D}, \quad F(c + c') = F(c) + F(c') \ .$$

In other words, additive $D$-dimensional CA over $G$ are endomorphisms of $G^{\mathbb{Z}^D}$.

The **sum of two additive CA** $F_1$ and $F_2$ over $G$ is naturally defined as the map on $G^{\mathbb{Z}^D}$ denoted by $F_1 + F_2$ and such that

$$\forall c \in G^{\mathbb{Z}^D}, \quad (F_1 + F_2)(c) = F_1(c) + F_2(c) \ .$$

## 2.1 Topological and measure theoretic properties

We now recall the definition of some topological and measure theoretical properties for general systems.

A **discrete time dynamical system (DTDS)** is a pair $(\mathcal{X}, \mathcal{F})$, where $\mathcal{X}$ is any set equipped with a distance $d$ and $\mathcal{F}$ is a transformation on $\mathcal{X}$ which is continuous with respect to $d$. Clearly, CA are DTDS. A DTDS $(\mathcal{X}, \mathcal{F})$ is **surjective**, resp., **open**, if $\mathcal{F}$ is surjective, resp., open. Open CA are surjective (for a proof see [25], for instance). Moreover, any open one-dimensional CA $F$ is characterized by the following property: there exists a natural $k > 0$ such that for every configuration $c \in G^{\mathbb{Z}^D}$ it holds that $|F^{-1}(c)| = k$. Two DTDS $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{X}', \mathcal{F}')$ are isomorphic if there exists an homeomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$ such that $\varphi \circ \mathcal{F} = \mathcal{F}' \circ \varphi$.

A DTDS $(\mathcal{X}, \mathcal{F})$ is **topologically transitive** (or, simply, transitive) if for all nonempty open subsets $U$ and $V$ of $\mathcal{X}$ there exists a natural number $t$ such that $\mathcal{F}^t(U) \cap V \neq \emptyset$, while it is said to be **topologically mixing** if for all nonempty open subsets $U$ and $V$ of $\mathcal{X}$ there exists a natural number $t_0$ such that the previous intersection condition holds for every $t \geq t_0$. Intuitively, a topologically transitive system $(\mathcal{X}, \mathcal{F})$ has elements of $\mathcal{X}$ which eventually move under iteration of $\mathcal{F}$ from one arbitrarily small neighbourhood to any other. As a consequence, the dynamical system cannot be decomposed into two disjoint open sets which are invariant under the map $\mathcal{F}$. Clearly, topological mixing is a stronger condition than transitivity. Further, $(\mathcal{X}, \mathcal{F})$ is **topologically weakly mixing** if the DTDS $(\mathcal{X} \times \mathcal{X}, \mathcal{F} \times \mathcal{F})$ is topologically transitive, while it is **totally transitive** if $(\mathcal{X}, \mathcal{F}^t)$ is topologically transitive for all $t \in \mathbb{N}$. We now recall another condition stronger than transitivity. A DTDS is **strongly transitive** if for any nonempty open subset $U$ of $\mathcal{X}$ it holds that $\bigcup_{t=1}^{\infty} \mathcal{F}^t(U) = \mathcal{X}$.

A DTDS $(\mathcal{X}, \mathcal{F})$ is **sensitive to initial conditions** if there exists $\epsilon > 0$ such that for any $\delta > 0$ and $x \in \mathcal{X}$ , there are an element $y \neq x$ with $d(y, x) < \delta$ and a natural number $t$ such that $d(\mathcal{F}^t(y), \mathcal{F}^t(x)) > \epsilon$. Roughly speaking, $(\mathcal{X}, \mathcal{F})$ is sensitive to initial conditions, or simply sensitive, if there exist elements arbitrarily close to $x$ which eventually separate from $x$ by at least $\epsilon$ under iteration of $\mathcal{F}$. If a DTDS is sensitive, then, for all practical purposes, the dynamics eventually defy numerical approximation. Small errors in computation which are introduced by round-off may become magnified upon iteration. The results of numerical computation of an orbit, no matter how accurate, may be completely different from the real orbit. In [13] it has been proven that, for CA, topological transitivity implies sensitivity. Thus, for CA, the notion of topological transitivity becomes central to chaos theory. A DTDS $(\mathcal{X}, \mathcal{F})$ is said to be **positively expansive** if there exists $\epsilon > 0$ such that for any pair of elements $x, y \in \mathcal{X}$ with $x \neq y$ there exists a natural number $t$ such that $d(\mathcal{F}^t(y), \mathcal{F}^t(x)) > \epsilon$. If $\mathcal{X}$ has infinite cardinality then expansivity is a stronger condition than sensitivity (see [31] for a study concerning expansivity and sensitivity in CA). While there are no positively expansive $D$-dimensional CA when $D \geq 2$, in dimension 1, expansivity implies both mixing, strong transitivity, and openness [41].

An element $x \in \mathcal{X}$ is a periodic point for a DTDS $(\mathcal{X}, \mathcal{F})$ if $\mathcal{F}^t(x) = x$ for some integer $t > 0$. A DTDS $(\mathcal{X}, \mathcal{F})$ has **dense periodic orbits** (DPO) if the set of its periodic points is dense in $\mathcal{X}$. The popular book by Devaney [27] isolates three components as being the essential features of chaos for DTDS: topological transitivity, sensitivity to initial conditions and denseness of periodic orbits (see [27, Def. 8.5]).

Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a probability space and let $(\mathcal{X}, \mathcal{F})$ be a DTDS where $\mathcal{F}$ is a measurable map which preserves $\mu$, i.e., $\mu(E) = \mu(\mathcal{F}^{-1}(E))$ for every $E \in \mathcal{M}$. The DTDS $(\mathcal{X}, \mathcal{F})$, or, the map $\mathcal{F}$, is **ergodic** with respect to $\mu$ if for every $E \in \mathcal{M}$

$$\left(E = \mathcal{F}^{-1}(E)\right) \Rightarrow (\mu(E) = 0 \text{ or } \mu(E) = 1)\,.$$

It is well known that $\mathcal{F}$ is ergodic iff for any pair of sets $A, B \in \mathcal{M}$ it holds that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(\mathcal{F}^{-i}(A) \cap B) = \mu(A)\mu(B)$$

The DTDS $(\mathcal{X}, \mathcal{F})$ is **(ergodic) mixing**, if for any pair of sets $A, B \in \mathcal{M}$ it holds that

$$\lim_{n\to\infty} \mu(\mathcal{F}^{-n}(A) \cap B) = \mu(A)\mu(B) \ ,$$

while it is **(ergodic) weak mixing**, if for any pair of sets $A, B \in \mathcal{M}$ it holds that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(\mathcal{F}^{-i}(A) \cap B) - \mu(A)\mu(B)| = 0$$

In order to apply ergodic theory to CA, we need to define the collection $\mathcal{M}$ of measurable subsets of $G^{\mathbb{Z}^D}$ and a probability measure $\mu : \mathcal{M} \to [0,1]$. We will use the normalized *Haar* measure $\mu_H$ defined over the $\sigma$-algebra generated by the cylinders which is, to our knowledge, one of the most widely used probabilistic setting in CA theory. The measure $\mu_H$ is defined as the product measure induced by the uniform probability distribution over $G$. In this way, for any $\ell \in \mathbb{N}$ and any pattern $P \in \mathcal{P}_\ell$, it holds that $\mu_H([P]) = \frac{1}{|G|^{(2\ell+1)^D}}$. Since in the rest of this paper we will only use the Haar measure, then we will write $\mu$ instead of $\mu_H$.

## 3 Statement of the main results

In this section, we state the main results of this paper. They allow us to simplify the relationships between the dynamical and measure theoretic properties of additive CA, as depicted in Figure 3.

The following is the main result of the paper. It builds a bridge between two pretty different ways of approaching the study of the dynamics of CA: measure-theoretic and topological. The arguments used in the proofs are closely crafted on the additivity property of the global rule and on the group structure of the alphabet, however, the overall impression is that this tight link between topology and measure theory shall be true in a much more general setting.

▶ **Theorem 1.** *Let $F$ be an additive $D$-dimensional CA over a finite abelian group. If $F$ is topologically transitive then $F$ is ergodic.*

Theorem 2 provides a new facet of the ergodicity property. This time ergodicity is related to set theoretic properties like surjectivity and aperiodicity of finite configurations.

▶ **Theorem 2.** *Any additive $D$-dimensional CA $F$ over a finite abelian group is ergodic if and only if $F$ is surjective and no finite configuration except $0$ is a periodic point for $F$.*

The following corollary collects all the known properties which are related to ergodicity in the context of additive CA over finite abelian groups.

▶ **Corollary 3.** *Let $F$ be an additive $D$-dimensional CA over a finite abelian group. The following properties are equivalent*

1. *ergodic mixing;*
2. *weak ergodic mixing;*
3. *ergodicity;*
4. *topological mixing;*
5. *total transitivity;*
6. *weak topological mixing;*
7. *topological transitivity;*
8. *$F$ is surjective and no finite configuration except $0$ is a periodic point of $F$.*

▶ **Corollary 4.** *Let $F$ be an additive $D$-dimensional CA over a finite abelian group. If $F$ is invertible and transitive then $F$ is isomorphic to a Bernoulli shift.*

Surjectivity has strong implications on the dynamics of general CA, the following proposition and its corollary prove that in the context of additive CA, those implications are even stronger.

▶ **Proposition 5.** *Let $F$ be an additive $D$-dimensional CA over a finite abelian group. If $F$ is surjective then it is open.*

▶ **Corollary 6.** *Surjectivity and openness are equivalent properties for additive $D$-dimensional CA over a finite abelian group. Furthermore, they are equivalent to DPO in dimension $D = 1$.*

The following theorem provides a first characterization of strong transitivity for additive CA over finite abelian groups. Roughly speaking, the theorem states that this property is conserved under translations and iterations. We wonder whether the same holds for general CA.

▶ **Theorem 7.** *Let $F$ be and additive $D$-dimensional CA over a finite abelian group. The following conditions are equivalent:*

1. *$F$ is strongly transitive;*
2. *for every $v \in \mathbb{Z}^D$, the CA $\sigma^v \circ F$ is strongly transitive;*
3. *for every $n \in \mathbb{N}$, the CA $F^n$ is strongly transitive;*

In the context of 1-dimensional CA, the following result characterizes the *roots* of powers of the shift map. Recall that a CA $F$ is a root of another CA $F'$ if there exists an integer $n > 0$ such that $F^n = F'$.

▶ **Theorem 8.** *[35, Thm. 18.1] Let $F$ be a $1$-dimensional CA over an alphabet $G$ of prime cardinality. If $F^n = \sigma^m$ for some naturals $n, m$ with $n \geq 1$, then $n|m$.*

In the case of linear CA over an alphabet of cardinality which is a power of a prime, the following weaker form of Theorem 8 can be proved.

▶ **Theorem 9.** *Let $G = \mathbb{Z}_{p^k}$ with $p$ prime and let $F$ be a 1-dimensional CA over $G$ defined by the neighborhood $\mathcal{N} = \{-r, \ldots, r\}$ and the local rule $f : \mathbb{Z}_{p^k}^{2r+1} \to \mathbb{Z}_{p^k}$ expressed by the linear combination with coefficients $a_{-r}, \ldots, a_r \in \mathbb{Z}_{p^k}$. If $F^n = \sigma^m$ for some naturals $n \geq 1$ and $m \geq 1$, then $m \geq n$.*

The following example shows that Theorem 9 is no longer true for additive CA over finite abelian groups.

▶ **Example 10.** Let $F$ be the 1-dimensional CA over $\mathbb{Z}_m^n$ defined by the neighbourhood $\mathcal{N} = \{0, 1\}$ and the local rule $f : (\mathbb{Z}_m^n)^2 \to \mathbb{Z}_m^n$ such that

$$\forall (x_0, x_1) \in (\mathbb{Z}_m^n)^2, \quad f(x_0, x_1) = M_0 x_0 + M_1 x_1 \ ,$$

where

$$M_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad M_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} .$$

It is easy to check that $F^n = \sigma$ contradicting the statement of Theorem 9.

Example 10 has also other important consequences. On the one hand, it provides examples of CA that are arbitrarily "small" roots of the shift map. On the other hand, it provides the basic building blocks for systems which are ergodic but have both arbitrarily low Lyapunov exponents and arbitrarily low topological entropies.

# 4 Proofs of the main results

## 4.1 Useful known results

Before going into the proofs of our main results let us recall some known facts which helped in shaping the situation depicted in Figure 2.

▶ **Theorem 11.** *[50, Thm. 1.28, pag. 50] Mixing, weak mixing and ergodicity are equivalent properties for endomorphisms of compact groups.*

▶ **Theorem 12.** *[4, Thm. 1] An ergodic automorphism of a compact metric abelian group is a Bernoulli shift.*

▶ **Proposition 13.** *[15, Prop. 6.7, pag. 32] Ergodic (resp., weak ergodic) mixing implies topological (resp., weak) mixing for endomorphisms of compact groups and measures with full support.*

The following will be fundamental for proving our main result.

▶ **Theorem 14.** *[48, Thm. 1] Let $F$ be any endomorphism of a compact abelian group with normalized Haar measure. Then, $F$ is ergodic if and only if $F$ is surjective and $F^n - I$ is surjective for all $n \in \mathbb{N}$ ($I$ is the identity map).*

## 4.2 Proofs of our results

In order to make the proof of the main result more readable, we cut it into several lemmata.

▶ **Lemma 15.** *Let $F$ be an additive $D$-dimensional CA over a finite abelian group $G$. For all $n \geq 1$, it holds that*

$$F^n - I = (F - I) \circ (I + F + \cdots + F^{n-1}) = (I + F + \cdots + F^{n-1}) \circ (F - I) \tag{2}$$

**Proof.** It is an immediate consequence of the fact that the composition operation on the set of additive CA is distributive over the sum operation (defined on the same set). ◀

▶ **Lemma 16.** *Let $F$ be an additive CA over a finite abelian group $G$. Let $\ell \in \mathbb{N}$ and $w \in \mathcal{P}_l$ such that $(F - I)(X) \cap [w] = \emptyset$. Then, it holds that*

$$\forall n \geq 1, \quad (F^n - I)(X) \cap [w] = \emptyset \ .$$

**Proof.** For every natural $n \geq 1$, by Lemma 15, we get

$$(F^n - I)(X) = (F - I)\left((I + F + \cdots + F^{n-1})(X)\right) \subseteq (F - I)(X) \ .$$

Since $(F - I)(X) \cap [w] = \emptyset$, it follows that $(F^n - I)(X) \cap [w] = \emptyset$. ◀

▶ **Lemma 17.** *Let $F$ be an additive $D$-dimensional CA over a finite abelian group $G$. If there exist $\ell \in \mathbb{N}$ and $w \in \mathcal{P}_l$ such that*

$$\forall n \geq 1, \quad (F^n - I)(X) \cap [w] = \emptyset \ ,$$

*then $F$ is not topologically transitive.*

**Proof.** For the sake of argument, assume that $F$ is topologically transitive. Choose arbitrarily two patterns $w_1, w_2 \in \mathcal{P}_l$ such that $w_2 - w_1 = w$. By transitivity, there exist a configuration $c \in [w_1]$ and a natural $n \geq 1$ such that $F^n(c) \in [w_2]$. Thus, $F^n(c) - c \in [w]$, or, equivalently, $(F^n - I)(c) \in [w]$, which is a contradiction. ◀

At present all the necessary pieces have been built to go through the proof of Theorem 1.

**Proof of Theorem 1.** For the sake of argument, assume that $F$ is transitive but not ergodic. Since transitive CA are surjective, then, by Theorem 14, there exists $n \geq 1$ such that $F^n - I$ is not surjective. Let $H = F^n$. Since $H - I$ is not surjective and $(H - I)(X)$ is closed, there exist a natural $\ell \in \mathbb{N}$ and a pattern $w \in \mathcal{P}_\ell$ such that $(H - I)(X) \cap [w] = \emptyset$. So, by Lemma 16, it follows that

$$\forall m \geq 1, \quad (H^m - I)(X) \cap [w] = \emptyset \ .$$

Therefore, by Lemma 17, $H$ is not topologically transitive. Since topologically transitive CA are totally transitive (see [46], where the proof involving 1-dimensional CA can be extended to any dimension $D$), we conclude that neither is $F$, which is a contradiction. ◀

**Proof of Theorem 2.** By Theorem 14, $F$ is ergodic if and only if it is surjective and for every natural $m \geq 1$ the CA $I - F^m$ is surjective. Fix an arbitrary natural $m \geq 1$ and define $H^{(m)} = I - F^m$. The Garden-of-Eden Theorem for CA [45, 47] guarantees that $H^{(m)}$ is surjective if and only if it is injective on finite configurations, i.e., $H^{(m)}(c) \neq H^{(m)}(c')$ for every pair of distinct finite configurations $c, c' \in G^{\mathbb{Z}^D}$. Set $d = c - c' \in G^{\mathbb{Z}^D}$. Clearly $d$ is a finite configuration. Furthermore, by additivity, it holds that $H^{(m)}(c) \neq H^{(m)}(c')$ if and only if $H^{(m)}(d) \neq 0$. By definition of $H^{(m)}$, the condition $H^{(m)}(d) \neq 0$ is true for every $m \geq 1$ if and only if $F^m(d) \neq d$, i.e., $d$ is not a periodic point of $F$. ◀

**Proof of Proposition 5.** We are going to prove that $F$ is open at the configuration $0 \in G^{\mathbb{Z}^D}$, i.e., equivalently, for every $\ell \in \mathbb{N}$, the set $F([0])$ is open, where $0 \in \mathcal{P}_\ell$. Since $F$ is an endomorphism of the topological group $G^{\mathbb{Z}^D}$, we conclude that $F$ is open.

To proceed, consider any arbitrary $\ell \in \mathbb{N}$. Clearly, it holds that $G^{\mathbb{Z}^D} = \bigcup_{w \in \mathcal{P}_\ell}[w]$ and, since $F$ is surjective, we get $G^{\mathbb{Z}^D} = \bigcup_{w \in \mathcal{P}_\ell} F([w])$. Thus, there exists $w' \in \mathcal{P}_\ell$ such that $F([w'])$ has non-empty interior. Let $c \in [w']$ be the configuration such that $c(v) = 0 \in G$ for every position $v \in \mathbb{Z}^D$ with $||v|| > \ell$. One gets $F([w']) = F([0] + c) = F([0]) + F(c)$ (where $[0] + c$ and $F([0]) + F(c)$ denote suitable cosets of $[0]$ and $F([0])$, respectively). Therefore, $F([0])$ must have non-empty interior, too. Since $F([0])$ is a subgroup of the topological group $G^{\mathbb{Z}^D}$, it follows that $F([0])$ is open. ◀

**Proof of Corollary 6.** It is well-known that any open $D$-dimensional CA is surjective. By Proposition 5, surjective additive $D$-dimensional CA over a finite abelian group are open. It is easy to see that DPO implies surjectivity in any dimension. While, by Proposition 5 and the fact that open 1-dimensional CA have DPO [7, Thm. 4.4], it follows that surjective additive 1-dimensional CA have DPO. ◀

**Proof of Corollary 4.** By Theorem 1, $F$ is ergodic. The thesis follows from Theorem 12. ◀

▶ **Lemma 18.** *An additive CA $F$ over a finite abelian group $G$ is strongly transitive if and only if the following condition holds: for any natural $\ell \in \mathbb{N}$ and any pattern $P \in \mathcal{P}_\ell$ there exists $t \in \mathbb{N}$ such that $0 \in F^t([P])$.*

**Proof.** The "only if" part trivially follows from the definition of strong transitivity. Let us prove the "if" part. Assume that $F$ is an additive CA over a finite abelian group $G$ and satisfying the condition in the statement. Consider an arbitrary natural $\ell \in \mathbb{N}$. For every $P \in \mathcal{P}_\ell$ let $n_{P,\ell}$ be the smallest natural $t$ such that $0 \in F^t([P])$. Define $n_\ell = \max\{n_{P,\ell} : P \in \mathcal{P}_\ell\}$. Since $F(0) = 0$ it holds that

$$\forall P \in \mathcal{P}_\ell, \quad 0 \in F^{n_\ell}([P]) \tag{3}$$

We now show that for any configuration $e \in G^{\mathbb{Z}^D}$, any natural $\ell$, and any pattern $P \in \mathcal{P}_\ell$ there exists $t' = n_\ell$ such that $e \in F^{t'}([P])$, that is, $F$ is strongly transitive. Choose arbitrarily a configuration $e \in G^{\mathbb{Z}^D}$, a natural $\ell$, and pattern $P \in \mathcal{P}_\ell$. Let $c$ be any configuration belonging to $F^{-n_\ell}(e)$. If $c \in [P]$ we are done. Otherwise, by (3), there exists $c' \in G^{\mathbb{Z}^D}$ such that $c + c' \in [P]$ and $F^{n_\ell}(c') = 0$. Thus, we get

$$F^{n_\ell}(c + c') = F^{n_\ell}(c) + F^{n_\ell}(c') = F^{n_\ell}(c) + 0 = F^{n_\ell}(c) = e \ ,$$

and this concludes the proof. ◀

**Proof of Theorem 7.** It is an immediate consequence of Lemma 18 and the fact that for every $v \in \mathbb{Z}^D$ and every $n \in \mathbb{N}$ both $\sigma^v(0) = 0$ and $F^n(0) = 0$ hold. ◀

**Proof of Theorem 9.** The CA $F$ can be represented by the Laurent polynomial $\mathbb{p}(x, x^{-1}) = \sum_{i=-r}^{r} a_i x^i \in \mathbb{Z}_{p^k}[x, x^{-1}]$, while $\sigma^1$ can be represented by the Laurent polynomial $x$. Assume that $F^n = \sigma^m$ for some naturals $n, m$ with $n \geq 1$ and $m \geq 1$. It is easy to verify that $F^n = \sigma^m$ if and only if $\left(\mathbb{p}(x, x^{-1})\right)^n = x^m$. We consider two cases. 1) If $p | a_i$ for each $i \in \mathcal{N}$ with $i \neq 0$ then, by [14, Lemma 5], putting $h = p^{k-1} \in \mathbb{N}$ we have that $\left(\mathbb{p}(x, x^{-1})\right)^h$ is a constant (i.e., it does not contain the formal variable $x$). Hence, for every $s \in \mathbb{N}$ it holds that $\left(\mathbb{p}(x, x^{-1})\right)^{sh} \neq x^{mh}$ and this contradicts that $F^n = \sigma^m$. 2) Otherwise, there exists $i \neq 0$ such that $\gcd(a_i, p) = 1$ (since $F^n = \sigma^m$ then $F$ must be injective and so for every other coefficient $a_j$ with $i \neq j$ we have $\gcd(a_j, p) = p$). First of all, we prove that there always exist $n'$ and $m'$ with $m' \geq n'$ such that $F^{n'} = \sigma^{m'}$. Indeed, by [14, Lemma 5], for $h = p^{k-1} \in \mathbb{N}$ we get $\left(\mathbb{p}(x, x^{-1})\right)^h = a_i^h x^{ih}$. Therefore, for $t = \varphi(p^k)$ (where $\varphi$ is the Euler's totient function) we get $\left(\mathbb{p}(x, x^{-1})\right)^{n'} = x^{m'}$, i.e., $F^{n'} = \sigma^{m'}$, with $n' = ht$ and $m' = iht$. To conclude, assume by contradiction that $F^n = \sigma^m$ with $m < n$. Then, we get $F^{n'n} = \sigma^{m'n}$ and $F^{n'n} = \sigma^{n'm}$. So, it follows that $m'n = n'm$, but this is not possible since $m'n \geq n'n > n'm$. Thus, it holds that $m \geq n$ and this concludes the proof. ◀

## 5 Conclusions and perspectives

Comparing Figure 2 and 3, one can immediately appreciate the impact of the results in the paper. All single arrow tips (i.e. the known relations for which the opposite implication was unknown) disappeared and several properties coalesced in the group of transitivity, total transitivity and weak topological mixing. However, what is more important is that many measure theoretical and topological properties coincide. These facts legitimate the following

▶ **Conjecture 1.** *Transitive CA are ergodic with respect to Bernoulli measures.*

Solving the previous conjecture will probably clarify also the status of the properties of ergodic mixing, weakly ergodic mixing and topological mixing, much like it happened for the case of endomorphisms of compact abelian groups in this paper. Investigating, the following well-known conjecture due to Blanchard and Tisseur (see [2, Conjecture 1] and [1] for more details) will definitively complete the overall picture.

▶ **Conjecture 2.** *All surjective CA have a dense set of periodic orbits.*

Another interesting research direction consists in establishing the decidability of the dynamical properties. In the framework of general CA, recent results from Ville Lukkarila have shown that both topological mixing and transitivity are undecidable properties [42]. Undecidability of sensitivity, surjectivity (for dimensions larger than 1) and openness (for dimensions larger than 1) were already known for years [38, 30]. It is therefore natural to conjecture that the remaining ones are also undecidable.

▶ **Conjecture 3.** *Ergodicity, weak ergodic mixing, ergodic mixing and strong transitivity are undecidable for CA.*

We want to remark that we intentionally left out the expansivity property from Conjecture 3, since it is so peculiar that we believe it might be decidable.

Finally, it would be very interesting to extend Theorem 7 to all CA, and not only to additive CA. In some way, this would turn strong transitivity into the "strongest" translation invariant property, since it is well-known that expansivity is not translations invariant and that there are no expansive CA for dimensions greater than one.

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