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GENERALIZED SAMUEL MULTIPLICITIES OF MONOMIAL IDEALS AND VOLUMES

R. ACHILLES AND M. MANARESI

ABSTRACT. We describe conjecturally the generalized Samuel multiplicities c_0, \ldots, c_{d-1} of a monomial ideal $I \subset K[x_1, \ldots, x_d]$ in terms of its Newton polyhedron $\Gamma(I)$. More precisely, we conjecture that c_i equals the sum of the normalized (d-i)-volumes of pyramids over the projections of the (d-i-1)-dimensional compact faces of $\Gamma(I)$ along the infinite-directions of *i*-unbounded facets in which they are contained. For c_0 proofs are known (Guibert, Jeffries and Montaño) and for c_{d-1} a proof will be given.

1. INTRODUCTION

In this paper, based on computations with the free softwares Germenes by A. Montesinos [11] and REDUCE [12] by A. C. Hearn [8] and REDUCE developers, we give a conjecture that in the case of monomial ideals links the generalized multiplicities defined algebraically in [3] with volumes derived from the Newton polyhedra of the ideals, thus extending a result of B. Teissier [14].

In 1988, B. Teissier [14, p. 131] proved that for an \mathfrak{m} -primary monomial ideal I of a local ring A the Samuel multiplicity is equal to the normalized volume of the complement of the Newton polyhedron of the ideal I. In 1999, G. Guibert [7] generalized Teissier's result. Precisely, Guibert defines the local Segre class of an ideal generated by a set of germs of holomorphic functions and, under a non-degeneracy condition, he describes such a class by Minkowski mixed volumes of polytopes. As a special case he obtains that for a certain class of monomial ideals the local Segre class is a normalized volume of the simplex generated by the origin and the vertices of the Newton polyhedron, see [7, 4.2]. By [4], the local Segre class is the so called j-multiplicity of the ideal. In 2013, J. Jeffries and J. Montaño [9] gave a different proof that the

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j-multiplicity of a monomial ideal is the normalized volume of the pyramid of the ideal.

The *j*-multiplicity of an ideal is different from zero if and only if its analytic spread is maximal, that is, equal to the Krull-dimension dof A. A result of C. Bivià-Ausina [5] states that the analytic spread diminished by one is the maximum of the dimensions of compact faces of the Newton polyhedron of I.

According to [3] the *j*-multiplicity is only the first coordinate of the generalized Samuel multiplicity vector $c(I) = (c_0(I), \ldots, c_d(I))$. Here we present and illustrate a conjecture which expresses the other components of c(I) in terms of the Newton polyhedron of I. Our conjecture extends the known result of G. Giubert and J. Jeffries and J. Montaño regarding $c_0(I)$, but we shall prove it here only for $c_{d-1}(I)$.

2. Generalized Samuel multiplicities

This section is a quick review of a generalization of Samuel's multiplicity by a sequence of numbers, the so-called generalized Samuel multiplicity, which we have introduced in [3].

Let A be a d-dimensional Noetherian local ring (A, \mathfrak{m}) with unique maximal ideal \mathfrak{m} or a standard graded algebra $A = \bigoplus_{i\geq 0} A_i$ such that A_0 is a field and $\mathfrak{m} = (A_1)A$ is the unique homogeneous maximal ideal of A. Let $I \subset A$ be an arbitrary ideal (not necessarily \mathfrak{m} -primary).

In order to define the generalized Samuel multiplicity c(I), consider $G_I(A) := \bigoplus_{j \ge 0} I^j / I^{j+1}$, the associated graded ring of A with respect to I and the bigraded ring

$$T = \bigoplus_{i,j\geq 0} T_{ij} = G_{\mathfrak{m}}(G_I(A)) = \bigoplus_{i,j\geq 0} \frac{\mathfrak{m}^i I^j + I^{j+1}}{\mathfrak{m}^{i+1} I^j + I^{j+1}} ,$$

where $T_{00} = A/\mathfrak{m} = K$ is a field.

Let $H^{(0,0)}(i,j) := \dim_K T_{ij}$ be the Hilbert function of the bigraded ring T and let

$$H^{(1,1)}(i,j) := \sum_{q=0}^{j} \sum_{p=0}^{i} H^{(0,0)}(p,q)$$

its twofold sum transform. For both $i, j \gg 1$ this function becomes a polynomial in (i, j), which can be written in the form

$$\sum_{k+l\leqslant d} a_{k,l}^{(1,1)} \binom{i+k}{k} \binom{j+l}{l}.$$

Following [3] define the generalized Samuel multiplicity to be the vector

$$\left(a_{0,d}^{(1,1)}, a_{1,d-1}^{(1,1)}, \dots, a_{d,0}^{(1,1)} \right) =: (c_0(T), c_1(T), \dots, c_d(T)) =: c(T)$$
$$=: (c_0(I), c_1(I), \dots, c_d(I)) =: c(I) .$$

The first coefficient $c_0(I)$ plays an important role as an intersection number and was introduced in [2]. It is called the *j*-multiplicity $j(I) := c_0(I)$.

The generalized Samuel multiplicities depend only on the highest dimensional components of T, see [15] or [3, Proposition 1.2]:

Proposition 1. With the preceding notation,

$$c(I) = c(T) = \sum_{P} \operatorname{length}(T_{P}) \cdot c(T/P),$$

where P runs through all highest dimensional prime ideals of T.

By analogy with the application of c(I) to intersection theory, we shall call $c_i(T/P) \neq 0$ a movable contribution to $c_i(I)$ if there is an integer k > i such that $c_k(T/P) \neq 0$.

3. A Conjecture and some results

Let I be an ideal in $R = K[x_1, \ldots, x_d] = K[\mathbf{x}]$ (K a field) minimally generated by the monomials

$$\mathbf{x}^{v_1} := x_1^{v_1(1)} \cdots x_d^{v_1(d)}, \ \dots, \ \mathbf{x}^{v_r} := x_1^{v_r(1)} \cdots x_d^{v_r(d)},$$

that is, $v_1 = (v_1(1), \ldots, v_1(d)), \ldots, v_r = (v_r(1) \ldots v_r(d))$ are the points of $\mathbb{Z}_{\geq 0}^d$ corresponding to the exponents of the generators of I.

The Newton polyhedron $\Gamma(I)$ of I is defined as the convex hull of $\{v \in \mathbb{Z}_{\geq 0}^d \mid \mathbf{x}^v\} \in I\}$ in \mathbb{R}^d , that is,

$$\Gamma(I) := \operatorname{conv}(\{v \in \mathbb{Z}_{\geq 0}^d \mid x_1^{v(1)} \cdots x_d^{v(d)} \in I\})$$
$$= \operatorname{conv}(\{v_1, \dots, v_r\}) + \mathbb{R}_{\geq 0}^d,$$

where + denotes the Minkowski sum (for the equality see [13, Lemma 4.3]). A hyperplane

$$H = \{ v \in \mathbb{R}^d \, | \, \langle v, a \rangle = b \} \quad (\text{with } a \in \mathbb{R}^d_{\geq 0}, \ b \in \mathbb{R})$$

is called a supporting hyperplane of the Newton polyhedron $\Gamma(I)$ if

$$\Gamma(I) \subset H^+ = \{ v \in \mathbb{R}^d \, | \, \langle v, a \rangle \ge b \} \text{ and } \Gamma(I) \cap H \neq \emptyset.$$

A subset $F \subset \Gamma(I)$ is called a *proper face* of $\Gamma(I)$ if there exists a supporting hyperplane H of $\Gamma(I)$ such that $F = \Gamma(I) \cap H$. The boundary of $\Gamma(I)$ is a set of faces of dimension d-1, called *facets* of $\Gamma(I)$, some of them compact.

The zero-dimensional faces are called *vertices* of $\Gamma(I)$. We shall denote the set of vertices by $\operatorname{vert}(I)$. Note that the monomials corresponding to the points in $\operatorname{vert}(I)$ are part of the set of minimal generators of I, so by renumbering we will assume that

$$\operatorname{vert}(I) = \{v_1, \dots, v_s\}$$
 with some $s \leq r$,

hence

$$\Gamma(I) = \operatorname{conv}(\{v_1, \dots, v_r\}) + \mathbb{R}^d_{\geq 0} = \operatorname{conv}(\{v_1, \dots, v_s\}) + \mathbb{R}^d_{\geq 0}.$$

Any face F can be described using its vertices and infinite-directions. Let e_j denote the unit vector with non-zero jth component, let H be a supporting hyperplane such that $F = \Gamma(I) \cap H$ and let a be a normal vector to H. Then the *infinite-directions* of F are given by those e_j such that the jth component of a is zero. If v_{i_1}, \ldots, v_{i_s} are the vertices of F, then

$$F = \operatorname{conv}(\{v_{i_1}, \dots, v_{i_s}\}) + \sum_{j: a(j)=0} \mathbb{R}_{\geq 0} e_j.$$

Of course, the compact faces are precisely those that do not have infinite directions e_i .

By the Minkowski-Weyl Theorem for convex polyhedra, there are uniquely determined finitely many closed half spaces

$$H_i^+ = \{ v \in \mathbb{R}^d \, | \, \langle v, a_i \rangle \ge b_i \} \text{ (with } a_i \in \mathbb{Z}_{\ge 0}^d, \ b_i \in \mathbb{Z}_{\ge 0}), \ i = 1, \dots, t,$$

such that

$$\Gamma(I) = H_1^+ \cap \dots \cap H_t^+.$$

Then $F_i := H_i \cap \Gamma(I)$, $i = 1, \ldots, t$, are the facets of $\Gamma(I)$. We will assume that H_1, \ldots, H_r are the hyperplanes corresponding to the unbounded facets and that H_{r+1}, \ldots, H_t are those corresponding to the compact facets.

To each bounded facet $F = \operatorname{conv}(\{v_{i1}, \ldots, v_{is}\})$ of $\Gamma(I)$ we associate the polytope (or pyramid)

$$\hat{F} := \operatorname{conv}(0, F) = \operatorname{conv}(\{0, v_{i1}, \dots, v_{is}\})$$

and denote by $\operatorname{vol}_d(\hat{F})$ its *d*-dimensional volume and by

$$\operatorname{Vol}_d(\hat{F}) := d! \operatorname{vol}_d(\hat{F})$$

its normalized volume.

A facet $F \subset \Gamma(I)$ is called an *h*-unbounded facet if the normal vector *a* to its supporting hyperplane has at least h > 0 coordinates a(j) which are zero, that is, the facet has *h* infinite-directions e_j .

Let $\mathcal{F}(k)$ be the set of all (d - (k + 1))-unbounded facets containing at least one k-dimensional compact face F^k , $0 \le k \le d - 2$. We define $\mathcal{F}(d-1)$ to be the set of all compact or bounded facets of $\Gamma(I)$.

To each couple (F^k, F^{d-1}) , with F^k a k-dimensional compact face and $F^{d-1} \in \mathcal{F}(k)$ containing F^k , we associate a (k + 1)-dimensional normalized volume $\operatorname{Vol}(F^k, F^{d-1})$ in the following way. A normal vector to the facet F^{d-1} lies on at least d - (k+1) coordinate hyperplanes. We project F^k on all linear subspaces $\mathbb{R}^{k+1} \subseteq \mathbb{R}^d$ obtained by intersecting d - (k + 1) of these coordinate hyperplanes, that is, we project F^k along all possible choices of d - (k + 1) infinite-directions of the facet F^{d-1} . We obtain polytopes of dimension at most k. We consider only the k-dimensional polytopes $\operatorname{pr}_{\mathbb{R}^{k+1}}(F^k) \subset \mathbb{R}^{k+1}$ obtained by the aforementioned projections and set

$$\hat{F}^k := \operatorname{conv}(\{0, \operatorname{pr}_{\mathbb{R}^{k+1}}(F^k)\}),$$

which has dimension k or k + 1. The volume associated to the couple (F^k, F^{d-1}) is

$$\operatorname{Vol}(F^k, F^{d-1}) := \min_{\mathbb{R}^{k+1}} \operatorname{Vol}_{k+1}(\hat{F}^k).$$

Conjecture 1. For each k = 0, ..., d - 1, the generalized Samuel multiplicity of a monomial ideal I is

$$c_{d-(k+1)}(I) = \sum_{F^{d-1} \in \mathcal{F}(k)} \min_{F^k} \left\{ \operatorname{Vol}(F^k, F^{d-1}) \right\},$$

where the minimum is taken over all compact faces F^k of $\Gamma(I)$ that are contained in the facet F^{d-1} .

Conjecture 2. Each summand in the formula of Conjecture 1 corresponds, in the sense of Proposition 1, to the contribution of a highest dimensional primary component of $T = G_{\mathfrak{m}}(G_I(R))$ to $c_{d-(k+1)}(I)$.

In particular, the number of compact facets of $\Gamma(I)$ is equal to the the number of d-dimensional associated prime ideals of T that contain $\mathfrak{m} = (x_1, \ldots, x_d)R$.

Note that in general the zero-ideal \mathfrak{n} of $T \cong K[x_1, \ldots, x_d, y_1, \ldots, y_r]/\mathfrak{n}$ is a binomial but not a monomial ideal, see [6].

Our conjectures are confirmed by many examples, but so far we do not have a proof except for Conjecture 1 in the extremal cases k = 0and k = d - 1, as it is stated in the following two theorems. **Theorem 1** (Jeffries and Montaño, [9]). If $I \subset K[x_1, \ldots, x_d]$ is a monomial ideal and F_{r+1}, \ldots, F_t are the compact facets of the Newton polyhedron $\Gamma(I)$, then

$$c_0(I) = \sum_{i=r+1}^t d! \operatorname{vol}(\hat{F}_i) = \sum_{i=r+1}^t \operatorname{Vol}(\hat{F}_i).$$

Theorem 2. Let I be a monomial ideal in $R = K[x_1, \ldots, x_d]$ generated by $x_1^{v_1(1)} \cdots x_d^{v_1(d)}, \ldots, x_1^{v_r(1)} \cdots x_d^{v_r(d)}$ and $m_j = \min\{v_1(j), \ldots, v_r(j)\}, j = 1, \ldots, d$. Then

$$c_{d-1}(I) = m_1 + \dots + m_d.$$

Proof. By [3, Proposition 2.3], $c_{d-1}(I) \neq 0$ if and only if dim R/I = d-1. If dim R/I < d-1, then none of the variables x_j appears in all monomials generating I, hence $m_j = 0$ for all j, and the result is true. If dim R/I = d-1, then again by [3, Proposition 2.3],

$$c_{d-1}(I) = \sum_{P} e(IR_{P}) \cdot e(R/P),$$

where P runs through all (d-1)-dimensional associated prime ideals of R/I, that is, prime ideals of the form (x_j) for some j, see [10, Satz 9]. Therefore $IR_P = (x_j^{m_j})R_P$ and $e(IR_P) = m_j$. By [10] the (d-1)-dimensional part of the primary decomposition of I is $(x_1^{m_1}) \cap (x_2^{m_2}) \cap \cdots \cap (x_d^{m_d})$, which is of degree $m_1 + \cdots + m_d$.

The following corollary of Theorem 2 states that the Conjecture 1 is true for k = 0.

Corollary 3. Using the preceding notations, for j = 1, ..., d set

$$F_j := \operatorname{conv}(\{v \in \operatorname{vert}(I) \mid v(j) = m_j\}) + \sum_{1 \le i \le d, i \ne j} \mathbb{R}_{\ge 0} e_i$$

and $\operatorname{vert}(F_j) := \operatorname{vert}(I) \cap F_j$.

Then $\mathcal{F}(0) = \{F_1, \ldots, F_d\}$. If $v \in vert(F_j)$, then $Vol(v, F_j) = m_j$ and it holds

$$c_{d-1}(I) = \sum_{j=1}^{d} \min_{v \in \operatorname{vert}(F_j)} \left\{ \operatorname{Vol}(v, F_j) \right\}.$$

Proof. Since each $v \in \Gamma(I)$ is the sum of a convex combination of the vertices v_1, \ldots, v_s of $\Gamma(I)$ and of some $w \in \mathbb{R}^d_{\geq 0}$, we have

$$v(j) \ge \min\{v_1(j), \dots, v_s(j)\} + w(j) \ge \min\{v_1(j), \dots, v_s(j)\},\$$

hence

$$m_j := \min\{v_1(j), \dots, v_r(j)\} = \min\{v_1(j), \dots, v_s(j)\} = \min_{v \in \Gamma(I)}\{v(j)\}.$$

It follows that F_1, \ldots, F_d are precisely the (d-1)-unbounded facets of $\Gamma(I)$, that is, $\mathcal{F}(0) = \{F_1, \ldots, F_d\}$.

Since the projection of $\mathbb{R}^d \to \mathbb{R}$ with center $\sum_{1 \leq i \leq d, i \neq j} \mathbb{R}e_i$ sends $v \in \operatorname{vert}(F_j)$ to $v(j) = m_j$, we have $\operatorname{Vol}(v, F_j) = v(j) = m_j$. Then, by Theorem 2, we obtain the desired formula for c_{d-1} , which finishes the proof.

4. Examples

We illustrate the theorems and the conjecture by examples of monomial ideals $I \subset R = K[x_1, \ldots, x_d]$, K an arbitrary field. We set $\mathfrak{m} := (x_1, \ldots, x_d)R$ and $T := G_{\mathfrak{m}}(G_I(R))$. All the examples will show a closed relation between the summands in the formula of Conjecture 1 and the highest dimensional primary components of T.

We begin with the simplest case of a monomial ideal generated by one monomial in two variables. The first two examples are covered by Theorems 1 and 2.

Example 1 (Figure 1). Let $I = (x^3y^2) \subset R = K[x, y]$. We have

$$c(I) = (c_0(I), c_1(I), c_2(I)) = (0, 5, 0) = 2 \cdot (0, 1, 0) + 3 \cdot (0, 1, 0),$$

where the summands are the contributions of the components of the bigraded ring $G_{\mathfrak{m}}(G_I(R))$, see Proposition 1.

The Newton polyhedron $\Gamma(I)$ has only one vertex v = (3, 2) and two (unbounded) facets $F_1 = v + R_{\geq 0}e_2$ and $F_2 = v + R_{\geq 0}e_1$ (see Figure 1), hence $\mathcal{F}(1) = \emptyset$ and $c_0(I) = 0$. We have $\mathcal{F}(0) = \{F_1, F_2\}$ and $\operatorname{Vol}(v, F_1) = 3$ and $\operatorname{Vol}(v, F_2) = 2$, hence $c_1(I) = 5$.

Example 2 (Figure 2). Let $I = (x^2y^5, x^3y^4, x^4y^2, x^6y) \subset R = K[x, y]$. We have

$$c(I) = (c_0(I), c_1(I), c_2(I))$$

= (24, 3, 0) = 16 \cdot (1, 0, 0) + 8 \cdot (1, 0, 0) + 2 \cdot (0, 1, 0) + (0, 1, 0),

where the summands are the contributions of the components of the bigraded ring $G_{\mathfrak{m}}(G_I(R))$, see Proposition 1.

The Newton polyhedron $\Gamma(I)$ has three vertices $v_1 = (2,5)$, $v_3 = (4,2)$, $v_4 = (6,1)$, two (unbounded) facets $F_1 = v_1 + R_{\geq 0}e_2$, $F_2 = v_4 + P_2$



FIGURE 1. Projection along the infinite-directions of the facets gives $c_1(I)$, which is the red distance.



FIGURE 2. The red area is $c_0(I)/2$, the red distance $c_1(I)$.

 $R_{\geq 0}e_1$ and two bounded facets: the line segments $F_3 = \operatorname{conv}(v_1, v_3)$, $F_4 = \operatorname{conv}(v_3, v_4)$ (see Figure 2), hence $\mathcal{F}(1) = \{F_3, F_4\}$ and

$$c_0(I) = \operatorname{Vol}(\operatorname{conv}(0, F_3)) + \operatorname{Vol}(\operatorname{conv}(0, F_4)) = \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} + \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 16 + 8.$$

We have $\mathcal{F}(0) = \{F_1, F_2\}$ and

$$c_1(I) = \operatorname{Vol}(v_1, F_1) + \operatorname{Vol}(v_4, F_2) = 2 + 1 = 3.$$

In the following example some of the compact faces of $\Gamma(I)$ do not contribute to the generalized Samuel multiplicity c(I). In this example there is also a movable contribution, therefore the number of the highest dimensional components of T is one less than the number of summands in the conjectured formula.



FIGURE 3. Infinite-directions (blue arrows) of the unbounded facets, $c_2(I)$ (red distance) and $c_1(I)/2$ (red area).



FIGURE 4. The volume of the red pyramid is $c_0/6$.

Example 3 (Figures 3, 4 and 5). Let $I = (x^2y, x^2z, xy^2, xz^2) \subset K[x, y, z]$. By a computer computation (using [1]) we have

$$c(I) = (c_0(I), c_1(I), c_2(I), c_3(I))$$

= (9, 3, 1, 0) = 3 \cdot (3, 0, 0, 0) + (0, 1, 0, 0) + (0, 2, 1, 0),

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1. The contribution 2 in the last vector is a movable contribution to $c_1(I)$. This can be read off also from the Newton polyhedron $\Gamma(I)$, see Figure 6.

According to the program Germenes [11], the compact faces of $\Gamma(I)$ are the vertices $v_1 = (2, 1, 0), v_2 = (2, 0, 1), v_3 = (1, 2, 0), v_4 = (1, 0, 2),$ the line segments $v_1v_2, v_1v_3, v_2v_4, v_3v_4$ and the quadrilateral facet



FIGURE 5. A movable contribution (to $c_1(I)/2$, red area) can be realized by at least two projections.

 $v_1v_2v_4v_3$. The unbounded facets are

$$F_1 = v_3 v_4 + \mathbb{R}_{\ge 0} e_2 + \mathbb{R}_{\ge 0} e_3, \quad F_2 = v_2 v_4 + \mathbb{R}_{\ge 0} e_1 + \mathbb{R}_{\ge 0} e_3,$$

$$F_3 = v_1 v_3 + \mathbb{R}_{\ge 0} e_1 + \mathbb{R}_{\ge 0} e_2, \quad F_4 = v_1 v_2 + \mathbb{R}_{\ge 0} e_1.$$

We observe that the set of bounded facets is $\mathcal{F}(2) = \{v_1v_2v_4v_3\}$ and

$$c_0(I) = \operatorname{Vol}(\operatorname{conv}(0, v_1, v_2, v_4, v_3)) = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 3 + 6 = 9,$$

see Figure 4.

The set of 1-unbounded facets that contain a compact one-dimensional face is $\mathcal{F}(1) = \{F_1, F_2, F_3, F_4\}$, and we have

$$\operatorname{Vol}(v_{1}v_{2}, F_{4}) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$
$$\operatorname{Vol}(v_{1}v_{3}, F_{3}) = \min\left\{ \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} \right\} = 0,$$
$$\operatorname{Vol}(v_{2}v_{4}, F_{2}) = \min\left\{ \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} \right\} = 0,$$
$$\operatorname{Vol}(v_{3}v_{4}, F_{1}) = \min\left\{ \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \right\} = 2$$



FIGURE 6. $\Gamma(I)$ with compact (black) and unbounded (blue) facets; projections of compact edges along infinitedirections (blue arrows) give $c_1(I)/2$ (red area).

(the last minimum is given by two different projections and is a movable contribution, see Figure 5), hence

$$c_1(I) = \operatorname{Vol}(v_1v_2, F_4) + \operatorname{Vol}(v_1v_3, F_3) + \operatorname{Vol}(v_2v_4, F_2) + \operatorname{Vol}(v_3v_4, F_1)$$

= 1 + 0 + 0 + 2 = 3.

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$. We have $Vol(v_3, F_1) = 1$, $Vol(v_4, F_1) = 1$, $Vol(v_4, F_2) = 0$, $Vol(v_1, F_3) = 0$, $Vol(v_3, F_3) = 0$, hence

$$c_2(I) = \min\{\operatorname{Vol}(v_3, F_1), \operatorname{Vol}(v_4, F_1)\} + \operatorname{Vol}(v_4, F_2) + \\ + \min\{\operatorname{Vol}(v_1, F_3), \operatorname{Vol}(v_3, F_3)\} = 1 + 0 + 0 = 1.$$

Example 4 (Figure 6). Let $I = (xy^4z^5, x^2y^5z^2, xy^5z^3, x^5yz^2, x^2yz^5, x^5y^2z) \subset K[x, y, z]$. By a computer computation we have

$$c(I) = (c_0(I), c_1(I), c_2(I), c_3(I)) = (168, 26, 3, 0) =$$

= 19 \cdot (1, 0, 0, 0) + 103 \cdot (1, 0, 0, 0) + 22 \cdot (1, 0, 0, 0) +
+ 24 \cdot (1, 0, 0, 0) + 7 \cdot (0, 1, 0, 0) + (0, 3, 1, 0) +
+ 8 \cdot (0, 1, 0, 0) + (0, 1, 0, 0) + 4 \cdot (0, 1, 0, 0) +
+ 3 \cdot (0, 1, 0, 0) + (0, 0, 1, 0) + (0, 0, 1, 0),

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1. The contribution 3 in the sixth vector is a movable contribution to $c_1(I)$.

The software Germenes [11] gives the following description of the Newton polyhedron $\Gamma(I)$. The compact faces of $\Gamma(I)$ are the 6 vertices

 $v_1 = (1, 4, 5), v_2 = (2, 5, 2), v_3 = (1, 5, 3), v_4 = (5, 1, 2), v_5 = (2, 1, 5), v_6 = (5, 2, 1), \text{the 9 line segments } v_4v_6, v_2v_6, v_5v_6, v_4v_5, v_3v_6, v_3v_2, v_3v_5, v_1v_5, v_1v_3 \text{ and the 4 triangles (bounded facets) } v_4v_5v_6, v_2v_3v_6, v_3v_5v_6, v_1v_3v_5.$ There are 7 unbounded facets:

$$\begin{split} F_1 &= v_1 v_3 + \mathbb{R}_{\ge 0} e_2 + \mathbb{R}_{\ge 0} e_3, \quad F_2 &= v_4 v_5 + \mathbb{R}_{\ge 0} e_1 + \mathbb{R}_{\ge 0} e_3, \\ F_3 &= v_6 + \mathbb{R}_{\ge 0} e_1 + \mathbb{R}_{\ge 0} e_2, \quad F_4 &= v_1 v_5 + \mathbb{R}_{\ge 0} e_3, \\ F_5 &= v_2 v_3 + \mathbb{R}_{\ge 0} e_2, \quad F_6 &= v_2 v_6 + \mathbb{R}_{\ge 0} e_2, \quad F_7 &= v_4 v_6 + \mathbb{R}_{\ge 0} e_1. \end{split}$$

From the set of bounded facets $\mathcal{F}(2) = \{v_4 v_5 v_6, v_2 v_3 v_6, v_3 v_5 v_6, v_1 v_3 v_5\}$ we get

$$c_{0}(I) = \operatorname{Vol}(\operatorname{conv}(0, v_{4}, v_{5}, v_{6})) + \operatorname{Vol}(\operatorname{conv}(0, v_{2}, v_{3}, v_{6}) + \operatorname{Vol}(\operatorname{conv}(0, v_{3}, v_{5}, v_{6}) + \operatorname{Vol}(\operatorname{conv}(0, v_{1}, v_{3}, v_{5}) = = \begin{vmatrix} 5 & 1 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 5 & 2 \\ 1 & 5 & 3 \\ 5 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 3 \\ 2 & 1 & 5 \\ 5 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 3 \\ 2 & 1 & 5 \\ 5 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 & 5 \\ 2 & 1 & 5 \\ 1 & 5 & 3 \end{vmatrix} = = 24 + 22 + 103 + 19 = 168.$$

We have $\mathcal{F}(1) = \{F_1, F_2, F_4, F_5, F_6, F_7\}$ and

$$\operatorname{Vol}(v_1v_3, F_1) = \min\left\{ \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} \right\} = \min\{1, 2\} = 1,$$
$$\operatorname{Vol}(v_4v_5, F_2) = \min\left\{ \begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} \right\} = \min\{3, 3\} = 3$$

(here the minimum is attained twice, that is, by two different projections which indicates a movable contribution),

$$\operatorname{Vol}(v_1v_5, F_4) = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7, \quad \operatorname{Vol}(v_2v_3, F_5) = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4,$$
$$\operatorname{Vol}(v_2v_6, F_6) = \begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix} = 8, \quad \operatorname{Vol}(v_4v_6, F_7) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3,$$

hence

$$c_1(I) = \operatorname{Vol}(v_1v_3, F_1) + \operatorname{Vol}(v_4v_5, F_2) + \operatorname{Vol}(v_1v_5, F_4) + + \operatorname{Vol}(v_2v_3, F_5) + \operatorname{Vol}(v_2v_6, F_6) + \operatorname{Vol}(v_4v_6, F_7) = = 1 + 3 + 7 + 4 + 8 + 3 = 26.$$

We observe that the compact 1-dimensional faces v_5v_6 , v_3v_6 , v_3v_5 , that is, the edges of the big triangle $v_3v_5v_6$, do not contribute to $c_1(I)$



FIGURE 7. The triangle defined by 3 affinely independent vertices is not a compact facet.

since they lie on no 1-unbounded facets. Moreover, as in the previous example, there is a movable contribution, namely $Vol(v_4v_5, F_2) = 3$.

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$, and we have $Vol(v_1, F_1) = 1$, $Vol(v_3, F_1) = 1$, $Vol(v_4, F_2) = 1$, $Vol(v_5, F_2) = 1$, $Vol(v_6, F_3) = 1$, hence

$$c_2(I) = \min\{\operatorname{Vol}(v_1, F_1), \operatorname{Vol}(v_1, F_1)\} + \min\{\operatorname{Vol}(v_4, F_2), \operatorname{Vol}(v_5, F_2)\} + \operatorname{Vol}(v_6, F_3) = 1 + 1 + 1 = 3.$$

Example 5 (Figure 7). Let $I = (xz, x^2y^2, yz^2) \subset K[x, y, z]$. By a computer computation we have

$$c(I) = (c_0(I), c_1(I), c_2(I), c_3(I)) = (0, 7, 0, 0) =$$

= 2 \cdot (0, 1, 0, 0) + 2 \cdot (0, 1, 0, 0) + 2 \cdot (0, 1, 0, 0) + (0, 1, 0, 0),

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1.

A computation with the program Germenes [11] shows the the compact faces of $\Gamma(I)$ are the vertices $v_1 = (1,0,1)$, $v_2 = (2,2,0)$, $v_3 = (0,1,2)$ and the line segments v_1v_2 , v_1v_3 . There are no compact or bounded facets, but 6 unbounded facets:

$$\begin{split} F_1 &= v_3 + \mathbb{R}_{\geq 0} \, e_2 + \mathbb{R}_{\geq 0} \, e_3, \quad F_2 &= v_1 + \mathbb{R}_{\geq 0} \, e_1 + \mathbb{R}_{\geq 0} \, e_3, \\ F_3 &= v_2 + \mathbb{R}_{\geq 0} \, e_1 + \mathbb{R}_{\geq 0} \, e_2, \quad F_4 &= v_1 v_2 + \mathbb{R}_{\geq 0} \, e_1, \\ F_5 &= v_1 v_3 + \mathbb{R}_{> 0} \, e_3, \quad F_6 &= v_1 v_2 v_3 + \mathbb{R}_{> 0} \, e_2. \end{split}$$

We observe that $\mathcal{F}(2) = \emptyset$, hence

$$c_0(I) = 0 \neq \operatorname{Vol}_3(\operatorname{conv}(\{0, v_1, v_2, v_3\})) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 6$$

This means that v_1 , v_2 , v_3 are affinely independent, but the local Segre class is zero and not equal to the normalized volume of the simplex generated by the origin and v_1 , v_2 , v_3 as claimed in [7, 4.2].

The set of 1-unbounded facets which contain a compact 1-dimensional face is $\mathcal{F}(1) = \{F_4, F_5, F_6\}$, and we have

$$\operatorname{Vol}(v_1v_2, F_4) = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2, \quad \operatorname{Vol}(v_1v_2, F_6) = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2,$$
$$\operatorname{Vol}(v_1v_3, F_5) = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2, \quad \operatorname{Vol}(v_1v_3, F_6) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

hence

$$c_1(I) = \operatorname{Vol}(v_1v_2, F_4) + \operatorname{Vol}(v_1v_2, F_6) + \operatorname{Vol}(v_1v_3, F_5) + \operatorname{Vol}(v_1v_3, F_6)$$

= 2 + 2 + 2 + 1 = 7.

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$ and we have

$$Vol(v_3, F_1) = 0$$
, $Vol(v_1, F_2) = 0$, $Vol(v_2, F_3) = 0$,

hence $c_2(I) = 0$.

In the following example there is only one 1-dimensional compact face, but it lies on three 2-unbounded facets.

Example 6. Let d = 4, $I = (x_1^3 x_2 x_3 x_4, x_1 x_2 x_3 x_4^2)$. By a computer computation we have

$$c(I) = (c_0(I), c_1(I), c_2(I), c_3(I), c_4(I)) = (0, 0, 7, 4, 0) =$$

= 5 \cdot (0, 0, 1, 0, 0) + (0, 0, 1, 0, 0) + (0, 0, 1, 0, 0) + (0, 0, 0, 1, 0) + (0, 0, 0, 1, 0) + (0, 0, 0, 1, 0) + (0, 0, 0, 1, 0),

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1.

The compact faces of the Newton polyhedron $\Gamma(I)$ are the vertices $v_1 = (3, 1, 1, 1), v_2 = (1, 1, 1, 2)$ and the line segment v_1v_2 . There are

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no compact facets, but 5 unbounded facets:

$$\begin{split} F_1 &= v_2 + \mathbb{R}_{\geq 0} \, e_2 + \mathbb{R}_{\geq 0} \, e_3 + \mathbb{R}_{\geq 0} \, e_4, \\ F_2 &= v_1 v_2 + \mathbb{R}_{\geq 0} \, e_1 + \mathbb{R}_{\geq 0} \, e_3 + \mathbb{R}_{\geq 0} \, e_4, \\ F_3 &= v_1 v_2 + \mathbb{R}_{\geq 0} \, e_1 + \mathbb{R}_{\geq 0} \, e_2 + \mathbb{R}_{\geq 0} \, e_4, \\ F_4 &= v_1 + \mathbb{R}_{\geq 0} \, e_1 + \mathbb{R}_{\geq 0} \, e_2 + \mathbb{R}_{\geq 0} \, e_3, \\ F_5 &= v_1 v_2 + \mathbb{R}_{\geq 0} \, e_2 + \mathbb{R}_{\geq 0} \, e_3. \end{split}$$

Obviously $\mathcal{F}(3) = \mathcal{F}(2) = \emptyset$, hence $c_0(I) = c_1(I) = 0$. We have $\mathcal{F}(1) = \{F_2, F_3, F_5\}$ and

$$\operatorname{Vol}(v_1 v_2, F_2) = \min\{\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\} = \min\{2, 1\} = 1,$$
$$\operatorname{Vol}(v_1 v_2, F_3) = \min\{\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\} = \min\{2, 1\} = 1,$$

$$\operatorname{Vol}(v_1 v_2, F_5) = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5,$$

We observe that in the computations of $\operatorname{Vol}(v_1v_2, F_2)$ and $\operatorname{Vol}(v_1v_2, F_3)$ the projection of the line segment v_1v_2 on the $\{x_2, x_3\}$ -plane gives the point (1, 1) and must not be considered. We obtain $c_2(I) =$ $\operatorname{Vol}(v_1v_2, F_2) + \operatorname{Vol}(v_1v_2, F_3) + \operatorname{Vol}(v_1v_2, F_5) = 1 + 1 + 5 = 7.$

The set of 3-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3, F_4\}$, and we have

$$Vol(v_2, F_1) = 1$$
, $Vol(v_1, F_2) = 1$, $Vol(v_2, F_2) = 1$,
 $Vol(v_1, F_3) = 1$, $Vol(v_2, F_3) = 1$, $Vol(v_1, F_4) = 1$,

hence

$$c_{3}(I) = \operatorname{Vol}(v_{2}, F_{1}) + \min\{\operatorname{Vol}(v_{1}, F_{2}), \operatorname{Vol}(v_{2}, F_{2})\} + \\ + \min\{\operatorname{Vol}(v_{1}, F_{3}), \operatorname{Vol}(v_{2}, F_{3})\} + \operatorname{Vol}(v_{1}, F_{3}) = \\ = 1 + 1 + 1 + 1 = 4.$$

We now present an example in dimension 5 in which several compact 1-dimensional faces lie on the same 3-unbounded facet. **Example 7.** Let d = 5, $I = (x_1^3 x_2 x_3 x_4 x_5, x_1 x_2^2 x_3 x_4 x_5, x_1 x_2 x_3 x_4 x_5^5)$. By a computer computation we have

$$\begin{split} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I), c_4(I), c_5(I)) = (0, 0, 26, 6, 5, 0) = \\ &= 22 \cdot (0, 0, 1, 0, 0, 0) + 2 \cdot (0, 0, 1, 0, 0, 0) + 2 \cdot (0, 0, 1, 0, 0, 0) + \\ &+ 2 \cdot (0, 0, 0, 1, 0, 0) + (0, 0, 0, 1, 0, 0) + (0, 0, 0, 1, 0, 0) + \\ &+ (0, 0, 0, 1, 0, 0) + (0, 0, 0, 1, 0, 0) + (0, 0, 0, 0, 1, 0) + \\ &+ (0, 0, 0, 0, 1, 0) + (0, 0, 0, 0, 1, 0) + (0, 0, 0, 0, 1, 0) + \\ &+ (0, 0, 0, 0, 1, 0), \end{split}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1.

The software Germenes [11] shows that the compact faces of $\Gamma(I)$ are the vertices $v_1 = (3, 1, 1, 1)$, $v_2 = (1, 2, 1, 1, 1)$, $v_3 = (1, 1, 1, 1, 5)$, the line segments v_1v_2 , v_1v_3 , v_2v_3 and the triangle $v_1v_2v_3$. The facets, all of them unbounded, are:

$$\begin{split} F_1 &= v_2 v_3 + \mathbb{R}_{\ge 0} e_2 + \mathbb{R}_{\ge 0} e_3 + \mathbb{R}_{\ge 0} e_4 + \mathbb{R}_{\ge 0} e_5, \\ F_2 &= v_1 v_3 + \mathbb{R}_{\ge 0} e_1 + \mathbb{R}_{\ge 0} e_3 + \mathbb{R}_{\ge 0} e_4 + \mathbb{R}_{\ge 0} e_5, \\ F_3 &= v_1 v_2 v_3 + \mathbb{R}_{\ge 0} e_1 + \mathbb{R}_{\ge 0} e_2 + \mathbb{R}_{\ge 0} e_4 + \mathbb{R}_{\ge 0} e_5, \\ F_4 &= v_1 v_2 v_3 + \mathbb{R}_{\ge 0} e_1 + \mathbb{R}_{\ge 0} e_2 + \mathbb{R}_{\ge 0} e_3 + \mathbb{R}_{\ge 0} e_5, \\ F_5 &= v_1 v_2 + \mathbb{R}_{\ge 0} e_1 + \mathbb{R}_{\ge 0} e_2 + \mathbb{R}_{\ge 0} e_3 + \mathbb{R}_{\ge 0} e_4, \\ F_6 &= v_1 v_2 v_3 + \mathbb{R}_{\ge 0} e_3 + \mathbb{R}_{\ge 0} e_4. \end{split}$$

Obviously $\mathcal{F}(4) = \mathcal{F}(3) = \emptyset$, hence $c_0(I) = c_1(I) = 0$. We have $\mathcal{F}(2) = \{F_3, F_4, F_6\}$ and

$$\operatorname{Vol}(v_1 v_2 v_3, F_3) = \operatorname{Vol}(v_1 v_2 v_3, F_4) = \min\left\{ \begin{vmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 5 \end{vmatrix} \right\} = \min\{2, 4\} = 2,$$

$$\operatorname{Vol}(v_1 v_2 v_3, F_6) = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 5 \end{vmatrix} = 22.$$

Observe that in the computations of $Vol(v_1v_2v_3, F_3)$ and $Vol(v_1v_2v_3, F_4)$ four projections of the compact triangle $v_1v_2v_3$ give a line segment and must not be considered. Summing up we get

$$c_2(I) = \operatorname{Vol}(v_1 v_2 v_3, F_3) + \operatorname{Vol}(v_1 v_2 v_3, F_4) + \operatorname{Vol}(v_1 v_2 v_3, F_6)$$

= 2 + 2 + 22 = 26.

The set of 3-unbounded facets containing 1-dimensional compact faces is $\mathcal{F}(1) = \{F_1, F_2, F_3, F_4, F_5\}$ and we have

$$\begin{aligned} \operatorname{Vol}(v_{2}v_{3}, F_{1}) &= \min\left\{ \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} &= \min\{1, 4\} = 1, \\ \operatorname{Vol}(v_{1}v_{3}, F_{2}) &= \min\left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} &= \min\{2, 4\} = 2, \\ \operatorname{Vol}(v_{1}v_{2}, F_{3}) &= \operatorname{Vol}(v_{1}v_{2}, F_{4}) &= \min\left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \right\} &= \min\{2, 1\} = 1, \\ \operatorname{Vol}(v_{1}v_{3}, F_{3}) &= \operatorname{Vol}(v_{1}v_{3}, F_{4}) &= \min\left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} &= \min\{2, 4\} = 2, \\ \operatorname{Vol}(v_{2}v_{3}, F_{3}) &= \operatorname{Vol}(v_{2}v_{3}, F_{4}) &= \min\left\{ \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} &= \min\{2, 4\} = 2, \\ \operatorname{Vol}(v_{2}v_{3}, F_{3}) &= \operatorname{Vol}(v_{2}v_{3}, F_{4}) &= \min\left\{ \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} &= \min\{1, 4\} = 1, \\ \operatorname{Vol}(v_{1}v_{2}, F_{5}) &= \min\left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \right\} &= \min\{2, 1\} = 1, \end{aligned}$$

hence

$$c_{3}(I) = \operatorname{Vol}(v_{2}v_{3}, F_{1}) + \operatorname{Vol}(v_{1}v_{3}, F_{2}) + \\ + \min\{\operatorname{Vol}(v_{1}v_{2}, F_{3}), \operatorname{Vol}(v_{1}v_{3}, F_{3}), \operatorname{Vol}(v_{2}v_{3}, F_{3})\} + \\ + \min\{\operatorname{Vol}(v_{1}v_{2}, F_{4}), \operatorname{Vol}(v_{1}v_{3}, F_{4}), \operatorname{Vol}(v_{2}v_{3}, F_{4})\} + \\ + \operatorname{Vol}(v_{1}v_{2}, F_{5}) = 1 + 2 + 1 + 1 + 1 = 6.$$

From the list of the facets we see that there are five 4-unbounded facets, precisely $\mathcal{F}(0) = \{F_1, F_2, F_3, F_4, F_5\}$ and we have

$$Vol(v_2, F_1) = 1, \quad Vol(v_3, F_1) = 1, \quad Vol(v_1, F_2) = 1, \quad Vol(v_3, F_2) = 1,$$

$$Vol(v_1, F_3) = 1, \quad Vol(v_2, F_3) = 1, \quad Vol(v_3, F_3) = 1, \quad Vol(v_1, F_4) = 1,$$

$$Vol(v_2, F_4) = 1, \quad Vol(v_3, F_4) = 1, \quad Vol(v_1, F_5) = 1, \quad Vol(v_2, F_5) = 1,$$

$$c_4(I) = \min\{\operatorname{Vol}(v_2, F_1), \operatorname{Vol}(v_3, F_1)\} + \min\{\operatorname{Vol}(v_1, F_2), \operatorname{Vol}(v_3, F_2)\} + \\ + \min\{\operatorname{Vol}(v_1, F_3), \operatorname{Vol}(v_2, F_3), \operatorname{Vol}(v_3, F_3)\} + \\ + \min\{\operatorname{Vol}(v_1, F_4), \operatorname{Vol}(v_2, F_4), \operatorname{Vol}(v_3, F_4)\} + \\ + \min\{\operatorname{Vol}(v_1, F_5), \operatorname{Vol}(v_2, F_5)\} = 1 + 1 + 1 + 1 + 1 = 5.$$

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