LOCAL REGULARITY FOR ELLIPTIC SYSTEMS WITH $\mathbf{p,q}$-GROWTH
REGOLARITÀ LOCALE PER I SISTEMI ELLITTICI CON CRESCITA $\mathbf{p,q}$

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Abstract. In this paper we consider quasilinear elliptic systems with p,q-growth. We discuss some aspects of the theory of regularity for systems and we state a local boundedness result for weak solutions, obtained in collaboration with P. Marcellini. Moreover, a new boundedness result is proved under weaker assumptions on the coefficients.

Sunto. In questo articolo consideriamo sistemi ellittici quasi lineari con crescita p,q. Illustriamo alcuni aspetti della teoria della regolarità per i sistemi ed enunciamo un risultato di limitatezza locale per soluzioni deboli, ottenuto in collaborazione con P. Marcellini. Inoltre, un nuovo risultato di limitatezza è dimostrato con ipotesi più deboli sui coefficienti.

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1. Introduction

It is well known that many mathematical models of physical phenomena take forms of partial differential equations or systems. As particular cases we have the Laplace Equation $\Delta u = 0$ and the Heat Equation $D_t u = \Delta u$.

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A partial differential system is a collection of equations involving m functions, $m > 1$, in n variables and their partial derivatives:

$$E(x, u(x), Du(x)) = 0 \quad \text{in } \Omega,$$

where $E : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}^m$, with $\Omega \subset \mathbb{R}^n$ open set, and $u : \Omega \to \mathbb{R}^m$ is a vector-valued map.

In this note we consider elliptic systems in divergence form:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(A_i^\alpha(x, u, Du)\right) = B^\alpha(x, u, Du) \qquad \alpha = 1, \dots, m, \tag{1}$$

with $A^\alpha : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \longrightarrow \mathbb{R}^n$, $B^\alpha : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \longrightarrow \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ is an open, bounded set. We observe that a significant case of system (1) is the Euler-Lagrange system of a variational integral. Indeed, if we consider the functional

$$I(u) := \int_\Omega f\left(x, u(x), Du(x)\right) dx$$

with $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}$ smooth enough, $f = f(x, s, \xi)$, then a minimizer u of I satisfies the Euler-Lagrange system; i.e.,

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(f_{\xi_i^\alpha}\left(x, u(x), Du(x)\right)\right) = f_{s^\alpha}\left(x, u(x), Du(x)\right) \qquad \alpha = 1, \dots, m.$$

Concerning system (1), the main problems are the proof of the existence in a given (and suitable) class of functions and under suitable boundary conditions, and the study of the regularity of the solutions.

More definitions of "weak" solutions to (1) can be given. Roughly, a "weak" solution of (1) is a function u in a suitable Sobolev space, such that for all the test functions φ in a suitable class, we have

$$\sum_{\alpha, i} \int_\Omega A_i^\alpha(x, u, Du)\, \varphi_{x_i}^\alpha(x)\, dx + \sum_\alpha \int_\Omega B^\alpha(x, u, Du)\, \varphi^\alpha(x)\, dx = 0.$$

Of course, to have well defined integrals, what is required is that the "pairing" is satisfied, e.g. $A(x, u, Du), B(x, u, Du) \in L^p$ requires $\varphi \in W^{1,p'}$, with $\frac{1}{p} + \frac{1}{p'} = 1$. The choice of a Sobolev space, rather than others, can affect the existence, the uniqueness or the regularity of the solutions.

Serrin in [51] provides an example of a second order elliptic equation with two solutions: due to standard results, there exists a unique solution in $W^{1,2}$, but, on the other hand, the author also exhibits another, and explicit, solution belonging to $W^{1,1}(\Omega) \setminus W^{1,2}(\Omega)$. With this example the author shows that the well-known properties of local boundedness, uniqueness for the Dirichlet problem, etc., cannot, in general, be extended from $p = 2$ to $1 \leq p < 2$.

Proving the existence of weak solutions is simpler than proving the existence of classical ones, but, as a consequence, the proof of *regularity* results becomes crucial: it is enough to note that two of the Hilbert's celebrated 23 problems at the International Congress of Mathematicians in Paris in 1900 are devoted to this:

- Hilbert's 19th Problem: *Are the solutions of regular variational problems always analytic?*

- Hilbert's 20th Problem: *Is it not the case that every regular variational problem has a solution, provided that certain assumptions on the boundary conditions are satisfied, and provided also (if necessary) that the concept of solution be suitable extended?*

As we will see there are, among others, two aspects that have to be taken into account when regularity is studied: the regularity of the solutions is strongly related to the integrability properties of the coefficients and to the dimension m ($m > 1$ is "worse" than $m = 1$).

In the next section we present some historical notes on the issue of the regularity of weak solutions for systems, in Section 3 we introduce the p, q-growth conditions and Section 4 is devoted to report a result obtained in collaboration with Marcellini in [11] (see Theorem 4.1 below) and to prove a new result (Theorem 4.2).

2. Historical notes

It is very difficult to cite and discuss all the many different contributions to the regularity problem, therefore we confine our historical presentation to the fundamental results only.

One of the first results on the regularity problem for nonlinear equations is due to Bernstein [4] (C^3-solutions of a nonlinear elliptic analytic second order equation in the plane are analytic functions) and, further, some contributions for linear systems are given by Caccioppoli [7], Douglis-Nirenberg [15], Morrey [44]. After these researches, no substantial progress was made (except for the two dimensional case) until the regularity result for elliptic equations by De Giorgi [12] in 1957. This celebrated theorem states that, given a second order linear elliptic equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} a_{ij}(x)\, u_{x_j} \right) = 0,$$

with essentially bounded measurable coefficients a_{ij} satisfying

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n$$

for some $\nu > 0$, then the $W^{1,2}$-weak solutions are Hölder continuous; i.e., there exists an exponent $\alpha \in (0,1)$ such that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$. This result is usually referred to as the De Giorgi-Nash Theorem, since this result was proved independently, and with different proofs, by Nash, too. The Nash paper [47], published in 1958, a year later than the De Giorgi's one, deals also with the parabolic equations. In his paper, Nash writes: *"P.R. Garabedian writes from London about a manuscript by Ennio De Giorgi containing such a result"*. It is worth remembering that later, in 1960, Moser in [46] extends the validity of the Harnack inequality to the solutions of general linear equations in divergence form and, from this, he obtains a different proof of the Hölder regularity.

The method used by De Giorgi is a very powerful one and it consists in three steps. Indeed, given a weak solution u, the De Giorgi's proof goes as follows:

(i) proof of Caccioppoli type inequalities: in particular, estimates of integrals of $|Du|$ with integrals of u on its super-(sub-)level sets,

(ii) proof of the local boundedness of u,

(iii) proof of the local Hölder continuity of u.

Thus, the proof of the local boundedness of u is, in the De Giorgi's proof, preliminary to further additional regularity.

Significant generalizations of the De Giorgi result are given by Stampacchia, we recall here [55] and [56], by Ladyzhenskaya and Ural'tseva in some papers and in the fundamental and celebrated book [30], by Serrin in two papers in 1964 and 1965 [52], [53] where a complete analysis of the nonlinear case is given. We also quote the contributions of Giaquinta and Giusti that in some papers in the early 80's, [22], [23], [24], generalized the Hölder regularity results to the minimizers of non-differentiable integral functionals with integrands satisfying the following growth conditions:

$$|Du|^p \leq f(x,u,Du) \leq c(1+|Du|)^p, \qquad p>1.$$

None of the proofs of the De Giorgi's result could be extended to cover the case of systems. At last De Giorgi himself, in an article published in 1968 [13], exhibits a counterexample of a second order linear elliptic system having a $W^{1,2}(\Omega,\mathbb{R}^n)$-weak solution, which is not only discontinuous, but even locally unbounded. The example is the following:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j,\beta=1}^{n} a_{ij}^{\alpha\beta}(x) u_{x_j}^{\beta}\right) = 0, \quad \alpha = 1,...,n,$$

with

$$a_{ij}^{\alpha\beta}(x) = \delta_{ij}\delta_{\alpha\beta} + \left((n-2)\delta_{i\alpha} + \frac{x_\alpha x_i}{|x|^2}\right)\left((n-2)\delta_{j\beta} + \frac{x_\beta x_j}{|x|^2}\right).$$

The function

$$u(x) = \frac{x}{|x|^\gamma} \text{ with } \gamma = \frac{n}{2}\left(1 - \frac{1}{\sqrt{4(n-1)^2+1}}\right) > 1$$

is a weak solution in $W^{1,2}(B_1(0))$ (and a solution in a classical sense in $B_1(0)\setminus\{0\}$), but it is not continuous, and even not locally bounded, since the solution has a singularity at $x=0$.

In matter of boundedness of solutions, we recall another classical counterexample in Giusti and Miranda's paper [26]. This article appeared immediately after De Giorgi's one, as one notices by the publication year (1968). Again, a quite simple system is considered, (now with bounded coefficients $a_{ij}^{\alpha\beta}(u)$ having analytic dependence on the solution u) and a bounded, but discontinuous, solution of this system is exhibited. What this example suggests is that there is a gap between local boundedness and (Hölder-)continuity. Since then a number of different counterexamples were exhibited during the late '60s, and even later: additional proofs that the regularity for systems is a delicate issue. We recall, among the most famous contributions, those by Maz'ja [40], Nečas [48] (Lipschitz, but not C^1 minimizers of integral functionals), Sveràk-Yan [57], [58], with examples of not Lipschitz minimizers of smooth uniformly convex functional and of an unbounded minimizer for $n = 5$, see also the recent paper [43] by Mooney-Savin for an example in lower dimension. We recall also, for the nonlinear case with different growth assumption and structure conditions, the papers by Freshe [17], [18], [19] and Hildebrandt-Widman [27] (bounded, discontinuous weak solutions).

By this brief discussion on the counterexamples we conclude that weak solutions to nonlinear elliptic systems or vector valued minimizers of regular integrals in general may be irregular. Moreover we observe that the adaptation of the methods from the theory of equations to systems is by no means obvious. For instance, notice that the De Giorgi method based on the truncation of the solution, now a usual trick to get regularity results, is delicate in the vectorial case: in the area of truncation the gradient is not vanishing (as it does in the scalar case) and can interfere, in a bad way, with the leading part.

Therefore, motivated by these counterexamples, we find in the mathematical literature at least two directions of research in regularity of generalized solutions of elliptic systems:

- *partial regularity*: i.e., regularity in an open set $\Omega_0 \subseteq \Omega$, $\text{meas}(\Omega \setminus \Omega_0) = 0$,
- *regularity in the interior of* Ω, when it is possible, under suitable structure assumptions and/or assuming *apriori* some regularity (e.g. the local boundedness).

Roughly, a typical condition that forces the regularity is the dependence of the operator/functional on the modulus of the gradient. For instance, let us consider the functional of the Calculus of Variations

$$\mathcal{F}(u) = \int_\Omega f(Du)\, dx, \qquad u : \Omega \to \mathbb{R}^m,\ m > 1.$$

In this case one expects a partial regularity result; if, instead, the functional is of *Uhlenbeck type*

$$\mathcal{F}(u) = \int_\Omega f(|Du|)\, dx, \qquad u : \Omega \to \mathbb{R}^m,\ m > 1,$$

one expects everywhere regularity. The counterpart of this last functional in the PDE setting is the system

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(h(|Du|)u^{\alpha}_{x_i}\right) = 0 \qquad \alpha = 1, \dots, m.$$

The pioneering result of everywhere regularity in dependence on the modulus of Du were proved by Uhlenbeck [60]. Uhlenbeck considered systems of the form

$$\operatorname{div}(|Du|^{p-2}Du) = 0$$

corresponding to the functional

$$\mathcal{F}(u) = \int |Du|^p \, dx,$$

with $p \geq 2$ and proved an everywhere regularity result: $u \in C^{1,\alpha}$. Further contributions for the equations are due to Evans [16], Di Benedetto [14], Lewis [33] and, for more general non linear systems, to Tolksdorf [59], Giaquinta-Modica [25] and, for the sub-quadratic case $1 < p < 2$, Acerbi-Fusco [1]. We recall also some recent results by Bulíček-Frehse [6], which prove the Hölder continuity for quite general structured systems.

We observe that there are also some Hölder continuity results for bounded solutions of quasilinear systems fulfilling the so-called *controllable* growth conditions and fixing among the several constants involved certain relations. More precisely, let us consider the system with principal part given by a diagonal matrix:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}\left(x, u, Du\right)\, u^{\alpha}_{x_j}\right) = B^{\alpha}(x, u(x), Du(x)), \quad \alpha = 1, \dots, m$$

with $0 < \lambda \leq \mu$ such that

$$\lambda|\zeta|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, u, \xi)\zeta_i\zeta_j \leq \mu|\zeta|^2 \qquad \forall \zeta \in \mathbb{R}^n,$$

and assume that there exist $a, b \geq 0$ such that

$$|B(x, u(x), \xi)| \leq a|\xi|^2 + b \qquad \forall \xi \in \mathbb{R}^{n\times m}. \tag{2}$$

Hildebrandt and Widman, see [27] and [28], prove that the bound $a < \lambda/\|u\|_{L^\infty}$ $(n = 2)$ and $a < \lambda/2\|u\|_{L^\infty}$ $(n > 2)$ are sufficient conditions to the Hölder continuity of bounded solutions and, by means of an adaptation of Frehse's counterexample, that the bound on a is optimal for $n = 2$. A similar result has been proved by Caffarelli in [8] with a different and geometric approach. His main idea is that the modulus of a solution is a supersolution of an associated linear elliptic equation and that it satisfies the weak Harnack inequality.

For what concerns the proof of the local boundedness of the solutions to elliptic systems there are not so many contributions. The result for linear elliptic systems

$$\begin{aligned} &\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} a_{ij}(x)\, u^{\alpha}_{x_j} + \sum_{\beta=1}^{m} b_i^{\alpha\beta}(x)\, u^{\beta} + f_i^{\alpha}(x) \right) + \\ &+ \sum_{i=1}^{n} \sum_{\beta=1}^{m} c_i^{\alpha\beta}(x)\, u^{\beta}_{x_i} + \sum_{\beta=1}^{m} d^{\alpha\beta}(x)\, u^{\beta} = f^{\alpha}(x) \qquad \alpha = 1, \ldots, m, \end{aligned} \tag{3}$$

$a_{ij}, b_i^{\alpha\beta}, c_i^{\alpha\beta}, d^{\alpha\beta}$ bounded measurable coefficients and given functions f_i^{α}, f^{α}, is in the book of Ladyzhenskaya and Ural'tseva, 1968 [30]. We stress that the leading part of the operator above, $\sum_i \frac{\partial}{\partial x_i}\left(\sum_j a_{ij}(x)\, u^{\alpha}_{x_j}\right)$, is far less general than the operator of the De Giorgi counterexample, $\sum_i \frac{\partial}{\partial x_i}\left(\sum_{j,\beta} a_{ij}^{\alpha\beta}(x)\, u^{\beta}_{x_j}\right)$.

In 1982 Meier, in his Ph.D. thesis, with supervisor Hildebrandt, and in a subsequent paper [41], studied the boundedness (and some integrability properties) of solutions to quasilinear elliptic systems:

$$\operatorname{div}\left(A^{\alpha}(x, u, Du)\right) = B^{\alpha}(x, u, Du) \quad \alpha = 1, ...m$$

under the following p-growth conditions ($p > 1$):

- $\sum_{\alpha} A^{\alpha}(x, u, \xi) \cdot \xi^{\alpha} \geq |\xi|^p - b|u|^p - c$,
- $|A^{\alpha}(x, u, \xi)| \leq C(|\xi|^{p-1} + |u|^{p-1} + 1)$,
- $|B^{\alpha}(x, u, \xi)| \leq C(|\xi|^{p-1} + |u|^{p-1} + 1)$.

The strategy of Meier's proof consists in a nontrivial generalization of that of Serrin [52], [53] for a single equation. The boundedness is obtained through the following pointwise crucial assumption for the so-called indicator function, that is, in a simplified and slightly more restrictive form,

$$I_A(x, u, Du) := \sum_{\alpha,\beta,i} A_i^{\alpha}(x, u, Du) u^{\beta}_{x_i} \frac{u^{\alpha} u^{\beta}}{|u|^2} \geq 0.$$

The linear case considered by Ladyzhenskaya and Ural'tseva is included, and also the more general case

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} a_{ij}\,(x,u,Du)\; u^{\alpha}_{x_j} \right) = 0,$$

if a_{ij} is positive semidefinite, because

$$\begin{aligned} I_A(x,u,Du) &= \sum_{\alpha,\beta,i} \left(\sum_{j=1}^{n} a_{ij}(x,u,Du) u^{\alpha}_{x_j} \right) u^{\beta}_{x_i} \frac{u^{\alpha} u^{\beta}}{|u|^2} \\ &= \sum_{i,j=1}^{n} \frac{a_{ij}(x,u,Du)}{|u|^2} \left(\sum_{\alpha} u^{\alpha} u^{\alpha}_{x_i} \right) \left(\sum_{\alpha} u^{\alpha} u^{\alpha}_{x_j} \right) \geq 0. \end{aligned}$$

We here recall also the geometric approach by Landes [31], [32] and an extension of the Meier's result due to Krömer [29].

To be precise, Meier deals with a weaker growth assumption on B, since the growth of B can be a *natural* growth ($|B^{\alpha}(x,u,\xi)| \sim |\xi|^p$), instead than a *controlled* one ($|B^{\alpha}(x,u,\xi)| \sim |\xi|^{p-1}$) as written above, but in case of *natural* growth this condition is not enough and has to be coupled with the *one-sided condition*:

$$\sum_{\alpha} u^{\alpha} B^{\alpha}(x,u,\xi) \geq -\left\{ (1-\theta)|\xi|^p + N|u| \left(c|\xi|^{p-1} + d|u|^{p-1} + f \right) \right\} \tag{4}$$

with $\theta \in (0,1]$, $c, d, f > 0$. Of course, the controlled growth implies this latter condition. As we have seen above, the growth of B is crucial also for the regularity of bounded weak solutions: in the case of natural growth without assuming (4), locally bounded weak solutions of systems with *natural growth* may be irregular as the examples by Frehse [18] and Hildebrandt-Widman [27] show.

For more details on the hystorical development of the regularity theory for systems we refer to the book by Giaquinta [20] and to the survey article by Mingione [42].

3. Regularity for elliptic systems with p,q-growth

In this section, we discuss a different aspect of how the growth conditions interact with the regularity. The systems may not satisfy standard p-growth conditions; for instance this happens if we consider the Euler-Lagrange system of functionals

$$I(u) = \int_\Omega f\left(x, u(x), Du(x)\right)\, dx,$$

with functions f having a p, q-growth; i.e.,

$$|\xi|^p \leq f(x, u, \xi) \leq (1 + |\xi|)^q, \qquad 1 \leq p < q.$$

Examples of energy densities of integral functionals with p, q-growth are the following:

- small perturbation of polynomial growth

$$f(\xi) = |\xi|^p \log^\alpha(1 + |\xi|), \quad \xi \in \mathbb{R}^{nm}, \quad p \geq 1,\ \alpha > 0;$$

- double phase functionals

$$f(x, \xi) = |\xi|^p + a(x)|\xi|^q, \qquad 0 \leq a(x)$$

(model for the study of the strongly anisotropic materials Zhikov [61], [62] for more detail and references);

- anisotropic growth

$$f(\xi) = \sum_{i=1}^{n} |\xi_i|^{p_i}$$

(here ξ_i, $i = 1, \ldots, n$, is the i-th column of the $m \times n$ matrix $\xi = (\xi_j^\alpha)$, $j = 1, \ldots, n$, $\alpha = 1, \ldots, m$);

- variable exponents

$$f(\xi) = |\xi|^{p(x)}, \quad f(\xi) = [h(|\xi|)]^{p(x)}, \quad p \leq p(x) \leq q.$$

(model proposed in 1996 in the theory of the electrorheological fluids by Rajagopal-Růžička [49], [50]; for the study of the image denoising by Chen et al. [9] and, recently, in theory of the growth of heterogeneous sandpiles by Bocea et al. [5]);

- anisotropic variable exponents

$$f(\xi) = \sum_{i=1}^{n} |\xi|^{p_i(x)}, \qquad p \leq p_i(x) \leq q.$$

We also mention an example of non-polynomial growth:

- exponential growth
$$f(x,\xi)\sim e^{|\xi|^{p(x)}}$$
(model proposed in the gas reaction by Aris [2] and in the combustion theory by Mosely [45]).

An important example of an integral functional with p,q-growth comes from the elasticity theory. Ball [3] considers a deformation of an elastic body that occupies a bounded domain $\Omega\subset\mathbb{R}^n$. If $u:\Omega\to\mathbb{R}^n$ is the displacement, then the total energy can be represented by an integral functional $\int_\Omega f(Du)\,dx$. One of the simplest, but typical, examples has integrand

$$f(Du)=g(Du)+h(\det Du)$$

where g,h are non-negative convex functions, that satisfy the growth conditions:

$$g(\xi)\geq c|\xi|^p \qquad \lim_{t\to+\infty}h(t)=+\infty$$

If $g(\xi)\sim|\xi|^p$ and $h(t)\sim t$ then

$$|\xi|^p\lesssim f(\xi)\lesssim(1+|\xi|)^n.$$

An example of system of PDE with p,q-growth is the following. Consider the sum of a *p-Laplacian* and a *degenerate q-Laplacian*:

$$\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(|Du|^{p-2}u^\alpha_{x_i}+m(x)|Du|^{q-2}u^\alpha_{x_i}\right)=0 \qquad \alpha=1,\dots,m$$

with $q>p$ and $0\leq m(x)\leq M$ (and $m=0$ in a set of positive measure). Denoting

$$A^\alpha_i(x,Du):=|Du|^{p-2}u^\alpha_{x_i}+m(x)|Du|^{q-2}u^\alpha_{x_i}$$

we have

$$|Du|^p\leq\sum_{i,\alpha}A^\alpha_i(x,Du)u^\alpha_{x_i}\leq(1+M)(1+|Du|)^q.$$

The first regularity result under the p,q-growth condition was proved by Marcellini [35], see also his papers [36] ,[37], [38], [39]. It is important to remark that two years before the first regularity result under p,q-growth ([35]) an example by Giaquinta [21] and Marcellini

[34] suggested that a bound on the gap between p and q is a necessary condition to the local regularity. For instance, the functional

$$\int_{B_1(0)} \left(\sum_{i=1}^{n-1} |u_{x_i}|^2 + c|u_{x_n}|^q \right) dx, \qquad n > 3$$

has an unbounded minimizer if $\frac{q}{2} > \frac{n-1}{n-3}$, or, equivalently, if $q > \overline{p}^*$. Here $\overline{p}$ denotes the harmonic mean of the n-vector $(2, \dots, 2, q)$ and $\overline{p}^*$ is its Sobolev exponent. We recall that the harmonic mean $\overline{p}$ of $(p_1, \dots, p_n)$ and $\overline{p}^*$ are defined as follows:

$$\frac{1}{\overline{p}} := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}, \qquad \overline{p}^* := \begin{cases} \frac{n\overline{p}}{n-\overline{p}} & \text{if } \overline{p} < n \\ \text{any } \mu > \overline{p} & \text{if } \overline{p} \geq n. \end{cases} \tag{5}$$

What is now well known is that, in general, to have the regularity of minimizers/solutions the gap between p and q must be not too large; in many cases this relation is expressed by an inequality of the type $q \leq c(n)p$ with $c(n) \to 1^+$ as n goes to infinity.

In the last years, there are many contributions on this subject; also in this case, for more details and references we refer to [42].

4. Local boundedness for elliptic systems with p, q-growth

In Cupini-Marcellini-Mascolo [11] we study the regularity for a particular class of quasilinear systems, which includes the *linear* case (3) considered by Ladyzhenskaya and Ural'tseva. Precisely, our systems are in the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} a_{ij}(x, u, Du)\, u_{x_j}^{\alpha} + b_i^{\alpha}(x, u, Du) \right) = f^{\alpha}(x, u, Du), \qquad \forall\, \alpha = 1, \dots, m. \tag{6}$$

Here, for semplicity, we consider the following simplified form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} a_{ij}(x, u, Du)\, u_{x_j}^{\alpha} \right) = 0, \quad \alpha = 1, \dots, m. \tag{7}$$

where $a_{ij} : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R}$ are Carathéodory functions, $i, j = 1, \dots, n$. Notice that this class (7) includes the general single equations in divergence form if $m = 1$. In fact any equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(a_i \left(x, u, Du \right) \right) = 0, \quad a_i(x,u,\cdot) \in C^1,$$

can be written as

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} a_{ij}(x,u,Du) u_{x_j} + a_i(x,u,0) \right) = 0,$$

(the presence of the term $a_i(x,u,0)$ is not difficult to manage, since it is a lower order term) as the following computation shows:

$$\begin{aligned} a_i\left(x,u,Du\right) - a_i\left(x,u,0\right) &= \int_0^1 \frac{d}{dt} a_i\left(x,u,t\,Du\right)\,dt \\ &= \int_0^1 \sum_{j=1}^{n} \frac{\partial a_i}{\partial \xi_j} \left(x,u,t\,Du\right) u_{x_j} dt \end{aligned}$$

and it suffices to denote

$$a_{ij}(x,u,Du) := \int_0^1 \sum_{j=1}^{n} \frac{\partial a_i}{\partial \xi_j} \left(x,u,t\,Du\right) dt$$

to conclude.

We assume that the following p-ellipticity condition holds

$$\sum_{i,j=1}^{n} \sum_{\alpha=1}^{m} a_{ij}(x,u,\xi) \zeta_i \zeta_j \geq \lambda \sum_{i=1}^{n} \zeta_i^2 |\xi_i|^{p-2} \qquad \forall \xi \in \mathbb{R}^{nm},\ \forall \zeta \in \mathbb{R}^n \tag{8}$$

together with the q-growth condition from above: there exists $q \geq p > 1$, and $\Lambda > 0$ such that

$$\left| \sum_j a_{ij}\left(x,u,\xi\right)\,\xi_j^\alpha \right| \leq \Lambda(|\xi|^{q-1} + 1) \qquad \forall \xi \in \mathbb{R}^{nm},\ \forall\, i,\ \alpha. \tag{9}$$

To be more explicit and precise, what we really use is not (8) itself, but two consequences implied by it:

- $(a_{ij}(x,u,\xi))$ is a positive semidefinite matrix; i.e.,

$$\sum_{i,j=1}^{n} a_{ij}(x,u,\xi) \zeta_i \zeta_j \geq 0 \qquad \forall \zeta \in \mathbb{R}^n, \tag{10}$$

  for a.e. x, every $u \in \mathbb{R}^m$ and every $\xi \in \mathbb{R}^{nm}$,

- p-coercivity: there exists $p>1$ and $\lambda>0$ such that

$$\sum_{\alpha=1}^{m}\sum_{i,j=1}^{n} a_{ij}(x,u,\xi)\xi_i^\alpha\xi_j^\alpha \geq \lambda\,|\xi|^p \tag{11}$$

Now, a vector-valued map $u\in W^{1,q}_{\rm loc}(\Omega;\mathbb{R}^m)$ is a weak solution of (7) if

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u,Du)\,u^\alpha_{x_j}\,\varphi^\alpha_{x_i}=0, \quad \alpha=1,\dots,m \tag{12}$$

for all $\varphi\in W^{1,q}(\Omega;\mathbb{R}^m)$ with $\operatorname{supp}\varphi\Subset\Omega$. We observe that the assumption $u\in W^{1,q}_{\rm loc}(\Omega;\mathbb{R}^m)$ gives a correct definition of weak solution since

$$\sum_{j=1}^{n} a_{ij}(\cdot,u,Du)u^\alpha_{x_j}\in L^{\frac{q}{q-1}}_{\rm loc}(\Omega), \quad \alpha=1,\dots,m.$$

Below, for the sake of simplicity, we denote $a(x,u,\xi)$ the vector in $\mathbb{R}^{nm}$

$$a(x,u,\xi):=\left(\sum_{j=1}^{n} a_{ij}(x,u,\xi)\xi_j^\alpha\right)_{i=1,\dots,n;\ \alpha=1,\dots,m}.$$

In [11] the following theorem is proved (Theorem 4.1).

Theorem 4.1. *Let* (8) *and* (9) *hold, with* $1<p<q$. *Assume also that either*

$$\langle a(x,u,\xi)-a(x,u,\eta),\xi-\eta\rangle\geq 0 \quad \forall\,\xi,\eta\in\mathbb{R}^{n\times m} \quad \text{and, if } p<n,\ q<p\frac{n-1}{n-p}, \tag{13}$$

or there exists a Carathéodory function $A:\Omega\times\mathbb{R}^m\times\mathbb{R}_+\to\mathbb{R}_+$, $t\to A(x,u,t)t$ *increasing, such that*

$$a_{ij}(x,u,\xi)=A(x,u,|\xi|)\delta_{ij} \quad \forall\, i,j=1,...,n,\ \forall\,\xi\in\mathbb{R}^{n\times m} \qquad \text{and} \qquad q<p^* \quad \text{if} \quad p<n. \tag{14}$$

Then any weak solution $u\in W^{1,q}_{\rm loc}(\Omega;\mathbb{R}^m)$ *to* (7) *is locally bounded. Moreover, for every* $B_R(x_0)\Subset\Omega$ *there exists a constant* $c>0$ *and* $\theta\geq 1$ *such that*

$$\sup_{B_{R/2}(x_0)}|u|\leq c\left\{\int_{B_R(x_0)}(|u|+1)^{p^*}\,dx\right\}^{\frac{\theta}{p^*}}.$$

Remark 4.1. *In* [11] *we obtain also another result: indeed we prove that if the p-coercivity* (11) *and the q-growth* (9) *are replaced by the corresponding* anisotropic *properties:*

- $\sum_{\alpha=1}^{m}\sum_{i,j=1}^{n} a_{ij}(x,u,\xi)\xi_i^\alpha\xi_j^\alpha \geq M_1 \sum_{i=1}^{n} |\xi_i|^{p_i}, \qquad p_1,...,p_n > 1$
- $\left|\sum_{j=1}^{n} a_{ij}\,(x,u,\xi)\;\xi_j^\alpha\right| \leq M \left\{\sum_{j=1}^{n} |\xi_j|^{p_j} + 1\right\}^{1-\frac{1}{p_i}} \quad \forall i,\,\alpha$

then the following result holds: if $\max\{p_1,p_2,\ldots,p_n\} < \overline{p}^*$*, where* $\overline{p}^*$ *is the Sobolev exponent of the harmonic mean* $\overline{p}$ *of the vector* $(p_1,\ldots,p_n)$*, then the solutions are locally bounded.*

We now give a new result of local boundedness of solutions to (7). We stress that now we get a result *without* the monotonicity assumption (13) or the dependence on the modulus in the gradient variable, see (14). Of course, a stronger assumption on the bound on q is needed to prove the regularity: p and q must be closer than before.

Theorem 4.2. *Let* (8) *and* (9) *hold.*

If $1 < p < q$ *and* $q < p\frac{n+1}{n}$ *(if* $p < n$*), then any weak solution* $u \in W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ *to* (7) *is locally bounded. Moreover, for every* $B_R(x_0) \Subset \Omega$ *there exist a constant* $c > 0$ *such that*

$$\sup_{B_{R/2}(x_0)} |u| \leq c\left\{\int_{B_R(x_0)} (|u|+1)^{p^*}\,dx\right\}^{\frac{1+\theta}{p^*}},$$

with $\theta = \frac{\tilde{q}}{p}\frac{\tilde{q}-p}{p^*-\tilde{q}}$*, where* $\tilde{q} := \frac{p}{p+1-q}$.

Remark 4.2. *Both in Theorem 4.1 and in Theorem 4.2 we prove the local boundedness of a weak solution in* $W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$*. One may wonder if such a solution exists due to the lack of coercivity in* $W^{1,q}$ *(indeed the coercivity/ellipticity assumption involves the exponent* p *and not* q*). In the recent paper written in colaboration with Leonetti* [10] *we prove an existence result of weak solutions in* $W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$*.*

Proof of Theorem 4.2. The scheme of the proof is similar to the proof of Theorem 4.1 in [11]. We split the proof into steps. From now on, the constants, often denoted with the letter c, may vary from line to line.

Step 1.

Fix a ball $B_{R_0}(x_0) \Subset \Omega$. Let us assume $0 < \rho < R \leq R_0$ and let $\eta \in C_c^\infty(\Omega)$ be a cut-off function, satisfying the following assumptions:

$$0 \leq \eta \leq 1, \qquad \eta \equiv 1 \text{ in } B_\rho, \qquad \operatorname{supp} \eta \Subset B_R, \qquad |D\eta| \leq \frac{2}{R-\rho}. \tag{15}$$

Let us approximate the identity function id : $\mathbb{R}_+ \to \mathbb{R}_+$ with an *increasing* sequence of C^1 functions $g_k : \mathbb{R}_+ \to \mathbb{R}_+$, such that

(16)

$$g_k(t) = \begin{cases} 0 & \text{for all } t \in [0, \frac{1}{k+1}] \\ k & \text{for all } t \geq k, \end{cases} \qquad 0 \leq g_k'(t) \leq 2 \quad \text{and} \quad g_k'(t)t \leq g_k(t) + \frac{2}{k} \quad \text{in } \mathbb{R}_+.$$

Notice that the last inequality can be assumed since the restriction of g_k to the interval $\left[\frac{1}{k+1}, k\right]$ can be seen as a smooth approximation of the linear funtion $G_k(t) = \frac{k(k+1)}{k(k+1)-1}\left(t - \frac{1}{k+1}\right)$, whose graph is the line of the plane connecting $(\frac{1}{k+1}, 0)$ and (k, k) and G_k satisfies $G_k'(t)t \leq G_k(t) + \frac{1}{k}$. Fixed $\nu > 0$, let $\Phi_{k,\nu} : \mathbb{R}_+ \to \mathbb{R}_+$ be the increasing function defined as

$$\Phi_{k,\nu}(t) := g_k(t^{p\nu}).$$

By (16) we obtain

$$(\Phi_{k,\nu})'(t)t \leq p\nu \left\{\Phi_{k,\nu}(t) + \frac{2}{k}\right\} \leq q\nu \left\{\Phi_{k,\nu}(t) + \frac{2}{k}\right\}. \tag{17}$$

Finally, we define

$$\varphi_{k,\nu}(x) := \Phi_{k,\nu}(|u(x)|)u(x)\eta^{\tilde{q}}(x) \qquad \text{for every } x \in B_{R_0}, \tag{18}$$

with $\tilde{q} := \frac{p}{p+1-q}$.

From now on, we write φ_k and Φ_k instead of $\varphi_{k,\nu}$ and $\Phi_{k,\nu}$. We claim that

$$\varphi_k \in W^{1,q}(B_{R_0}; \mathbb{R}^m), \quad \operatorname{supp} \varphi_k \Subset B_R.$$

Indeed, Φ_k is in $C^1(\mathbb{R}_+)$, bounded, because $\|\Phi_k\|_{L^\infty(\mathbb{R}_+)} \leq k$, and with bounded derivative. Precisely, if $a_k = (k+1)^{-\frac{1}{p\nu}}$ and $b_k = k^{\frac{1}{p\nu}}$, then

$$(\Phi_k)'(s) = \begin{cases} 0 & \text{if } s \in \mathbb{R}_+ \setminus [a_k, b_k] \\ p\nu g_k'(s^{p\nu})s^{p\nu-1} & \text{if } s \in [a_k, b_k] \end{cases}$$

and

$$\|(\Phi_k)'\|_{L^\infty(\mathbb{R}_+)} \le 2p\nu \max\left\{a_k^{p\nu-1}, b_k^{p\nu-1}\right\} < \infty.$$

As a consequence, taking also into account that $u \in W^{1,q}(B_{R_0})$ we have that $\Phi_k(|u|)u$ is in $W^{1,q}(B_{R_0})$ and the claim follows.

Step 2.

Using φ_k as a test function in (12) we get

$$\begin{aligned} I_1 + I_2 := &\int_{B_R} \sum_{i,j=1}^n \sum_{\alpha=1}^m a_{ij}(x,u,Du)\, u^\alpha_{x_j} u^\alpha_{x_i} \Phi_k(|u|)\, \eta^{\tilde q}\, dx \\ &+ \int_{B_R} \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^m a_{ij}(x,u,Du) u^\alpha_{x_j} u^\alpha \frac{u^\beta}{|u|}\, u^\beta_{x_i}\, (\Phi_k)'(|u|)\, \eta^{\tilde q}\, dx \\ =&-\tilde q \int_{B_R} \sum_{i,j=1}^n \sum_{\alpha=1}^m a_{ij}(x,u,Du)\, u^\alpha_{x_j} u^\alpha \eta_{x_i} \Phi_k(|u|)\, \eta^{\tilde q-1}\, dx =: I_3. \end{aligned} \tag{19}$$

Now, we separately consider and estimate I_i, $i = 1,2,3$.

Estimate of I_1: By (11)

$$I_1 \ge \lambda \int_{B_R} |Du|^p \Phi_k(|u|)\, \eta^{\tilde q}\, dx. \tag{20}$$

Estimate of I_2: By (10), with $\zeta_i = \sum_{\alpha=1}^m u^\alpha\, u^\alpha_{x_i}$, we have that for a.e. $x \in \{|u| > 0\}$

$$\sum_{i,j=1}^n \sum_{\alpha,\beta=1}^m a_{ij}(x,u,Du) u^\alpha_{x_j} u^\alpha u^\beta\, u^\beta_{x_i} = \sum_{i,j=1}^n a_{ij}(x,u,Du) \left\{\sum_{\alpha=1}^m u^\alpha\, u^\alpha_{x_j}\right\} \left\{\sum_{\alpha=1}^m u^\alpha\, u^\alpha_{x_i}\right\} \ge 0.$$

Thus, since $\Phi_k' \ge 0$ we have

$$I_2 := \int_{B_R} \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^m a_{ij}(x,u,Du) u^\alpha_{x_j} u^\alpha \frac{u^\beta}{|u|}\, u^\beta_{x_i} \Phi_k'(|u|)\, \eta^{\tilde q}\, dx \ge 0.$$

Estimate of I_3: By (9) we have

$$\begin{aligned} I_3 &:= -\tilde q \int_{B_R} \sum_{i,j=1}^n \sum_{\alpha=1}^m a_{ij}(x,u,Du)\, u^\alpha_{x_j} u^\alpha \eta_{x_i} \Phi_k(|u|)\, \eta^{\tilde q-1}\, dx \\ &\le c(\tilde q,\Lambda) \int_{B_R} \eta^{\tilde q-1} \left(|Du|^{q-1} + 1\right) |u| \Phi_k(|u|) |D\eta|\, dx \\ &= c(\tilde q,\Lambda) \int_{B_R} \eta^{\tilde q-1} |Du|^{q-1} |u| \Phi_k(|u|) |D\eta|\, dx + c(\tilde q,\Lambda) \int_{B_R} \eta^{\tilde q-1} |u| \Phi_k(|u|) |D\eta|\, dx. \end{aligned} \tag{21}$$

Notice that $\tilde{q}-1=\tilde{q}\frac{q-1}{p}>0$. Since $q<p+1$, by the Young inequality with exponents $\frac{p}{q-1}$ and $\frac{p}{p+1-q}$ we have that for all $\epsilon>0$ there exists $c_\epsilon>0$ such that

$$\eta^{\tilde{q}-1}|Du|^{q-1}|u||D\eta| = \left(\eta^{\tilde{q}\frac{q-1}{p}}|Du|^{q-1}\right)(|u||D\eta|) \le \epsilon\eta^{\tilde{q}}|Du|^p + c_\epsilon\,(|u||D\eta|)^{\frac{p}{p+1-q}}\,. \tag{22}$$

Moreover, taking into account that $0\le\eta\le 1$ and $p\ge p+1-q$,

$$\eta^{\tilde{q}-1}|u||D\eta| \le |u||D\eta| \le 1+|u||D\eta| \le (1+|u||D\eta|)^{\frac{p}{p+1-q}}\,. \tag{23}$$

Collecting (21), (22) and (23) we have

$$I_3 \le c(\tilde{q},\Lambda)\epsilon\int_{B_R}|Du|^p\eta^{\tilde{q}}\Phi_k(|u|)\,dx + \tilde{c}(\tilde{q},\Lambda,\epsilon)\int_{B_R}(1+|u||D\eta|)^{\frac{p}{p-q+1}}\,\Phi_k(|u|)\,dx.$$

Choosing $\epsilon=\frac{\lambda}{2c(\tilde{q},\Lambda)}$ and taking into account that

$$(1+|u||D\eta|)^{\frac{p}{p-q+1}} \le \frac{c}{(R-\rho)^{\frac{p}{p-q+1}}}\left(1+|u|^{\frac{p}{p-q+1}}\right) \le \frac{2c}{(R-\rho)^{\frac{p}{p-q+1}}}[\max\{|u|,1\}]^{\frac{p}{p-q+1}}$$

with c possibly depending on $\operatorname{diam}(\Omega)$, we have that the estimates of I_1, I_2 and I_3 proved above imply that there exists $C>0$ such that

$$\int_{B_R}|Du|^p\Phi_k(|u|)\eta^{\tilde{q}}\,dx \le \frac{C}{(R-\rho)^{\frac{p}{p-q+1}}}\int_{B_R}[\max\{|u|,1\}]^{\frac{p}{p-q+1}}\,\Phi_k(|u|)\,dx.$$

Since $\Phi_k(|u|)\to|u|^{p\nu}$ as k goes to $+\infty$, passing to the limit we obtain

$$\int_{B_R}|Du|^p|u|^{p\nu}\,\eta^{\tilde{q}}\,dx \le \frac{C}{(R-\rho)^{\frac{p}{p-q+1}}}\int_{B_R}[\max\{|u|,1\}]^{\frac{p}{p-q+1}+p\nu}\,dx. \tag{24}$$

By $q\ge p$ we have $\tilde{q}\ge p$ therefore $\eta^{\tilde{q}-p}\le 1$. Hence, using the above inequality,

$$\begin{aligned}
&\int_{B_R}\left|D\left[\eta^{\frac{\tilde{q}}{p}}\left(|u|^{\nu+1}+1\right)\right]\right|^p\,dx\\
&\le\; c(\tilde{q},p)\int_{B_R}\eta^{\tilde{q}-p}|D\eta|^p\left(|u|^{\nu+1}+1\right)^p\,dx + c(\nu+1)^p\int_{B_R}\eta^{\tilde{q}}|u|^{p\nu}|Du|^p\,dx\\
&\le \frac{2^pc(\tilde{q},p)}{(R-\rho)^p}\int_{B_R}[\max\{|u|,1\}]^{p+p\nu}\,dx + \frac{c(\nu+1)^p}{(R-\rho)^{\frac{p}{p-q+1}}}\int_{B_R}[\max\{|u|,1\}]^{\frac{p}{p-q+1}+p\nu}\,dx,
\end{aligned}$$

that implies

$$\int_{B_R}\left|D\left[\eta^{\frac{\tilde{q}}{p}}\left(|u|^{\nu+1}+1\right)\right]\right|^p\,dx \le \frac{c(\nu+1)^p}{(R-\rho)^{\frac{p}{p-q+1}}}\int_{B_R}[\max\{|u|,1\}]^{\frac{p}{p-q+1}+p\nu}\,dx. \tag{25}$$

Notice that if $p < n$, the assumption $q < p\frac{n+1}{n}$ is equivalent to $\tilde{q} < p^*$, (this last inequality can be always assumed if $p \geq n$ choosing p^* great enough), thus the Sobolev embedding theorem implies that $u \in L^{\tilde{q}}_{\text{loc}}(\Omega)$. By Hölder inequality,

$$\int_{B_R} [\max\{|u|, 1\}]^{\tilde{q}+p\nu}\, dx = \int_{B_R} [\max\{|u|, 1\}]^{\tilde{q}-p} [\max\{|u|, 1\}]^{p(1+\nu)}\, dx$$
$$\leq \left\{ \int_{B_{R_0}} [\max\{|u|, 1\}]^{\tilde{q}} \right\}^{1-\frac{p}{\tilde{q}}} \left\{ \int_{B_R} [\max\{|u|, 1\}]^{\tilde{q}(\nu+1)} \right\}^{\frac{p}{\tilde{q}}}$$
$$\leq c \left\{ \int_{B_{R_0}} [\max\{|u|, 1\}]^{p^*} \right\}^{\frac{\tilde{q}-p}{p^*}} \left\{ \int_{B_R} [\max\{|u|, 1\}]^{\tilde{q}(\nu+1)} \right\}^{\frac{p}{\tilde{q}}}.$$

Therefore, (25) becomes

$$\int_{B_R} \left| D\left[\eta^{\frac{\tilde{q}}{p}} \left(|u|^{\nu+1} + 1 \right) \right] \right|^p dx$$
$$\leq \frac{c(\nu+1)^p}{(R-\rho)^{\tilde{q}}} \left\{ \int_{B_{R_0}} [\max\{|u|, 1\}]^{p^*} \right\}^{\frac{\tilde{q}-p}{p^*}} \left\{ \int_{B_R} [\max\{|u|, 1\}]^{\tilde{q}(\nu+1)} \right\}^{\frac{p}{\tilde{q}}}. \tag{26}$$

By the classical Sobolev imbedding theorem,

$$\left(\int_{B_\rho} (\max\{|u|, 1\})^{p^*(\nu+1)}\, dx \right)^{\frac{p}{p^*}} \leq \left(\int_{B_R} \left| \eta^{\frac{\tilde{q}}{p}} \left(|u|^{\nu+1} + 1 \right) \right|^{p^*} dx \right)^{\frac{p}{p^*}}$$
$$\leq c \int_{B_R} \left| D\left[\eta^{\frac{\tilde{q}}{p}} \left(|u|^{\nu+1} + 1 \right) \right] \right|^p dx,$$

and (26) implies

$$\left(\int_{B_\rho} [\max\{|u|, 1\}]^{p^*(\nu+1)}\, dx \right)^{\frac{1}{p^*}}$$
$$\leq \frac{c(\nu+1)}{(R-\rho)^{\frac{\tilde{q}}{p}}} \left\{ \int_{B_{R_0}} [\max\{|u|, 1\}]^{p^*} \right\}^{\frac{\tilde{q}-p}{pp^*}} \left\{ \int_{B_R} [\max\{|u|, 1\}]^{\tilde{q}(\nu+1)} \right\}^{\frac{1}{\tilde{q}}}. \tag{27}$$

Step 3.

The inequality (27) allows to use the Moser's iteration procedure, so obtaining that $\max\{|u|, 1\} \in L^\infty_{\text{loc}}$ and, therefore, $u \in L^\infty_{\text{loc}}$. Since this is a standard argument, we give only a sketch of the proof. Denote $v(x) := \max\{|u(x)|, 1\}$. For all $h \in \mathbb{N}$ define

$\nu_h = -1 + \left(\frac{p^*}{\tilde{q}}\right)^h$, $\rho_h = R_0/2 + R_0/2^{h+1}$ and $R_h = R_0/2 + R_0/2^h$. By (27), replacing ν, R and ρ with ν_h, R_h and ρ_h, respectively, we have

$$\|v\|^{\nu_h+1}_{L^{\tilde{q}(\nu_{h+1}+1)}(B_{R_{h+1}})} \leq c \cdot \left(\frac{2p^*}{q}\right)^h \|v\|^{\frac{\tilde{q}-p}{p}}_{L^{p^*}(B_{R_0})} \|v\|^{\nu_h+1}_{L^{\tilde{q}(\nu_h+1)}(B_{R_h})} \tag{28}$$

or, equivalently,

$$\|v\|_{L^{\tilde{q}(\nu_{h+1}+1)}(B_{R_{h+1}})} \leq c^{\left(\frac{\tilde{q}}{p^*}\right)^h} \cdot \left(\frac{2p^*}{\tilde{q}}\right)^{h\left(\frac{\tilde{q}}{p^*}\right)^h} \|v\|^{\frac{\tilde{q}-p}{p}\left(\frac{\tilde{q}}{p^*}\right)^h}_{L^{p^*}(B_{R_0})} \|v\|_{L^{\tilde{q}(\nu_h+1)}(B_{R_h})} \tag{29}$$

So, we have that $v \in L^{\tilde{q}(\nu_h+1)}(B_{R_h})$ implies $v \in L^{\tilde{q}(\nu_{h+1}+1)}(B_{R_{h+1}})$.

Taking into account that $\tilde{q}(\nu_1+1) = \overline{p}^*$ and that $\frac{\tilde{q}-p}{p}\sum_{h=1}^{\infty}\left(\frac{\tilde{q}}{p^*}\right)^h = \frac{\tilde{q}}{p}\frac{\tilde{q}-p}{p^*-\tilde{q}}$, an iterated use of the above inequality implies the existence of a positive constant c such that

$$\|v\|_{L^\infty(B_{R_0/2}(x_0))} \leq c\, \|v\|^{\frac{\tilde{q}}{p}\frac{\tilde{q}-p}{p^*-\tilde{q}}+1}_{L^{p^*}(B_{R_0}(x_0))}.$$

The claim follows. □

We remark that the previous proof holds also for the more general system (6) under suitable growth assumptions on b_i^α and f^α.

We like to conclude this paper with a citation by J. Serrin in [54] about the issue of regularity:

".....what in 1900 was a shy branch has blossomed in the twentieth century and developed in ways that Hilbert could never have imagined and now covers such a vast area of researches that just a few years ago would have seemed amazing."

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