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A cluster expansion for interacting spin-flip processes

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Abstract. We consider a system of spin flip processes, one-for each point of $\mathbb{Z}$, interacting through an Ising type interaction. We construct a cluster expansion and prove that it is convergent when the intensity h of the spin-flip processes is sufficiently high. The system is relevant in the study of the ground state of a quantum Ising process with transverse magnetic field.

1 Introduction

The ground state of the Quantum Ising model with a transverse magnetic field can be represented as a classical Ising model with one added continuous dimension ([DLP]). In its turn this classical Ising model can be represented via a suitable FK random cluster model ([F] [CKP]). This last representation has been used for example in [GOS] to study the entaglement of the ground state.

Here we consider an Ising model in 1+1 dimensions, where the second dimension is continuous. We introduce a representation of this model as a Gibbs random field in $\mathbb{Z}^2$, in which the spins take values in a space of trajectories. We construct a cluster expansion based on this representation and prove that it satisfies the conditions for convergence (see [KP] when the parameter h corresponding to the transverse magnetic field is sufficiently large.

2 Definition of the system

We consider a Gibbs random field on $\mathbb{Z}$ in which the spins take values in a space of processes with values in $\{-1, 1\}$.

Let us first define the finite volume distribution. We consider on $\mathbb{R}$ a Poisson point process with intensity h. Given an interval $\left[-\frac{\beta}{2}, \frac{\beta}{2}\right]$, the Poisson point process induces a measure on piecewise constant functions on $\left[-\frac{\beta}{2}, \frac{\beta}{2}\right]$ with values in $\{-1, 1\}$. The spin configuration $\sigma(t)$ at $t = \pm\frac{\beta}{2}$ is obtained by taking the values equal to -1 or 1 with probability $\frac{1}{2}$ and switching the value of $\sigma(t)$ at each point of the point process configuration. It is not important which extreme of $\left[-\frac{\beta}{2}, \frac{\beta}{2}\right]$ one chooses and which value one assigns to the spin at switwing times..

Let us denote by X_I the set of piecewise constant functions with values in $\{-1, 1\}$ defined on the interval I and by μ_I the measure just described on X_I.

Let Λ a finite subinterval of $\mathbb{Z}$. We define a boundary condition for $\Lambda \times \left[-\frac{\beta}{2}, \frac{\beta}{2}\right]$ as two configurations η_1, η_2 in $X_{\left[-\frac{\beta}{2}, \frac{\beta}{2}\right]}$ for the horizontal boundary and two configuration $\xi_1, \xi_2 \in \{-1, 1\}$ for the lower and the upper boundary.

The conditional Gibbs measure is a measure on the space $X^{\Lambda}_{\left[-\frac{\beta}{2}, \frac{\beta}{2}\right]}$ whose density w.r.t. $\bigotimes_{i\in\Lambda} \mu^{(i)}_{\left[-\frac{\beta}{2}, \frac{\beta}{2}\right]}$ is given by

$$
\begin{aligned}
& Z^{-1}_{\eta_1,\eta_2,\xi_1,\xi_2} \times \\
& \exp(-J \sum_{x,y\in\Lambda:|x-y|=1} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt \sigma_x(t)\, \sigma_y(t) \\
& - J \sum_{x\in\Lambda} \sum_{y\in\partial\Lambda:|x-y|=1} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt \sigma_x(t)\, \eta_y(t)) \\
& \times \prod_{x\in\Lambda} \delta_{\sigma_x\left(-\frac{\beta}{2}\right),\xi_1(x)} \delta_{\sigma_x\left(\frac{\beta}{2}\right),\xi_2(x)}
\end{aligned}
$$

where $Z_{\eta_1,\eta_2,\xi_1,\xi_2}$ is the normalizing constant.

We first define a conditional measure on the trajectories $\sigma(t)$ with values in $\{-1, 1\}$ for t belonging to some interval $[a, b]$ and boundary conditions $\sigma(a) = \varepsilon_1$ and $\sigma(b) = \varepsilon_2$ with $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$.

Let μ be a Poisson point process in the interval $[a, b]$ with intensity h. If $\varepsilon_1 = \varepsilon_2, \mu^{\varepsilon_1,\varepsilon_2}$ is the measure μ conditioned to the presence of an even number of jumps, while if $\varepsilon_1 \neq \varepsilon_2$ then $\mu^{\varepsilon_1,\varepsilon_2}$ is the measure μ conditioned to the presence of an odd number of jumps. Hence the trajectory corresponding to this realization is a piecewise constant function which is equal to ε_1 in a and to ε_2 in b and has jumps in each point of the point process configuration.

It doesn't matter the value at the jump points since it has no consequence.

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3 The cluster expansion

We perform a cluster expansion on the model and verify that, when h is sufficiently large, we can ensure that, for a suitable choice of the parameter, the condition of Kotezky and Preiss are satisfied and the cluster expansion is therefore convergent.

Set $\delta = \frac{\beta}{N}$ for some N. We subdivide each "vertical segment" into closed subintervals of lenght δ. We associate a lattice to the subdivision and to each site of the lattice a *spin* in the space of the piecewise function on the interval $[0, \delta]$ with values in $\{-1, 1\}$. The *spins* on neighbouring sites in the vertical directions must satisfy the compatibility condition that the final value of the trajectory of one *spin* must agree with the initial value of the trajectory of the other.

Therefore, we consider a spin model on $\mathbb{Z} \times \delta\mathbb{Z}$. If $x = (x_1, x_2)$ and $y = (x_1, x_2)$ with $x_2 \in \delta\mathbb{Z}, x_1, y_1 \in \mathbb{Z}$ with $|x_1 - y_1| = 1$, then

$$W\left(\sigma_x, \sigma_y\right) := J \int_0^\delta \sigma_x(t)\, \sigma_y(t)\, dt. \tag{1}$$

If $x = (x_1, x_2)$ and $y = (x_1, y_2)$ with $y_2 = x_2 + \delta$, then

$$W\left(\sigma_x, \sigma_y\right) := \delta_{\sigma_x(t), \sigma_y(t)} . \tag{2}$$

The finite volume distribution on a volume Λ with boundary conditions η on $\partial\Lambda$ has density w.r.t. the reference measure $\bigotimes_{x\in\Lambda} \mu(d\sigma_x)$ given by

$$Z_\eta^{-1} \exp\left(- \sum_{x,y\in\Lambda: |x-y|=1} W\left(\sigma_x, \sigma_y\right) - \sum_{x\in\Lambda} \sum_{y\in\partial\Lambda} W\left(\sigma_x, \eta_y\right)\right) \tag{3}$$

with Z_η a normalizing constant.

We perform a first expansion: if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $|x_2 - y_2| = 1$, we write

$$e^{-W(\sigma_x, \sigma_y)} = 1 + \left[e^{-W(\sigma_x, \sigma_y)} - 1\right] \tag{4}$$

and perform the expansion

$$Z = \sum_{\ell\subset\Lambda^n} \int \left(\prod_{e\in\ell\ :\ e=\{x_2, y_2\}} \left[e^{-W(\sigma_x, \sigma_y)} - 1\right]\right) \bigotimes_{x\in\Lambda} \mu(d\sigma_x) . \tag{5}$$

Given $\ell \subset \Lambda^n$ on each vertical segment we look at those sites A that belong to some $e \in \ell$. If two sites in A are not consecutive we integrate over over the intermediate sites. This integral can be explicitely performed: if the sites are $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $y_1 > x_1$, the integral is

$$\begin{cases} \frac{1+e^{-2h(y_1-x_1)}}{2} & \text{if } \sigma_x(\delta) = \sigma_y(0) \\ \frac{1-e^{-2h(y_1-x_1)}}{2} & \text{if } \sigma_x(\delta) \neq \sigma_y(0) \end{cases} . \tag{6}$$

We take $\delta = \frac{1}{\sqrt{h}}$. Therefore, the activity of a polymer R

can be estimated by

$$\left(e^{\frac{1}{\sqrt{h}}} - 1\right)^{\#\{horizontal\ bonds\}} \frac{e^{-2\sqrt{h}\#\{vertical\ bonds\}}}{2} . \tag{7}$$

If we denote by

$$V(R) := \#\{horizontal\ bonds\ of\ R\} + \#\{vertical\ bonds\ of\ R\} , \tag{8}$$

We have proved the following theorem.

Theorem. The activity of a polymer $\zeta(R)$ can be bounded by

$$\zeta(R) \leq c(h)^{V(R)} \tag{9}$$

where

$$c(h) := \max\left\{e^{\frac{1}{\sqrt{h}}} - 1, e^{-2\sqrt{h}}\right\}. \tag{10}$$

Remark. The bounds in (9) and (10) imply that cluster expansion (see [KP]) is convergent. In particular they imply that two point correlation functions decay exponentially with the distance when h is large with decay constant going to ∞ as $h \to \infty$.

4 Conclusions

The result of this paper can be extended without problems to $\mathbb{Z}^d$ and should be appliable to study the ground state of the quantum Ising model with a transverse magnetic flield and in particular the entanglement.

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