Inversion Symmetry of the Euclidean Group: Theory and Application to Robot Kinematics

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

*Published Version:*
Inversion Symmetry of the Euclidean Group: Theory and Application to Robot Kinematics / Wu, Yuanqing; Lowe, Harald; Carricato, Marco; Li, Zexiang. - In: IEEE TRANSACTIONS ON ROBOTICS. - ISSN 1552-3098. - STAMPA. - 32:2(2016), pp. 312-326. [10.1109/TRO.2016.2522442]

This version is available at: https://hdl.handle.net/11585/536714 since: 2020-01-30

*Published:*

DOI: http://doi.org/10.1109/TRO.2016.2522442

# Inversion Symmetry of the Euclidean Group: Theory and Application in Robot Kinematics

Yuanqing Wu, Harald Löwe, Marco Carricato, Zexiang Li

*Abstract*—**Just as the three-dimensional (3D) Euclidean space can be inverted through any of its points, the special Euclidean group $\mathbf{SE(3)}$ admits an inversion symmetry through any of its elements, and is known to be a symmetric space. In this paper, we show that the symmetric submanifolds of $\mathbf{SE(3)}$ can be systematically exploited to study the kinematics of a variety of kinesiological and mechanical systems, and therefore has many potential applications in robot kinematics. Unlike Lie subgroups of $\mathbf{SE(3)}$, symmetric submanifolds inherit distinct geometric properties from inversion symmetry. They can be generated by kinematic chains with symmetric joint twists. The main contribution of this paper is: (i) to give a complete classification of symmetric submanifolds of $\mathbf{SE(3)}$; (ii) to investigate their geometric properties for robotics applications; and (iii) to develop a generic method for synthesizing their kinematic chains.**

*Index Terms*—**Euclidean group, geodesic symmetry, symmetric space, totally geodesic submanifold, Lie triple system (LTS), Constant-Velocity (CV) coupling, kinesiology, parallel manipulator, type synthesis.**

## I. Introduction

The special Euclidean group $\mathrm{SE}(3)$ refers to the 6D *Lie group* of proper rigid displacements of 3D Euclidean space. It is Hervé [2,3] and Brockett [4] who initiated application of $\mathrm{SE}(3)$ and its Lie subgroups (e.g. the special orthogonal group $\mathrm{SO}(3)$; see [5]) in robotics (kinematics and dynamics [6]–[8], estimation and control [9]–[11], etc.).

Recent advances in type synthesis of parallel manipulators [12]–[18] can be attributed to the successful exploitation of the Lie algebra $\mathfrak{se}(3)$ of $\mathrm{SE}(3)$. Central to the synthesis problem is the *exponential map*, denoted $\exp$, which maps $\mathfrak{se}(3)$, locally diffeomorphically, into $\mathrm{SE}(3)$. In this paper, we shall adopt the homogeneous matrix representation for $\mathrm{SE}(3)$ [6], so that the exponential map is identified with the matrix exponential $e^{(\cdot)}$. Consider a serial manipulator comprising revolute ($\mathcal{R}$), helical ($\mathcal{H}_p$; $p$ denotes pitch) or prismatic ($\mathcal{P}$) joints, with linearly independent joint twists $\{\hat{\boldsymbol{\xi}}_1,\ldots,\hat{\boldsymbol{\xi}}_k\}$, $\hat{\boldsymbol{\xi}}_i\in\mathfrak{se}(3)$. Its direct kinematics map is given by the *product of exponentials* (POE) formula [6]. The set of end-effector motions generated by the serial manipulator is given by:

$$\prod_{i=1}^{k}\exp\{\hat{\boldsymbol{\xi}}_i\}_{\mathrm{sp}}\triangleq\left\{e^{\theta_1\hat{\boldsymbol{\xi}}_1}\cdots e^{\theta_k\hat{\boldsymbol{\xi}}_k}\,\middle|\,\theta_i\in\mathbb{R},i=1,\ldots,k\right\}\tag{1}$$

which coincides with a $k$D submanifold of $\mathrm{SE}(3)$ in an open neighborhood of the identity $\mathbf{I}\in\mathrm{SE}(3)$. We shall refer to (1) as a *POE-submanifold*, and we say it is the *motion type* [18] of the serial manipulator. In particular, when the linear span of $\hat{\boldsymbol{\xi}}_i$'s, denoted by $\{\hat{\boldsymbol{\xi}}_1,\ldots,\hat{\boldsymbol{\xi}}_k\}_{\mathrm{sp}}$, is a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{se}(3)$, $\prod_{i=1}^{k}\exp\{\hat{\boldsymbol{\xi}}_i\}_{\mathrm{sp}}$ is locally an open submanifold of the corresponding (connected) Lie subgroup G ([19, pp. 299, Lemma 9.2.6]). We simply say the motion type is G.

The element-wise product of two Lie subgroups [20,21] can also be locally identified with a POE-submanifold [18], but is in general not a Lie subgroup. Carricato *et al.* showed that the tangent spaces of such submanifolds are all mutually congruent, thus defining what is called a *persistent screw system* of the end-effector [22,23]. When a persistent screw system exists, the corresponding submanifold may be generated by the envelop of a tangent space smoothly moving in $\mathrm{SE}(3)$ like a rigid body [22]–[26]. In most previous studies on type synthesis of parallel manipulators, the motion type of a parallel manipulator can be identified with a POE-submanifold (see for example, category I/II submanifolds [18], virtual chain [27], displacement manifold [28]).

Since $\exp:\mathfrak{se}(3)\to\mathrm{SE}(3)$ is a local diffeomorphism of an open neighborhood of the origin $\hat{o}\in\mathfrak{se}(3)$ onto an open neighborhood of the identity $\mathbf{I}\in\mathrm{SE}(3)$, the exponential image of the $k$D vector subspace $\{\hat{\boldsymbol{\xi}}_1,\ldots,\hat{\boldsymbol{\xi}}_k\}_{\mathrm{sp}}\subset\mathfrak{se}(3)$:

$$\exp\{\hat{\boldsymbol{\xi}}_1,\ldots,\hat{\boldsymbol{\xi}}_k\}_{\mathrm{sp}}\triangleq\left\{e^{\theta_1\hat{\boldsymbol{\xi}}_1+\cdots+\theta_k\hat{\boldsymbol{\xi}}_k}\,\middle|\,\theta_i\in\mathbb{R},i=1,\ldots,k\right\}$$

is locally a $k$D submanifold of $\mathrm{SE}(3)$, which we refer to as an *Exp-submanifold* (Exp for "exponential") [1]. It is clear from the Baker-Campbell-Hausdorff formula ([19, pp. 57, Prop. 3.4.4]) that in general Exp-submanifolds are not POE-submanifolds (except for example, when $\{\hat{\boldsymbol{\xi}}_1,\ldots,\hat{\boldsymbol{\xi}}_k\}_{\mathrm{sp}}$ is a Lie subalgebra $\mathfrak{g}\subset\mathfrak{se}(3)$, $\exp\{\hat{\boldsymbol{\xi}}_1,\ldots,\hat{\boldsymbol{\xi}}_k\}_{\mathrm{sp}}$ is a subset and contains an open neighborhood of the corresponding connected Lie subgroup G ([19, pp. 299, Lemma 9.2.6])). Therefore, in general, Exp-submanifolds can only be generated by closed-loop manipulators [1,29].

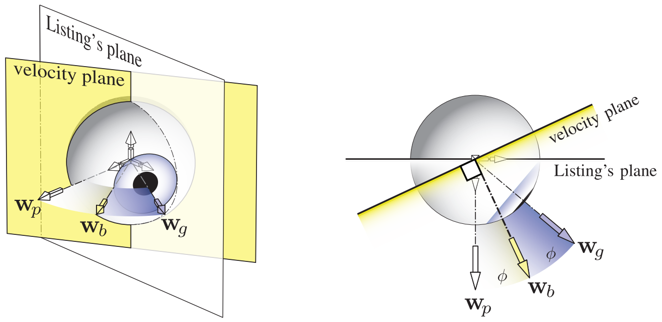
This work was supported by RGC Grant No.616509, No.615610 and No.615212, Shenzhen Municipal Science and Technology Planning Program for Basic Research No. JCYJ20120618150931822, and by National Natural Science Foundation Grant No.51375413.

This paper was partly presented at the 2013 IEEE International Conference on Robotics and Automation (ICRA2013) [1]. The current manuscript was originally submitted on September 21, 2012, and then, in a revised form, on September 24, 2014.

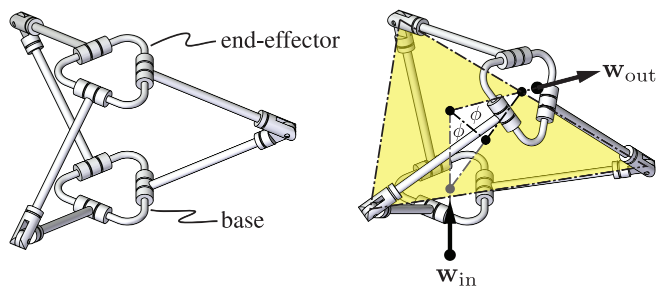
Yuanqing Wu (corresponding author) was and Zexiang Li is with Electronic and Computer Engineering, Hong Kong University of Science and Technology (including Shenzhen Research Institute), Hong Kong, P.R. China (e-mail: troy.woo@gmail.com yuanqing.wu@unibo.it, eezxli@ust.hk). Dr.Wu is currently at the Department of Industrial Engineering (DIN) of the University of Bologna, financially supported by the PRIN grant no. 20124SMZ88 of the Italian Ministry of Education, Universities and Research.

Harald Löwe is with Institut Computational Mathematics, Technical University of Braunschweig, Deutschland (e-mail: h.loewe@tu-bs.de).

Marco Carricato is with Department of Industrial Engineering (DIN), University of Bologna, Italy (e-mail: marco.carricato@unibo.it).




(a)




(b)

Fig. 1. (a) Listing's law of eye saccade: $\mathbf{w}_p, \mathbf{w}_g$ and $\mathbf{w}_b$ denote the primary or initial direction (perpendicular to the Listing's plane), the gaze direction and their angle bisector (perpendicular to the velocity plane). (b) reflected tripod [30].

Exp-submanifolds have been used in robotics and biomechanics without being explicitly recognized or systematically studied. Bonev *et al.* [31] analyzed the rotational motion of several constant-velocity (CV) couplings using a modified Euler angle parametrization (tilt and torsion angles, [32]), which is equivalent to the following parametrization for $\mathrm{SO}(3)$ [29]:

$$(\theta_1, \theta_2, \sigma) \mapsto e^{\theta_1 \hat{\mathbf{x}} + \theta_2 \hat{\mathbf{y}}} e^{\sigma \hat{\mathbf{z}}} \in \mathrm{SO}(3) \tag{2}$$

where $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ denotes the canonical basis for $\mathbb{R}^3$ and $\hat{\mathbf{w}}$ defines a $3 \times 3$ skew-symmetric matrix such that $\hat{\mathbf{w}}\mathbf{v} = \mathbf{w} \times \mathbf{v},\ \forall \mathbf{w}, \mathbf{v} \in \mathbb{R}^3$. Bonev observed that the torsion angle $\sigma$ for a CV coupling is always zero, leading to a motion type $\exp\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}_{\mathrm{sp}}$, a 2D Exp-submanifold [1,29]. In kinesiology and biomechanics, the same Exp-submanifold (under quaternion representation) is identified to be the motion type of human eye saccade [33,34], and is known to obey Listing's half-angle law: as the gaze direction rotates away from the initial primary direction (normal of the Listing's plane), the instantaneous velocity plane rotates away from the Listing's plane along the same rotation axis but half in magnitude (see Fig. 1(a)).

Another example of Exp-submanifolds arises in three degree-of-freedom (DoF) parallel-architecture CV couplings for intersecting shafts [35,36]. The instantaneous velocity of such a coupling always lies in the bisecting plane[1], demanding the joint twists in each connecting chain to be mirror symmetric about the bisecting plane in all configurations. Typical

[1]The yellow plane shown in Fig. 1(b): (i) it is perpendicular to the plane of input and output velocity $\mathbf{w}_{\text{in}}$ and $\mathbf{w}_{\text{out}}$; and (ii) it bisects the complement of the working angle formed by $\mathbf{w}_{\text{in}}$ and $\mathbf{w}_{\text{out}}$.

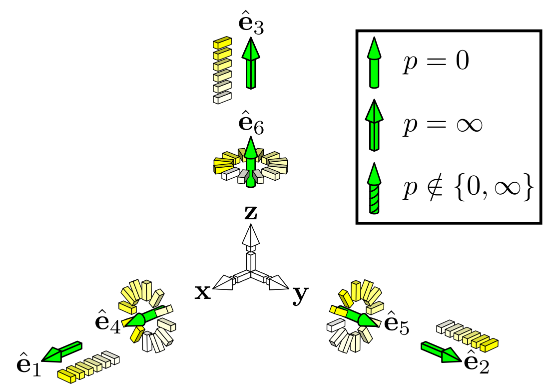



Fig. 2. Graphical representation of the canonical basis of $\mathfrak{se}(3)$ and notation for twists with different pitch value $p$.

CV connecting chains include $\mathcal{RSR}$ ($\mathcal{S}$ for spherical joint) chain and $\mathcal{RER}$ ($\mathcal{E}$ for planar gliding joint) chain [35]. An in-parallel assembly of three $\mathcal{RSR}$ chains that are mirror symmetric about a common bisecting plane results in the "reflected tripod" [30,31] (see Fig. 1(b)), which found applications in robotic wrists [37] and hyperredundant robots [38]. We showed that its motion type is the 3D Exp-submanifold $\exp\{\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}_{\mathrm{sp}}$ [29], where $\{\hat{\mathbf{e}}_i\}_{i=1}^6$ denotes the canonical basis of $\mathfrak{se}(3)$ (also see Fig. 2):

$$\hat{\mathbf{e}}_1 \triangleq \begin{bmatrix} \hat{\mathbf{0}} & \mathbf{x} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix}, \ \hat{\mathbf{e}}_2 \triangleq \begin{bmatrix} \hat{\mathbf{0}} & \mathbf{y} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix}, \ \hat{\mathbf{e}}_3 \triangleq \begin{bmatrix} \hat{\mathbf{0}} & \mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix},$$
$$\hat{\mathbf{e}}_4 \triangleq \begin{bmatrix} \hat{\mathbf{x}} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix}, \ \hat{\mathbf{e}}_5 \triangleq \begin{bmatrix} \hat{\mathbf{y}} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix}, \ \hat{\mathbf{e}}_6 \triangleq \begin{bmatrix} \hat{\mathbf{z}} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix}$$

If each 5-DoF $\mathcal{RSR}$ chain of the reflected tripod is reduced to a 4-DoF mirror symmetric $\mathcal{UU}$ ($\mathcal{U}$ for Cardan or universal joint) chain with the $\mathcal{U}$ joints of all chains on each side of the mirror sharing the same center of rotation [36], we have the UNITRU coupling [39] (see Fig. 9(a)), which has the motion type of a 2D surface in $\exp\{\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}_{\mathrm{sp}}$ [29]. This 2-DoF parallel manipulator is later used by Rosheim in the Omni-Wrist III [40] and reportedly to mimic human shoulder complex movement in terms of an extraordinary orientation range [41]. So far, no further results on general Exp-submanifolds of $\mathrm{SE}(3)$ are available in the literature.

It turns out that both the Listing's law of eye saccade and mirror symmetry of CV couplings can be attributed to *inversion (or geodesic) symmetry*, a class of diffeomorphism maps associated with symmetric spaces [42]. In fact, $\mathrm{SE}(3)$ is an *(affine) symmetric space* with the inversion symmetry at each point $\mathbf{g}$ defined by [43]:

$$\forall \mathbf{g} \in \mathrm{SE}(3) \ \Rightarrow \ S_{\mathbf{g}}(\mathbf{h}) \triangleq \mathbf{g}\mathbf{h}^{-1}\mathbf{g} \in \mathrm{SE}(3), \forall \mathbf{h} \in \mathrm{SE}(3) \tag{3}$$

and that both $\exp\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}_{\mathrm{sp}}$ (or equivalently $\exp\{\hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}_{\mathrm{sp}}$) and $\exp\{\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}_{\mathrm{sp}}$ can be extended via inversions to a unique symmetric space. Such a submanifold $\mathrm{M}$ is closed under inversions:

$$\forall \mathbf{g}, \mathbf{h} \in \mathrm{M} \ \Rightarrow \ \mathbf{g}\mathbf{h}^{-1}\mathbf{g} \in \mathrm{M} \tag{4}$$

and will be referred to as *symmetric submanifolds* of $\mathrm{SE}(3)$ (they are more often referred to as symmetric subspaces in mathematical literature [42,43]). Note that all connected Lie subgroups of $\mathrm{SE}(3)$ are automatically closed under inversions, and are *trivial* symmetric submanifolds. We exclude Lie subgroups from our study, since their symmetric space structure can be studied in a similar way to that for $\mathrm{SE}(3)$.

In this paper, we will show: (i) there are seven conjugacy classes of symmetric submanifolds of $\mathrm{SE}(3)$, all of which can be locally represented by $\exp\mathfrak{m}$, with $\mathfrak{m}$ being a *Lie triple (sub)system* (LTS) of $\mathfrak{se}(3)$ [42,43], i.e. a vector subspace of $\mathfrak{se}(3)$ that is closed under double Lie brackets:

$$\forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3 \in \mathfrak{m} \ \Rightarrow\ [[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2], \boldsymbol{\xi}_3] \in \mathfrak{m}$$

or simply $[[\mathfrak{m},\mathfrak{m}],\mathfrak{m}] \subset \mathfrak{m}$, where the Lie bracket is defined by $[\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2] \triangleq \hat{\boldsymbol{\xi}}_1\hat{\boldsymbol{\xi}}_2 - \hat{\boldsymbol{\xi}}_2\hat{\boldsymbol{\xi}}_1, \forall \hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2 \in \mathfrak{se}(3)$; (ii) these symmetric submanifolds can be systematically studied to derive several common geometric properties, which include Listing's law and mirror symmetry of CV couplings as special cases (for $\exp\{\hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}_{\mathrm{sp}}$ and $\exp\{\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}_{\mathrm{sp}}$ respectively); and (iii) we can develop a generic method for synthesizing their kinematic chains.

The only revelation that comes close to our discovery is Selig's attempt to study full cycle mobility using totally geodesic submanifolds of $\mathrm{SE}(3)$ (see [44, Ch. 15.2]). Only Lie subgroups are considered in [44], but both Lie subgroups and symmetric submanifolds are totally geodesic (for the concept of totally geodesic submanifolds of a symmetric space, see [42,45]; the totally geodesic submanifolds of a symmetric space are exactly the symmetric submanifolds [43, pp. 121, Coro., Lemma 1.3]).

Recently, during the review process of this manuscript, Selig [46] submitted and published a paper that, among other things, presents a classification of the LTSs of $\mathfrak{se}(3)$. However, we published a systematic classification of the LTSs of $\mathfrak{se}(3)$ in an earlier conference paper [1], which this manuscript relies upon and extends. In comparison to our previous and current results, Selig's classification has the following differences: it includes several Lie subalgebras into the classification of LTSs (all Lie subalgebras are, in fact, trivial LTSs), but omits the 5D LTS $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}_{\mathrm{sp}}$, because its does not meet the requirements of the application on which Ref. [46] focuses to, namely motion planning.

This paper is organized as follows. Section II gives a brief review of the symmetric space $\mathrm{SE}(3)$ and a systematic classification of its symmetric submanifolds. Section III summarizes a list of geometric properties common to all symmetric submanifolds, and shows how they can be applied to the study of eye saccade and CV coupling motion. Section IV proposes a systematic approach for synthesizing kinematic chains for symmetric submanifolds. Finally, Section V concludes our work.

## II. Inversion Symmetry of $\mathrm{SE}(3)$ and Symmetric Submanifolds

In this section, we first give a brief introduction to the symmetric space theory of $\mathrm{SE}(3)$ following the elementary treatment of Loos [43]. Then we give a systematic classification of symmetric submanifolds or $\mathrm{SE}(3)$. We assume the readers are familiar with basic Lie group theory of $\mathrm{SE}(3)$ [6].

### A. $\mathrm{SE}(3)$ *as a symmetric space*

We associate to each $\mathbf{g} \in \mathrm{SE}(3)$ an inversion symmetry $S_{\mathbf{g}}$ as defined in (3). $S_{\mathbf{g}}$ is involutive, i.e. $S_{\mathbf{g}} \circ S_{\mathbf{g}} = \mathrm{id}_{\mathrm{SE}(3)}$, and reverses the exponential map $\exp \mathbf{g}\hat{\boldsymbol{\xi}} \triangleq \mathbf{g}e^{\hat{\boldsymbol{\xi}}}$ for any tangent vector $\mathbf{g}\hat{\boldsymbol{\xi}} \in \mathrm{T}_{\mathbf{g}}\mathrm{SE}(3)$ at $\mathbf{g} \in \mathrm{SE}(3)$:

$$S_{\mathbf{g}}(\exp \mathbf{g}\hat{\boldsymbol{\xi}}) = \mathbf{g}(\mathbf{g}e^{\hat{\boldsymbol{\xi}}})^{-1}\mathbf{g} = \mathbf{g}e^{-\hat{\boldsymbol{\xi}}} = \exp(-\mathbf{g}\hat{\boldsymbol{\xi}}).$$

$\mathrm{SE}(3)$ equipped with the inversion symmetry is called a *symmetric space*. A *quadratic representation* $Q(\mathbf{g})$ of $\mathbf{g} \in \mathrm{SE}(3)$ is a diffeomorphism defined by:

$$Q(\mathbf{g}) \triangleq S_{\mathbf{g}} \circ S_{\mathbf{I}} : \mathrm{SE}(3) \to \mathrm{SE}(3), \quad Q(\mathbf{g})(\mathbf{h}) = \mathbf{ghg}$$

The group generated by $\{Q(\mathbf{g})|\mathbf{g} \in \mathrm{SE}(3)\}$ under composition of maps is called the *group of displacements* ([43, pp. 64]) of $\mathrm{SE}(3)$ and shall be denoted by $\tilde{\mathrm{G}}$. Since $\mathrm{SE}(3)$ is connected, $\tilde{\mathrm{G}}$ acts transitively on $\mathrm{SE}(3)$ ([43, pp. 91, Th. 3.1 a)]). See Appendix A for more details.

For any $\hat{\boldsymbol{\xi}} \in \mathfrak{se}(3)$, we denote the corresponding right- and left-invariant vector fields on $\mathrm{SE}(3)$ by $\hat{\boldsymbol{\xi}}^r$ and $\hat{\boldsymbol{\xi}}^l$ respectively, i.e.:

$$\begin{cases} \hat{\boldsymbol{\xi}}^r(\mathbf{g}) \triangleq \hat{\boldsymbol{\xi}}\mathbf{g} \\ \hat{\boldsymbol{\xi}}^l(\mathbf{g}) \triangleq \mathbf{g}\hat{\boldsymbol{\xi}} \end{cases} \qquad \forall \mathbf{g} \in \mathrm{SE}(3)$$

There are two special classes of vector fields relevant to the inversion symmetry of $\mathrm{SE}(3)$: the $(-)$-*derivations* $\mathfrak{D}_-$ and $(+)$-*derivations* $\mathfrak{D}_+$ ([43, pp. 81]). Every twist $\hat{\boldsymbol{\xi}} \in \mathfrak{se}(3)$ defines a $(-)$-*derivation* $\hat{\boldsymbol{\xi}}_-$:

$$\hat{\boldsymbol{\xi}}_-(\mathbf{g}) \triangleq \frac{1}{2}(\hat{\boldsymbol{\xi}}^r + \hat{\boldsymbol{\xi}}^l)(\mathbf{g}) = \frac{1}{2}(\hat{\boldsymbol{\xi}}\mathbf{g} + \mathbf{g}\hat{\boldsymbol{\xi}}),\ \forall \mathbf{g} \in \mathrm{SE}(3) \qquad (5)$$

and a $(+)$-*derivation* $\hat{\boldsymbol{\xi}}_+$:

$$\hat{\boldsymbol{\xi}}_+(\mathbf{g}) \triangleq \frac{1}{2}(\hat{\boldsymbol{\xi}}^r - \hat{\boldsymbol{\xi}}^l)(\mathbf{g}) = \frac{1}{2}(\hat{\boldsymbol{\xi}}\mathbf{g} - \mathbf{g}\hat{\boldsymbol{\xi}}),\ \forall \mathbf{g} \in \mathrm{SE}(3) \qquad (6)$$

In this case, the $\mathbb{R}$-vector spaces $\mathfrak{D}_-$ and $\mathfrak{D}_+$ can be identified with $\mathfrak{se}(3)$ respectively. The integral curves of $\hat{\boldsymbol{\xi}}_-$ and $\hat{\boldsymbol{\xi}}_+$ passing through $\mathbf{g} \in \mathrm{SE}(3)$ are given by $e^{\frac{t}{2}\hat{\boldsymbol{\xi}}}\mathbf{g}e^{\frac{t}{2}\hat{\boldsymbol{\xi}}} = Q(\exp\frac{t}{2}\hat{\boldsymbol{\xi}})(\mathbf{g})$ and $e^{\frac{t}{2}\hat{\boldsymbol{\xi}}}\mathbf{g}e^{-\frac{t}{2}\hat{\boldsymbol{\xi}}} = C(\exp\frac{t}{2}\hat{\boldsymbol{\xi}})(\mathbf{g})$ respectively ($C(\mathbf{h})$ denotes conjugation by $\mathbf{h} \in \mathrm{SE}(3)$), since:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\, Q(\exp\dfrac{t}{2}\hat{\boldsymbol{\xi}})(\mathbf{g})\Big|_{t=0} = \hat{\boldsymbol{\xi}}_-(\mathbf{g}) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\, C(\exp\dfrac{t}{2}\hat{\boldsymbol{\xi}})(\mathbf{g})\Big|_{t=0} = \hat{\boldsymbol{\xi}}_+(\mathbf{g}) \end{cases} \qquad \forall \mathbf{g} \in \mathrm{SE}(3) \qquad (7)$$

For any $\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\zeta}} \in \mathfrak{se}(3)$, from the fact that (see Appendix B for proof):

$$[\hat{\boldsymbol{\xi}}^r, \hat{\boldsymbol{\zeta}}^r] = -[\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\zeta}}]^r, \quad [\hat{\boldsymbol{\xi}}^l, \hat{\boldsymbol{\zeta}}^l] = [\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\zeta}}]^l, \quad [\hat{\boldsymbol{\xi}}^r, \hat{\boldsymbol{\zeta}}^l] = 0 \qquad (8)$$

we have:

$$\begin{cases} [\hat{\boldsymbol{\xi}}_-, \hat{\boldsymbol{\zeta}}_-] = -\dfrac{1}{2}[\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\zeta}}]_+ \\ [\hat{\boldsymbol{\xi}}_+, \hat{\boldsymbol{\zeta}}_+] = -\dfrac{1}{2}[\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\zeta}}]_+ \\ [\hat{\boldsymbol{\xi}}_+, \hat{\boldsymbol{\zeta}}_-] = -\dfrac{1}{2}[\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\zeta}}]_- \end{cases} \Rightarrow \begin{cases} [\mathfrak{D}_-, \mathfrak{D}_-] \subset \mathfrak{D}_+ \\ [\mathfrak{D}_+, \mathfrak{D}_+] \subset \mathfrak{D}_+ \\ [\mathfrak{D}_+, \mathfrak{D}_-] \subset \mathfrak{D}_- \end{cases} \qquad (9)$$

Therefore $\mathfrak{D}_+$ is a Lie algebra and $\mathfrak{D}_-$ is a LTS since:

$$[[\mathfrak{D}_-, \mathfrak{D}_-], \mathfrak{D}_-] \subset [\mathfrak{D}_+, \mathfrak{D}_-] \subset \mathfrak{D}_-$$

TABLE I
CONJUGACY CLASSES OF LTSS OF $\mathfrak{se}(3)$ (EXCLUDING ALL LIE SUBALGEBRAS OF $\mathfrak{se}(3)$, WHICH ARE TRIVIAL LTSS).

| dim | LTS $\mathfrak{m}$ (normal form) | screw sys. [30] | $\mathfrak{h}_{\mathfrak{m}} = [\mathfrak{m}, \mathfrak{m}]$ | $\mathfrak{g}_{\mathfrak{m}} = \mathfrak{h}_{\mathfrak{m}} + \mathfrak{m}$ | isotropy group |
|---|---|---|---|---|---|
| 2 | $\mathfrak{m}_{2A} \triangleq \{\hat{e}_3, \hat{e}_4\}_{sp}$ | 2nd special 2-sys. | $\{\hat{e}_2\}_{sp}$ | $\{\hat{e}_2, \hat{e}_3, \hat{e}_4\}_{sp}$ | $\exp\{\hat{e}_1, \hat{e}_2\}_{sp}$ |
| | $\mathfrak{m}_{2A}^p \triangleq \{\hat{e}_3, \hat{e}_4 + p\hat{e}_1\}_{sp}$ | | | $\{\hat{e}_2, \hat{e}_3, \hat{e}_4 + p\hat{e}_1\}_{sp}$ | |
| | $\mathfrak{m}_{2B} \triangleq \{\hat{e}_4, \hat{e}_5\}_{sp}$ | 1st special 2-sys. | $\{\hat{e}_6\}_{sp}$ | $\{\hat{e}_4, \hat{e}_5, \hat{e}_6\}_{sp}$ | $\exp\{\hat{e}_6\}_{sp}$ |
| 3 | $\mathfrak{m}_{3A} \triangleq \{\hat{e}_1, \hat{e}_3, \hat{e}_4\}_{sp}$ | 10th special 3-sys. | $\{\hat{e}_2\}_{sp}$ | $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}_{sp}$ | $\exp\{\hat{e}_1, \hat{e}_2\}_{sp}$ |
| | $\mathfrak{m}_{3B} \triangleq \{\hat{e}_3, \hat{e}_4, \hat{e}_5\}_{sp}$ | 4th special 3-sys. | $\{\hat{e}_1, \hat{e}_2, \hat{e}_6\}_{sp}$ | $\mathfrak{se}(3)$ | $\exp\{\hat{e}_1, \hat{e}_2, \hat{e}_6\}_{sp}$ |
| 4 | $\mathfrak{m}_4 \triangleq \{\hat{e}_1, \hat{e}_2, \hat{e}_4, \hat{e}_5\}_{sp}$ | 5th special 4-sys. | $\{\hat{e}_3, \hat{e}_6\}_{sp}$ | | $\exp\{\hat{e}_3, \hat{e}_6\}_{sp}$ |
| 5 | $\mathfrak{m}_5 \triangleq \{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{e}_5\}_{sp}$ | special 5-sys. | $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_6\}_{sp}$ | | $\exp\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_6\}_{sp}$ |

Moreover, $\mathfrak{D}_+ \oplus \mathfrak{D}_-$ is a Lie algebra, and is identified with the Lie algebra of $\tilde{G}$ ([43, pp. 91, Th. 3.1 d)]). In reference to the identification of $\mathfrak{se}(3)$ with $\mathfrak{D}_-$ via (5), we also say that $\mathfrak{se}(3)$ is a LTS.

### *B. Symmetric submanifolds of* $\mathrm{SE}(3)$

A *symmetric submanifold* (containing the identity $\mathbf{I}$) of $\mathrm{SE}(3)$ is a submanifold M which is closed under inversion symmetry (eq. (4)). Since for $\forall \mathbf{g_0}, \mathbf{g}, \mathbf{h} \in \mathrm{SE}(3)$:

$$\begin{aligned} C(\mathbf{g}_0)(\mathbf{g}\mathbf{h}^{-1}\mathbf{g}) &= \mathbf{g}_0(\mathbf{g}\mathbf{h}^{-1}\mathbf{g})\mathbf{g}_0^{-1} \\ &= \mathbf{g}_0\mathbf{g}\mathbf{g}_0^{-1}\mathbf{g}_0\mathbf{h}^{-1}\mathbf{g}_0^{-1}\mathbf{g}_0\mathbf{g}\mathbf{g}_0^{-1} \\ &= C(\mathbf{g}_0)(\mathbf{g})(C(\mathbf{g}_0)(\mathbf{h}))^{-1}C(\mathbf{g}_0)(\mathbf{g}) \end{aligned}$$

the conjugation $C(\mathbf{g}_0)(\mathrm{M})$ of a symmetric submanifold M is still a symmetric submanifold. Therefore, the systematic classification of symmetric submanifolds reduces to that of their conjugacy classes.

Let $\mathfrak{m}$ denote the tangent space $\mathrm{T_I M}$ of M at $\mathbf{I}$. $\mathfrak{m}$ is identified with the $(-)$-derivations $\mathfrak{D}_-(\mathrm{M})$ as in (5), and is a LTS (subsystem) of $\mathfrak{se}(3)$ ([43, pp. 121]). From the definition of Lie group and Lie algebra ([6, Appendix A]), all Lie subgroups of $\mathrm{SE}(3)$ are trivial symmetric submanifolds, with their corresponding Lie subalgebras being trivial LTSs. Their symmetric space structure can be understood in the same way as that of $\mathrm{SE}(3)$. By the definition of LTS and *Jacobi identity* ([6, Appendix A]), one easily verifies that both $\mathfrak{h}_{\mathfrak{m}} \triangleq [\mathfrak{m}, \mathfrak{m}]$ and $\mathfrak{g}_{\mathfrak{m}} \triangleq \mathfrak{h}_{\mathfrak{m}} + \mathfrak{m}$ are Lie subalgebras of $\mathfrak{se}(3)$. $\mathfrak{h}_{\mathfrak{m}}$ is identified with $\mathfrak{D}_+(\mathrm{M})$ as in (6). A relation similar to (9) holds for $\mathfrak{m}$ and $\mathfrak{h}_{\mathfrak{m}}$:

$$[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}_{\mathfrak{m}}, \quad [\mathfrak{h}_{\mathfrak{m}}, \mathfrak{h}_{\mathfrak{m}}] \subset \mathfrak{h}_{\mathfrak{m}}, \quad [\mathfrak{h}_{\mathfrak{m}}, \mathfrak{m}] \subset \mathfrak{m} \tag{10}$$

The following theorem establishes the fundamental relation between a symmetric submanifold M and its tangent space at identity.

**Theorem 1.** *A (connected) symmetric submanifold* $\mathrm{M} \subset \mathrm{SE}(3)$ *(containing* $\mathbf{I}$*) is generated (via inversion symmetry) by any open neighborhood (containing* $\mathbf{I}$*) of* $\exp \mathfrak{m}$*, where the tangent space* $\mathfrak{m} = \mathrm{T_I M}$ *is a LTS of* $\mathfrak{se}(3)$*:*

$$\mathrm{M} = \{\mathbf{ghg} \in \mathrm{SE}(3) | \forall \mathbf{g}, \mathbf{h} \in \exp \mathfrak{m}\} \tag{11}$$

*Proof.* See Appendix C. ☐

According to Theorem 1, there is a one-to-one correspondence between LTSs of $\mathfrak{se}(3)$ and symmetric submanifolds of $\mathrm{SE}(3)$ (containing $\mathbf{I}$). Therefore, the systematic classification of symmetric submanifolds of $\mathrm{SE}(3)$ up to conjugation is equivalent to that of conjugacy classes of LTSs of $\mathfrak{se}(3)$. Starting from a screw system of $\mathfrak{se}(3)$ (see for example [30, Ch. 12] or [44, Ch. 8]), we can determine if it is a LTS by verifying closure under double Lie brackets for an arbitrarily chosen basis (the complete computation cannot be provided here due to space limitations and will be presented in [47].). A total of seven conjugacy classes of LTSs are found and listed in Table I. The subscripts $_{m\mathrm{A}}$ and $_{n\mathrm{B}}$ for the LTSs in the second column denote an $m$D LTS containing one rotational/helical DoF and an $n$D LTS containing two rotational DoFs respectively. The same subscript convention is used for the corresponding symmetric submanifolds. Essentially the same classification was reported in [1]. The only difference is that the 2D LTS $\{\hat{e}_3, \hat{e}_4 + p\hat{e}_1\}_{sp}$, though given here, was omitted in [1], since it can be essentially studied in the same way as $\{\hat{e}_3, \hat{e}_4\}_{sp}$.

The screw systems corresponding to the LTSs (normal form) are depicted in Fig. 3. A generic member of each conjugacy class $\mathfrak{m}$ is of the form $\mathrm{Ad}_{\mathbf{g}}\mathfrak{m}, \mathbf{g} \in \mathrm{SE}(3)$, where $\mathrm{Ad}_{\mathbf{g}}$ is the Adjoint transformation:

$$\mathrm{Ad}_{\mathbf{g}}\hat{\boldsymbol{\xi}} \triangleq \mathbf{g}\hat{\boldsymbol{\xi}}\mathbf{g}^{-1}, \hat{\boldsymbol{\xi}} \in \mathfrak{se}(3)$$

It is also clear from Theorem 1 that the local Exp-submanifold $\exp \mathfrak{m}$ we discussed in Section I is actually an open neighborhood (about $\mathbf{I}$) of a unique symmetric submanifold $\mathrm{M} \subset \mathrm{SE}(3)$ when $\mathfrak{m}$ is a LTS of $\mathfrak{se}(3)$. We shall say M is generated (via inversion symmetry) by $\exp \mathfrak{m}$, which we denote by $\mathrm{M} = \langle \exp \mathfrak{m} \rangle$[2]. A generic member of each conjugacy class M is given by:

$$C(\mathbf{g})(\mathrm{M}) = \langle \exp \mathrm{Ad}_{\mathbf{g}}\mathfrak{m} \rangle.$$

It is proved that $\exp : \mathfrak{g} \to \mathrm{G}$ of a Lie subalgebra $\mathfrak{g}$ into its corresponding Lie subgroup G may not be surjective (surjectivity fails for the conjugacy class of $\{\hat{e}_2, \hat{e}_3, \hat{e}_4 + p\hat{e}_1\}_{sp}$) [48]. The following proposition shows $\exp : \mathfrak{m} \to \mathrm{M}$ may not be surjective either.

[2]For convenience, we shall also denote a (connected) Lie subgroup $\mathrm{G} \subset \mathrm{SE}(3)$ with lie subalgebra $\mathfrak{g} \subset \mathfrak{se}(3)$ by $\langle \exp \mathfrak{g} \rangle$, since Lie subgroups are trivial symmetric submanifolds.

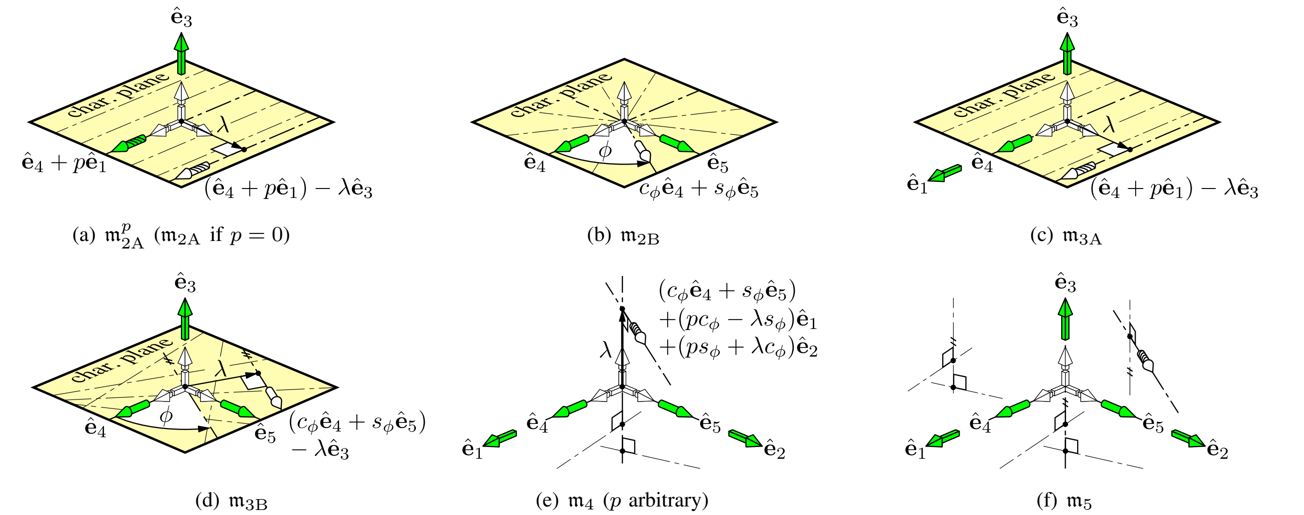



Fig. 3. Screw systems of Lie triple systems. The basis screws of each LTS are denoted by green arrows, and a generic screw in each LTS is denoted by a white arrow. Except for $\mathfrak{m}_{2\mathrm{A}}^p$, the pitch value $p$ of the generic screw takes an arbitrary finite value. The characteristic (char.) plane of $\mathfrak{m}_{2\mathrm{A}}^p$ ($\mathfrak{m}_{2\mathrm{A}}$), $\mathfrak{m}_{2\mathrm{B}}$, $\mathfrak{m}_{3\mathrm{A}}$, $\mathfrak{m}_{3\mathrm{B}}$ is defined to be the plane containing all axes of their screw systems. $c_{(\cdot)}$ and $s_{(\cdot)}$ denote $\cos(\cdot)$ and $\sin(\cdot)$ respectively.

**Proposition 1.** $\exp : \mathfrak{m} \to \mathrm{M}$ *is surjective for all LTSs except* $\mathfrak{m}_{2\mathrm{A}}^p = \{\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4 + p\hat{\mathbf{e}}_1\}_{\mathrm{sp}}$. *Every point of* $\mathrm{M}_{2\mathrm{A}}^p$ *can be reached from* $\exp \mathfrak{m}_{2\mathrm{A}}^p$ *by one inversion.*

*Proof.* See Appendix D. □

## III. Geometric Properties of Symmetric Submanifolds

In Section II-B, we use a LTS $\mathfrak{m} \subset \mathfrak{se}(3)$ (identified with $\mathfrak{D}_-(\mathrm{M})$) to recover the corresponding symmetric submanifold $\mathrm{M} = \langle \exp \mathfrak{m} \rangle \subset \mathrm{SE}(3)$. A list of useful properties common to all symmetric submanifolds is given as follows.

**Proposition 2.** *The following statements are true for all LTSs* $\mathfrak{m}$ *and symmetric submanifolds* $\mathrm{M} = \langle \exp \mathfrak{m} \rangle$*:*

a) *The right translation (back to the identity* $\mathbf{I}$*) of the tangent space at* $\mathbf{g} = e^{\hat{\boldsymbol{\xi}}}, \hat{\boldsymbol{\xi}} \in \mathfrak{m}$ *is given by:*

$$(\mathrm{T}_{\mathbf{g}}\mathrm{M})\mathbf{g}^{-1} = \mathrm{Ad}_{\mathbf{g}^{\frac{1}{2}}}\mathfrak{m} = \mathrm{Ad}_{e^{\hat{\boldsymbol{\xi}}/2}}\mathfrak{m}$$

*In the case of* $\mathbf{g} = \mathbf{e}^{\hat{\boldsymbol{\xi}}_1}\mathbf{e}^{\hat{\boldsymbol{\xi}}_2}\mathbf{e}^{\hat{\boldsymbol{\xi}}_1}$ *(for* $\mathfrak{m}_{2\mathrm{A}}^p$*):*

$$(\mathrm{T}_{\mathbf{g}}\mathrm{M})\mathbf{g}^{-1} = \mathrm{Ad}_{e^{\hat{\boldsymbol{\xi}}_1}e^{\hat{\boldsymbol{\xi}}_2/2}}\mathfrak{m}$$

b) $\mathfrak{m}$ *is Adjoint invariant by elements of the* ***isotropy group*** $\mathrm{H}_\mathrm{M} \triangleq \langle \exp \mathfrak{h}_\mathfrak{m} \rangle$ *with* $\mathfrak{h}_\mathfrak{m} \triangleq [\mathfrak{m}, \mathfrak{m}]$

$$\forall \mathbf{h} \in \mathrm{H}_\mathrm{M} \quad \Rightarrow \quad \mathrm{Ad}_{\mathbf{h}}\mathfrak{m} \equiv \mathfrak{m}$$

$\mathrm{M}$ *is conjugation invariant by elements of* $\mathrm{H}_\mathrm{M}$

$$\forall \mathbf{h} \in \mathrm{H}_\mathrm{M} \quad \Rightarrow \quad C(\mathbf{h})(\mathrm{M}) \equiv \mathrm{M}$$

c) $\mathfrak{g}_\mathfrak{m} = \mathfrak{m} + \mathfrak{h}_\mathfrak{m}$ *is the* ***completion algebra*** *of* $\mathfrak{m}$ *(i.e. the smallest Lie subalgebra containing* $\mathfrak{m}$*), and* $\mathrm{G}_\mathrm{M} \triangleq \langle \exp \mathfrak{g}_\mathfrak{m} \rangle$ *is the* ***completion group*** *of* $\mathrm{M}$ *(i.e. the smallest connected Lie subgroup containing* $\mathrm{M}$*); all LTSs except* $\mathfrak{m}_5 = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}_{\mathrm{sp}}$ *satisfy* $\mathfrak{h}_\mathfrak{m} \cap \mathfrak{m} = \{\hat{o}\}$ *and admit a local parametrization for* $\mathrm{G}_\mathrm{M}$*:*

$$\forall \mathbf{g} \in \mathrm{G}_\mathrm{M} \quad \Rightarrow \quad \begin{aligned} \widetilde{\exp}_{\mathbf{g}} &: \mathfrak{m} \times \mathfrak{h}_\mathfrak{m} \to \mathrm{G}_\mathrm{M} \\ &(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\zeta}}) \mapsto \mathbf{g}e^{\hat{\boldsymbol{\xi}}}e^{\hat{\boldsymbol{\zeta}}} \end{aligned}$$

*in an open neighborhood of* $\mathbf{g} \in \mathrm{G}_\mathrm{M}$.

*Proof.* See Appendix E. □

First, Prop.2 a) is indeed a generalization to Listing's law of human eye saccade for the special case of $\mathrm{M}_{2\mathrm{B}} = \langle \exp \mathfrak{m}_{2\mathrm{B}} \rangle$, and may therefore be referred to as the *half-angle property*. It is a direct consequence of the inversion symmetry of the symmetric submanifolds (see the proof of Prop.2 a)). Other than correctly predicting human eye saccade behavior, the half-angle property also prescribes the location of the bisecting plane of a CV coupling, and is therefore a convenient tool for position and constraint analysis of CV connecting chains [36].

Second, Prop.2 b) can be referred to as the *conjugation invariance property* of the corresponding LTS $\mathfrak{m}$ and symmetric submanifold $\mathrm{M} = \langle \exp \mathfrak{m} \rangle$. Physically, it corresponds to the fact that the kinematics of the underlying symmetric submanifold is uniform in the directions specified by the isotropy group $\mathrm{H}_\mathrm{M}$. In the case of a 2-DoF ($\mathrm{M}_{2\mathrm{B}}$) CV coupling, this uniformity ensures uniform (or constant) velocity transmission between two intersecting shafts. The conjugation invariance of $\mathrm{M}_{3\mathrm{B}}$, for example, can also be exploited to synthesize its parallel manipulators using identical and conjugate kinematic chains. From a more mathematical viewpoint, conjugation invariance implies that symmetric submanifolds should be quotient (or homogeneous) spaces of certain transitive Lie transformation group action, which we briefly summarized in Appendix A.

Third, in Prop.2 c), the completion algebra $\mathfrak{g}_\mathfrak{m}$ is very useful for type synthesis of kinematic chains for symmetric submanifolds (see Section IV). In the case $\mathfrak{h}_\mathfrak{m} \cap \mathfrak{m} = \{\hat{\mathbf{0}}\}$, the local parametrization $\widetilde{\exp}_{\mathbf{I}}$ for an open neighborhood of $\mathbf{I}$ in $\mathrm{G}_\mathrm{M}$ is an immediate generalization of the tilt and torsion angle parametrization for $\mathrm{SO}(3)$ [32].

We give two full computation examples with applications for a better understanding of the aforesaid geometric properties of symmetric submanifolds of $\mathrm{SE}(3)$.

**Example 1** ($\mathrm{M}_{2\mathrm{B}}$)**.** It is obvious from:

$$\begin{cases} [[\hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5], \hat{\mathbf{e}}_4] = [\hat{\mathbf{e}}_6, \hat{\mathbf{e}}_4] = \hat{\mathbf{e}}_5 \\ [[\hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5], \hat{\mathbf{e}}_5] = [\hat{\mathbf{e}}_6, \hat{\mathbf{e}}_5] = -\hat{\mathbf{e}}_4 \end{cases}$$

that $\mathfrak{m}_{2\mathrm{B}} = \{\hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}_{\mathrm{sp}} \subset \mathfrak{se}(3)$ (or $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}_{\mathrm{sp}} \subset \mathfrak{so}(3)$, with $\mathfrak{so}(3)$ the Lie algebra of $\mathrm{SO}(3)$) is a LTS. Since $\exp$ is surjective for $\mathfrak{m}_{2\mathrm{B}}$ by Prop.1, $\mathrm{M}_{2\mathrm{B}} = \exp \mathfrak{m}_{2\mathrm{B}}$ (see also [34]):

$$\mathrm{M}_{2\mathrm{B}} = \{e^{\hat{\mathbf{w}}} | \mathbf{w} \in \{\mathbf{x}, \mathbf{y}\}_{\mathrm{sp}}\}$$

It can be verified by straightforward computation (preferably using unit quaternions) that $\mathrm{M}_{2\mathrm{B}}$ is closed under inversion symmetry:

$$\forall \mathbf{w}_1, \mathbf{w}_2 \in \{\mathbf{x}, \mathbf{y}\}_{\mathrm{sp}} \quad \Rightarrow \quad e^{\hat{\mathbf{w}}_1} e^{-\hat{\mathbf{w}}_2} e^{\hat{\mathbf{w}}_1} \in \mathrm{M}_{2\mathrm{B}}$$

A computational verification of the half-angle property for $\mathrm{M}_{2\mathrm{B}} = \langle \exp \mathfrak{m}_{2\mathrm{B}} \rangle$ can be given as follows: according to [49, eq. (20)],

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}t} e^{\hat{\mathbf{w}}(t)}\right) e^{-\hat{\mathbf{w}}(t)} &= \int_0^1 e^{u\hat{\mathbf{w}}(t)} \dot{\hat{\mathbf{w}}}(t) e^{-u\hat{\mathbf{w}}(t)} \mathrm{d}u \\ &= \left(\int_0^1 e^{u\hat{\mathbf{w}}(t)} \dot{\mathbf{w}}(t) \mathrm{d}u\right)^{\wedge} \end{aligned} \tag{12}$$

for $\hat{\mathbf{w}} \in \mathfrak{m}_{2\mathrm{B}} = \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}_{\mathrm{sp}}$ (which is equivalent to $\{\hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}_{\mathrm{sp}}$). Since both $\mathbf{w}$ and $\dot{\mathbf{w}}$ lie in the $\mathbf{xy}$-plane, $\dot{\mathbf{w}}(t)$ can be written as $\lambda_1 \mathbf{w} + \lambda_2 \mathbf{w}^{\perp}$ for some real numbers $\lambda_1, \lambda_2$ and $\mathbf{w}^{\perp} = \hat{\mathbf{z}}\mathbf{w}$. Then (12) gives:

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}t} e^{\hat{\mathbf{w}}(t)}\right) e^{-\hat{\mathbf{w}}(t)} = \left(\int_0^1 e^{u\hat{\mathbf{w}}(t)} (\lambda_1 \mathbf{w} + \lambda_2 \mathbf{w}^{\perp}) \mathrm{d}u\right)^{\wedge} \\ &= \left(\int_0^1 (\lambda_1 \mathbf{w} + \lambda_2 (c_{u\|\mathbf{w}\|} \mathbf{w}^{\perp} + s_{u\|\mathbf{w}\|} \mathbf{z})) \mathrm{d}u\right)^{\wedge} \\ &= \left(e^{\hat{\mathbf{w}}(t)/2} \left(\lambda_1 \mathbf{w} + \lambda_2 \operatorname{sinc}\left(\tfrac{\|\mathbf{w}\|}{2}\right) \mathbf{w}^{\perp}\right)\right)^{\wedge} \\ &\in \left(e^{\hat{\mathbf{w}}(t)/2} \{\mathbf{x}, \mathbf{y}\}_{\mathrm{sp}}\right)^{\wedge} \end{aligned} \tag{13}$$

which verifies that $(\mathrm{T}_{e^{\hat{\mathbf{w}}}} \mathrm{M}_{2\mathrm{B}}) e^{-\hat{\mathbf{w}}} = \mathrm{Ad}_{e^{\hat{\mathbf{w}}/2}} \mathfrak{m}_{2\mathrm{B}}$.

$\mathrm{M}_{2\mathrm{B}}$ is conjugation invariant by elements of the isotropy group $\mathrm{H}_{\mathrm{M}_{2\mathrm{B}}}$ of $\mathrm{M}_{2\mathrm{B}}$, which is given by $\langle \exp \mathfrak{h}_{\mathfrak{m}_{2\mathrm{B}}} \rangle = \langle \exp\{\hat{\mathbf{z}}\}_{\mathrm{sp}} \rangle = \mathrm{SO}(2)$:

$$\begin{aligned} \forall \phi \in \mathbb{R} \quad \Rightarrow \quad & e^{\phi \hat{\mathbf{z}}} \cdot \langle \exp\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}_{\mathrm{sp}} \rangle \cdot e^{-\phi \hat{\mathbf{z}}} \\ & = \langle \exp\{c_\phi \hat{\mathbf{x}} + s_\phi \hat{\mathbf{y}}, -s_\phi \hat{\mathbf{x}} + c_\phi \hat{\mathbf{y}}\}_{\mathrm{sp}} \rangle \\ & = \langle \exp\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}_{\mathrm{sp}} \rangle \end{aligned}$$

A direct consequence of the conjugation invariance is that $\langle \exp \mathfrak{m}_{2\mathrm{B}} \rangle$ may be generated by a parallel manipulator with identical and axially symmetric kinematic chains [50]. To see this, if a submanifold $\mathrm{M}$ with conjugation invariance prescribed by a Lie subgroup $\mathrm{H}_{\mathrm{M}}$ is locally contained in a POE-submanifold $\prod_{i=1}^k \exp\{\hat{\boldsymbol{\xi}}_i\}_{\mathrm{sp}}$ generated by a kinematic chain $(\hat{\boldsymbol{\xi}}_1, \ldots, \hat{\boldsymbol{\xi}}_k)$ (generation of symmetric submanifolds using kinematic chains will be systematically studied in Sec. IV),

$$\mathrm{M} \subset_{\mathrm{local}} \prod_{i=1}^k \exp\{\hat{\boldsymbol{\xi}}_i\}_{\mathrm{sp}}$$

Then we have:

$$\forall \mathbf{g} \in \mathrm{H}_{\mathrm{M}} \quad \Rightarrow \quad \mathrm{M} = C(\mathbf{g})(\mathrm{M}) \subset_{\mathrm{local}} C(\mathbf{g})(\prod_{i=1}^k \exp\{\hat{\boldsymbol{\xi}}_i\}_{\mathrm{sp}})$$

where the conjugate submanifold $C(\mathbf{g})(\prod_{i=1}^k \exp\{\hat{\boldsymbol{\xi}}_i\}_{\mathrm{sp}})$ still contains $\mathrm{M}$ and is generated by an identical but rigidly displaced chain $(\mathrm{Ad}_{\mathbf{g}} \hat{\boldsymbol{\xi}}_1, \ldots, \mathrm{Ad}_{\mathbf{g}} \hat{\boldsymbol{\xi}}_k)$. A parallel manipulator comprising two or more such chains may generate $\mathrm{M}$ with appropriate velocity or force matching conditions [18, Prop. 6].

Finally, $\widetilde{\exp}_{\mathbf{I}} : \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}_{\mathrm{sp}} \times \{\hat{\mathbf{z}}\}_{\mathrm{sp}} \to \mathrm{SO}(3)$ is simply a slight variation of the tilt and torsion angle parametrization [32] given earlier in (2). Its parametrization singularity may be easily investigated using half-angle property: the spatial Jacobian of $e^{\hat{\mathbf{w}}} e^{\sigma \hat{\mathbf{z}}}, \hat{\mathbf{w}} = \theta_1 \hat{\mathbf{x}} + \theta_2 \hat{\mathbf{y}}$ is given by:

$$\left[ (\textstyle\int_0^1 e^{u\hat{\mathbf{w}}(t)} \mathrm{d}u) \mathbf{x} \quad (\int_0^1 e^{u\hat{\mathbf{w}}(t)} \mathrm{d}u) \mathbf{y} \quad e^{\hat{\mathbf{w}}} \mathbf{z} \right] \in \mathbb{R}^{3 \times 3}$$

where the first two columns always span the half tilted $\mathbf{xy}$-plane $e^{\hat{\mathbf{w}}/2} \{\mathbf{x}, \mathbf{y}\}_{\mathrm{sp}}$. Therefore,

$$\begin{aligned} &\left\{ (\textstyle\int_0^1 e^{u\hat{\mathbf{w}}(t)} \mathrm{d}u) \mathbf{x}, (\int_0^1 e^{u\hat{\mathbf{w}}(t)} \mathrm{d}u) \mathbf{y}, e^{\hat{\mathbf{w}}} \mathbf{z} \right\}_{\mathrm{sp}} \\ &= e^{\hat{\mathbf{w}}/2} \{\mathbf{x}, \mathbf{y}, e^{\hat{\mathbf{w}}/2} \mathbf{z}\}_{\mathrm{sp}} \end{aligned}$$

and parametrization singularity is reached when $e^{\hat{\mathbf{w}}/2} \mathbf{z}$ lies in the $\mathbf{xy}$-plane, or when the tilt angle reaches 180 degrees. □

**Example 2** ($\mathrm{M}_{3\mathrm{B}}$)**.** It is obvious from:

$$\begin{cases} [[\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4], \hat{\mathbf{e}}_3] = [\hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3] = \hat{0} \\ [[\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4], \hat{\mathbf{e}}_4] = [\hat{\mathbf{e}}_2, \hat{\mathbf{e}}_4] = -\hat{\mathbf{e}}_3 \\ [[\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4], \hat{\mathbf{e}}_5] = [\hat{\mathbf{e}}_2, \hat{\mathbf{e}}_5] = \hat{0} \\ [[\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_5], \hat{\mathbf{e}}_3] = [-\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_3] = \hat{0} \\ [[\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_5], \hat{\mathbf{e}}_4] = [-\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_4] = \hat{0} \\ [[\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_5], \hat{\mathbf{e}}_5] = [-\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_5] = -\hat{\mathbf{e}}_3 \\ [[\hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5], \hat{\mathbf{e}}_3] = [\hat{\mathbf{e}}_6, \hat{\mathbf{e}}_3] = \hat{0} \\ [[\hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5], \hat{\mathbf{e}}_4] = [\hat{\mathbf{e}}_6, \hat{\mathbf{e}}_4] = \hat{\mathbf{e}}_5 \\ [[\hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5], \hat{\mathbf{e}}_5] = [\hat{\mathbf{e}}_6, \hat{\mathbf{e}}_5] = -\hat{\mathbf{e}}_4 \end{cases}$$

that $\mathfrak{m}_{3\mathrm{B}} = \{\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}_{\mathrm{sp}}$ is a LTS. Since $\exp$ is surjective for $\mathfrak{m}_{3\mathrm{B}}$ by Prop.1, $\mathrm{M}_{3\mathrm{B}} = \exp \mathfrak{m}_{3\mathrm{B}}$. A typical element $\mathbf{g} \in \mathrm{M}_{3\mathrm{B}}$ is given by:

$$\mathbf{g} = \exp \begin{bmatrix} 2\hat{\mathbf{w}} & 2\lambda \mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} e^{2\hat{\mathbf{w}}} & 2\lambda \operatorname{sinc}(\|\mathbf{w}\|) e^{\hat{\mathbf{w}}} \mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}$$

and

$$\mathbf{g}^{\frac{1}{2}} = \exp \begin{bmatrix} \hat{\mathbf{w}} & \lambda \mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} e^{\hat{\mathbf{w}}} & \lambda \operatorname{sinc}\left(\frac{\|\mathbf{w}\|}{2}\right) e^{\hat{\mathbf{w}}/2} \mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}$$

where $\mathbf{w} \in \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}_{\mathrm{sp}}$ and $\lambda \in \mathbb{R}$.

It can be verified by straightforward computation that:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{g}\right) \mathbf{g}^{-1} = \mathbf{g}^{\frac{1}{2}} \begin{bmatrix} 2(\lambda_1 \hat{\mathbf{w}} + \lambda_2 \operatorname{sinc}(\|\mathbf{w}\|) \hat{\mathbf{w}}^{\perp}) & \lambda' \mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} \mathbf{g}^{-\frac{1}{2}}$$

for some $\lambda' \in \mathbb{R}$, where one needs to use both (13) and the following similar equation:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} e^{2\hat{\mathbf{w}}}\right) e^{-2\hat{\mathbf{w}}} = \left(2 e^{\hat{\mathbf{w}}(t)} \left(\lambda_1 \mathbf{w} + \lambda_2 \operatorname{sinc}(\|\mathbf{w}\|) \mathbf{w}^{\perp}\right)\right)^{\wedge}$$

and therefore $(\mathrm{T_g M_{3B}})\mathbf{g}^{-1} = \mathrm{Ad}_{\mathbf{g}^{\frac{1}{2}}}\mathfrak{m}_{3B}, \forall \mathbf{g} \in \mathrm{M_{3B}}$. The half-angle property immediately leads to a closed-form direct kinematic solutions for the reflected tripod [31].

$\mathrm{M_{3B}}$ is conjugation invariant by elements of the isotropy group $\mathrm{H}_{\mathrm{M_{3B}}}$, which is given by $\langle \exp \mathfrak{h}_{\mathfrak{m}_{3B}} \rangle = \langle \exp\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_6\}_{\mathrm{sp}} \rangle = \mathrm{SE}(2)$: given any $\mathbf{g} \in \mathrm{SE}(2)$ of the form

$$\mathbf{g} = \begin{bmatrix} e^{\theta \hat{\mathbf{z}}} & \lambda_1 \mathbf{x} + \lambda_2 \mathbf{y} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}$$

we have:

$$\begin{cases} \mathrm{Ad}_{\mathbf{g}} \hat{\mathrm{e}}_3 = \hat{\mathrm{e}}_3 \\ \mathrm{Ad}_{\mathbf{g}} \hat{\mathrm{e}}_4 = (\lambda_1 s_\theta - \lambda_2 c_\theta)\hat{\mathrm{e}}_3 + c_\theta \hat{\mathrm{e}}_4 + s_\theta \hat{\mathrm{e}}_5 \\ \mathrm{Ad}_{\mathbf{g}} \hat{\mathrm{e}}_5 = (\lambda_1 c_\theta + \lambda_2 s_\theta)\hat{\mathrm{e}}_3 - s_\theta \hat{\mathrm{e}}_4 + c_\theta \hat{\mathrm{e}}_5 \end{cases}$$

and therefore $\mathrm{Ad}_{\mathbf{g}} \mathfrak{m}_{3B} \equiv \mathfrak{m}_{3B}$, $\forall \mathbf{g} \in \mathrm{SE}(2)$ and $C(\mathbf{g})(\mathrm{M_{3B}}) \equiv \mathrm{M_{3B}}$, $\forall \mathbf{g} \in \mathrm{SE}(2)$. Using the same argument as in Example 1, we see that $\mathrm{M_{3B}}$ may be generated by identical kinematic chains with arbitrary planar displacement (see the $\mathrm{M_{3B}}$ modules of The BROMMI hyperredundant robotic arm [38]).

Finally, $\widetilde{\exp}_{\mathbf{I}} : \{\hat{\mathrm{e}}_3, \hat{\mathrm{e}}_4, \hat{\mathrm{e}}_5\}_{\mathrm{sp}} \times \{\hat{\mathrm{e}}_1, \hat{\mathrm{e}}_2, \hat{\mathrm{e}}_6\}_{\mathrm{sp}} \to \mathrm{SE}(3)$ gives a local parametrization of $\mathrm{SE}(3)$. □

## IV. Generating Symmetric Submanifolds Using Symmetric Kinematic Chains

### A. *Symmetric chains of symmetric submanifolds*

Consider a $k$D symmetric submanifold $\mathrm{M} = \langle \exp \mathfrak{m} \rangle$ and a basis of unit twists $\{\hat{\boldsymbol{\xi}}_1, \ldots, \hat{\boldsymbol{\xi}}_k\} \subset \mathfrak{m}$ for the LTS $\mathfrak{m}$. In general, M cannot be generated by the POE-submanifold $\prod_{i=1}^{k} \exp\{\hat{\boldsymbol{\xi}}_i\}_{\mathrm{sp}}$ since it is not closed under group product. In view of the inversion symmetry, M can be locally generated by a $(2k-1)$-DoF *symmetric twist chain* (SC) $(\hat{\boldsymbol{\xi}}_1, \ldots, \hat{\boldsymbol{\xi}}_{k-1}, \hat{\boldsymbol{\xi}}_k, \hat{\boldsymbol{\xi}}_{k-1}, \ldots, \hat{\boldsymbol{\xi}}_1)$ with symmetric joint variables:

$$\begin{gathered} e^{\theta_1 \hat{\boldsymbol{\xi}}_1} \cdots e^{\theta_{k-1} \hat{\boldsymbol{\xi}}_{k-1}} \cdot e^{\theta_k \hat{\boldsymbol{\xi}}_k} \cdot e^{\theta_{k-1} \hat{\boldsymbol{\xi}}_{k-1}} \cdots e^{\theta_1 \hat{\boldsymbol{\xi}}_1} = \\ Q(e^{\theta_1 \hat{\boldsymbol{\xi}}_1}) \circ \cdots \circ Q(e^{\theta_{k-1} \hat{\boldsymbol{\xi}}_{k-1}})(e^{\theta_k \hat{\boldsymbol{\xi}}_k}) \in \mathrm{M} \\ \theta_i \in \mathbb{R}, \ i = 1, \ldots, k \end{gathered}$$

We emphasize that the motion generated by a SC with arbitrary joint variables:

$$\begin{gathered} e^{\theta_1^+ \hat{\boldsymbol{\xi}}_1} \cdots e^{\theta_{k-1}^+ \hat{\boldsymbol{\xi}}_{k-1}} \cdot e^{\theta_k \hat{\boldsymbol{\xi}}_k} \cdot e^{\theta_{k-1}^- \hat{\boldsymbol{\xi}}_{k-1}} \cdots e^{\theta_1^- \hat{\boldsymbol{\xi}}_1} \\ \theta_i^{\pm} \in \mathbb{R}, \ i = 1, \ldots, k-1 \end{gathered}$$

is in general not contained in M (but in $\mathrm{G_M}$) unless the joint variables are also symmetric:

$$\theta_i^+ \equiv \theta_i^-, \ i = 1, \ldots, k-1. \tag{14}$$

In this case, we say the SC undergoes a *symmetric movement*. The following proposition shows that the joint twists in a SC can take general values in the completion algebra $\mathfrak{g}_{\mathfrak{m}} = \mathfrak{h}_{\mathfrak{m}} + \mathfrak{m}$.

**Proposition 3.** *Given a $k$D LTS $\mathfrak{m}$ of a symmetric submanifold $\mathrm{M} = \langle \exp \mathfrak{m} \rangle$, a pair of unit twists $(\hat{\boldsymbol{\xi}}^+, \hat{\boldsymbol{\xi}}^-)$ of $\mathfrak{g}_{\mathfrak{m}}$ satisfies:*

$$\forall \mathbf{g} \in \mathrm{M}, \theta \in \mathbb{R} \quad \Rightarrow \quad e^{\theta \hat{\boldsymbol{\xi}}^+} \mathbf{g} e^{\theta \hat{\boldsymbol{\xi}}^-} \in \mathrm{M}$$

*if:*

$$\begin{cases} \hat{\boldsymbol{\xi}}^+ = \hat{\boldsymbol{\xi}} + \hat{\boldsymbol{\zeta}}, \\ \hat{\boldsymbol{\xi}}^- = \hat{\boldsymbol{\xi}} - \hat{\boldsymbol{\zeta}} \end{cases}, \quad \hat{\boldsymbol{\xi}} \in \mathfrak{m}, \hat{\boldsymbol{\zeta}} \in \mathfrak{h}_{\mathfrak{m}} \tag{15}$$

*in which case $(\hat{\boldsymbol{\xi}}^+, \hat{\boldsymbol{\xi}}^-)$ is referred to as a **symmetric twist pair** (SP). The decomposition (15) is unique except for $\mathfrak{m}_5$ where $\mathfrak{h}_{\mathfrak{m}_5} \cap \mathfrak{m}_5 \neq \{\hat{\mathrm{o}}\}$. A SC formed by $k$ nesting SPs $(\hat{\boldsymbol{\xi}}_i^+, \hat{\boldsymbol{\xi}}_i^-), \hat{\boldsymbol{\xi}}_i^{\pm} \triangleq \hat{\boldsymbol{\xi}}_i \pm \hat{\boldsymbol{\zeta}}_i, \hat{\boldsymbol{\xi}}_i \in \mathfrak{m}, \hat{\boldsymbol{\zeta}}_i \in \mathfrak{h}_{\mathfrak{m}}, i = 1, \ldots, k$ locally generates $\mathrm{M} = \langle \exp \mathfrak{m} \rangle$ if:*

$$\{\hat{\boldsymbol{\xi}}_1, \ldots, \hat{\boldsymbol{\xi}}_k\}_{\mathrm{sp}} = \mathfrak{m}$$

*Condition (15) is equivalent to*

$$\{\hat{\boldsymbol{\xi}}_1^{\pm}, \ldots, \hat{\boldsymbol{\xi}}_k^{\pm}\}_{\mathrm{sp}} \oplus \mathfrak{h}_{\mathfrak{m}} = \mathfrak{g}_{\mathfrak{m}} \tag{16}$$

*if $\mathfrak{h}_{\mathfrak{m}} \cap \mathfrak{m} = \{\hat{\mathrm{o}}\}$.*

*Proof.* See Appendix F. □

We emphasize that in the above proposition, both $\{\hat{\boldsymbol{\xi}}_i^+\}_{i=1}^k$ and $\{\hat{\boldsymbol{\xi}}_i^-\}_{i=1}^k$ are linearly independent sets of twists, twists of the SC $(\hat{\boldsymbol{\xi}}_1^+, \ldots, \hat{\boldsymbol{\xi}}_k^+; \hat{\boldsymbol{\xi}}_k^-, \ldots, \hat{\boldsymbol{\xi}}_1^-)$ may become linearly dependent.

**Corollary 1.** *Given a SP $(\hat{\boldsymbol{\xi}}^+, \hat{\boldsymbol{\xi}}^-)$ of a LTS $\mathfrak{m} \subset \mathfrak{se}(3)$ and a twist $\hat{\boldsymbol{\eta}} \in \mathfrak{m}$, the pair $(\hat{\boldsymbol{\xi}'}^+, \hat{\boldsymbol{\xi}'}^-)$ defined by:*

$$\begin{cases} \hat{\boldsymbol{\xi}'}^+ \triangleq \mathrm{Ad}_{e^{+\hat{\boldsymbol{\eta}}}} \hat{\boldsymbol{\xi}}^+ \\ \hat{\boldsymbol{\xi}'}^- \triangleq \mathrm{Ad}_{e^{-\hat{\boldsymbol{\eta}}}} \hat{\boldsymbol{\xi}}^- \end{cases} \tag{17}$$

*is also a SP of $\mathfrak{m}$. In particular, $(\mathrm{Ad}_{e^{+\hat{\boldsymbol{\eta}}}} \hat{\boldsymbol{\xi}}, \mathrm{Ad}_{e^{+\hat{\boldsymbol{\eta}}}} \hat{\boldsymbol{\xi}}), \hat{\boldsymbol{\xi}} \in \mathfrak{m}$ is a SP of $\mathfrak{m}$.*

*Proof.* See Appendix G. □

### B. *Symmetry type of symmetric chains*

We shall use both the *geometric condition* (17) and the *algebraic condition* (15) to study the particular symmetry type of SPs and SCs for each LTS.

*1) $\mathfrak{m}_{2B}$:* Consider the conjugacy class of 2D LTS $\mathfrak{m}_{2B} = \{\hat{\mathrm{e}}_4, \hat{\mathrm{e}}_5\}_{\mathrm{sp}}$ as shown in Fig. 3(b). It consists of a pencil of zero-pitch twists in the char. plane (Hunt's 1st special 2-system, [30]). A SP $(\hat{\boldsymbol{\xi}}^+; \hat{\boldsymbol{\xi}}^-) = (\mathrm{Ad}_{e^{+\psi\hat{\boldsymbol{\eta}}}} \hat{\boldsymbol{\xi}}; \mathrm{Ad}_{e^{-\psi\hat{\boldsymbol{\eta}}}} \hat{\boldsymbol{\xi}}), \hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}} \in \mathfrak{m}_{2B}$ is generated by a pair of rotational displacements $(+\psi, -\psi)$ of $\hat{\boldsymbol{\xi}}$ about the unit twist $\hat{\boldsymbol{\eta}}$. Pictorially, $\hat{\boldsymbol{\xi}}^+$ and $\hat{\boldsymbol{\xi}}^-$ are mirror symmetric about the char. plane of $\mathfrak{m}_{2B}$ (see Fig. 4 left). Equivalently, the SP is given by the algebraic condition:

$$\begin{cases} \hat{\boldsymbol{\xi}}^+ = c_\psi \hat{\boldsymbol{\xi}} + s_\psi \hat{\mathrm{e}}_6 \\ \hat{\boldsymbol{\xi}}^- = c_\psi \hat{\boldsymbol{\xi}} - s_\psi \hat{\mathrm{e}}_6 \end{cases}$$

where $\hat{\boldsymbol{\xi}} \in \mathfrak{m}_{2B}$ and $\hat{\mathrm{e}}_6 \in \mathfrak{h}_{\mathfrak{m}_{2B}} = \{\hat{\mathrm{e}}_6\}_{\mathrm{sp}}$. Since $\mathfrak{g}_{\mathfrak{m}_{2B}} = \{\hat{\mathrm{e}}_4, \hat{\mathrm{e}}_5, \hat{\mathrm{e}}_6\}_{\mathrm{sp}}$, members of the $\mathfrak{m}_{2B}$-SC can be arbitrary twists not equal to scalar multiples of $\hat{\mathrm{e}}_6$; a $\mathfrak{m}_{2B}$-SC consists of two nesting SPs:

$$\begin{cases} \hat{\boldsymbol{\xi}}_1^+ = \hat{\boldsymbol{\xi}}_1 + \hat{\boldsymbol{\zeta}}_1 \\ \hat{\boldsymbol{\xi}}_1^- = \hat{\boldsymbol{\xi}}_1 - \hat{\boldsymbol{\zeta}}_1 \end{cases} \quad \begin{cases} \hat{\boldsymbol{\xi}}_2^+ = \hat{\boldsymbol{\xi}}_2 + \hat{\boldsymbol{\zeta}}_2 \\ \hat{\boldsymbol{\xi}}_2^- = \hat{\boldsymbol{\xi}}_2 - \hat{\boldsymbol{\zeta}}_2 \end{cases}$$

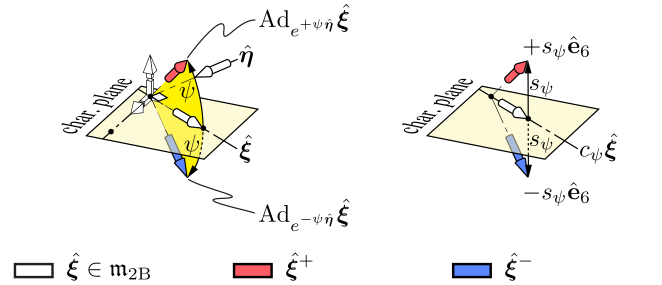



Fig. 4. Symmetric twist pairs of $\mathfrak{m}_{2\mathrm{B}}$. On the left: geometric condition, and on the right: algebraic condition.

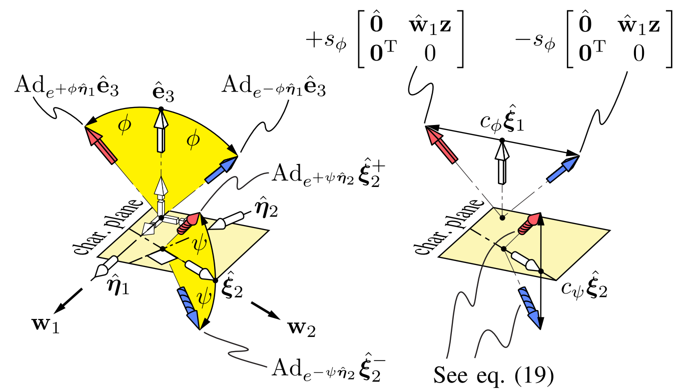



Fig. 5. Symmetric twist pairs of $\mathfrak{m}_{3\mathrm{B}}$. On the left: geometric condition, and on the right: algebraic condition.

where $\hat{\xi}_1, \hat{\xi}_2 \in \mathfrak{m}_{2\mathrm{B}} = \{\hat{e}_4, \hat{e}_5\}_{\mathrm{sp}}$ and $\hat{\zeta}_1, \hat{\zeta}_2 \in \mathfrak{h}_{\mathfrak{m}_{2\mathrm{B}}} = \{\hat{e}_6\}_{\mathrm{sp}}$, and such that:

$$\{\hat{\xi}_1, \hat{\xi}_2\}_{\mathrm{sp}} = \mathfrak{m} = \{\hat{e}_4, \hat{e}_5\}_{\mathrm{sp}}.$$

We shall refer to $(\hat{\xi}_1^+, \hat{\xi}_2^+; \hat{\xi}_2^-, \hat{\xi}_1^-)$ as an *even* SC. When $\hat{\zeta}_2 = 0$ and $\hat{\xi}_2^+ = \hat{\xi}_2^- = \hat{\xi}_2$, we can lump the two twists together and have an *odd* $\mathfrak{m}_{2\mathrm{B}}$-SC $(\hat{\xi}_1^+; \hat{\xi}_2; \hat{\xi}_1^-)$.

Therefore, a $\mathfrak{m}_{2\mathrm{B}}$-SC is a concentric $\mathcal{RRRR}$ or $\mathcal{RRR}$ chain with bilateral symmetry about the char. plane. As we have pointed out earlier, without the symmetric movement condition (14), a $\mathfrak{m}$-SC may generate a general subset of the completion group $\mathrm{G_M} = \langle \exp \mathfrak{g}_{\mathfrak{m}} \rangle$ instead of $\langle \exp \mathfrak{m} \rangle$. Twists of a SC need not be linearly independent (or non-redundant) either. Since $\mathfrak{g}_{\mathfrak{m}_{2\mathrm{B}}}$ is the three dimensional spherical Lie algebra $\mathfrak{so}(3)$ of $\mathrm{SO}(3)$, the concentric 4-$\mathcal{R}$ chain is necessarily redundant. When $\hat{\zeta}_1 \neq 0$, the 3-$\mathcal{R}$ chain is a non-redundant $\mathfrak{so}(3)$-chain. When $\hat{\zeta}_1 = \hat{\zeta}_2 = 0$, $\hat{\xi}_i^+ = \hat{\xi}_i^- = \hat{\xi}_i \in \mathfrak{m}_{2\mathrm{B}}, i = 1, 2$ and we have a singular $\mathfrak{so}(3)$-chain. Both the odd $\mathfrak{m}_{2\mathrm{B}}$-SC and the even $\mathfrak{m}_{2\mathrm{B}}$-SC can be found in the design of novel CV joints [51,52]. □

*2) $\mathfrak{m}_{3\mathrm{B}}$:* Consider the conjugacy class of 3D LTS $\mathfrak{m}_{3\mathrm{B}} = \{\hat{e}_3, \hat{e}_4, \hat{e}_5\}_{\mathrm{sp}}$ as shown in Fig. 3(d), with $\mathfrak{h}_{\mathfrak{m}_{3\mathrm{B}}} = \{\hat{e}_1, \hat{e}_2, \hat{e}_6\}_{\mathrm{sp}}$. It consists of a field of zero-pitch twists in, and an infinite-pitch twist perpendicular to, the char. plane (Hunt's 4th special 3-system, [30]). Its SP can be one of the following cases (see Fig. 5):

a) A pair of infinite-pitch twists $(\hat{\xi}_1^+, \hat{\xi}_1^-)$ given by:

$$\begin{cases} \hat{\xi}_1^+ \triangleq \mathrm{Ad}_{e^{+\phi\hat{\eta}_1}}\hat{e}_3 = c_\phi \hat{e}_3 + s_\phi \begin{bmatrix} \hat{\mathbf{0}} & \hat{\mathbf{w}}_1\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} \\ \hat{\xi}_1^- \triangleq \mathrm{Ad}_{e^{-\phi\hat{\eta}_1}}\hat{e}_3 = c_\phi \hat{e}_3 - s_\phi \begin{bmatrix} \hat{\mathbf{0}} & \hat{\mathbf{w}}_1\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} \end{cases} \tag{18}$$

where $\hat{\eta}_1 \in \mathfrak{m}_{3\mathrm{B}}$, with unit direction vector $\mathbf{w}_1$. The second half of (18) gives the algebraic condition for the same SP, where

$$\begin{bmatrix} \hat{\mathbf{0}} & \hat{\mathbf{w}}_1\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} \in \mathfrak{h}_{\mathfrak{m}_{3\mathrm{B}}} = \{\hat{e}_1, \hat{e}_2, \hat{e}_6\}_{\mathrm{sp}}$$

The SP corresponds to a pair of prismatic joints that are symmetric about the $\mathbf{z}$-axis in a plane containing them, and is the same as a mirror symmetry about the char. plane if we flip the direction of $\hat{\xi}_1^-$. This will make no difference in type synthesis but will reverse the joint variable for $\hat{\xi}_1^-$ in the symmetric movement condition (14).

b) A pair of finite-pitch twists $(\hat{\xi}'^+_2, \hat{\xi}'^-_2)$ with pitch $(p, -p)$, given by:

$$\begin{cases} \hat{\xi}'^+_2 \triangleq \mathrm{Ad}_{e^{+\psi\hat{\eta}_2}}\hat{\xi}_2^+ \\ \hat{\xi}'^-_2 \triangleq \mathrm{Ad}_{e^{-\psi\hat{\eta}_2}}\hat{\xi}_2^- \end{cases}$$

with $\hat{\eta}_2 \in \mathfrak{m}_{3\mathrm{B}}$ having a unit direction vector of $\mathbf{w}_2$, and $(\hat{\xi}_2^+, \hat{\xi}_2^-)$ is another SP defined by algebraic condition:

$$\begin{cases} \hat{\xi}_2^+ \triangleq \hat{\xi}_2 + p \begin{bmatrix} \hat{\mathbf{0}} & \mathbf{w}_2 \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} \\ \hat{\xi}_2^- \triangleq \hat{\xi}_2 - p \begin{bmatrix} \hat{\mathbf{0}} & \mathbf{w}_2 \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} \end{cases}$$

$(\hat{\xi}'^+_2, \hat{\xi}'^-_2)$ can also be directly derived from the algebraic condition:

$$\begin{cases} \hat{\xi}'^+_2 = \left(c_\psi \hat{\xi}_2 + s_\psi p \hat{e}_3\right) + \left(s_\psi \hat{e}_6 + c_\psi p \begin{bmatrix} \hat{\mathbf{0}} & \mathbf{w}_2 \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix}\right) \\ \hat{\xi}'^-_2 = \left(c_\psi \hat{\xi}_2 + s_\psi p \hat{e}_3\right) - \left(s_\psi \hat{e}_6 + c_\psi p \begin{bmatrix} \hat{\mathbf{0}} & \mathbf{w}_2 \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix}\right) \end{cases} \tag{19}$$

The SP corresponds to a pair of helical joints with equal and opposite pitches $p$ and $-p$ (or a pair of revolute joints if $p = 0$), and is mirror symmetric about the char. plane. This gives an explicit proof of Hunt's observation that mirror symmetric helical joints in a CV kinematic chain should have equal and opposite pitches [35].

Note that when $\hat{\eta}_2 \in \mathfrak{m}_{3B}$ is a zero-pitch twist, the axes of the two helical joints intersects at a point on the char. plane; when $\hat{\eta}_2$ is chosen to be the infinite-pitch twist $\hat{e}_3$, the axes of the two helical joints become parallel.

A $\mathfrak{m}_{3\mathrm{B}}$-SC consists of three nesting SPs $\{\hat{\xi}_i^+; \hat{\xi}_i^-\}_{i=1}^3$, each being one of the aforementioned cases. According to (16), the three twists $\hat{\xi}_1^+, \hat{\xi}_2^+, \hat{\xi}_3^+ \in \mathfrak{g}_{\mathfrak{m}_{3\mathrm{B}}} = \mathfrak{se}(3)$ must be chosen such that:

$$\{\hat{\xi}_1^+, \hat{\xi}_2^+, \hat{\xi}_3^+\}_{\mathrm{sp}} \oplus \{\hat{e}_1, \hat{e}_2, \hat{e}_6\}_{\mathrm{sp}} = \mathfrak{se}(3)$$

The enumeration of eligible candidates is studied in our earlier work [53]. The two most commonly seen $\mathfrak{m}_{3\mathrm{B}}$-SCs [35] are the mirror symmetric $\mathcal{RER}$ chain, which is equivalent to a mirror symmetric $\mathcal{RRPRR}$ chain, and the mirror symmetric $\mathcal{RSR}$ chain, which is equivalent to a mirror symmetric 5-$\mathcal{R}$

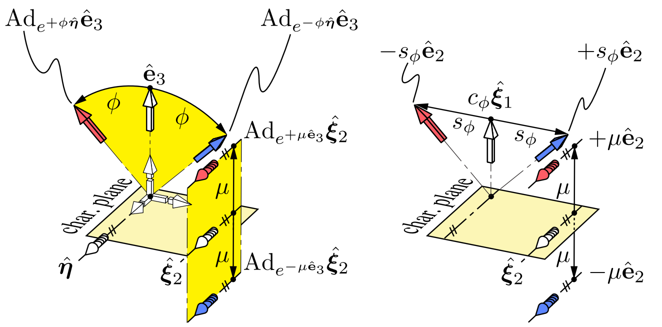



Fig. 6. Symmetric twist pairs of $\mathfrak{m}_{2\mathrm{A}}^p$ (or $\mathfrak{m}_{2\mathrm{A}}$ by letting $p = 0$). On the left: geometric condition, and on the right: algebraic condition.

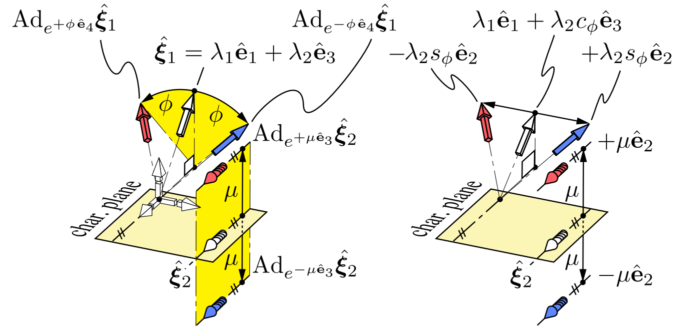



Fig. 7. Symmetric twist pairs of $\mathfrak{m}_{3\mathrm{A}}$. On the left: geometric condition, and on the right: algebraic condition.

chain. When (14) is not enforced, these chains generate 5D submanifolds of SE(3). $\square$

The above results corroborate Hunt's exhaustive classification of 3D CV chains in [35]. It also confirms our earlier conclusion that $\mathrm{M}_{3B} \triangleq \langle \exp \mathfrak{m}_{3\mathrm{B}} \rangle$ is indeed the motion type of 3-DoF CV couplings. Moreover, the mirror symmetry is a manifestation of the inversion symmetry of the underlying symmetric submanifold.

*3) $\mathfrak{m}_{2\mathrm{A}}$ **and** $\mathfrak{m}_{2\mathrm{A}}^p$:* Consider the conjugacy class of 2D LTS $\mathfrak{m}_{2\mathrm{A}}^p = \{\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4 + p\hat{\mathbf{e}}_1\}_{\mathrm{sp}}$ (and also $\mathfrak{m}_{2\mathrm{A}} = \{\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}_{\mathrm{sp}}$ by letting $p = 0$) as shown in Fig. 3(a), with $\mathfrak{h}_{\mathfrak{m}_{2\mathrm{A}}^p} = \{\hat{\mathbf{e}}_2\}_{\mathrm{sp}}$. It consists of all twists with pitch $p$ parallel to the $\mathbf{x}$-axis in the char. plane, and also a twist of infinite pitch perpendicular to the char. plane (Hunt's 2nd special 2-system, [30]). Note that $\mathfrak{m}_{2\mathrm{A}}$ is a LTS (subsystem) of $\mathfrak{m}_{3\mathrm{B}}$; its SPs can be synthesized in a similar manner and therefore have the same type of symmetry: as shown in Fig. 6, $\mathfrak{m}_{2\mathrm{A}}$-SPs are mirror symmetric about the char. plane, with typical SPs given by:

$$\begin{cases} \hat{\boldsymbol{\xi}}_1^+ \triangleq \mathrm{Ad}_{e^{+\phi\hat{\boldsymbol{\eta}}}}\hat{\mathbf{e}}_3 = c_\phi\hat{\mathbf{e}}_3 + s_\phi\hat{\mathbf{e}}_2 \\ \hat{\boldsymbol{\xi}}_1^- \triangleq \mathrm{Ad}_{e^{-\phi\hat{\boldsymbol{\eta}}}}\hat{\mathbf{e}}_3 = c_\phi\hat{\mathbf{e}}_3 - s_\phi\hat{\mathbf{e}}_2 \end{cases}$$

$$\begin{cases} \hat{\boldsymbol{\xi}}_2^+ \triangleq \mathrm{Ad}_{e^{+\mu\hat{\mathbf{e}}_3}}\hat{\boldsymbol{\xi}}_2 = \hat{\boldsymbol{\xi}}_2 + \mu\hat{\mathbf{e}}_2 \\ \hat{\boldsymbol{\xi}}_2^- \triangleq \mathrm{Ad}_{e^{-\mu\hat{\mathbf{e}}_3}}\hat{\boldsymbol{\xi}}_2 = \hat{\boldsymbol{\xi}}_2 - \mu\hat{\mathbf{e}}_2 \end{cases}$$

$\mathfrak{m}_{2\mathrm{A}}^p$-SPs admit exactly the same symmetry type, with the zero-pitch SP replaced by a SP of pitch $p$. Unlike the case of $\mathfrak{m}_{3\mathrm{B}}$, the two finite-pitch twists in $(\hat{\boldsymbol{\xi}}_2^+; \hat{\boldsymbol{\xi}}_2^-)$ in Fig. 6 have equal but not opposite pitches for the obvious reason that the LTS $\mathfrak{m}_{2\mathrm{A}}^p$ itself admits finite-pitch twists.

Since $\mathfrak{g}_{\mathfrak{m}_{2\mathrm{A}}} = \{\hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}_{\mathrm{sp}}$ is the Lie algebra of the 3D planar Euclidean group, a $\mathfrak{m}_{2\mathrm{A}}$-SC $(\hat{\boldsymbol{\xi}}_1^+, \hat{\boldsymbol{\xi}}_2^+; \hat{\boldsymbol{\xi}}_2^-, \hat{\boldsymbol{\xi}}_1^-)$ should satisfy:

$$\{\hat{\boldsymbol{\xi}}_1^+, \hat{\boldsymbol{\xi}}_2^+\}_{\mathrm{sp}} \oplus \{\hat{\mathbf{e}}_2\}_{\mathrm{sp}} = \{\hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}_{\mathrm{sp}}$$

A $\mathfrak{m}_{2\mathrm{A}}$-SC can be one of the following:

a) A mirror symmetric $\mathcal{RRRR}$ or $\mathcal{RRR}$ chain with parallel axes;
b) A mirror symmetric $\mathcal{PRRP}$ or $\mathcal{PRP}$ chain with $\mathcal{R}$ perpendicular to the two $\mathcal{P}$'s;
c) A mirror symmetric $\mathcal{RPPR}$ or $\mathcal{RPR}$ with parallel $\mathcal{R}$'s both perpendicular to the $\mathcal{P}$'s.

These are all planar motion generators if the symmetric motion condition (14) is not enforced. Synthesis of $\mathfrak{m}_{2\mathrm{A}}^p$ follows exactly the same approach and have exactly the same result with all revolute joints replaced by helical joints with pitch $p$. Therefore $\mathfrak{m}_{2\mathrm{A}}^p$-SCs are planar helical motion generators when (14) is not enforced. $\square$

*4) $\mathfrak{m}_{3\mathrm{A}}$:* Consider the conjugacy class of 3D LTS $\mathfrak{m}_{3\mathrm{A}} = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}_{\mathrm{sp}}$ as shown in Fig. 3(c), with $\mathfrak{h}_{\mathfrak{m}_{3\mathrm{A}}} = \{\hat{\mathbf{e}}_2\}_{\mathrm{sp}}$. It consists of twists of all pitches on all lines parallel to the $\mathbf{x}$-axis in the char. plane ($\mathbf{xy}$-plane), and a twist of infinite-pitch perpendicular to the $\mathbf{xy}$-plane (Hunt's 10th special 3-system, [30]). From the fact that $\mathfrak{m}_{2\mathrm{A}}^{(p)} \subset \mathfrak{m}_{3\mathrm{A}}$, we see that any $\mathfrak{m}_{2\mathrm{A}}$-SPs and $\mathfrak{m}_{2\mathrm{A}}^p$-SPs are also $\mathfrak{m}_{3\mathrm{A}}$-SPs (compare Fig. 7 with Fig. 6). Besides, $\mathfrak{m}_{3\mathrm{A}}$ admits the following SP by the algebraic condition:

$$\begin{cases} \hat{\boldsymbol{\xi}}_1^+ \triangleq \mathrm{Ad}_{e^{+\phi\hat{\mathbf{e}}_4}}\hat{\boldsymbol{\xi}}_1 = \lambda_1\hat{\mathbf{e}}_1 + c_\phi\lambda_2\hat{\mathbf{e}}_3 - \lambda_2 s_\phi\hat{\mathbf{e}}_2 \\ \hat{\boldsymbol{\xi}}_1^- \triangleq \mathrm{Ad}_{e^{-\phi\hat{\mathbf{e}}_4}}\hat{\boldsymbol{\xi}}_1 = \lambda_1\hat{\mathbf{e}}_1 + c_\phi\lambda_2\hat{\mathbf{e}}_3 + \lambda_2 s_\phi\hat{\mathbf{e}}_2 \end{cases}$$

where $\hat{\boldsymbol{\xi}}_1 = \lambda_1\hat{\mathbf{e}}_1 + \lambda_2\hat{\mathbf{e}}_3$ for some real constants $\lambda_1, \lambda_2$ (see Fig. 7). The prismatic SP is no longer mirror symmetric about the $\mathbf{xy}$-plane, but instead becomes mirror symmetric about the $\mathbf{xz}$-plane.

Since $\mathfrak{g}_{\mathfrak{m}_{3\mathrm{A}}} = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}_{\mathrm{sp}}$ is the Lie algebra of the 4D Schönflies group, a $\mathfrak{m}_{3\mathrm{A}}$-SC $(\hat{\boldsymbol{\xi}}_1^+, \hat{\boldsymbol{\xi}}_2^+, \hat{\boldsymbol{\xi}}_3^+; \hat{\boldsymbol{\xi}}_3^-, \hat{\boldsymbol{\xi}}_2^-, \hat{\boldsymbol{\xi}}_1^-)$ should satisfy:

$$\{\hat{\boldsymbol{\xi}}_1^+, \hat{\boldsymbol{\xi}}_2^+, \hat{\boldsymbol{\xi}}_3^+\}_{\mathrm{sp}} \oplus \{\hat{\mathbf{e}}_2\}_{\mathrm{sp}} = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}_{\mathrm{sp}}$$

The enumeration of eligible candidates of $(\hat{\boldsymbol{\xi}}_1^+, \hat{\boldsymbol{\xi}}_2^+, \hat{\boldsymbol{\xi}}_3^+)$ can be found in [53]. Since $\mathfrak{m}_{3\mathrm{A}}$-SCs are 5 or 6-DoF chains (in comparison to the dimension of the Schönflies group being 4), they are redundant Schönflies motion generators in the absence of the symmetric motion condition (14). $\square$

*5) $\mathfrak{m}_4$:* Consider the conjugacy class of 4D LTS $\mathfrak{m}_4 = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}_{\mathrm{sp}}$ as shown in Fig. 3(e), with $\mathfrak{h}_{\mathfrak{m}_4} = \{\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_6\}_{\mathrm{sp}}$. It consists of twists of all pitches along the lines of pencils in each plane normal to the $\mathbf{z}$-axis, and that the centers of the pencils all lie on the $\mathbf{z}$-axis (Hunt's 5th special 4-system, [30]). Its SP can be one of the following cases (see Fig. 8):

a) A pair of finite-pitch twists $(\hat{\boldsymbol{\xi}}_1^+, \hat{\boldsymbol{\xi}}_1^-)$, both with pitch $p$, given by:

$$\begin{cases} \hat{\boldsymbol{\xi}}_1^+ \triangleq \mathrm{Ad}_{e^{+\phi\hat{\boldsymbol{\eta}}_1}e^{+\lambda\hat{\boldsymbol{\eta}}_1'}}\hat{\boldsymbol{\xi}}_1 \\ \hat{\boldsymbol{\xi}}_1^- \triangleq \mathrm{Ad}_{e^{-\phi\hat{\boldsymbol{\eta}}_1}e^{-\lambda\hat{\boldsymbol{\eta}}_1'}}\hat{\boldsymbol{\xi}}_1 \end{cases}$$

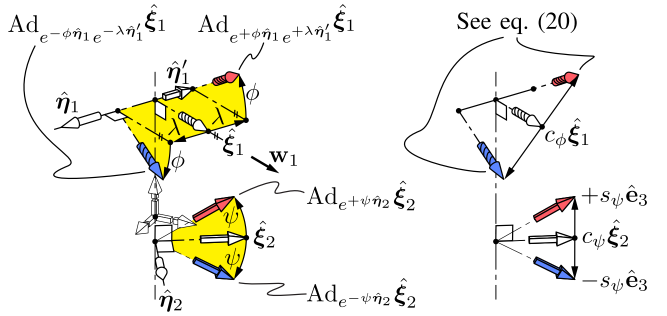



Fig. 8. Symmetric twist pairs of $\mathfrak{m}_4$. On the left: geometric condition, and on the right: algebraic condition.

where $\hat{\boldsymbol{\eta}}_1, \hat{\boldsymbol{\eta}}_1' \in \mathfrak{m}_4$ have pitches equal to $0$ and $\infty$ respectively, and $\hat{\boldsymbol{\xi}}_1 \in \mathfrak{m}_4$ has unit direction vector $\mathbf{w}_1$ and pitch $p$. It can also be given by the algebraic condition:

$$\begin{cases} \hat{\boldsymbol{\xi}}_1^+ = c_\phi \hat{\boldsymbol{\xi}}_1 + \lambda s_\phi \begin{bmatrix} \hat{\mathbf{0}} & \mathbf{w}_1 \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} + (pc_\phi - \lambda c_\phi)\hat{\mathbf{e}}_3 + s_\phi \hat{\mathbf{e}}_6 \\ \hat{\boldsymbol{\xi}}_1^- = c_\phi \hat{\boldsymbol{\xi}}_1 + \lambda s_\phi \begin{bmatrix} \hat{\mathbf{0}} & \mathbf{w}_1 \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix} - (pc_\phi - \lambda c_\phi)\hat{\mathbf{e}}_3 - s_\phi \hat{\mathbf{e}}_6 \end{cases} \tag{20}$$

b) A pair of infinite-pitch twists $(\hat{\boldsymbol{\xi}}_2^+, \hat{\boldsymbol{\xi}}_2^-)$, given by:

$$\begin{cases} \hat{\boldsymbol{\xi}}_2^+ \triangleq \mathrm{Ad}_{e^{+\psi\hat{\boldsymbol{\eta}}_2}} \hat{\boldsymbol{\xi}}_2 = c_\psi \hat{\boldsymbol{\xi}}_2 + s_\psi \hat{\mathbf{e}}_3 \\ \hat{\boldsymbol{\xi}}_2^- \triangleq \mathrm{Ad}_{e^{-\psi\hat{\boldsymbol{\eta}}_2}} \hat{\boldsymbol{\xi}}_2 = c_\psi \hat{\boldsymbol{\xi}}_2 - s_\psi \hat{\mathbf{e}}_3 \end{cases}$$

where $\hat{\boldsymbol{\eta}}_2 \in \mathfrak{m}_4$ has pitch $0$, and $\hat{\boldsymbol{\xi}}_2 \in \mathfrak{m}_4$ has infinite-pitch.

If we flip the direction of $\hat{\boldsymbol{\xi}}_i^-$'s, $(\hat{\boldsymbol{\xi}}_i^+, \hat{\boldsymbol{\xi}}_i^-), i = 1, 2$ both admit a 2-fold rotational symmetry about the $\mathbf{z}$-axis, with the two twists $(\hat{\boldsymbol{\xi}}_1^+, \hat{\boldsymbol{\xi}}_1^-)$ having equal finite pitch of $p$ (or in particular $0$). Therefore, a $\mathfrak{m}_4$-SP may be a pair of prismatic, helical or revolute joints.

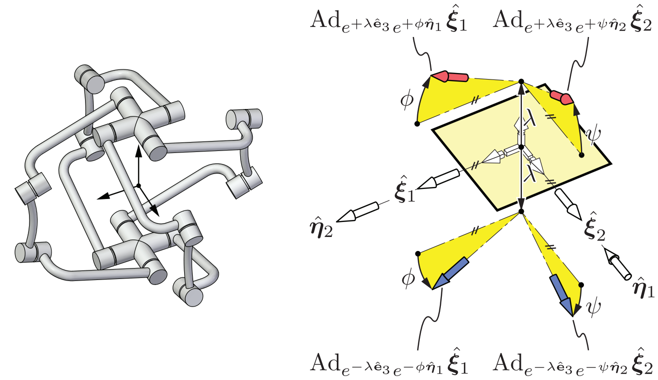
Since $\mathfrak{g}_{\mathfrak{m}_4} = \mathfrak{se}(3)$, a $\mathfrak{m}_4$-SC $(\hat{\boldsymbol{\xi}}_1^+, \ldots, \hat{\boldsymbol{\xi}}_4^+; \hat{\boldsymbol{\xi}}_4^-, \ldots, \hat{\boldsymbol{\xi}}_1^-)$ should satisfy:

$$\{\hat{\boldsymbol{\xi}}_1^+, \ldots, \hat{\boldsymbol{\xi}}_4^+\}_{\mathrm{sp}} \oplus \{\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_6\}_{\mathrm{sp}} = \mathfrak{se}(3)$$

Eligible candidates of $(\hat{\boldsymbol{\xi}}_1^+, \ldots, \hat{\boldsymbol{\xi}}_4^+)$ can be found in [53]. Since $\mathfrak{m}_4$-SCs are 7 or 8-DoF chains, they are redundant (and possibly, singular) $\mathrm{SE}(3)$ motion generators in the absence of the symmetric motion condition (14). □

We remark that since $\mathfrak{m}_5$ contains all other LTSs, its SPs can be any of the SPs of the aforementioned LTSs and do not possess a particular symmetry type. The derivation of its SCs can be conducted in a similar manner to the previous cases. Due to space limitations, a complete treatment of $\mathfrak{m}_5$-SPs and SCs will not be given here.

Finally, when a SC of a $k$D LTS $\mathfrak{m}$ has less than $k$ nesting SPs, its symmetric movement generates a submanifold of the symmetric submanifold $\mathrm{M} = \langle \exp \mathfrak{m} \rangle$. Such *incomplete SCs* are also prevalent in practice.



(a) (b)

Fig. 9. (a) a UNITRU coupling with four $\mathcal{UU}$-SCs; (b) twists of a $\mathcal{UU}$ chain as an incomplete $\mathfrak{m}_{3\mathrm{B}}$-SC.

**Example 3** ($\mathcal{UU}$ SC)**.** A $\mathcal{UU}$ SC (as shown in Fig. 9(b)) is briefly mentioned in [35] and studied in [36]. Its SPs are given by:

$$\begin{cases} \hat{\boldsymbol{\xi}}_1^+ = \mathrm{Ad}_{e^{+\lambda\hat{\mathbf{e}}_3}} \circ \mathrm{Ad}_{e^{+\phi\hat{\boldsymbol{\eta}}_1}} \hat{\boldsymbol{\xi}}_1 \triangleq \mathrm{Ad}_{e^{+\lambda\hat{\mathbf{e}}_3}} \hat{\boldsymbol{\zeta}}_1^+ \\ \hat{\boldsymbol{\xi}}_1^- = \mathrm{Ad}_{e^{-\lambda\hat{\mathbf{e}}_3}} \circ \mathrm{Ad}_{e^{-\phi\hat{\boldsymbol{\eta}}_1}} \hat{\boldsymbol{\xi}}_1 \triangleq \mathrm{Ad}_{e^{-\lambda\hat{\mathbf{e}}_3}} \hat{\boldsymbol{\zeta}}_1^- \end{cases}$$
$$\begin{cases} \hat{\boldsymbol{\xi}}_2^+ = \mathrm{Ad}_{e^{+\lambda\hat{\mathbf{e}}_3}} \circ \mathrm{Ad}_{e^{+\psi\hat{\boldsymbol{\eta}}_2}} \hat{\boldsymbol{\xi}}_2 \triangleq \mathrm{Ad}_{e^{+\lambda\hat{\mathbf{e}}_3}} \hat{\boldsymbol{\zeta}}_2^+ \\ \hat{\boldsymbol{\xi}}_2^- = \mathrm{Ad}_{e^{-\lambda\hat{\mathbf{e}}_3}} \circ \mathrm{Ad}_{e^{-\psi\hat{\boldsymbol{\eta}}_2}} \hat{\boldsymbol{\xi}}_2 \triangleq \mathrm{Ad}_{e^{-\lambda\hat{\mathbf{e}}_3}} \hat{\boldsymbol{\zeta}}_2^- \end{cases}$$

where $\lambda \in \mathbb{R}$, $\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2, \hat{\boldsymbol{\eta}}_1, \hat{\boldsymbol{\eta}}_2 \in \mathfrak{m}_{2\mathrm{B}}$. In other words, the $\mathcal{UU}$ SC is generated from a $\mathfrak{m}_{2\mathrm{B}}$ SC $(\hat{\boldsymbol{\zeta}}_1^+, \hat{\boldsymbol{\zeta}}_2^+; \hat{\boldsymbol{\zeta}}_2^-, \hat{\boldsymbol{\zeta}}_1^-)$ using a geometric condition (17) for $\mathfrak{m}_{3\mathrm{B}}$.

The SC generates a 2D submanifold of $\mathrm{M}_{3\mathrm{B}}$ under symmetric motion condition (14). In comparison to a $\mathfrak{m}_{3\mathrm{B}}$-SC, it has not a free but a dependent translational DoF [29]. Rosheim used this phenomenon to characterize the human shoulder complex movement [41].

The UNITRU coupling can be synthesized using three or more identical $\mathcal{UU}$ SCs with axial symmetry, since the generated 2D submanifold is conjugation invariant by elements of $\exp\{\hat{\mathbf{e}}_6\}_{\mathrm{sp}}$ [29]. □

## V. Conclusion

In this paper, we have systematically studied inversion symmetry of the special Euclidean group $\mathrm{SE}(3)$ and its symmetric submanifolds that arise from kinesiology and robot mechanical systems. Besides sharing many similarities with Lie subgroups of $\mathrm{SE}(3)$, symmetric submanifolds expand our general knowledge of motion types for the analysis and synthesis of many kinesiological joints or robot mechanical generators that defy a Lie group explanation.

The main contribution of our work is as follows. First, we have identified, for the first time, seven conjugacy classes of symmetric submanifolds of $\mathrm{SE}(3)$ (Table I), which are generated via inversion symmetry by the exponential image of Lie triple subsystems of $\mathfrak{se}(3)$. So far as the authors are aware of, this is also the first time the inversion symmetry of $\mathrm{SE}(3)$ is studied. Second, we found that these symmetric submanifolds share both a list of geometric properties and also a universal type synthesis method for their symmetric twist chains, which are kinematic chains comprising nesting symmetric twist pairs. Third, the symmetry type of the symmetric pairs and symmetric chains for each LTS is systematically studied, thereby systematically extending (if not completing) Hunt's observation made more than forty years ago.

Our ongoing work comprises: systematic type synthesis of parallel manipulators generating symmetric submanifolds and systematic kinematics and singularity analysis of such parallel manipulators (some preliminary case study is available in [50]); application of symmetric submanifolds of SE(3) in robot dynamics, planning and control; study on general Exp-submanifolds $\exp\mathfrak{n}$ with $\mathfrak{n}$ a general vector subspace of $\mathfrak{se}(3)$ that is neither a Lie subalgebra nor a Lie triple subsystem.

ACKNOWLEDGEMENTS

The first author thanks Dr.Jon Selig and Dr.Liu Guanfeng for some fruitful discussions.

## APPENDIX A
## SYMMETRIC SPACE AS HOMOGENEOUS SPACE [42,43]

The group of displacements $\tilde{\mathrm{G}}$ generated by $Q(\mathrm{SE}(3))$ is the Cartesian product $\mathrm{SE}(3)\times\mathrm{SE}(3)$ which acts on SE(3) via:

$$\tau:\tilde{\mathrm{G}}\times\mathrm{SE}(3)\to\mathrm{SE}(3),\ \ ((\mathbf{g}_1,\mathbf{g}_2),\mathbf{h})\mapsto\mathbf{g}_1\mathbf{h}\mathbf{g}_2^{-1} \tag{21}$$

where the quadratic representation $Q(\mathbf{g})$ of $\mathbf{g}\in\mathrm{SE}(3)$ is given by $(\mathbf{g},\mathbf{g}^{-1})\in\tilde{\mathrm{G}}$. The associated involution $\sigma:\tilde{\mathrm{G}}\to\tilde{\mathrm{G}}$ is given by:

$$\sigma(\mathbf{g}_1,\mathbf{g}_2)=S_{\mathbf{I}}\circ(\mathbf{g}_1,\mathbf{g}_2)\circ S_{\mathbf{I}}=(\mathbf{g}_2,\mathbf{g}_1)$$

the stabilizer $\mathrm{H}^\sigma$ of $\sigma$ equals $\{(\mathbf{g},\mathbf{g})|\mathbf{g}\in\mathrm{SE}(3)\}$, and is the same as the isotropy group $\tilde{\mathrm{H}}$ of $\mathbf{I}\in\mathrm{SE}(3)$.

The homogeneous space $\tilde{\mathrm{G}}/\tilde{\mathrm{H}}$ is diffeomorphic to SE(3) via:

$$\pi:\tilde{\mathrm{G}}/\tilde{\mathrm{H}}\to\mathrm{SE}(3),\ \ (\mathbf{g}_1,\mathbf{g}_2)\tilde{\mathrm{H}}\mapsto\mathbf{g}_1\mathbf{g}_2^{-1}$$

and $\mathrm{d}\pi_{\tilde{\mathrm{H}}}(\hat{\boldsymbol{\xi}}_1,\hat{\boldsymbol{\xi}}_2)=\hat{\boldsymbol{\xi}}_1-\hat{\boldsymbol{\xi}}_2$.

$\sigma$ induces an involution $\mathrm{d}\sigma_{(\mathbf{I},\mathbf{I})}$ on $\mathfrak{se}(3)\times\mathfrak{se}(3)$:

$$\mathrm{d}\sigma_{(\mathbf{I},\mathbf{I})}:(\hat{\boldsymbol{\xi}}_1,\hat{\boldsymbol{\xi}}_2)\mapsto(\hat{\boldsymbol{\xi}}_2,\hat{\boldsymbol{\xi}}_1)$$

Its two eigenspaces $\tilde{\mathfrak{h}}$ (of eigenvalue 1) and $\tilde{\mathfrak{m}}$ (of eigenvalue $-1$) are given by:

$$\tilde{\mathfrak{h}}\triangleq\{(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\xi}})|\hat{\boldsymbol{\xi}}\in\mathfrak{se}(3)\},\ \ \tilde{\mathfrak{m}}\triangleq\{(\hat{\boldsymbol{\xi}},-\hat{\boldsymbol{\xi}})|\hat{\boldsymbol{\xi}}\in\mathfrak{se}(3)\}$$

and satisfy:

$$[\tilde{\mathfrak{h}},\tilde{\mathfrak{h}}]\subset\tilde{\mathfrak{h}},\quad[\tilde{\mathfrak{h}},\tilde{\mathfrak{m}}]\subset\tilde{\mathfrak{m}},\quad[\tilde{\mathfrak{m}},\tilde{\mathfrak{m}}]\subset\tilde{\mathfrak{h}}$$

$\mathrm{d}\pi_{\tilde{\mathrm{H}}}$ maps the Lie subalgebra $\tilde{\mathfrak{h}}$ and LTS $\tilde{\mathfrak{m}}$ to $\{\hat{o}\}$ and $\mathfrak{se}(3)$ respectively, thus identifying $\tilde{\mathfrak{m}}$ with $\mathfrak{se}(3)$.

Given a symmetric submanifold M with $\mathrm{T}_{\mathbf{I}}\mathrm{M}=\mathfrak{m}$ and $\mathfrak{h}_{\mathfrak{m}}\triangleq[\mathfrak{m},\mathfrak{m}]$, define $\tilde{\mathfrak{m}}$ and $\tilde{\mathfrak{h}}$ by:

$$\tilde{\mathfrak{m}}\triangleq\{(\hat{\boldsymbol{\xi}},-\hat{\boldsymbol{\xi}})|\hat{\boldsymbol{\xi}}\in\mathfrak{m}\},\ \ \tilde{\mathfrak{h}}\triangleq\{(\hat{\boldsymbol{\zeta}},\hat{\boldsymbol{\zeta}})|\hat{\boldsymbol{\zeta}}\in\mathfrak{h}_{\mathfrak{m}}\}$$

The group of displacements of M is generated by the Lie algebra $\tilde{\mathfrak{g}}$ given by:

$$\tilde{\mathfrak{g}}\triangleq\tilde{\mathfrak{h}}\oplus\tilde{\mathfrak{m}}=\{(\hat{\boldsymbol{\xi}}+\hat{\boldsymbol{\zeta}},-\hat{\boldsymbol{\xi}}+\hat{\boldsymbol{\zeta}})|\hat{\boldsymbol{\xi}}\in\mathfrak{m},\hat{\boldsymbol{\zeta}}\in\mathfrak{h}_{\mathfrak{m}}\}$$

Therefore, $\tilde{\mathrm{G}}$ is generated by elements of the form $(e^{\hat{\boldsymbol{\xi}}+\hat{\boldsymbol{\zeta}}},e^{-\hat{\boldsymbol{\xi}}+\hat{\boldsymbol{\zeta}}})$ and a generic element of M is given by $\tau((e^{\hat{\boldsymbol{\xi}}+\hat{\boldsymbol{\zeta}}},e^{-\hat{\boldsymbol{\xi}}+\hat{\boldsymbol{\zeta}}}),\mathbf{g})=e^{\hat{\boldsymbol{\xi}}+\hat{\boldsymbol{\zeta}}}\mathbf{g}e^{\hat{\boldsymbol{\xi}}-\hat{\boldsymbol{\zeta}}},\forall\mathbf{g}\in\mathrm{M}$. The vector subspaces $\mathfrak{m},\mathfrak{h}_{\mathfrak{m}}$ and $\mathfrak{g}_{\mathfrak{m}}$ in Prop.2 b), c) are in fact the projection of $\tilde{\mathfrak{m}},\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{g}}$ into the first component of $\mathfrak{se}(3)\times\mathfrak{se}(3)$.

## APPENDIX B
## PROOF OF EQ. (8)

Please refer to [54] for basic concepts in differential geometry and Lie groups used in this proof.

Given any smooth function $f:\mathrm{SE}(3)\to\mathbb{R}$ and two right-invariant vector fields $\hat{\boldsymbol{\xi}}^r,\hat{\boldsymbol{\zeta}}^r$ defined by $\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\zeta}}\in\mathfrak{se}(3)$, we have for any $\mathbf{g}\in\mathrm{SE}(3)$:

$$\begin{aligned}[\hat{\boldsymbol{\xi}}^r,\hat{\boldsymbol{\zeta}}^r]f(\mathbf{g})&=(\hat{\boldsymbol{\xi}}^r\hat{\boldsymbol{\zeta}}^r-\hat{\boldsymbol{\zeta}}^r\hat{\boldsymbol{\xi}}^r)f(\mathbf{g})\\&=\frac{\mathrm{d}}{\mathrm{d}u}\hat{\boldsymbol{\zeta}}^rf(e^{u\hat{\boldsymbol{\xi}}}\mathbf{g})|_0-\frac{\mathrm{d}}{\mathrm{d}u}\hat{\boldsymbol{\xi}}^rf(e^{u\hat{\boldsymbol{\zeta}}}\mathbf{g})|_0\\&=\frac{\mathrm{d}}{\mathrm{d}u}\hat{\boldsymbol{\zeta}}^r(e^{u\hat{\boldsymbol{\xi}}}\mathbf{g})f|_0-\frac{\mathrm{d}}{\mathrm{d}u}\hat{\boldsymbol{\xi}}^r(e^{u\hat{\boldsymbol{\zeta}}}\mathbf{g})f|_0\\&=\frac{\mathrm{d}}{\mathrm{d}u}(\hat{\boldsymbol{\zeta}}e^{u\hat{\boldsymbol{\xi}}}\mathbf{g})f|_0-\frac{\mathrm{d}}{\mathrm{d}u}(\hat{\boldsymbol{\xi}}e^{u\hat{\boldsymbol{\zeta}}}\mathbf{g})f|_0\\&=(\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\xi}}\mathbf{g}-\hat{\boldsymbol{\xi}}\hat{\boldsymbol{\zeta}}\mathbf{g})f=-[\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\zeta}}]^rf(\mathbf{g})\end{aligned}$$

and therefore $[\hat{\boldsymbol{\xi}}^r,\hat{\boldsymbol{\zeta}}^r]=-[\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\zeta}}]^r$.

Similarly,

$$\begin{aligned}[\hat{\boldsymbol{\xi}}^l,\hat{\boldsymbol{\zeta}}^l]f(\mathbf{g})&=(\hat{\boldsymbol{\xi}}^l\hat{\boldsymbol{\zeta}}^l-\hat{\boldsymbol{\zeta}}^l\hat{\boldsymbol{\xi}}^l)f(\mathbf{g})\\&=\frac{\mathrm{d}}{\mathrm{d}u}\hat{\boldsymbol{\zeta}}^lf(\mathbf{g}e^{u\hat{\boldsymbol{\xi}}})|_0-\frac{\mathrm{d}}{\mathrm{d}u}\hat{\boldsymbol{\xi}}^lf(\mathbf{g}e^{u\hat{\boldsymbol{\zeta}}})|_0\\&=\frac{\mathrm{d}}{\mathrm{d}u}\hat{\boldsymbol{\zeta}}^l(\mathbf{g}e^{u\hat{\boldsymbol{\xi}}})f|_0-\frac{\mathrm{d}}{\mathrm{d}u}\hat{\boldsymbol{\xi}}^l(\mathbf{g}e^{u\hat{\boldsymbol{\zeta}}})f|_0\\&=\frac{\mathrm{d}}{\mathrm{d}u}(\mathbf{g}e^{u\hat{\boldsymbol{\xi}}}\hat{\boldsymbol{\zeta}})f|_0-\frac{\mathrm{d}}{\mathrm{d}u}(\mathbf{g}e^{u\hat{\boldsymbol{\zeta}}}\hat{\boldsymbol{\xi}})f|_0\\&=(\mathbf{g}\hat{\boldsymbol{\xi}}\hat{\boldsymbol{\zeta}}-\mathbf{g}\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\xi}})f=[\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\zeta}}]^lf(\mathbf{g})\end{aligned}$$

and therefore $[\hat{\boldsymbol{\xi}}^l,\hat{\boldsymbol{\zeta}}^l]=[\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\zeta}}]^l$.

Finally,

$$\begin{aligned}[\hat{\boldsymbol{\xi}}^r,\hat{\boldsymbol{\zeta}}^l]f(\mathbf{g})&=(\hat{\boldsymbol{\xi}}^r\hat{\boldsymbol{\zeta}}^l-\hat{\boldsymbol{\zeta}}^l\hat{\boldsymbol{\xi}}^r)f(\mathbf{g})\\&=\frac{\mathrm{d}}{\mathrm{d}u}\hat{\boldsymbol{\zeta}}^lf(e^{u\hat{\boldsymbol{\xi}}}\mathbf{g})|_0-\frac{\mathrm{d}}{\mathrm{d}u}\hat{\boldsymbol{\xi}}^rf(\mathbf{g}e^{u\hat{\boldsymbol{\zeta}}})|_0\\&=\frac{\mathrm{d}}{\mathrm{d}u}\hat{\boldsymbol{\zeta}}^l(e^{u\hat{\boldsymbol{\xi}}}\mathbf{g})f|_0-\frac{\mathrm{d}}{\mathrm{d}u}\hat{\boldsymbol{\xi}}^r(\mathbf{g}e^{u\hat{\boldsymbol{\zeta}}})f|_0\\&=\frac{\mathrm{d}}{\mathrm{d}u}(e^{u\hat{\boldsymbol{\xi}}}\mathbf{g}\hat{\boldsymbol{\zeta}})f|_0-\frac{\mathrm{d}}{\mathrm{d}u}(\hat{\boldsymbol{\xi}}\mathbf{g}e^{u\hat{\boldsymbol{\zeta}}})f|_0\\&=(\hat{\boldsymbol{\xi}}\mathbf{g}\hat{\boldsymbol{\zeta}}-\hat{\boldsymbol{\xi}}\mathbf{g}\hat{\boldsymbol{\zeta}})f=0\end{aligned}$$

and therefore $[\hat{\boldsymbol{\xi}}^r,\hat{\boldsymbol{\zeta}}^l]=0$. □

## APPENDIX C
## PROOF OF THEOREM 1

In reference to (7), the group of displacements $\tilde{\mathrm{G}}(\mathrm{M})$ of M is generated by $Q(\exp\mathfrak{m})\triangleq\{Q(e^{\hat{\boldsymbol{\xi}}})|\hat{\boldsymbol{\xi}}\in\mathfrak{m}\}$. Since $\tilde{\mathrm{G}}(\mathrm{M})$ acts transitively on M, M is recovered by letting $\tilde{\mathrm{G}}(\mathrm{M})$ act on the identity $\mathbf{I}\in\mathrm{M}$:

$$Q(e^{\hat{\boldsymbol{\xi}}})(\mathbf{I})=e^{\hat{\boldsymbol{\xi}}}\mathbf{I}e^{\hat{\boldsymbol{\xi}}}=e^{2\hat{\boldsymbol{\xi}}}\in\mathrm{M},\ \ \hat{\boldsymbol{\xi}}\in\mathfrak{m}$$

Therefore $\exp\mathfrak{m}\subset\mathrm{M}$ and also contains an open neighborhood of M, which generates M via inversion symmetry, i.e. M =

$\{\mathbf{g}\mathbf{h}^{-1}\mathbf{g} \in \mathrm{SE}(3) | \forall \mathbf{g}, \mathbf{h} \in \exp \mathfrak{m}\}$. For more details, see [43, pp. 95, Prop. 3.2].

To see that $\{\mathbf{g}\mathbf{h}\mathbf{g} \in \mathrm{SE}(3) | \forall \mathbf{g}, \mathbf{h} \in \exp \mathfrak{m}\}$ is the same as $\{\mathbf{g}\mathbf{h}^{-1}\mathbf{g} \in \mathrm{SE}(3) | \forall \mathbf{g}, \mathbf{h} \in \exp \mathfrak{m}\}$, note that if $\mathbf{h} = e^{\hat{\boldsymbol{\xi}}} \in \exp \mathfrak{m}$, so is $\mathbf{h}^{-1} = e^{-\hat{\boldsymbol{\xi}}} \in \exp \mathfrak{m}$. □

## APPENDIX D
## PROOF OF PROPOSITION 1

The proposition is proved by straightforward computation, as follows. Consider first the 2D LTS $\mathfrak{m}_{2\mathrm{A}} = \{\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}_{\mathrm{sp}}$. $\exp \mathfrak{m}_{2\mathrm{A}}$ comprises homogeneous matrices of the form:

$$\exp(\theta_1 \hat{\mathbf{e}}_4 + \theta_2 \hat{\mathbf{e}}_3) = \begin{bmatrix} e^{\theta_1 \hat{\mathbf{x}}} & \theta_2' e^{\frac{\theta_1}{2}\hat{\mathbf{x}}}\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}$$

where $\theta_1, \theta_2 \in \mathbb{R}$ and $\theta_2' = 2 s_{\theta_1/2} \theta_2$, and also

$$\exp \theta_2 \hat{\mathbf{e}}_3 = \begin{bmatrix} \mathbf{I} & \theta_2 \mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}$$

Combining the two cases, $\exp \mathfrak{m}_{2\mathrm{A}}$ comprises elements of the form:

$$\begin{bmatrix} e^{\theta_1 \hat{\mathbf{x}}} & \theta_2 e^{\frac{\theta_1}{2}\hat{\mathbf{x}}}\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}, \ \theta_1, \theta_2 \in \mathbb{R}$$

To see these are the only elements of $\mathrm{M}_{2\mathrm{A}}$, compute $\mathbf{g}\mathbf{h}\mathbf{g}$ for any $\mathbf{g}, \mathbf{h} \in \exp \mathfrak{m}_{2\mathrm{A}}$ (see eq. (11)):

$$\underbrace{\begin{bmatrix} e^{\theta_1 \hat{\mathbf{x}}} & \theta_2 e^{\frac{\theta_1}{2}\hat{\mathbf{x}}}\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}}_{\mathbf{g}} \underbrace{\begin{bmatrix} e^{\theta_1' \hat{\mathbf{x}}} & \theta_2' e^{\frac{\theta_1'}{2}\hat{\mathbf{x}}}\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}}_{\mathbf{h}} \underbrace{\begin{bmatrix} e^{\theta_1 \hat{\mathbf{x}}} & \theta_2 e^{\frac{\theta_1}{2}\hat{\mathbf{x}}}\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}}_{\mathbf{g}}$$
$$= \begin{bmatrix} e^{(2\theta_1 + \theta_1')\hat{\mathbf{x}}} & (\theta_2' + 2c_{(\theta_1+\theta_1')/2}) e^{\frac{2\theta_1+\theta_1'}{2}\hat{\mathbf{x}}}\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}$$
$$\in \exp\ \mathfrak{m}_{2\mathrm{A}}$$

This verifies that $\mathrm{M}_{2\mathrm{A}} = \exp \mathfrak{m}_{2\mathrm{A}}$, or $\exp : \mathfrak{m}_{2\mathrm{A}} \to \mathrm{M}_{2\mathrm{A}}$ is surjective. In a similar way, we can verify that exp is also surjective for $\mathfrak{m}_{2\mathrm{B}}, \mathfrak{m}_{3\mathrm{A}}, \mathfrak{m}_{3\mathrm{B}}, \mathfrak{m}_4$ and $\mathfrak{m}_5$.

For $\mathfrak{m}_{2\mathrm{A}}^p = \{\hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4 + p\hat{\mathbf{e}}_1\}_{\mathrm{sp}}$, it can be verified that the only elements in $\mathrm{M}_{2\mathrm{A}}^p$ that do not correspond to elements in $\exp \mathfrak{m}_{2\mathrm{A}}^p$ are of the form (c.f. [48, Prop. 2.2]):

$$\begin{bmatrix} \mathbf{I} & 2ip\pi\mathbf{x} + \theta\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}, \ i \in \mathbb{Z} - \{0\}, \theta \in \mathbb{R} - \{0\}$$

which can be generated by $\exp \mathfrak{m}_{2\mathrm{A}}^p$ via only one inversion:

$$\begin{bmatrix} \mathbf{I} & 2ip\pi\mathbf{x} + \theta\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} =$$
$$\begin{bmatrix} \mathbf{I} & \frac{\theta}{2}\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 2ip\pi\mathbf{x} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \frac{\theta}{2}\mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}$$
$$= e^{\frac{\theta}{2}\hat{\mathbf{e}}_3} e^{2i\pi(\hat{\mathbf{e}}_4 + p\hat{\mathbf{e}}_1)} e^{\frac{\theta}{2}\hat{\mathbf{e}}_3}$$

In particular exp is not surjective for $\mathfrak{m}_{2\mathrm{A}}^p$. □

## APPENDIX E
## PROOF OF PROPOSITION 2

**Proof of a):** Since $Q(e^{\hat{\boldsymbol{\xi}}/2})$ is a diffeomorphism sending $\mathbf{I}$ to $\mathbf{g} = e^{\hat{\boldsymbol{\xi}}}$, it induces an isomorphism from $\mathfrak{m}$ onto $\mathrm{T}_{e^{\hat{\boldsymbol{\xi}}}}\mathrm{M}$. Therefore:

$$\begin{aligned} (\mathrm{T}_{\mathbf{g}}\mathrm{M})\mathbf{g}^{-1} &= (e^{\hat{\boldsymbol{\xi}}/2}\mathfrak{m}e^{\hat{\boldsymbol{\xi}}/2})e^{-\hat{\boldsymbol{\xi}}} \\ &= e^{\hat{\boldsymbol{\xi}}/2}\mathfrak{m}e^{-\hat{\boldsymbol{\xi}}/2} = \mathrm{Ad}_{e^{\hat{\boldsymbol{\xi}}/2}}\mathfrak{m} \end{aligned}$$

Similarly, $Q(e^{\hat{\boldsymbol{\xi}}_1}) \circ Q(e^{\hat{\boldsymbol{\xi}}_2/2})$ sends $\mathbf{I}$ to $\mathbf{g} = e^{\hat{\boldsymbol{\xi}}_1}e^{\hat{\boldsymbol{\xi}}_2}e^{\hat{\boldsymbol{\xi}}_1}$ diffeomorphically, and

$$\begin{aligned} (\mathrm{T}_{\mathbf{g}}\mathrm{M})\mathbf{g}^{-1} &= (e^{\hat{\boldsymbol{\xi}}_1}e^{\hat{\boldsymbol{\xi}}_2/2}\mathfrak{m}e^{\hat{\boldsymbol{\xi}}_2/2}e^{\hat{\boldsymbol{\xi}}_1})e^{-\hat{\boldsymbol{\xi}}_1}e^{-\hat{\boldsymbol{\xi}}_2}e^{-\hat{\boldsymbol{\xi}}_1} \\ &= (e^{\hat{\boldsymbol{\xi}}_1}e^{\hat{\boldsymbol{\xi}}_2/2})\mathfrak{m}(e^{\hat{\boldsymbol{\xi}}_1}e^{\hat{\boldsymbol{\xi}}_2/2})^{-1} \\ &= \mathrm{Ad}_{e^{\hat{\boldsymbol{\xi}}_1}e^{\hat{\boldsymbol{\xi}}_2/2}}\mathfrak{m} \end{aligned}$$

**Proof of b):** For any $[\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2] \in \mathfrak{h}_{\mathfrak{m}}$ and $\hat{\boldsymbol{\xi}} \in \mathfrak{m}$:

$$\mathrm{Ad}_{e^{[\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2]}}\hat{\boldsymbol{\xi}} = e^{\mathrm{ad}_{[\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2]}}\hat{\boldsymbol{\xi}} = \sum_{i=0}^{\infty} \frac{\mathrm{ad}^i_{[\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2]}}{i!}\hat{\boldsymbol{\xi}} \in \mathfrak{m}. \qquad (22)$$

since $\mathrm{ad}_{[\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2]}\hat{\boldsymbol{\xi}} \triangleq [[\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2], \hat{\boldsymbol{\xi}}] \in \mathfrak{m}$ (since $\mathfrak{m}$ is a LTS). Here we have used the adjoint map $\mathrm{ad}_{(\cdot)}$ defined by:

$$\mathrm{ad}_{\hat{\boldsymbol{\xi}}_1}\hat{\boldsymbol{\xi}}_2 \triangleq [\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2], \quad \forall \hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2 \in \mathfrak{se}(3)$$

and the fact that ([19, pp. 54, Lemma 3.4.1]):

$$\mathrm{Ad}_{e^{\hat{\boldsymbol{\xi}}}} = e^{\mathrm{ad}_{\hat{\boldsymbol{\xi}}}}, \ \forall \hat{\boldsymbol{\xi}} \in \mathfrak{se}(3)$$

Now we have $\mathrm{Ad}_{e^{[\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2]}}\mathfrak{m} \subseteq \mathfrak{m}$. The equality holds because $\mathrm{Ad}_{e^{[\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2]}}$ is a linear isomorphism. Since $\mathrm{H}_{\mathrm{M}}$ is generated by $\exp \mathfrak{h}_{\mathfrak{m}}$ and $\mathrm{Ad}_{\mathbf{g}\mathbf{h}} = \mathrm{Ad}_{\mathbf{g}} \circ \mathrm{Ad}_{\mathbf{h}}, \forall \mathbf{g}, \mathbf{h} \in \mathrm{SE}(3)$, we have

$$\forall \mathbf{h} \in \mathrm{H}_{\mathrm{M}} \quad \Rightarrow \quad \mathrm{Ad}_{\mathbf{h}}\mathfrak{m} \equiv \mathfrak{m} \qquad (23)$$

Next, in light of Prop.1, for any $\forall \mathbf{g} \in \mathrm{M}$, either $\mathbf{g} = e^{\hat{\boldsymbol{\xi}}}, \hat{\boldsymbol{\xi}} \in \mathfrak{m}$ or $\mathbf{g} = e^{\hat{\boldsymbol{\xi}}}e^{\hat{\boldsymbol{\eta}}}e^{\hat{\boldsymbol{\xi}}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\xi}} \in \mathfrak{m}$. In the former case,

$$C(\mathbf{h})(e^{\hat{\boldsymbol{\xi}}}) = \mathbf{h}e^{\hat{\boldsymbol{\xi}}}\mathbf{h}^{-1} = e^{\mathrm{Ad}_{\mathbf{h}}\hat{\boldsymbol{\xi}}} \in \exp \mathfrak{m} \subseteq \mathrm{M}$$

by the previous equation (23). The latter case:

$$\begin{aligned} C(\mathbf{h})(e^{\hat{\boldsymbol{\xi}}}e^{\hat{\boldsymbol{\eta}}}e^{\hat{\boldsymbol{\xi}}}) &= \mathbf{h}e^{\hat{\boldsymbol{\xi}}}e^{\hat{\boldsymbol{\eta}}}e^{\hat{\boldsymbol{\xi}}}\mathbf{h}^{-1} \\ &= \mathbf{h}e^{\hat{\boldsymbol{\xi}}}\mathbf{h}^{-1}\mathbf{h}e^{\hat{\boldsymbol{\eta}}}\mathbf{h}^{-1}\mathbf{h}e^{\hat{\boldsymbol{\xi}}}\mathbf{h}^{-1} \\ &= C(\mathbf{h})(e^{\hat{\boldsymbol{\xi}}}) \cdot C(\mathbf{h})(e^{\hat{\boldsymbol{\eta}}}) \cdot C(\mathbf{h})(e^{\hat{\boldsymbol{\xi}}}) \end{aligned}$$

is reduced to the former. Therefore we have $C(\mathbf{h})(\mathrm{M}) \subseteq \mathrm{M}$. The equality holds since $C(\mathbf{h})$ is a diffeomorphism for any $\mathbf{h} \in \mathrm{SE}(3)$, i.e.

$$\forall \mathbf{h} \in \mathrm{H}_{\mathrm{M}} \quad \Rightarrow \quad C(\mathbf{h})(\mathrm{M}) \equiv \mathrm{M}$$

**Proof of c):** $\mathfrak{g}_{\mathfrak{m}} \triangleq \mathfrak{h}_{\mathfrak{m}} + \mathfrak{m}$ is a Lie subalgebra and therefore contains the completion algebra of $\mathfrak{m}$. On the other hand, the completion algebra should contain both $\mathfrak{m}$ and $\mathfrak{h}_{\mathfrak{m}} \triangleq [\mathfrak{m}, \mathfrak{m}]$. Therefore, $\mathfrak{h}_{\mathfrak{m}}$ is the completion algebra of $\mathfrak{m}$.

Next, since $\mathfrak{m} \subset \mathfrak{g}_{\mathfrak{m}}$, we have

$$\mathrm{M} = \langle \exp \mathfrak{m} \rangle \subset \langle \exp \mathfrak{g}_{\mathfrak{m}} \rangle = \mathrm{G}_{\mathrm{M}}$$

On the other hand, The Lie algebra of the (connected) completion group of M will be a Lie subalgebra of $\mathfrak{se}(3)$ containing

$\mathfrak{m}$ and therefore $\mathfrak{g}_{\mathfrak{m}}$. Therefore, $\mathrm{G_M}$ is indeed the completion group of $\mathrm{M}$.

If $\mathfrak{h}_{\mathfrak{m}} \cap \mathfrak{m} = \{\hat{o}\}$, we have $\mathfrak{g}_{\mathfrak{m}} = \mathfrak{h}_{\mathfrak{m}} \oplus \mathfrak{m}$ and

$$\widetilde{\exp}_{\mathbf{g}} : \mathfrak{m} \times \mathfrak{h}_{\mathfrak{m}} \to \mathrm{G_M}$$

defines a local diffeomorphism by the inverse function theorem [54]. $\square$

## Appendix F
## Proof of Proposition 3

The first claim is immediate, since according to Appendix A, $e^{\hat{\boldsymbol{\xi}}^+}\mathbf{g}e^{\hat{\boldsymbol{\xi}}^-}$ with

$$\begin{cases} \hat{\boldsymbol{\xi}}^+ = \hat{\boldsymbol{\xi}} + \hat{\boldsymbol{\zeta}} \\ \hat{\boldsymbol{\xi}}^- = \hat{\boldsymbol{\xi}} - \hat{\boldsymbol{\zeta}} \end{cases}, \qquad \hat{\boldsymbol{\xi}} \in \mathfrak{m}, \hat{\boldsymbol{\zeta}} \in \mathfrak{h}_{\mathfrak{m}}, \mathbf{g} \in \mathrm{M}$$

is a generic element of $\mathrm{M}$.

The second claim is the same as saying that when $\{\hat{\boldsymbol{\xi}}_1, \ldots, \hat{\boldsymbol{\xi}}_k\}_{\mathrm{sp}} = \mathfrak{m}$,

$$\{e^{\theta_1 \hat{\boldsymbol{\xi}}_1^+} \cdots e^{\theta_k \hat{\boldsymbol{\xi}}_k^+} e^{\theta_k \hat{\boldsymbol{\xi}}_k^-} \cdots e^{\theta_1 \hat{\boldsymbol{\xi}}_1^-} | \theta_1, \ldots, \theta_k \in \mathbb{R}\}$$

with $\hat{\boldsymbol{\xi}}_i^{\pm} = \hat{\boldsymbol{\xi}}_i \pm \hat{\boldsymbol{\zeta}}_i, \hat{\boldsymbol{\xi}}_i \in \mathfrak{m}, \hat{\boldsymbol{\zeta}}_i \in \mathfrak{h}_{\mathfrak{m}}$ is locally identical to $\mathrm{M} = \langle \exp \mathfrak{m} \rangle$. The Jacobian of the map $(\theta_1, \ldots, \theta_k) \mapsto e^{\theta_1 \hat{\boldsymbol{\xi}}_1^+} \cdots e^{\theta_k \hat{\boldsymbol{\xi}}_k^+} e^{\theta_k \hat{\boldsymbol{\xi}}_k^-} \cdots e^{\theta_1 \hat{\boldsymbol{\xi}}_1^-}$ at $(0, \ldots, 0)$ is given by:

$$\begin{aligned} (\dot{\theta}_1, \ldots, \dot{\theta}_k) \mapsto & \dot{\theta}_1 (\hat{\boldsymbol{\xi}}_1^+ + \hat{\boldsymbol{\xi}}_1^-) + \cdots + \dot{\theta}_k (\hat{\boldsymbol{\xi}}_k^+ + \hat{\boldsymbol{\xi}}_k^-) \\ &= 2\dot{\theta}_1 \hat{\boldsymbol{\xi}}_1 + \cdots + 2\dot{\theta}_k \hat{\boldsymbol{\xi}}_k \end{aligned}$$

The claims follows immediate from the inverse function theorem [54].

Finally, if $\mathfrak{h}_{\mathfrak{m}} \cap \mathfrak{m} = \{\hat{o}\}$ and $\{\hat{\boldsymbol{\xi}}_1, \ldots, \hat{\boldsymbol{\xi}}_k\}_{\mathrm{sp}} = \mathfrak{m}$, we have:

$$\{\hat{\boldsymbol{\xi}}_1 \pm \hat{\boldsymbol{\zeta}}_1, \ldots, \hat{\boldsymbol{\xi}}_k \pm \hat{\boldsymbol{\zeta}}_k\}_{\mathrm{sp}} + \mathfrak{h}_{\mathfrak{m}} = \mathfrak{g}_{\mathfrak{m}}$$

The sum is direct by a simple dimension argument. On the other hand, if

$$\{\hat{\boldsymbol{\xi}}_1 \pm \hat{\boldsymbol{\zeta}}_1, \ldots, \hat{\boldsymbol{\xi}}_k \pm \hat{\boldsymbol{\zeta}}_k\}_{\mathrm{sp}} \oplus \mathfrak{h}_{\mathfrak{m}} = \mathfrak{g}_{\mathfrak{m}}$$

A change of basis leads to

$$\{\hat{\boldsymbol{\xi}}_1, \ldots, \hat{\boldsymbol{\xi}}_k\}_{\mathrm{sp}} \oplus \mathfrak{h}_{\mathfrak{m}} = \mathfrak{g}_{\mathfrak{m}}$$

The claim follows by a simple dimension argument. $\square$

## Appendix G
## Proof of Corollary 1

Given a SP $(\hat{\boldsymbol{\xi}}^+, \hat{\boldsymbol{\xi}}^-)$:

$$\begin{cases} \hat{\boldsymbol{\xi}}^+ = \hat{\boldsymbol{\xi}} + \hat{\boldsymbol{\zeta}} \\ \hat{\boldsymbol{\xi}}^- = \hat{\boldsymbol{\xi}} - \hat{\boldsymbol{\zeta}} \end{cases} \quad \hat{\boldsymbol{\xi}} \in \mathfrak{m}, \hat{\boldsymbol{\zeta}} \in \mathfrak{h}_{\mathfrak{m}}$$

and some $\hat{\boldsymbol{\eta}} \in \mathfrak{m}$, we have:

$$\begin{aligned} \mathrm{Ad}_{e^{\pm \hat{\boldsymbol{\eta}}}} \hat{\boldsymbol{\xi}}^{\pm} &= e^{\pm \mathrm{ad}_{\hat{\boldsymbol{\eta}}}} (\hat{\boldsymbol{\xi}} \pm \hat{\boldsymbol{\zeta}}) \\ &= \left( \sum_{i=0}^{\infty} \frac{\mathrm{ad}_{\hat{\boldsymbol{\eta}}}^{2i}}{(2i)!} \pm \sum_{i=0}^{\infty} \frac{\mathrm{ad}_{\hat{\boldsymbol{\eta}}}^{2i+1}}{(2i+1)!} \right) (\hat{\boldsymbol{\xi}} \pm \hat{\boldsymbol{\zeta}}) \\ &= \left( \sum_{i=0}^{\infty} \frac{\mathrm{ad}_{\hat{\boldsymbol{\eta}}}^{2i}}{(2i)!} \hat{\boldsymbol{\xi}} + \sum_{i=0}^{\infty} \frac{\mathrm{ad}_{\hat{\boldsymbol{\eta}}}^{2i+1}}{(2i+1)!} \hat{\boldsymbol{\zeta}} \right) \pm \\ & \left( \sum_{i=0}^{\infty} \frac{\mathrm{ad}_{\hat{\boldsymbol{\eta}}}^{2i}}{(2i)!} \hat{\boldsymbol{\zeta}} + \sum_{i=0}^{\infty} \frac{\mathrm{ad}_{\hat{\boldsymbol{\eta}}}^{2i+1}}{(2i+1)!} \hat{\boldsymbol{\xi}} \right) \end{aligned}$$

with

$$\begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{\mathrm{ad}_{\hat{\boldsymbol{\eta}}}^{2i}}{(2i)!} \hat{\boldsymbol{\xi}} + \sum_{i=0}^{\infty} \frac{\mathrm{ad}_{\hat{\boldsymbol{\eta}}}^{2i+1}}{(2i+1)!} \hat{\boldsymbol{\zeta}} \in \mathfrak{m} \\ \displaystyle\sum_{i=0}^{\infty} \frac{\mathrm{ad}_{\hat{\boldsymbol{\eta}}}^{2i}}{(2i)!} \hat{\boldsymbol{\zeta}} + \sum_{i=0}^{\infty} \frac{\mathrm{ad}_{\hat{\boldsymbol{\eta}}}^{2i+1}}{(2i+1)!} \hat{\boldsymbol{\xi}} \in \mathfrak{h}_{\mathfrak{m}} \end{cases}$$

by (10). Thus $(\mathrm{Ad}_{e^{+\hat{\boldsymbol{\eta}}}} \hat{\boldsymbol{\xi}}^+, \mathrm{Ad}_{e^{-\hat{\boldsymbol{\eta}}}} \hat{\boldsymbol{\xi}}^-)$ defines a SP by the algebraic condition (15). $\square$

## References


[1] Y. Wu, G. Liu, H. Löwe, and Z. Li, "Exponential submanifolds: A new kinematic model for mechanism analysis and synthesis," in *Robotics and Automation (ICRA), 2013 IEEE International Conference on*, May 2013, pp. 4177–4182.

[2] J. M. Hervé, "Analyse structurelle des mécanismes par groupe des déplacements," *Mechanism and Machine Theory*, vol. 13, no. 4, pp. 437–450, 1978.

[3] ——, "Lie group of rigid body displacements, a fundamental tool for mechanism design," *Mechanism and Machine Theory*, vol. 34, no. 5, pp. 719–730, 1999.

[4] R. W. Brockett, "Robotic manipulators and the product of exponentials formula," in *In P.A. Fuhrman, editor, Mathematical Theory of Networks and Systems*. Springer-Verlag, 1984, pp. 120–129.

[5] J. Beckers, J. Patera, M. Perroud, and P. Winternitz, "Subgroups of the Euclidean group and symmetry breaking in nonrelativistic quantum mechanics," *Journal of mathematical physics*, vol. 18, no. 1, pp. 72–83, 1977.

[6] Z. Li, R. M. Murray, and S. S. Sastry, *A mathematical introduction to robotic manipulation*. CRC Press, 1994.

[7] F. C. Park, J. E. Bobrow, and S. R. Ploen, "A Lie group formulation of robot dynamics," *The International journal of robotics research*, vol. 14, no. 6, pp. 609–618, 1995.

[8] A. Müller and P. Maisser, "A Lie-group formulation of kinematics and dynamics of constrained mbs and its application to analytical mechanics," *Multibody System Dynamics*, vol. 9, no. 4, pp. 311–352, 2003.

[9] J. Kwon, M. Choi, F. C. Park, and C. Chun, "Particle filtering on the Euclidean group: framework and applications," *Robotica*, vol. 25, no. 06, pp. 725–737, 2007.

[10] R. Mahony, T. Hamel, and J.-M. Pflimlin, "Nonlinear complementary filters on the special orthogonal group," *Automatic Control, IEEE Transactions on*, vol. 53, no. 5, pp. 1203–1218, 2008.

[11] F. Bullo and R. M. Murray, "Proportional derivative (PD) control on the Euclidean group," in *European Control Conference*, vol. 2, 1995, pp. 1091–1097.

[12] J. M. Hervé and F. Sparacino, "Structural synthesis of parallel robots generating spatial translation," in *Proc. 5th Int. Conf. Advanced Robotics*, vol. 1, 1991, pp. 808–813.

[13] A. Frisoli, D. Checcacci, F. Salsedo, and M. Bergamasco, "Synthesis by screw algebra of translating in-parallel actuated mechanisms," in *Advances in Robot Kinematics*. Springer, 2000, pp. 433–440.

[14] M. Carricato and V. Parenti-Castelli, "Singularity-free fully-isotropic translational parallel mechanisms," *The International Journal of Robotics Research*, vol. 21, no. 2, pp. 161–174, 2002.

[15] ——, "A family of 3-dof translational parallel manipulators," *Journal of mechanical design*, vol. 125, no. 2, pp. 302–307, 2003.

[16] Z. Huang and Q. Li, "Type synthesis of symmetrical lower-mobility parallel mechanisms using the constraint-synthesis method," *International Journal of Robotics Research*, vol. 22, no. 1, pp. 59–79, 2003.

[17] X. Kong and C. Gosselin, "Type synthesis of 3-dof spherical parallel manipulators based on screw theory," *Journal of Mechanical Design*, vol. 126, no. 1, pp. 101–108, 2004.

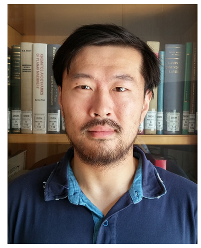
[18] J. Meng, G. Liu, and Z. Li, "A geometric theory for analysis and synthesis of sub-6 dof parallel manipulators," *Robotics, IEEE Transactions on*, vol. 23, no. 4, pp. 625 –649, aug. 2007.

[19] J. Hilgert and K.-H. Neeb, *Structure and geometry of Lie groups*. Springer Science & Business Media, 2011.

[20] J. M. Hervé. (2003) The planar spherical kinematic bond: Implementation in parallel mechanisms. [Online]. Available: http://www.parallemic.org/Reviews/Review013.html

[21] C. C. Lee and J. M. Herve, "Translational parallel manipulator with doubly planar limbs," *Mechanism and Machine Theory*, vol. 41, no. 4, pp. 359–486, 2006.

[22] M. Carricato and J. M. R. Martínez, "Persistent screw systems," in *Advances in Robot Kinematics: Motion in Man and Machine*. Springer, 2010, pp. 185–194.
[23] M. Carricato and D. Zlatanov, "Persistent screw systems," *Mechanism and Machine Theory*, vol. 73, pp. 296–313, 2014.
[24] M. Carricato and J. M. R. Martınez, "Persistent screw systems of dimension three," in *Proc. of 13th World Congress in Mechanism and Machine Science, Guanajuato, Mexico*, 2011, pp. 1–12.
[25] M. Carricato, "Persistent screw systems of dimension four," in *Latest Advances in Robot Kinematics*. Springer, 2012, pp. 147–156.
[26] ——, "Four-dimensional persistent screw systems of the general type," in *Computational Kinematics*, ser. Mechanisms and Machine Science. Springer Netherlands, 2014, vol. 15, pp. 299–306.
[27] X. Kong and C. Gosselin, *Type synthesis of parallel mechanisms*. Springer-Verlag, Berlin, Heidelberg, 2007.
[28] Q. Li, Z. Huang, and J. M. Herve, "Type synthesis of 3R2T 5-dof parallel mechanisms using the Lie group of displacements," *IEEE Transactions on Robotics*, vol. 20, no. 2, pp. 173–180, 2004.
[29] Y. Wu, Z. Li, and J. Shi, "Geometric properties of zero-torsion parallel kinematics machines," in *Intelligent Robots and Systems (IROS), 2010 IEEE/RSJ International Conference on*. IEEE, 2010, pp. 2307–2312.
[30] K. H. Hunt, *Kinematic geometry of mechanisms*. Oxford University Press, 1978.
[31] I. A. Bonev, D. Zlatanov, and C. M. Gosselin, "Advantages of the modified euler angles in the design and control of pkms," *2002 Parallel Kinematic Machines International Conference*, pp. 171–188, 2002.
[32] I. A. Bonev and J. Ryu, "New approach to orientation workspace analysis of 6-dof parallel manipulators," *Mechanism and Machine Theory*, vol. 36, no. 1, pp. 15–28, 2001.
[33] D. Tweed and T. Vilis, "Geometric relations of eye position and velocity vectors during saccades," *Vision Research*, vol. 30, no. 1, pp. 111–127, 1990.
[34] A. D. Polpitiya, W. P. Dayawansa, C. F. Martin, and B. K. Ghosh, "Geometry and control of human eye movements," *Automatic Control, IEEE Transactions on*, vol. 52, no. 2, pp. 170–180, 2007.
[35] K. H. Hunt, "Constant-velocity shaft couplings: a general theory," *ASME J. Eng. Ind*, vol. 95, no. 2, pp. 455–464, 1973.
[36] M. Carricato, "Decoupled and homokinetic transmission of rotational motion via constant-velocity joints in closed-chain orientational manipulators," *Journal of Mechanisms and Robotics*, vol. 1, no. 4, pp. 1–14, 2009.
[37] S. L. Canfield, A. J. Ganino, C. F. Reinholtz, and R. J. Salerno, "Spatial, parallel-architecture robotic carpal wrist," U.S. Patent 5 699 695, Dec. 23, 1997.
[38] R. Behrens, M. Poggendorf, E. Schulenburg, and N. Elkmann, "An elephant's trunk-inspired robotic arm: Trajectory determination and control," in *Robotics; Proceedings of ROBOTIK 2012; 7th German Conference on*. VDE, 2012, pp. 1–5.
[39] I. H. Culver, "Constant velocity universal joint," U.S. Patent 3 477 249, Nov. 13, 1969.
[40] M. E. Rosheim and G. F. Sauter, "New high-angulation omni-directional sensor mount," in *Proceedings of SPIE - The International Society for Optical Engineering*, vol. 4821, 2002, pp. 163–174.
[41] M. E. Rosheim, *Leonardo's lost robots*. Springer, 2006.
[42] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*. New York, 1963, vol. 2.
[43] O. Loos, *Symmetric spaces*. WA Benjamin, 1969, vol. 1.
[44] J. M. Selig, *Geometrical Fundamental of Robotics*, ser. Monographs in Computer Science. New York ; Hong Kong: Springer, 2005.
[45] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*. New York, 1963, vol. 1, no. 2.
[46] J. Selig, "A class of explicitly solvable vehicle motion problems," *Robotics, IEEE Transactions on*, vol. 31, no. 3, pp. 766–777, June 2015.
[47] Y. Wu and M. Carricato, "Identification and geometric characterization of Lie triple screw systems," in *14th IFTOMM World Congress*, Taipei, Taiwan, October 25–30, 2015.
[48] R. Chhabra and M. R. Emami, "A generalized exponential formula for forward and differential kinematics of open-chain multi-body systems," *Mechanism and Machine Theory*, vol. 73, pp. 61–75, 2014.
[49] F. C. Park and K. Okamura, "Kinematic calibration and the product of exponential formula," *Advances in Robot Kinematics and Computational Geometry*, pp. 119–128, 1994.
[50] Y. Wu and M. Carricato, "Design of a novel 3-dof serial-parallel robotic wrist: A symmetric space approach," in *Proceedings of International Symposium on Robotics Research*, September 12 – 15, Sestri Levante, Italy, 2015.
[51] G. A. Thompson, "Constant velocity coupling and control system therefor," U.S. Patent 7 144 326, Dec. 5, 2006.
[52] H. Kocabas, "Design and analysis of a spherical constant velocity coupling mechanism," *Journal of Mechanical Design*, vol. 129, no. 9, pp. 991–998, 2007.
[53] Y. Wu, H. Wang, and Z. Li, "Quotient kinematics machines: Concept, analysis, and synthesis," *ASME Journal of Mechanisms and Robotics*, vol. 3, no. 4, pp. 041 004–1 – 041 004–1, Nov 2011.
[54] W. M. Boothby, *An introduction to differentiable manifolds and Riemannian geometry*, 2nd ed. Academic Press, 2003.



**Yuanqing Wu** received his B.Sc. degree in mechanical engineering in 2001 and Ph.D. degree in mechatronics engineering in 2009, both from Shanghai Jiaotong University, China.

He was a research assistant and research associate at the Electronic and Computer Engineering Department, Hong Kong University of Science and Technology, China. He is currently a contract researcher with the Department of Industrial Engineering, University of Bologna. His research interests includes kinematic geometry, mechanism theory, parallel robots, geometric control and geometric algebra computing.

**Harald Löwe** received his Diploma (with honors) in mathematics in 1990 and the degree of a Dr. rer. nat. (summa cum laude) in 1994, both from the Technical University of Braunschweig, Germany. Since his Habilitation in 2001, he is working as a professor at the Department of Mathematics of the Technical University of Braunschweig.

He was visiting professor at the University of La Laguna (Spain), at the University Henri Poincaré (France). Recently, he is guest professor at the Beijing Union University (China). He is chief editor of the journal "Mathematikinformation". He received the Heinrich-Büssing Price for outstanding research in 2002. Currently, he is concerned with projects in mathematics with high schools. His research interests are geometry, geometric methods in robotics, and the design of spiral bevel gears.

**Marco Carricato** received the M.Sc. degree (with honors) in mechanical engineering in 1998 and the Ph.D. degree in mechanics of machines in 2002. He has been with the University of Bologna since 2004. He is currently an associate professor.

He was visiting researcher at the University of Florida (USA), at Laval University (Canada), at the University of Guanajuato (Mexico), at INRIA–Sophia Antipolis (France), at the Ecole Centrale of Nantes (France), and at the Hong Kong University of Science and Technology. He is an associate editor of the journal Mechanism and Machine Theory. He was awarded the AIMETA Junior Prize 2011 by the Italian Association of Theoretical and Applied Mechanics for outstanding results in the field of mechanics of machines. His research interests include robotic systems, servo-actuated automatic machinery and the theory of mechanisms, with a particular emphasis on parallel manipulators, efficiency and optimization of servomechanisms, the theory of screws, and the theory of homokinetic couplings.

**Zexiang Li** (M'83–F'08) received the B.S. degree in electrical engineering and economics (with hons.) from Carnegie Mellon University, Pittsburgh, PA, in 1983, the M.A. degree in mathematics and the Ph.D. degree in electrical engineering and computer science from the University of California, Berkeley, in 1985 and 1989, respectively.

He worked at ALCOA, the Robotics Institute of CMU and the AI Lab of MIT (89–90). He was an assistant professor at the Courant Institute of New York University (90–92). In 1992, he joined the Department of Electronic and Computer Engineering of the Hong Kong University of Science and Technology (HKUST) and is currently a professor of the department. He founded the Automation Technology Center (ATC) and co-founded the Robotics Institute (RI) at HKUST. He received the ALCOA Foundation Fellowship in 1979, and the E. Anthony Fellowship in 1983. He was a recipient of the University Scholar award from CMU in 1983, the E.I. Jury award from UC Berkeley in 1989, the Research Initiation award from NSF (US) in 1990, the Outstanding Young Researcher award (Class B) from NSF China in 2000, the LEAD award from AMI, USA in 2001, and the Natural Science award (3rd class) from China in 1997. His research interests include multi-fingered robotic hand, parallel manipulators, workpiece localization and inspection, motion control, precision assembly, and unmanned aerial vehicles (UAVs). He is a co-founder of Googol Technology and DJI.