Physics Letters B 802 (2020) 135219

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Singularity avoidance in quantum gravity

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ARTICLE INFO

Article history:
Received 23 November 2019
Received in revised form 9 January 2020
Accepted 13 January 2020
Available online 15 January 2020
Editor: A. Ringwald

ABSTRACT

The purpose of this work is to investigate the consequences of quantum gravity for the singularity problem. We study the higher-derivative terms that invariably appear in any quantum field theoretical model of gravity, handling them both non-perturbatively and perturbatively. In the former case, by computing the contributions of the additional degrees of freedom to a congruence of geodesics, we show that the appearance of singularities is no longer a necessity. In the latter, which corresponds to treating the quantised general relativity as an effective field theory, we generalise the Hawking-Penrose theorems to include one-loop corrections of both massless matter and graviton fluctuations.

1. Introduction

Infinities in physics usually signal the existence of unknown phenomena taking place at new scales, at which point one should replace our current theories (and possibly formalisms) for a more fundamental description of Nature. In general relativity, these infinities come in the form of singularities [1] and they point to the breakdown of our current understanding of gravity. The standard view in the community is that quantum gravity should be able to resolve this issue by smoothing out singularities. Nonetheless, despite the many existing approaches to quantum gravity, there is no consensus about what it is and how one should construct a quantum theory of spacetime, thus a proof of principle for the singularity avoidance is yet to be found.

Partial success has been obtained in the linear approximation, where one expands the metric around some background vacuum solution and quantises only the vacuum fluctuation. While this approach is background-dependent and not really a quantum theory of spacetime, it allows for the quantisation of the gravitational field by employing standard methods of quantum field theory. Regardless of the gravitational bare action one starts with, one then invariably needs to deal with higher-derivative terms to be able to handle ultraviolet (UV) divergences. Whether these can be absorbed into a finite set of free parameters will however largely depend on the choice of the bare action.

In this work, we look at the singularity problem using two different models, both within the limits of quantum field theory, and which differ from each other depending on whether we treat higher-derivative terms perturbatively or non-perturbatively. We first consider quadratic gravity, a higher derivative theory containing terms up to quadratic curvatures in the action and which has proven to be renormalisable [2] and asymptotically free [3]. Quadratic gravity could in principle be seen as a fundamental theory of gravity if it was not for the ghost present in its spectrum, i.e. a particle with the wrong sign in front of its kinetic term, which could lead to instabilities in the theory [2,4] (see also Ref. [5] for a review). But it can nonetheless be considered as a first approximation of a more fundamental theory of quantum gravity.

We will also explore quantum general relativity, treated as an effective field theory. In this scenario, the theory is non-renormalisable [6] and ghost-free [7–9], the only degree of freedom being the usual graviton. The idea of effective field theories is to use the known degrees of freedom in the infrared (IR), namely the graviton in our case, and compute radiative corrections to the interactions perturbatively with respect to inverse powers of the Planck mass M_p. It is very important to stress that this effective field theory does not constitute yet another approach to quantum gravity. It rather consists of a systematical study of quantum gravity in the IR,

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https://doi.org/10.1016/j.physletb.2020.135219

regardless of what happens in the UV regime. Any respectful UV completion of gravity, including quadratic gravity described in the last paragraph, should produce the same results of the effective theory at low energies.

To understand whether the Hawking-Penrose singularity theorems [1] can be evaded in quantum gravity, we make use of the celebrated Raychaudhuri equation [10][1]

$$\dot{\theta} = -\sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{1}{3}\theta^2 - \Delta \ , \tag{1}$$

which describes the divergence θ of a family of geodesics whose tangent vectors are k^μ, where $\sigma_{\mu\nu}$ is the shear tensor and

$$\Delta = R_{\mu\nu}k^\mu k^\nu \tag{2}$$

is the discriminant (also known as Raychaudhuri scalar). Since the first two terms in Eq. (1) are non-positive, the analysis of the convergence of geodesics ultimately reduces to the study of the sign of Δ. Positive contributions to Δ (or negative contributions to $\dot{\theta}$) leads to focusing geodesics, which reflects the attractive character of classical gravity. Negative contributions to Δ, on the other hand, would be interpreted as repulsive interactions, which would be able to defocus geodesics, possibly preventing the formation of singularities.

This paper is organized as follows. In Sec. 2, we investigate higher-derivative gravity by making the scalar and the ghost fields explicit in the action and studying their contribution to the formation of singularities. In Sec. 3, we use the quantum action of general relativity to explore generalisations of the Hawking-Penrose theorem at one-loop order. We then discuss our findings in Sec. 4.

2. Quadratic gravity

The divergence structure of general relativity at one-loop reveals the appearance of quadratic curvature terms in the action. The fact that the original Einstein-Hilbert action does not contain these terms indicates that general relativity is non-renormalisable. This motivated the inclusion of squared curvature terms in the bare action, leading to a modified theory of gravity. This theory turns out to be renormalisable to all loop orders, but a ghost is present in the spectrum. Several ideas have been proposed to get rid of this ghost [11–15]. Here we stick to the position that the ghost is actually fictitious as it only appears as a byproduct of the truncation of an action containing infinitely many terms [15]. One can therefore project the ghost out by a suitable choice of the boundary conditions.

The action of quadratic gravity reads

$$S = \frac{M_p^2}{2}\int d^4x\sqrt{-\tilde{g}}\left(\tilde{R} + \alpha\tilde{R}^2 + \beta\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + \gamma\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right) \ , \tag{3}$$

where $\tilde{R}$, $\tilde{R}_{\mu\nu}$ and $\tilde{R}_{\mu\nu\rho\sigma}$ are the Ricci scalar, Ricci tensor and Riemann tensor of the metric $\tilde{g}_{\mu\nu}$, respectively.[2] The above action contains massive spin-0 and spin-2 fields in addition to the usual graviton (massless spin-2). They can be made explicit via successive field redefinitions of the metric [26],

$$\bar{g}_{\mu\nu} = e^\chi\,\tilde{g}_{\mu\nu} \tag{4}$$

$$g^{\mu\nu} = [\det A(\phi_{\sigma\tau})]^{-1/2}\,\bar{g}^{\mu\alpha}\,A_\alpha^{\ \nu} \ , \tag{5}$$

such that the action (3) in the $g_{\mu\nu}$ frame reads

$$\begin{aligned} S = \frac{M_p^2}{2}\int\sqrt{-g}\Big\{ & R - \frac{3}{2}\left[A^{-1}(\phi_{\sigma\tau})\right]_\mu^{\ \nu}\nabla^\mu\chi\,\nabla_\nu\chi - \frac{3}{2}m_0^2[\det A(\phi_{\sigma\tau})]^{-1/2}\left(1-e^{-\chi}\right)^2 \\ & - g^{\mu\nu}\left[C^\lambda{}_{\mu\rho}(\phi_{\sigma\tau})C^\rho{}_{\nu\lambda}(\phi_{\sigma\tau}) - C^\lambda{}_{\mu\nu}(\phi_{\sigma\tau})C^\rho_{\rho\lambda}(\phi_{\sigma\tau})\right] \\ & + \frac{1}{4}m_2^2[\det A(\phi_{\sigma\tau})]^{-1/2}\left(\phi_{\mu\nu}\,\phi^{\mu\nu} - \phi^2\right)\Big\} \ , \end{aligned} \tag{6}$$

where $\phi \equiv g^{\mu\nu}\,\phi_{\mu\nu}$, $m_0 = (6\alpha + 2\beta + 2\gamma)^{-1/2}$ is the mass of the scalar field χ and $m_2 = (-\beta - 4\gamma)^{-1/2}$ is the mass of the massive spin-2 particle $\phi_{\mu\nu}$. The tensors $A_\lambda^{\ \nu}$ and $C^\lambda{}_{\mu\nu}$ are defined by

$$A_\lambda^{\ \nu} = \left(1 + \frac{1}{2}\phi\right)\delta_\lambda^{\ \nu} - \phi_\lambda^{\ \nu} \ , \tag{7}$$

and

$$C^\lambda{}_{\mu\nu} = \frac{1}{2}\left(\tilde{g}^{-1}\right)^{\lambda\rho}\left(\nabla_\mu\tilde{g}_{\nu\rho} + \nabla_\nu\tilde{g}_{\mu\rho} - \nabla_\rho\tilde{g}_{\mu\nu}\right) \ , \tag{8}$$

where the connection ∇ is compatible with the new metric $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ is seen as a function of $g_{\mu\nu}$. The field $\phi_{\mu\nu}$ must satisfy the spin-2 consistent conditions

[1] As usual, we discard the contribution due to the twist $\omega_{\mu\nu}$.

[2] Note that the square of the Riemann tensor is usually eliminated in favour of the other two curvature invariants by invoking Gauss-Bonnet theorem. Here we choose to leave it explicit in the action just to follow the same notations commonly used in the literature.

$$\nabla^{\mu}\left(\phi_{\mu\nu}-g_{\mu\nu}\,\phi\right)-g^{\lambda\mu}\,C^{\rho}_{\ \lambda\mu}\left(\phi_{\rho\nu}-g_{\rho\nu}\,\phi\right)-g^{\lambda\mu}\,C^{\rho}_{\ \lambda\nu}\left(\phi_{\rho\mu}-g_{\rho\mu}\,\phi\right)=0 \tag{9}$$

and

$$\phi-m_2^{-2}\left\{\left[\det A(\phi_{\sigma\tau})\right]^{1/2}\left[A^{-1}(\phi_{\sigma\tau})\right]_{\mu}^{\ \nu}\nabla^{\mu}\chi\,\nabla_{\nu}\chi+2\,m_0^2\left(1-e^{-\chi}\right)^2\right\}=0\,. \tag{10}$$

The equations of motion are quite involved, thus we consider two different approximations. First, let us assume that the massive spin-2 field $\phi_{\mu\nu}$ is isotropic, that is

$$\phi_{\mu\nu}=\tfrac{1}{4}\,\phi\,g_{\mu\nu}\,, \tag{11}$$

and the equations of motion for the metric reads

$$\begin{aligned}R_{\mu\nu}-\frac{1}{2}\,g_{\mu\nu}\,R=&\,\frac{3}{2}\left(1+\frac{\phi}{4}\right)^{-1}\left(\partial_{\mu}\chi\,\partial_{\nu}\chi-\frac{1}{2}\,g_{\mu\nu}\,\partial_{\rho}\chi\,\partial^{\rho}\chi\right)\\&+\frac{3}{32}\left(1+\frac{\phi}{4}\right)^{-2}\left(\partial_{\mu}\phi\,\partial_{\nu}\phi-\frac{1}{2}\,g_{\mu\nu}\,\partial_{\rho}\phi\,\partial^{\rho}\phi\right)\\&-\frac{3}{2}\,g_{\mu\nu}\,V(\chi,\phi)\,,\end{aligned} \tag{12}$$

where

$$V(\chi,\phi)=\frac{m_0^2\left(1-e^{-\chi}\right)^2}{2\left(1+\frac{\phi}{4}\right)}+\frac{m_2^2\,\phi^2}{16\left(1+\frac{\phi}{4}\right)^2}\,. \tag{13}$$

By trace-reversing Eq. (12), we obtain

$$R_{\mu\nu}=\frac{3}{2}\left(1+\frac{\phi}{4}\right)^{-1}\partial_{\mu}\chi\,\partial_{\nu}\chi+\frac{3}{32}\left(1+\frac{\phi}{4}\right)^{-2}\partial_{\mu}\phi\,\partial_{\nu}\phi+\frac{3}{2}\,g_{\mu\nu}\,V(\chi,\phi)\,. \tag{14}$$

Contracting the above with a time-like vector u^{μ} gives the discriminant defined in Eq. (2) with $k^{\mu}=u^{\mu}$,

$$\Delta=\left[\frac{3}{2}\left(1+\frac{\phi}{4}\right)^{-1}\partial_{\mu}\chi\,\partial_{\nu}\chi+\frac{3}{32}\left(1+\frac{\phi}{4}\right)^{-2}\partial_{\mu}\phi\,\partial_{\nu}\phi+\frac{3}{2}\,g_{\mu\nu}\,V(\chi,\phi)\right]u^{\mu}\,u^{\nu}\,. \tag{15}$$

From the Penrose-Hawking theorems, singularities are unavoidable when $\Delta\geq 0$. By defining the trajectory's proper time $u^{\mu}=dx^{\mu}/d\tau$, we have $u^{\mu}\,\partial_{\mu}\chi=d\chi/d\tau\equiv\dot{\chi}$, and noting that on time-like physical trajectories $g_{\mu\nu}\,u^{\mu}\,u^{\nu}=-1$, leads to

$$\Delta=\frac{3}{2}\left(1+\frac{\phi}{4}\right)^{-1}\dot{\chi}^2+\frac{3}{32}\left(1+\frac{\phi}{4}\right)^{-2}\dot{\phi}^2-\frac{3}{2}\,V(\chi,\phi)\,. \tag{16}$$

Thus we conclude that singularities are no longer a clear necessity as we now have negative contributions to Δ.

For example, singularities can be avoided in situations where the potential V dominates over the kinetic terms,[3] such as in the early universe. In the vicinity of the Big Bang epoch, the potential is expected to dominate so that inflation can take place, thus leading to the possible avoidance of the initial singularity. Note also that singularities can be avoided even when the kinetic terms dominate due to the coupling between the spin-0 and spin-2 fields if $\phi<-4$. This should be confronted with Eq. (10). Upon using (11), we obtain

$$A_{\lambda}^{\ \nu}=\left(1+\frac{\phi}{4}\right)\delta_{\lambda}^{\ \nu}\,, \tag{17}$$

and (10) will turn into a condition for χ,

$$\begin{aligned}m_2^2\,\phi&=\left[\det A(\phi_{\sigma\tau})\right]^{1/2}\left[A^{-1}(\phi_{\sigma\tau})\right]_{\mu}^{\ \nu}\nabla^{\mu}\chi\,\nabla_{\nu}\chi+2\,m_0^2\left(1-e^{-\chi}\right)^2\\&=\left(1+\frac{\phi}{4}\right)\partial_{\mu}\chi\,\partial^{\mu}\chi+2\,m_0^2\left(1-e^{-\chi}\right)^2\end{aligned} \tag{18}$$

or

$$\phi=\frac{\partial^{\mu}\chi\,\partial_{\mu}\chi+2m_0^2(1-e^{-\chi})^2}{m_2^2-\frac{1}{4}\partial^{\mu}\chi\,\partial_{\mu}\chi}. \tag{19}$$

The condition $\phi<-4$ then translates into

$$2\,m_0^2\,(1-e^{-\chi})^2+4\,m_2^2>0, \tag{20}$$

[3] We are assuming $m_i^2>0$ $(i=0,2)$ to avoid tachyonic instabilities.

for $m_2^2 < \frac{1}{4}\partial^\mu\chi\,\partial_\mu\chi$ (note that $\phi < -4$ has no solution for $m_2^2 > \frac{1}{4}\partial^\mu\chi\,\partial_\mu\chi$). In particular, the condition $\phi < -4$ is satisfied whenever the kinetic energy of χ dominates. Let us see a concrete example of this. Consider an explicit isotropic metric of the form

$$ds^2 = -A\,dt^2 + B\,dr^2 + r^2\,d\Omega^2\ , \tag{21}$$

where A and B only depend on r and χ is static. From Eq. (18), we obtain

$$m_2^2\,\phi = -\left(1+\frac{\phi}{4}\right)\frac{(\chi')^2}{B} + 2\,m_0^2\left(1-e^{-\chi}\right)^2\ . \tag{22}$$

Therefore, if $m_i^2 > 0$ $(i = 0, 2)$, $\phi < -4$ requires that $B < 0$. Note that B becomes negative, for example, inside the Schwarzschild radius. This means that the defocussing of time-like geodesics is switched on as soon as they cross the horizon, leading to the possible avoidance of the singularity at $r = 0$ of a Schwarzschild black hole.

Now let us investigate the scenario where $\phi_{\mu\nu}$ is not isotropic, but both massive fields are seen as small perturbations around an arbitrary spacetime. In this case, the action reads

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R + \mathcal{L}_\chi + \mathcal{L}_\phi\right], \tag{23}$$

where the spin-0 sector is given by

$$\mathcal{L}_\chi = -\frac{3}{2}\partial^\mu\chi\,\partial_\mu\chi - \frac{3}{2}m_0^2(1-e^{-\chi})^2 \tag{24}$$

and the spin-2 sector reads

$$\begin{aligned}\mathcal{L}_\phi = &-\frac{1}{4}\left(\nabla_\mu\phi\nabla^\mu\phi - \nabla_\mu\phi_{\nu\rho}\nabla^\mu\phi^{\nu\rho} + 2\nabla^\mu\phi_{\mu\nu}\nabla^\nu\phi - 2\nabla_\mu\phi_{\nu\rho}\nabla^\rho\phi^{\mu\nu}\right)\\ &+\frac{1}{4}m_2^2\left(\phi_{\mu\nu}\phi^{\mu\nu} - \phi^2\right)\ .\end{aligned} \tag{25}$$

The spin-2 conditions (9) and (10) become

$$\nabla^\mu\phi_{\mu\nu} = \phi = 0\ . \tag{26}$$

Varying Eq. (23) with respect to $g^{\mu\nu}$ and imposing the conditions (26), leads to

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{27}$$

where $T_{\mu\nu} = T^\chi_{\mu\nu} + T^\phi_{\mu\nu}$ and

$$8\pi G T^\chi_{\mu\nu} = g_{\mu\nu}\left[-\frac{3}{2}\partial_\rho\chi\,\partial^\rho\chi - \frac{3}{2}m_0^2(1-e^{-\chi})^2\right] + 3\partial_\mu\chi\,\partial_\nu\chi\,, \tag{28}$$

$$8\pi G T^\phi_{\mu\nu} = \frac{1}{4}g_{\mu\nu}\left[\nabla_\rho\phi_{\alpha\beta}\nabla^\rho\phi^{\alpha\beta} + m_2^2\phi_{\alpha\beta}\phi^{\alpha\beta}\right]2\nabla_\mu\phi_{\alpha\beta}\nabla_\nu\phi^{\alpha\beta} \tag{29}$$

are the energy-momentum tensors of each field. The discriminant is then given by

$$\Delta = 3\dot\chi^2 - \frac{3}{2}m_0^2(1-e^{-\chi})^2 - \frac{3}{4}\nabla_\gamma\phi_{\alpha\beta}\nabla^\gamma\phi^{\alpha\beta} + \frac{1}{4}m_2^2\phi_{\alpha\beta}\phi^{\alpha\beta} - 2(\nabla_0\phi^\alpha_{\ \beta})^2\ . \tag{30}$$

Note that, while nothing can be said about the sign of the third term, we see that the first and fourth terms are positive while the second and fifth terms are negative. Therefore, as before, singularities can be avoided in situations where the negative terms are dominant. We should stress that the negative contribution to Δ due to the last term in Eq. (30) corresponds to the fact that $\phi_{\mu\nu}$ is a ghost and thus mediates a repulsive interaction. We can therefore ameliorate the singularity problem at the price of having ghost instabilities in the theory. However, as it was shown in Refs. [15–17], this ghost can be projected out of asymptotic states by means of a contour in the Fourier space, which is justified as the Ostrogradskian ghost is absent in a complete theory containing infinitely many diffeomorphism invariants [15,16]. It should be noted that projecting out the ghost particle from the theory only prevents it from appearing in external lines of Feynman diagrams, but it can still contribute to internal lines, which has precisely the desired effect of mediating the repulsive interaction that leads to the last term in Eq. (30). We should also mention that the energy conditions [1,10] appearing among the hypotheses of the singularity theorems correspond to properties one expects for classical matter fields. Hence, it is not unconceivable that they can be averted by the energy-momentum tensors (28) and (29) given the purely quantum nature of the fields χ and $\phi_{\mu\nu}$. We note that similar conclusions to the above were recently reached in [18] using different methods.

Finally, we must confront our results with [19], where it was proven that Ricci-flat spacetimes are exact solutions of general local and non-local higher derivative theories, indicating that singularities of Einstein's vacuum solutions should exist even beyond general relativity. In our approach, one must make a distinction between the Ricci tensor in the original frame $\tilde R_{\mu\nu}$ and the Ricci tensor in the transformed frame $R_{\mu\nu}$. Essentially, when one splits up the original metric $\tilde g_{\mu\nu}$ into the new background metric $g_{\mu\nu}$ and the additional massive fields $\{\chi, \phi_{\mu\nu}\}$, the curvature is also split into different pieces. One of these pieces is identified as the Ricci tensor of the new metric, while the others are the strengths corresponding to the massive fields χ and $\phi_{\mu\nu}$. For example, considering only the spin-2 sector, the relation between the Ricci tensor in different frames reads

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \nabla_\mu C^\lambda{}_{\lambda\nu} + \nabla_\lambda C^\lambda{}_{\mu\nu} - C^\lambda{}_{\mu\rho} C^\rho{}_{\nu\lambda} + C^\lambda{}_{\mu\nu} C^\rho{}_{\rho\lambda}. \tag{31}$$

For $\phi_{\mu\nu} = 0$, both the strength $C^\lambda{}_{\mu\nu}$ and the new Ricci tensor $R_{\mu\nu}$ vanish upon using the equation of motion, which implies $\tilde{R}_{\mu\nu} = 0$. However, $\tilde{R}_{\mu\nu} = 0$ does not imply $\phi_{\mu\nu} = 0$. More generally, although the vanishing of the massive fields implies the vanishing of the original Ricci tensor $\tilde{R}_{\mu\nu}$, the opposite is not true. Therefore, Ricci flatness is not a preserved property under field redefinitions. If one then chooses to interpret the theory in the original frame, then the singularities of Einstein's vacuum solutions remain present as pointed out in [19]. But they are just a reflection of a bad choice of field variables. In fact, if one interprets the theory in the new frame, where gravity (described by the new metric $g_{\mu\nu}$) is inherently accompanied by two other fields, then the singularities can be unraveled. This seems to indicate a profound link between singularities and the choice of field variables, whose details will be worked out in the future.

3. Quantum general relativity

General relativity is known to be non-renormalisable, which until a few decades ago had been considered as a long-standing problem since quantum general relativity would require an infinite number of observations to make the theory predictive, should one proceed naively by using the old-fashioned methods of quantum field theory. However, within the realm of effective field theories, the non-renormalisability of general relativity is actually a positive result as it shows the reason why classical general relativity is so successful in describing the Universe as we see it: new physics comes suppressed by the extremely huge Planck scale $M_p \sim 10^{19}$ GeV.

The idea of effective field theories is to organise all possible terms in the action in powers of E/M, where E is the typical energy of the problem under consideration and M is a cutoff. The zeroth order term is the basic action, the one which defines the degrees of freedom and classical interactions. Higher powers of E/M contributes only to the latter, thus leading to corrections to vertices in the Feynman diagrams, but not to the propagators. For general relativity, one has $M \sim M_p$ and the bare action reads

$$S = \int d^4x \sqrt{-g} \left(\frac{M_p^2}{2} R + \tilde{b}_1 R^2 + \tilde{b}_2 R_{\mu\nu} R^{\mu\nu} + \mathcal{O}(\partial^6 g) \right) , \tag{32}$$

where $\tilde{b}_i$ are bare coefficients. The basic action is the Einstein-Hilbert action, which has the graviton as the only degree of freedom. One must not confuse the actions (3) and (32). Although they look the same, they are treated differently and have different features. In (3), all terms are treated on the same foot, leading to a renormalisable theory that contains other degrees of freedom besides the graviton. The action (32), on the other hand, is the result of a non-renormalisable theory (i.e. general relativity) and must be treated perturbatively to comply with the effective field theory approach, which is predictive at energies below M_p and contains the graviton as the only particle in the spectrum [7–9].

The quantisation of (32) can be performed in the background field formalism, in which the metric is split as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the classical background metric, and the perturbation $h_{\mu\nu}$ is quantised. Barvinsky et al. developed a very general formalism to obtain the effective action Γ of a gauge theory (including gravity) as an expansion in curvatures [20–23]. If we restrict to the cases where the only non-vanishing vacuum expectation value is the metric $g_{\mu\nu}$ and only letting massless particles run in the loops, we obtain

$$\begin{aligned}\Gamma = \int \mathrm{d}^4x \sqrt{-g} \Bigg[& \frac{M_p^2}{2} R + b_1 R^2 + b_2 R_{\mu\nu} R^{\mu\nu} + c_1 R \log\left(\frac{-\Box}{\mu^2}\right) R + c_2 R_{\mu\nu} \log\left(\frac{-\Box}{\mu^2}\right) R^{\mu\nu} \\ & + c_3 R_{\mu\nu\rho\sigma} \log\left(\frac{-\Box}{\mu^2}\right) R^{\mu\nu\rho\sigma} \Bigg] .\end{aligned} \tag{33}$$

Note that the coefficients of the local operators acquire a running $b_i = b_i(\mu)$ after renormalisation. The coefficients of the non-local operators, on the other hand, are fully specified and depend solely on the spin of the fields that have been integrated out.

The equation of motion for the perturbation $h_{\mu\nu}$ around the background $\bar{g}_{\mu\nu}$ can be obtained in perturbation theory by evaluating the equation of motion from (33) order by order in $h_{\mu\nu}$:

$$G_{\mu\nu}[\bar{g}] = 0, \tag{34}$$

$$\frac{\delta G_{\mu\nu}[\bar{g}]}{\delta g^{\rho\sigma}} h^{\rho\sigma} + \Delta G_{\mu\nu}[\bar{g}] = 0, \tag{35}$$

where $G_{\mu\nu}$ is the Einstein tensor and $\Delta G_{\mu\nu}$ denotes the quantum corrections to Einstein's equation coming from the higher derivatives in (33). At zeroth order, we obtain Eq. (34) which is simply Einstein's equation for the background $\bar{g}_{\mu\nu}$. The leading order Eq. (35), on the other hand, gives precisely the equation of motion for $h_{\mu\nu}$:

$$\Box \bar{h}_{\mu\nu} = -\bar{R} \left[b_1 + (c_1 - c_3) \log\left(\frac{-\Box}{\mu^2}\right) \right] \bar{R}_{\mu\nu} + \frac{1}{2} \bar{g}_{\mu\nu} \bar{R}_{\rho\sigma} \left[b_2 + (c_2 + 4c_3) \log\left(\frac{-\Box}{\mu^2}\right) \right] \bar{R}^{\rho\sigma} , \tag{36}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} h$, the bar is used for background objects and the order in h will be denoted by a superscript bracketed number.

Let us start by looking at time-like congruences. First, we expand the discriminant up to linear order,

$$\Delta = \bar{R}_{\mu\nu} k^\mu k^\nu + R^{(1)}_{\mu\nu} k^\mu k^\nu , \tag{37}$$

where $R^{(1)}_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} + \frac{1}{4}\bar{g}_{\mu\nu}\Box\bar{h}$. From (36) and (37), we obtain

$$\Delta = \bar{R}_{\mu\nu}k^\mu k^\nu + \frac{1}{2}\bar{R}\left[b_1 + (c_1 - c_3)\log\left(\frac{-\Box}{\mu^2}\right)\right]\bar{R}_{\mu\nu}k^\mu k^\nu + \frac{1}{4}\bar{R}\left[b_1 + (c_1 - c_3)\log\left(\frac{-\Box}{\mu^2}\right)\right]\bar{R} - \frac{1}{4}\bar{R}_{\rho\sigma}\left[b_2 + (c_2 + 4c_3)\log\left(\frac{-\Box}{\mu^2}\right)\right]\bar{R}^{\rho\sigma} . \tag{38}$$

Given that the background metric satisfies Einstein's field equations in the absence of a cosmological constant, i.e. $\bar{R}_{\mu\nu} = 0$, all terms in Eq. (38) vanish and we obtain a generalisation of the Hawking-Penrose theorem for singularities along time-like geodesics at one-loop order. This is a direct consequence of the fact that the right hand side of Eq. (36) vanishes for a Ricci flat spacetime and it obviously extends to null-like trajectories as we will confirm below. Our finding is in agreement with the results of Refs. [24,25] for the Schwarzschild black hole.

Note, however, that the vanishing of the discriminant Δ up to one-loop order only indicates that nothing can be said regarding the formation of singularities at this order. Since we are working in perturbation theory, $\Delta = 0$ represents a marginal result and it is therefore inconclusive. One must go beyond one-loop corrections to study the sign of Δ. Likewise, within perturbation theory, Δ is clearly dominated by the classical contribution $\bar{R}_{\mu\nu}k^\mu k^\nu$, thus the sign of Δ is not changed by the loop contributions unless the tree level result is marginal, i.e. $\bar{R}_{\mu\nu}k^\mu k^\nu = 0$.

We should also stress two important points regarding this generalisation. First, the background can be completely arbitrary and need not be described by an Einstein manifold. In this case, one would obtain an additional contribution on the right-hand side of Eq. (36) given by $-G_{\mu\nu}[\bar{g}]$. We chose an Einstein background to simplify our analysis. Secondly, even if it is described by Einstein's equations (as we assumed above), the background is Ricci flat only in the absence of a cosmological constant and for vanishing matter vacuum expectation values, in which case the macroscopic energy-momentum tensor is zero. We have indeed assumed that all background matter fields are zero in order to obtain the quantum action (33). In the presence of a non-zero cosmological constant Λ, for example, we can have both positive and negative contributions according to the sign of the coupling constants c_i, which ultimately depend on the spin of the integrated particles, and $b_i(\mu)$, whose sign is dictated by their renormalisation group.

With the above points in mind, we can now state the one-loop generalisation of the Hawking-Penrose theorem for time-like congruences. Let us suppose that $\bar{R}_{\mu\nu}k^\mu k^\nu = 0$, otherwise the standard Hawking-Penrose result holds. Then, a necessary (but not sufficient) condition for the avoidance of time-like singularities is that

$$\bar{R}\left[b_1 + (c_1 - c_3)\log\left(\frac{-\Box}{\mu^2}\right)\right]\bar{R} - \bar{R}_{\rho\sigma}\left[b_2 + (c_2 + 4c_3)\log\left(\frac{-\Box}{\mu^2}\right)\right]\bar{R}^{\rho\sigma} > 0 . \tag{39}$$

Observe that the quadratic theory studied in Sec. 2, when treated perturbatively, should agree with the results of quantum general relativity at low energies. In fact, treating higher-order curvature terms as perturbations corresponds to handling the fields χ and $\phi_{\mu\nu}$ perturbatively as well. One can then compare both approaches order by order. For example, the scalar field in Eq. (6) is defined by

$$\chi = \log(1 + 3m^2 R) = 3m^2 R + \mathcal{O}(R^2) , \tag{40}$$

thus the mass term $\chi^2 \sim R^2$ reproduces the square of the Ricci scalar in Eq. (38). This in fact corresponds to putting the field χ on shell (at tree level), which makes physical sense because the masses of χ and $\phi_{\mu\nu}$ are supposedly of the order of the Planck mass, thus both fields decouple from the theory at energies below M_p.

Let us now look at null-like congruences. From $\bar{g}_{\mu\nu}k^\mu k^\nu = 0$, we find

$$\Delta = \bar{R}_{\mu\nu}k^\mu k^\nu + \frac{1}{2}\bar{R}\left[b_1 + (c_1 - c_3)\log\left(\frac{-\Box}{\mu^2}\right)\right]\bar{R}_{\mu\nu}k^\mu k^\nu , \tag{41}$$

and we again conclude that the Hawking-Penrose theorem is fulfilled for a Ricci flat spacetime in the case of null-like singularities as it should be expected from Eq. (36). Nonetheless, for null-like vectors k^μ, the discriminant Δ in Eq. (41) vanishes even in the presence of a cosmological constant because the Ricci tensor $\bar{R}_{\mu\nu}$ is proportional to the metric $\bar{g}_{\mu\nu}$. Non-trivial contributions to Δ are only possible when either classical matter is present or on non-instanton backgrounds. Unfortunately, as discussed earlier, one-loop corrections are only sizeable when $\bar{R}_{\mu\nu}k^\mu k^\nu = 0$, making all terms in Eq. (41) vanish and leading to $\Delta = 0$. As we explained before, this means that the study of formation of singularities along null-like geodesics is inconclusive at one-loop order and one must go beyond this approximation in order to be able to determine the sign of Δ.

4. Conclusions

In this paper, we have shown how quantum corrections to gravity can make the singularity problem less severe by giving positive contributions to $\dot{\theta}$ via the Raychaudhuri equation.

We first considered the fourth-derivative extension of general relativity, which is renormalisable but contains a ghost field in the Lagrangian, i.e. a field whose kinetic term has a negative norm, and which can be interpreted as a repulsive force. It is precisely this repulsive feature of the ghost, together with the potential terms, that could be able to prevent the formation of singularities in the spacetime. On the other hand, the very same feature is also responsible for vacuum instabilities in the theory should it be present in asymptotic states. We argued that one can take advantage of the repulsive character of the ghost without facing instability issues by projecting the ghost particle out of the asymptotic spectrum. We also found out theoretical evidence suggesting the existence of an interesting link between singularities and the choice of field variables. The details of this new finding will be investigated in a future project.

In the second part of the paper, we looked at the problem from an effective field theory perspective, thus treating the action perturbatively as an expansion in inverse powers of the Planck mass. We showed that the Hawking-Penrose theorems can be generalised to include one-loop corrections when the background is Ricci flat, but the conditions for the formation of singularities are modified otherwise. Within perturbation theory, the tree level contribution naturally dominates over the loop corrections, thus the one-loop correction will dictate the fate of the discriminant only in the marginal case in which the classical contribution vanishes.

In the present work we have tried to reach general conclusions by analysing the actions (3) and (33) without considering specific physical systems. It will be interesting to further study this issue by employing explicit cosmological models or descriptions of the gravitational collapse of compact objects.

Acknowledgements

I.K. and R.C. are partially supported by the INFN grant FLAG. The work of R.C. has also been carried out in the framework of activities of the National Group of Mathematical Physics (GNFM, INdAM) and COST action Cantata.

References

[1] S.W. Hawking, G.F.R. Ellis, The Large Scale Structure of Space-Time, Cambridge University Press, Cambridge, 1973.
[2] K.S. Stelle, Phys. Rev. D 16 (1977) 953.
[3] I.G. Avramidi, A.O. Barvinsky, Phys. Lett. B 159 (1985) 269.
[4] K.S. Stelle, Gen. Relativ. Gravit. 9 (1978) 353.
[5] F. de, O. Salles, I.L. Shapiro, Universe 4 (9) (2018) 91, arXiv:1808.09015 [gr-qc].
[6] G. 't Hooft, M.J.G. Veltman, Ann. Inst. Henri Poincaré A, Phys. Théor. 20 (1974) 69.
[7] J.Z. Simon, Phys. Rev. D 43 (1991) 3308.
[8] J.F. Donoghue, Phys. Rev. D 50 (1994) 3874, arXiv:gr-qc/9405057.
[9] C.P. Burgess, Living Rev. Relativ. 7 (2004) 5, arXiv:gr-qc/0311082.
[10] R.M. Wald, General Relativity, Chicago University Press, 1984.
[11] T.D. Lee, G.C. Wick, Nucl. Phys. B 9 (1969) 209.
[12] R.E. Cutkosky, P.V. Landshoff, D.I. Olive, J.C. Polkinghorne, Nucl. Phys. B 12 (1969) 281.
[13] E.T. Tomboulis, Phys. Lett. B 70 (1977) 361.
[14] E.T. Tomboulis, Phys. Rev. Lett. 52 (1984) 1173.
[15] I. Kuntz, Exorcising ghosts in quantum gravity, arXiv:1909.11072 [hep-th].
[16] N. Barnaby, N. Kamran, J. High Energy Phys. 0802 (2008) 008, arXiv:0709.3968 [hep-th].
[17] I. Kuntz, R. da Rocha, Eur. Phys. J. C 79 (6) (2019) 447, arXiv:1903.10642 [hep-th].
[18] F. Kuipers, X. Calmet, Singularity theorems in the effective field theory for quantum gravity at second order in curvature, arXiv:1911.05571 [gr-qc].
[19] Y.D. Li, L. Modesto, L. Rachwał, J. High Energy Phys. 1512 (2015) 173, https://doi.org/10.1007/JHEP12(2015)173, arXiv:1506.08619 [hep-th].
[20] A.O. Barvinsky, G.A. Vilkovisky, Phys. Rep. 119 (1985) 1.
[21] A.O. Barvinsky, G.A. Vilkovisky, Nucl. Phys. B 282 (1987) 163.
[22] A.O. Barvinsky, G.A. Vilkovisky, Nucl. Phys. B 333 (1990) 471.
[23] G.A. Vilkovisky, Nucl. Phys. B 234 (1984) 125.
[24] X. Calmet, B.K. El-Menoufi, Eur. Phys. J. C 77 (4) (2017) 243, arXiv:1704.00261 [hep-th].
[25] X. Calmet, R. Casadio, F. Kuipers, Phys. Rev. D 100 (8) (2019) 086010, arXiv:1909.13277 [hep-th].
[26] A. Hindawi, B.A. Ovrut, D. Waldram, Phys. Rev. D 53 (1996) 5583, arXiv:hep-th/9509142.