Interpolation in Singular Geometric Theories

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Abstract

We present a generalization of Maehara's lemma to show that the extensions of classical and intuitionistic first-order logic with a special type of geometric axioms, called singular geometric axioms, have Craig's interpolation property. As a corollary, we obtain a direct proof of interpolation for (classical and intuitionistic) first-order logic with identity, as well as interpolation for several mathematical theories, including the theory of equivalence relations, (strict) partial and linear orders, and various intuitionistic order theories such as apartness and positive partial and linear orders.

Craig's interpolation theorem [1] is a central result in first-order logic. It asserts that for any theorem $A \to B$ there exists a formula C, called *interpolant*, such that $A \to C$ and $C \to B$ are also theorems and C only contains non-logical symbols that are contained in both A and B (and if A and B have no non-logical symbols in common, then either $\neg A$ is a theorem or B is). The aim of this paper is to extend interpolation beyond first-order logic. In particular, we show how to prove interpolation in extensions of intuitionistic and classical sequent calculi with *singular geometric rules*, a special case of geometric rules investigated in [7]. Interpolation for singular geometric rules will be obtained by generalizing a standard result, reportedly due to Maehara in [12] and known as "Maehara's lemma" [5].

Proving Maehara-style interpolation for extensions of first-order logic is not at all straightforward, since the standard proof normally relies on the fact that such extensions admit a cut-free systematization in Gentzen's sequent calculus – which is in general not the case. To overcome this obstacle we shall build on previous work by Negri and von Plato who have shown (in a series of papers starting from [8]) how to recover cut elimination (as well as the admissibility of other structural rules) for extensions of the calculi G3c and m-G3i for classical and intuitionistic first-order logic. Of particular interest for the present work are the extensions with geometric rules, investigated in [7]. Once cut elimination is recovered in this way, we impose a singularity condition on geometric rules to isolate those containing at most one non-logical predicate (identity will be counted as logical). Our main result is to show that Maehara's lemma holds when G3c and G3i are extended with singular geometric rules (Lemma 9). Then interpolation follows easily from the generalized Maehara's lemma (Theorem 10). Finally, we consider applications of Theorem 10 and we show that singular geometric rules include many interesting extensions of intuitionistic and classical first-order logic, especially (classical and intuitionistic) first-order logic with identity, the theory of equivalence relations, (strict) partial and linear orders, the theory of apartness and the theory of positive partial and linear orders. We shall omit the proofs altogether and indicate reference to the existing literature when necessary.

1 Classical and intuitionistic sequent calculi

The language $\mathcal{L}$ is a first-order language with individual constants and no functional symbols. Moreover, let $\mathsf{FV}(A)$ be the set of free variables of a formula A and let $\mathsf{Con}(A)$ be the set of its individual constants. We agree that the set of terms $\mathsf{Ter}(A)$ of A is $\mathsf{FV}(A) \cup \mathsf{Con}(A)$. Moreover, if $\mathsf{Rel}(A)$ is the set of non-logical predicates of A then we define the language $\mathcal{L}(A)$ of A as $\mathsf{Ter}(A) \cup \mathsf{Rel}(A)$. Notice that $= \notin \mathcal{L}(A)$, for all A. Such notions are immediately extended to multisets of formulas Γ, by letting $\mathsf{FV}(\Gamma)$ to be defined as $\bigcup_{A\in\Gamma} \mathsf{FV}(A)$, and analogously for $\mathsf{Con}(\Gamma)$, $\mathsf{Ter}(\Gamma)$, $\mathsf{Rel}(\Gamma)$ and $\mathcal{L}(\Gamma)$.

The calculus Gc (Gi) is a variant of LK (LI) for classical (intuitionistic, respectively) first-order logic, originally introduced by Gentzen in [2]. In the literature, especially in [13] and [9], Gc and Gi are commonly referred to as G3c and G3i but we will omit '3' in the interest of readability. Moreover, we will write G to refer to either Gc or Gi. The rules are the standard ones and the reader interested is referred to [13] and [9].

The key feature of G is that the structural rules, including cut, are all admissible in it.

Theorem 1. *Cut elimination holds in* G.

2 Geometric theories

Extensions of G are not, in general, cut free; this means that Theorem 1 does not necessarily hold in the presence of new initial sequents or rules. It does hold, however, in the presence of rules following a certain pattern. To see this, we recall basic results from [7].

A *geometric axiom* is a formula following the *geometric axiom scheme* below:

$$\forall \bar{x}(P_1 \wedge \ldots \wedge P_n \rightarrow \exists \bar{y}_1 M_1 \vee \ldots \vee \exists \bar{y}_m M_m)$$

where each P_j is an atom and each M_i is a conjunction of a list of atoms $Q_{i_1}, \ldots, Q_{i_\ell}$ and none of the variables in any $\bar{y}_i$ are free in the P_js. We shall conveniently abbreviate $Q_{i_1}, \ldots, Q_{i_\ell}$ in $\boldsymbol{Q}_i$. In a geometric axiom, if $m = 0$ then the consequent of $\rightarrow$ becomes $\perp$, whereas if $n = 0$ the antecedent of $\rightarrow$ becomes $\top$. A *geometric theory* is a theory containing only geometric axioms. An m-premise *geometric rule*, for $m \geq 0$, is a rule following the *geometric rule scheme* below:

$$\frac{\boldsymbol{Q}_1^*, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta \quad \cdots \quad \boldsymbol{Q}_m^*, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}\ R$$

where each $\boldsymbol{Q}_i^*$ is obtained from $\boldsymbol{Q}_i$ by replacing every variable in $\bar{y}_i$ with a variable which does not occur free in the conclusion. Such variables will be called the *eigenvariables* of R. Without loss of generality, we assume that each $\bar{y}_i$ consists of a single variable. In sequent calculus a geometric theory can be formulated by adding on top of G finitely many geometric rules (recall that Δ contains exactly one formula in Gi). Moreover, geometric rules are assumed to satisfy the well-known closure property for contraction (see [9, 6.1.7]). Let $\mathsf{G^g}$ be any extension of G with finitely many geometric rules satisfying the closure condition (from now on, we will tacitly assume that the closure condition is always met). Cut elimination and the admissibility of the structural rules hold in $\mathsf{G^g}$. Although we will heavily rely on [7], we start by introducing a more general notion of substitution that allows an arbitrary term u (possibly a constant) to be replaced by a term t. In the presence of such general substitutions, special care is needed in order to maintain the height-preserving admissibility of substitutions. In particular, general substitutions are height-preserving admissible, provided that the replaced term u does not occur essentially in the calculus. Intuitively, a term u occurs essentially in a rule R when u cannot be replaced (by an arbitrary term), namely when u is a constant and u already occurs in the axiom from which R is obtained. More precisely,

Definition 2. A constant u occurs essentially in a geometric axiom A if and only if, for some $t \not\equiv u$, $A[{}^t_u]$ is not an instance of the axiom A.

We also agree that a term u occurs essentially in a geometric rule R when it does so in the corresponding axiom. For example, in the geometric axiom $\neg 1 \leqslant 0$ of non-degenerate partial orders (see [10, p. 116]) both 1 and 0 occur essentially; hence they also occur essentially in the corresponding geometric rule *Non-deg*:

$$\frac{}{1 \leqslant 0, \Gamma \Rightarrow \Delta}\ \textit{Non-deg}$$

The general substitution $[\,{}^t_u\,]$ is height-preserving admissible in G^{g}, provided that u occurs essentially in none of its geometric rules.

Lemma 3. *In* G^{g}, *if* $\vdash^n \Gamma \Rightarrow \Delta$, *$t$ is free for u in* Γ, Δ, *and u does not occur essentially in any rule of* G^{g}, *then* $\vdash^n \Gamma[\,{}^t_u\,] \Rightarrow \Delta[\,{}^t_u\,]$.

It is also easy to show that:

Theorem 4. *Cut elimination holds for* G^{g}.

Proof. See [7] for Gc^{g} and [3] for Gi^{g}. QED

3 Singular geometric theories

To prove interpolation in extensions of first-order logic, the class of geometric rules seems too large. Thus, we restrict our attention to a proper sub-class of it and we introduce the class of singular geometric theories. In the next section we will state (Lemma 9) that Maehara's lemma holds for singular geometric extensions of first-order logic.

A *singular geometric axiom* is a geometric axiom with at most one non-logical predicate and no constant occurring essentially. A *singular geometric theory* is a theory containing only singular geometric axioms. In sequent calculus a singular geometric theory can be formulated by extending G with finitely many geometric rules of form:

$$\frac{\boldsymbol{Q}_1^*, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta \quad \cdots \quad \boldsymbol{Q}_m^*, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}\ R$$

where no constant occurs essentially and that satisfy the following singularity condition:

$$|\mathsf{Rel}(\boldsymbol{Q}_1^*, \ldots, \boldsymbol{Q}_m^*, P_1, \ldots, P_n)| \leq 1 \qquad (\star)$$

It is evident that a number of important classical and intuitionistic mathematical theories are singular geometric. Regarding the classical ones, the theory of partial orders (R is reflexive, transitive and anti-symmetric), the theory of linear orders (R is a linear partial order), as well as the theories of strict partial orders (R is irreflexive and transitive) and strict linear orders (R is a trichotomic strict partial order) are singular geometric. Constructive singular geometric theories, on the other hand, include von Plato's theories of positive partial orders [11] (R is irreflexive and co-transitive) and positive linear orders (R is an asymmetric positive partial order), as well as the theory of apartness (R is irreflexive and splitting). Also the theory of equivalence relations (R is reflexive, transitive and symmetric) falls within the class of singular geometric theories. Finally, the fact that a relation R is functional (total and right-unique) can be axiomatized using singular geometric axioms. Singular geometric axioms are important in logic, too. Specifically, the axioms of identity are singular geometric.

$=$ is reflexive	$\forall x(x = x)$
$=$ satisfies the indiscernibility of identicals	$\forall x \forall y(x = y \wedge P[\frac{x}{z}] \to P[\frac{x}{y}])$

Notice that the indiscernibility of identicals satisfies the singularity condition ($\star$) because identity is a logical predicate. Hence, first-order logic with identity is a singular geometric theory.

Cut elimination for singular geometric rules clearly follows from cut elimination for geometric rules. More precisely, let G^{s} be any extension of G with a finite set of singular geometric rules. Then:

Theorem 5. *Cut elimination holds in* G^{s}.

4 Interpolation with singular geometric rules

The standard proof of interpolation in sequent calculi rests on a result due to Maehara which appeared (in Japanese) in [5] and was later made available to international readership by Takeuti in his [12]. While interpolation is a result about logic, regardless the formal system (sequent calculus, natural deduction, axiom system, etc), Maehara's lemma is a "sequent-calculus version" of interpolation. Although originally Maehara proved his lemma for LK, it is easy to adapt the proof so that it holds also in G (cf. [13, §4.4]). We recall from [13] some basic definitions.

Definition 6 (partition, split-interpolant). A *partition of a sequent* $\Gamma \Rightarrow \Delta$ is an expression $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$, where $\Gamma = \Gamma_1, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$ (for $=$ the multiset-identity). A *split-interpolant* of a partition $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$ is a formula C such that:

I $\quad \vdash \Gamma_1 \Rightarrow \Delta_1, C$

II $\quad \vdash C, \Gamma_2 \Rightarrow \Delta_2$

III $\quad \mathcal{L}(C) \subseteq \mathcal{L}(\Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2)$

We use $\Gamma_1 \ ; \ \Gamma_2 \overset{C}{\Rightarrow} \Delta_1 \ ; \ \Delta_2$ to indicate that C is a split-interpolant for $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$.

Lemma 7 (Maehara). *In* Gc *every partition* $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$ *of a derivable sequent* $\Gamma \Rightarrow \Delta$ *has a split-interpolant. In* Gi *every partition* $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \ ; \ A$ *of a derivable sequent* $\Gamma \Rightarrow A$ *has a split-interpolant.*

From Maehara's lemma it is immediate to prove Craig's interpolation theorem.

Theorem 8 (Craig). *If $A \Rightarrow B$ is derivable in* G *then there exists a C such that $\vdash A \Rightarrow C$ and $\vdash C \Rightarrow B$ and $\mathcal{L}(C) \subseteq \mathcal{L}(A) \cap \mathcal{L}(B)$.*

Of any calculus for which Theorem 8 holds, we say that it has the interpolation property. Now we extend Lemma 7 to extensions of G with singular geometric rules.

Lemma 9. *In* $\mathsf{Gc^s}$ *every partition $\Gamma_1 \; ; \; \Gamma_2 \Rightarrow \Delta_1 \; ; \; \Delta_2$ of a derivable sequent $\Gamma \Rightarrow \Delta$ has a split-interpolant. In* $\mathsf{Gi^s}$ *every partition $\Gamma_1 \; ; \; \Gamma_2 \Rightarrow \; ; \; A$ of a derivable sequent $\Gamma \Rightarrow A$ has a split-interpolant.*

Proof. The reader is referred to [3]. QED

From Lemma 9 it is immediate to conclude that singular geometric extensions of classical and intuitionistic logic satisfy the interpolation theorem, namely:

Theorem 10. $\mathsf{G^s}$ *has the interpolation property.*

5 Applications

We now consider some corollaries of Theorem 10 in which the strategy for building interpolants provided in Lemma 9 is applied. Notice that in the theories considered in this section all contracted instances are admissible and, hence, we can ignore them.

5.1 First-order logic with identity

We start with first-order logic with identity. Recall that a cut-free calculus for classical first-order logic with identity has been presented in [8] by adding on top of Gc the rules $Ref_=$ and $Repl_=$ corresponding to the reflexivity of $=$ and Leibniz's principle of indiscernibility of identicals, respectively.

$$\frac{s = s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,Ref_= \qquad \frac{P[{}^{t}_{x}], P[{}^{s}_{x}], t = s, \Gamma \Rightarrow \Delta}{P[{}^{s}_{x}], t = s, \Gamma \Rightarrow \Delta}\,Repl_=$$

In intuitionistic theories, on the other hand, identity is often treated differently and we will provide a constructively more acceptable treatment of identity later in dealing with apartness. In general, however, nothing prevents us from building intuitionistic first-order logic with identity in a parallel fashion to the classical case. This is, for example, the route taken in [13] and we will follow suit.

Corollary 11. $\mathsf{G^=}$ *has the interpolation property.*

5.2 Equivalence relations

In a perfectly parallel fashion, we obtain the theory of equivalence relations by adding to G the rules corresponding to the reflexivity, transitivity and symmetry of a binary relation $\sim$. Thus, $\mathsf{EQ} = \mathsf{G} + \{Ref_{\sim}, Trans_{\sim}, Sym_{\sim}\}$.

$$\frac{s \sim s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref_{\sim} \qquad \frac{s \sim u, s \sim t, t \sim u, \Gamma \Rightarrow \Delta}{s \sim t, t \sim u, \Gamma \Rightarrow \Delta}\ Trans_{\sim}$$

$$\frac{t \sim s, s \sim t, \Gamma \Rightarrow \Delta}{s \sim t, \Gamma \Rightarrow \Delta}\ Sym_{\sim}$$

From the fact that these rules are singular geometric, it follows that:

Corollary 12. EQ *has the interpolation property.*

5.3 Partial and linear orders

Now we consider some well-known order theories. We start with partial orders. In sequent calculus, the theory of partial orders is obtained by extending $\mathsf{Gc}^{=}$ with the following rules corresponding to the axioms of reflexivity, transitivity and anti-symmetry of a binary relation $\leqslant$. Thus, let $\mathsf{PO} = \mathsf{Gc}^{=} + \{Ref_{\leqslant}, Trans_{\leqslant}, Anti\text{-}sym_{\leqslant}\}$:

$$\frac{s \leqslant s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref_{\leqslant} \qquad \frac{s \leqslant u, s \leqslant t, t \leqslant u, \Gamma \Rightarrow \Delta}{s \leqslant t, t \leqslant u, \Gamma \Rightarrow \Delta}\ Trans_{\leqslant}$$

$$\frac{s = t, s \leqslant t, t \leqslant s, \Gamma \Rightarrow \Delta}{s \leqslant t, t \leqslant s, \Gamma \Rightarrow \Delta}\ Anti\text{-}sym_{\leqslant}$$

Linear orders are obtained by assuming that the partial order $\leqslant$ is also linear, i.e. $\mathsf{LO} = \mathsf{PO} + \{Lin_{\leqslant}\}$.

$$\frac{s \leqslant t, \Gamma \Rightarrow \Delta \quad t \leqslant s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Lin_{\leqslant}$$

Both PO and LO are singular geometric theories, hence:

Corollary 13. LO *(hence,* PO*) has the interpolation property.*

Unlike $\mathsf{G}^{=}$ and EQ, the underlying logical calculus of both PO and LO is the classical one. The reason is that linearity is intuitionistically contentious and normally it requires a different, more constructively acceptable, axiomatization that will be considered in Section 5.6.

5.4 Strict partial and linear orders

The theory of strict partial orders consists of the axioms of first-order logic with identity plus the irreflexivity and transitivity of $<$. As we did for PO and LO, we consider this theory to be based on classical logic, i.e. by adding on top of $\mathsf{Gc}^=$ the following rules:

$$\frac{}{s < s, \Gamma \Rightarrow \Delta}\ \mathit{Irref}_< \qquad \frac{s < u, s < t, t < u, \Gamma \Rightarrow \Delta}{s < t, t < u, \Gamma \Rightarrow \Delta}\ \mathit{Trans}_<$$

Let SPO be $\mathsf{Gc}^= + \{\mathit{Irref}_<, \mathit{Trans}_<\}$. Total strict partial orders are then obtained assuming that $<$ is also trichotomic, i.e. $\mathsf{SLO} = \mathsf{SPO} + \{\mathit{Trich}_<\}$:

$$\frac{s = t, \Gamma \Rightarrow \Delta \quad s < t, \Gamma \Rightarrow \Delta \quad t < s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Trich}_<$$

Corollary 14. SLO *(hence,* SPO*) has the interpolation property.*

5.5 Apartness

We noticed earlier that in intuitionistic theories the identity relation is not always treated as in classical logic. In particular, identity is defined in terms of the more constructively acceptable relation of apartness. Apartness was originally introduced by Brouwer (and later axiomatized by Heyting in [4]) to express inequality between real numbers in the constructive analysis of the continuum: whereas saying that two real numbers a and b are unequal only means that the assumption $a = b$ is contradictory, to say that a and b are apart expresses the constructively stronger requirement that their distance on the real line can be effectively measured, i.e. that $\mid a - b \mid > 0$ has a constructive proof. Classically, inequality and apartness coincide, but intuitionistically two real numbers can be unequal without being apart. The theory of apartness consists of intuitionistic first-order logic plus the irreflexivity and splitting of $\neq$. Following [6], the theory of apartness is formulated by adding on top of Gi the following rules:[1]

$$\frac{}{s \neq s, \Gamma \Rightarrow A}\ \mathit{Irref}_{\neq} \qquad \frac{s \neq u, s \neq t, \Gamma \Rightarrow A \quad t \neq u, s \neq t, \Gamma \Rightarrow A}{s \neq t, \Gamma \Rightarrow A}\ \mathit{Split}_{\neq}$$

Let $\mathsf{AP} = \mathsf{Gi} + \{\mathit{Irref}_{\neq}, \mathit{Split}_{\neq}\}$. Given that these two rules are singular geometric rules, it follows that:

Corollary 15. AP *has the interpolation property.*

[1] Notice that Negri's underlying calculus is a quantifier-free version of Gi.

5.6 Positive partial and linear orders

Just like apartness is a positive version of inequality, so excess $\nleqslant$ is a positive version of the negation of a partial order $\leqslant$. The excess relation was introduced by von Plato in [11] and has been further investigated by Negri in [6]. The theory of positive partial orders consists of intuitionistic first-order logic plus the irreflexivity and co-transitivity of $\nleqslant$.[2] Let $\mathsf{PPO} = \mathsf{Gi} + \{Irref_{\nleqslant}, \ Co\text{-}trans_{\nleqslant}\}$

$$\frac{}{s \nleqslant s, \Gamma \Rightarrow A}\ Irref_{\nleqslant} \qquad \frac{s \nleqslant u, s \nleqslant t, \Gamma \Rightarrow A \quad u \nleqslant t, s \nleqslant t, \Gamma \Rightarrow A}{s \nleqslant t, \Gamma \Rightarrow A}\ Co\text{-}trans_{\nleqslant}$$

The theory of positive linear orders extends the theory of positive partial orders with the asymmetry of $\nleqslant$. Specifically, let $\mathsf{PLO} = \mathsf{PPO} + \{Asym_{\nleqslant}\}$:

$$\frac{}{s \nleqslant t, t \nleqslant s, \Gamma \Rightarrow A}\ Asym_{\nleqslant}$$

Given that all these rules are singular geometric, from Theorem 10 it follows that

Corollary 16. PPO *and in* PLO *have the interpolation property.*

To conclude, we have shown (Lemma 9) how to extend Maehara's lemma to extensions of classical and intuitionistic sequent calculi with singular geometric rules and provided a number of interesting examples of singular geometric rules that are important both in logic and mathematics, especially in order theories. In particular, we have shown that Lemma 9 covers first-order logic with identity and its extension with the theory of (strict) partial and linear orders. We have also seen that the same holds for the intuitionistic theories of apartness, as well as for positive partial and linear order.

References

[1] W. Craig. Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory. *The Journal of Symbolic Logic*, 22(3):269–285, 1957.

[2] G. Gentzen. Investigation into logical deductions. In M. E. Sazbo, editor, *The Collected Papers of Gerhard Gentzen*, chapter 3, pages 68–131. North-Holland, 1969.

[3] G. Gherardi, P. Maffezioli, and E. Orlandelli. Interpolation in extensions of first-order logic. *Studia Logica*, 2019.

[4] A. Heyting. *Intuitionism. An Introduction*. North-Holland, 1956.

[5] S. Maehara. On the interpolation theorem of Craig. *Suugaku*, 12:235–237, 1960 (in Japanese).

[2] Co-transitivity and splitting should not be confused. In particular, splitting (along with irreflexivity) gives symmetry, whereas co-transitivity does not. This is what distinguishes apartness (which is symmetric) from excess (which in general is not).

[6] S. Negri. Sequent calculus proof theory of intuitionistic apartness and order relations. *Archive for Mathematical Logic*, 38(8):521–547, 1999.

[7] S. Negri. Contraction-free sequent calculi for geometric theories with an application to Barr's theorem. *Archive for Mathematical Logic*, 42(4):389–401, 2003.

[8] S. Negri and J. von Plato. Cut elimination in the presence of axioms. *The Bulletin of Symbolic Logic*, 4(4):418–435, 1998.

[9] S. Negri and J. von Plato. *Structural Proof Theory*. Cambridge University Press, 2001.

[10] S. Negri and J. von Plato. *Proof Analysis: A Contribution to Hilbert's Last Problem*. Cambridge University Press, 2011.

[11] J. von Plato. Positive lattices. In P. Schuster, U. Berger, and H. Osswald, editors, *Reuniting the Antipodes – Constructive and Nonstandard Views of the Continuum*, volume 306 of *Synthese Library*, pages 185–197. Kluwer, 2001.

[12] G. Takeuti. *Proof Theory*, volume 81 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, 2nd edition, 1987.

[13] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 2nd edition, 2000.