

# Discrete symmetries in Heterotic/F-theory duality and mirror symmetry

Mirjam Cvetič,<sup>a,b,c</sup> Antonella Grassi<sup>b</sup> and Maximilian Poretschkin<sup>a</sup>

<sup>a</sup>*Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, PA 19104-6396, U.S.A.*

<sup>b</sup>*Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6396, U.S.A.*

<sup>c</sup>*Center for Applied Mathematics and Theoretical Physics, University of Maribor, Maribor, Slovenia*

*E-mail:* [cvetic@cvetic.hep.upenn.edu](mailto:cvetic@cvetic.hep.upenn.edu), [grassi@math.upenn.edu](mailto:grassi@math.upenn.edu), [mporet@sas.upenn.edu](mailto:mporet@sas.upenn.edu)

**ABSTRACT:** We study aspects of Heterotic/F-theory duality for compactifications with Abelian discrete gauge symmetries. We consider F-theory compactifications on genus-one fibered Calabi-Yau manifolds with  $n$ -sections, associated with the Tate-Shafarevich group  $\mathbb{Z}_n$ . Such models are obtained by studying first a specific toric set-up whose associated Heterotic vector bundle has structure group  $\mathbb{Z}_n$ . By employing a conjectured Heterotic/F-theory mirror symmetry we construct dual geometries of these original toric models, where in the stable degeneration limit we obtain a discrete gauge symmetry of order two and three, for compactifications to six dimensions. We provide explicit constructions of mirror-pairs for symmetric examples with  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$ , in six dimensions. The Heterotic models with symmetric discrete symmetries are related in field theory to a Higgsing of Heterotic models with two symmetric abelian U(1) gauge factors, where due to the Stückelberg mechanism only a diagonal U(1) factor remains massless, and thus after Higgsing only a diagonal discrete symmetry of order  $n$  is present in the Heterotic models and detected via Heterotic/F-theory duality. These constructions also provide further evidence for the conjectured mirror symmetry in Heterotic/F-theory at the level of fibrations with torsional sections and those with multi-sections.

**KEYWORDS:** Discrete Symmetries, F-theory, String Duality, Superstrings and Heterotic Strings

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## 1 Introduction and summary of results

Recent years have witnessed important advances in F-theory compactifications [1–3]. While the study of non-Abelian gauge symmetries has been extensively studied in the past, the study of Abelian and discrete gauge symmetries has been advanced only lately.

F-theory compactifications with Abelian gauge symmetries  $U(1)^n$  are based on constructions of elliptically fibered Calabi-Yau manifolds with rank- $n$  Mordell-Weil (MW) group of rational sections. A systematic approach was initiated in [4] for constructions with rank-one MW group, and generalized to rank-two [5–7] and rank-three [8] MW groups, respectively.

Recently, there has also been progress in F-theory compactifications with discrete gauge symmetries  $\mathbb{Z}_n$ , initiated in [9] and further advanced in [10–14]. A natural object attached to these compactifications is given by the Tate-Shafarevich (TS) group of the genus-one fibration which is a discrete group that organizes inequivalent genus-one geometries which share the same associated Jacobian fibration. In F-theory discrete symmetries are also well understood as a result of a Higgsing process of F-theory compactifications with  $U(1)$  symmetries, where the matter field with charge  $n$  acquires a vacuum expectation value and breaks the Abelian  $U(1)$  gauge symmetry to the discrete symmetry group  $\mathbb{Z}_n$ . This field theoretical process is geometrically interpreted as a conifold transition [11, 15, 16]. Most of the past works primarily focused on  $\mathbb{Z}_2$  gauge symmetry. However, new insights into aspects of the TS group, and its relations to M-theory vacua, as well as the study of the Higgsing process in the case of  $\mathbb{Z}_3$  were addressed in [14]. See also [17] for related work.

Heterotic/F-theory duality plays an important role in shedding light on the origin of gauge symmetries in Heterotic gauge theory from the geometric perspective of F-theory. In the past aspects of non-Abelian gauge symmetries have been studied extensively [1–3, 18–20]. In particular, Heterotic/F-theory duality allows for making statements about the Heterotic vector bundle  $V$ , which is typically hard to control, in terms of the controllable geometry of the Calabi-Yau manifold on the F-theory side. On the other hand, only recently key steps towards developing the geometrical duality map between Heterotic and F-theory compactifications with Abelian gauge symmetries were taken in [21].

The purpose of this paper is to study discrete gauge symmetries in Heterotic/F-theory duality. Since in Heterotic string theory the gauge group is given as the commutant of the structure group of the respective vector bundles within the two  $E_8$  factors, in order to engineer a discrete gauge group in a Heterotic string compactification, one needs a pair of vector bundles  $V_1, V_2$  whose structure group is the centralizer of the desired discrete gauge group, e.g., [19].

In order to construct explicitly such models we employ the methods of [21] which provide a semi-stable degeneration for genus-one fibrations. We then generalize a conjecture which connects toric mirror symmetry and Heterotic/F-theory mirror duality<sup>1</sup> in eight dimensions [20]. In particular this relates the gauge symmetry (structure group of the bundle) of a K3 surface to the structure group of the Heterotic bundle (gauge symmetry) which

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<sup>1</sup>See [22–24] for the use of mirror symmetry in the context of Heterotic/F-theory duality in order to compute  $F^4$  couplings in eight dimensions.

is associated to the rational elliptic surfaces which arise from the dual K3 surface in the stable degeneration limit. The dual pairs of K3 surfaces can be constructed by employing toric geometry techniques and are highlighted in section 2.3. For our purposes we construct the mirror models where the mirror symmetry in the fiber exchanges torsional sections of order  $n$  with the  $n$ -section. Thus, such a construction relates models with discrete structure group of the bundle to the model with a discrete gauge symmetry.

We therefore start with the construction of toric models where in the stable degeneration limit the the structure groups of the Heterotic bundles take the form  $G/D$  where  $G$  is a group of ADE type and  $D$  is a discrete group. By employing the conjectured F-theory/Heterotic mirror-symmetry we construct dual toric models, where in the stable degeneration limit we obtain a discrete gauge symmetry of order  $n$ , for compactifications in six dimensions. The explicit examples are based on the symmetric constructions where the two bundles  $V_1$  and  $V_2$  are the same. In six dimensions for these models only a diagonal gauge symmetry is realized explicitly, and in particular for the dual polytope only a diagonal discrete symmetry is realized in the effective theory. We also demonstrate at the level of an effective six-dimensional field theory how such symmetric Heterotic models with discrete symmetry are related to un-Higgsing to Heterotic models with only a diagonal  $U(1)$  massless gauge symmetry in the effective theory. It is important to stress that this description therefore only works in complex codimensions greater than one.

As concrete examples, we construct and analyse the mirror dual pairs for the case of symmetric  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$  symmetry. These constructions also provide further evidence for the conjectured mirror symmetry in Heterotic/F-theory at the level of fibrations with torsional sections and those with multi-sections.

This paper is organized in the following way. In section 2, we provide a brief review of the key aspects of Heterotic/F-theory duality, the origin of discrete symmetries in F-theory and Heterotic theory, and a discussion of the mirror pairs of K3 surfaces, as it is key in the study of conjectured mirror Heterotic/F-theory duality. In section 3.1 we summarize a conjectured Heterotic mirror symmetry where the Heterotic background bundle of the original K3 surface is interchanged with the gauge symmetry of the mirror K3 one. In section 3.2 we provide supporting evidence for this conjecture by studying mirror two-dimensional fiber ambient spaces where the torsional sections (associated with the discrete bundle structure group) and multi-sections (associated with the discrete gauge symmetry) surfaces are interchanged. In section 3.3 we elucidate the theory aspects, how six-dimensional Heterotic models with discrete symmetry are related to those with  $U(1)$  gauge symmetry via Higgsing by matter fields with conjectured charges; in the symmetric case only a diagonal  $U(1)$  is massless, and thus after Higgsing only a diagonal discrete symmetry is present. In section 4.1 we present the construction of models with symmetric  $\mathbb{Z}_2$  gauge symmetry which we demonstrate explicitly in the geometry of the mirror polytope. In section 4.2 we elaborate on aspects of six-dimensional geometry. In section 4.3 we repeat the construction for models with the symmetric  $\mathbb{Z}_3$  gauge symmetry. Concluding remarks in section 5 highlight the key insights of the paper and possible future directions. In the appendix we present explicit results for the Weierstrass map of the models studied in the main text.

## 2 Heterotic/F-theory duality and U(1)-factors

This section is divided into two parts: in the first part, we review basic facts about Heterotic/F-theory duality and discrete symmetries in F-theory and the Heterotic string. In addition, in order to set the stage for the construction of certain Heterotic background bundles, we also review the construction of mirror pairs of Calabi-Yau manifolds. The review part is mainly based on [18, 25, 26], to which we refer for further details.

The next part formulates two important conjectures which are not rigorously proven but given strong evidence. The first conjecture is concerned with the construction of background bundles that have structure group  $(E_7 \times SU(2))/\mathbb{Z}_2$  and  $(E_6 \times SU(3))/\mathbb{Z}_3$ . The second conjecture establishes a field theory connection between six-dimensional models with massive U(1) symmetries and models with discrete symmetries and geometrically corresponds to an analogue of conifold transitions on the Heterotic side.

### 2.1 Heterotic/F-theory duality in eight dimensions

The basic statement of Heterotic/F-theory duality is that the Heterotic  $E_8 \times E_8$  String compactified on a torus, which we denote by  $Z_1$ , is equivalent to F-theory compactified on an elliptically fibered K3 surface  $X_2$ .

#### 2.1.1 The standard stable degeneration limit

The moduli of both theories are matched in the stable degeneration limit. In this limit, the K3 surface  $X_2$  degenerates into two half K3 surfaces  $X_2^+$ ,  $X_2^-$  as

$$X_2 \rightarrow X_2^+ \cup_{Z_1} X_2^- . \tag{2.1}$$

$X_2^+$  as well as  $X_2^-$  are elliptic fibrations  $\pi_{\pm} : X_2^{\pm} \rightarrow \mathbb{P}^1$  over a  $\mathbb{P}^1$ . These two  $\mathbb{P}^1$  intersect in precisely one point so that the two half K3 surfaces intersect in a common elliptic fiber which is identified with the Heterotic elliptic curve,  $X_2^+ \cap X_2^- = Z_1$ .

Traditionally, this has been formulated for K3 surfaces which are given as elliptic Weierstrass fibrations over a  $\mathbb{P}^1$ . More recently, it has been realized [17, 21] that the study of U(1)s requires more general descriptions of the stable degeneration limit, in particular for fiber ambient spaces different from  $\mathbb{P}^{(1,2,3)}$ .

#### 2.1.2 Matching the continuous gauge groups

The non-Abelian part  $G$  of the F-theory gauge group is given by the singularities of the elliptic fibration of  $X_2$ , while the Abelian part is given by  $U(1)^{16-\text{rk}(G)} \times U(1)^4$  [1, 2, 27]. These are inherited by the two half K3 surfaces  $X_2^{\pm}$  in the stable degeneration limit as follows.

The homology lattice of a half K3 surface  $X_2^{\pm}$  is given in general by

$$H_2(X_2^{\pm}, \mathbb{Z}) = \Gamma_8 \oplus U . \tag{2.2}$$

Here,  $\Gamma_8$  denotes the root lattice of  $E_8$ , while  $U$  contains the classes of the elliptic fiber as well as of the zero section. For a non-generic half K3 surface, where the the curves in a

sublattice  $\mathcal{R}(G^\pm)$  of  $\Gamma_8$ , denoting the root lattice of some ADE group  $G^\pm$ , are shrunken to zero size, the half K3 surface develops a singularity of type  $G^\pm$ . In rational homology,  $\Gamma_8$  further splits as

$$\Gamma_8 = \text{MW}(X_2^\pm) \oplus \mathcal{R}(G^\pm), \tag{2.3}$$

where  $\text{MW}(X_2^\pm)$  denotes the Mordell-Weil group of  $X_2^\pm$ .

The latter fact provides the connection to the Heterotic description of the gauge group. According to [18], the moduli space of semi-stable  $E_8$ -bundles on an elliptic curve  $Z_1$  corresponds to the complex structure moduli space of a half K3 surface  $X_2$  whose anti-canonical class is given by  $Z_1$ . Furthermore, if  $X_2$  has an ADE singularity of type  $\tilde{G}_\pm$  then the structure group of  $V_1, V_2$  is reduced to the centralizer  $H_\pm$  of  $\tilde{G}_\pm$  within  $E_8$ , respectively. In contrast, the abelian  $U(1)$  symmetries are translated as follows. Sections of  $\text{MW}(X_2^+)$  that glue with a section of  $\text{MW}(X_2^-)$  give rise to a global section of the K3 surface  $X_2$  [21].<sup>2</sup>

The six-dimensional duality is obtained by fibering the eight-dimensional duality over a common  $\mathbb{P}^1$ . Thus, on the F-theory side one deals with a Calabi-Yau threefold  $Z_3$  which is elliptically fibered<sup>3</sup> over a rational surface, for example a Hirzebruch surface  $\mathbb{F}_n$ . In contrast, the Heterotic string is compactified on an elliptically fibered K3 surface whose base  $\mathbb{P}^1$  is to be identified with the base of the rational surface. In general, due to monodromies on the base also non-simply laced gauge groups can occur [18, 27]. In addition, only those eight-dimensional sections that promote to rational six-dimensional sections give rise to  $U(1)$  symmetries in six dimensions.<sup>4</sup>

Finally, another six-dimensional effect is the non-perturbative enhancement of the gauge group [27] due to singularities which map to some of the fibers of the rational surface. In other words, these are singularities which are visible both on the Heterotic side as well as on the F-theory side.

## 2.2 Discrete symmetries in Heterotic string theory and F-theory

As the main focus of this work is the investigation of discrete symmetries, we review their appearance on the F-theory side as well as on the Heterotic side in this subsection.

### 2.2.1 Discrete symmetries in F-theory

Discrete symmetries are best understood within F-theory as result of a Higgsing process of continuous  $U(1)$  symmetries. It is important to stress that this description therefore only works in complex co-dimensions greater than one. As a well-known fact,  $U(1)$  symmetries within F-theory are detected by the generators of the Mordell Weil group [4]. Each such generator gives rise to an additional globally well-defined embedding of the base manifold

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<sup>2</sup>For a detailed discussion of the role of the Mordell-Weil group in the context of half K3 surfaces and Heterotic bundles, see [28].

<sup>3</sup>This is the classical approach. More generally, one is led to consider genus-one fibered three-folds as well.

<sup>4</sup>If a half K3 surface has a singularity of rank  $r$ , there are  $9-r$  linear independent sections in its Mordell-Weil group. However, once the half K3 surface is promoted to a six-dimensional rational three-fold, these sections are not necessarily preserved.

into the elliptic fibration. Reducing the  $C_3$ -form field along the corresponding divisors identifies the corresponding  $U(1)$  symmetry.

Such a  $U(1)$  symmetry can be higgsed to a discrete  $\mathbb{Z}_n$  symmetry, if there is a matter field of charge  $n$ . Such matter fields arise from M2-branes that wrap components of  $I_2$ -fibers which appear at co-dimension two loci. Their corresponding charges are determined from the number of intersections of the corresponding sections with the respective component of the  $I_2$ -fiber. These fields become massless if the corresponding component of the  $I_2$ -fiber shrinks to zero size. From a mathematical perspective, this shrinking can be viewed as part of a conifold transition [11, 15, 16]. Here, the shrunken component of the  $I_2$ -fiber, which is topologically a two-sphere, gets replaced by a three-sphere. The physical interpretation of the latter deformation is to give a vacuum expectation value to the Higgs field. In addition, for an  $\mathbb{Z}_n$ -symmetry, the conifold transition glues  $n$  rational sections to an  $n$ -section. In fact, genus-one fibered Calabi-Yau manifolds with an  $n$ -section have an element of order  $n$  in their Tate-Shafarevich (TS) group which is the geometrical analogue of the  $\mathbb{Z}_n$ -symmetry that occurs in the fields theory and labels in-equivalent geometries that share the same Jacobian fibration [9, 10, 14].

### 2.2.2 Discrete symmetries in the Heterotic string

As discussed in a previous subsection 2.1.2, the Heterotic gauge group is given as the commutant of the structure group of the respective vector bundles within the two  $E_8$  factors. Thus, in order to engineer a discrete gauge group in a Heterotic string compactification, one needs a pair of vector bundles  $V_1, V_2$  whose structure group is the centralizer of the desired discrete gauge group. These centralizers have been determined for different discrete groups in, e.g., [19]. That is, in order to realize a discrete gauge group  $\mathbb{Z}_2$ , one needs a background bundle with structure group  $(E_7 \times SU(2))/\mathbb{Z}_2$ .

### 2.3 Constructing mirror pairs of K3 surfaces

In the following, we recall the construction of toric mirror pairs of K3 surfaces. As outlined in section 3.1, we will eventually use mirror symmetry techniques in order to construct background bundles with structure groups  $(E_7 \times SU(2))/\mathbb{Z}_2$  and  $(E_6 \times SU(3))/\mathbb{Z}_3$ . To set the geometrical stage, we review in the following Batyrev's formalism [29] to construct mirror pairs of Calabi-Yau manifolds using pairs of reflexive polyhedra. A more detailed review on this subject and further references can be found in, e.g., [30].

It is a well-known fact (see, e.g., [31]), that given a reflexive  $n$ -dimensional polyhedron  $\Delta$ , there is a natural simplicial fan associated to it which will be denoted by  $\Sigma$ .  $\Sigma$  defines a toric Fano variety which is denoted by  $\mathbb{P}_\Delta$ . In particular, if a fine triangulation of  $\Delta$  has been chosen, the associated variety  $\mathbb{P}_\Delta$  is Gorenstein and terminal. We also note that a general section  $\chi$  of the anti-canonical bundle  $\mathcal{O}(-K_{\mathbb{P}_\Delta})$  of  $\mathbb{P}_\Delta$  defines a Calabi-Yau  $(n - 1)$ -fold. Finally, there is a mirror Calabi-Yau  $(n - 1)$ -fold which is given by a section of the anticanonical bundle of the Fano toric variety associated to the dual polytope of  $\Delta$ , denoted by  $\Delta^\circ$ . In particular, the defining equation for  $\chi$  is given by

$$\chi = \sum_{P^\circ \in \Delta^\circ} \prod_{P \in \Delta} a_P x_P^{\langle P, P^\circ \rangle + 1}. \tag{2.4}$$

Here,  $P$  and  $P^\circ$  label the integer points of  $\Delta$  and  $\Delta^\circ$ , respectively. In addition,  $x_P$  denotes the coordinate associated to the ray determined by the point  $P$  and  $\langle, \rangle$  denotes the natural product between the dual lattices into which  $\Delta$  and  $\Delta^\circ$  are embedded. In addition, one can calculate the rank of the Picard lattice from Batyrev's formula which has been generalized to (very general) toric K3 surfaces in [32]

$$\text{rk Pic}(X) = l(\Delta) - n - 1 - \sum_{\Gamma} l^*(\Gamma) + \sum_{\Theta} l^*(\Theta)l^*(\hat{\Theta}). \tag{2.5}$$

Recall that  $\text{rk Pic}(Y) = \dim_{\mathbb{Q}} H^2(X, \mathbb{Q}) \cap H^{1,1}(Y, \mathbb{C})$ ; when  $X$  is a Calabi-Yau with  $\dim(X) \geq 3$ ,  $\text{rk Pic}(X) = h^{1,1}(X)$ . Here  $l(\Delta)$  ( $l^*(\Delta)$ ) denote the number of (inner) points of the  $n$ -dimensional polytope  $\Delta$ . In addition,  $\Gamma$  ( $\Gamma^\circ$ ) denote the codimension one faces of  $\Delta$  ( $\Delta^\circ$ ), while  $\Theta$  denotes a codimension two face with  $\hat{\Theta}$  being its dual. In the following, we focus on K3 surfaces which are given as elliptic and genus-one fibrations over  $\mathbb{P}^1$  and whose corresponding ambient space is given by the direct product  $\mathbb{P}^1 \times \mathbb{P}_{\Delta_2}$ , where  $\Delta_2$  denotes any two-dimensional reflexive polytope.

### 3 Conjectures in field theory and geometry

This section is devoted to the discussion of two conjectures. The first one is concerned with the construction of Heterotic vector bundles that exhibit structure groups whose commutant within  $E_8$  gives rise to a discrete symmetry. The second one discusses the relation of six-dimensional Heterotic field theories with  $U(1)$ 's to those with discrete symmetries. Both conjectures are supported by a number of convincing observations.

#### 3.1 Constructing background bundles using mirror symmetry

In [20] Berglund and Mayr provide explicit descriptions of vector bundles with structure algebra of type ABCDE. In particular they consider K3 surfaces which are elliptic fibrations over  $\mathbb{P}^1$  and the elliptic fiber is specified by the ambient space  $\mathbb{P}^{(1,2,3)}$ . Calling the affine base coordinate  $z$ , they conjecture the following: if a K3 surface has singularities of type  $G_1, G_2$  at  $z = 0$  and  $z = \infty$ , respectively, its mirror K3 will have singularities of type  $H_1$  and  $H_2$ , where  $H_i = [E_8, G_i]$ .

This statement can be traced back to the fact that mirror symmetry for K3 surfaces has an interpretation in terms of orthogonal lattices. It can be proven in the cases considered in by [20] by considering the stable degeneration limit as discussed in section 2.1.1 and further elaborated on in [21]; it is clear in fact that the points  $z = 0$  and  $z = \infty$  map to different half K3 surfaces. Thus, each half K3 surface inherits precisely one singularity, whose commutant within  $E_8$  gives rise to a singularity within the mirror dual K3 surface. This conjecture should hold true more generally, i.e. in particular also for singularities that are inherited by both half K3 surfaces, as follows.

Consider a (genus-one) K3 surface  $X_2$  whose mirror K3 surface  $\tilde{X}_2$  has also a genus-one fibration; the semi-stable degeneration of both K3 surfaces can then be computed with the toric symplectic cut techniques of [21] (see also [33]); note that these methods also extend semi-stable degeneration to genus-one fibrations. Assume that  $X_2$  has a couple of



ADE singularities which correspond to a gauge factor  $G$ . Let  $G_1$  (respectively  $G_2$ ) be the product of the singularities inherited by the first half K3 surface  $X_2^+$  (respectively  $X_2^-$ ). Then the mirror K3 surface  $\tilde{X}_2$  will degenerate into two rational surfaces  $\tilde{X}_2^+ \cup \tilde{X}_2^-$  and the singularities of  $\tilde{X}_2^+$  and  $\tilde{X}_2^-$  correspond to gauge algebras  $H_1$  and  $H_2$ , respectively and will be given as  $H_i = [E_8, G_i]$ . It is important to stress that for this generalization, we can assume any toric ambient space  $\mathbb{P}_{\Delta_2}$  — in contrast to only considering  $\mathbb{P}^{(1,2,3)}$  — for a two-dimensional polytope  $\Delta_2$  for the corresponding genus-one fibration. In this way, one also obtains structure groups which do not appear in the classification by Berglund and Mayr.

### 3.2 Mirror symmetry in the fiber: trading multi- for torsional sections

In this sub-section we provide supporting evidence for the conjecture made above. In particular, as demonstrated in the concrete examples in section 4, mirror symmetry exchanges different two-dimensional fiber ambient spaces, which leads to the exchange of multi-sections and torsional sections in the corresponding mirror geometries.

In section 4 we will use this conjecture in order to construct background bundles with structure group  $(E_7 \times SU(2))/\mathbb{Z}_2$  and  $(E_6 \times SU(3))/\mathbb{Z}_3$ . We construct an elliptic K3 surface with gauge group  $(E_7 \times E_7 \times SU(2) \times SU(2))/\mathbb{Z}_2$  and ambient space of the elliptic fiber  $\mathbb{P}^{(1,1,2)}/\mathbb{Z}_2$ . The dual K3 surface is also elliptic with ambient space of the elliptic fiber given by  $\mathbb{P}^{(1,1,2)}$  which generically has a bi-section. The latter is a clear signal of a  $\mathbb{Z}_2$  symmetry. The same observation can be made for the case of a discrete  $\mathbb{Z}_3$  symmetry where the fiber ambient spaces  $\mathbb{P}^2/\mathbb{Z}_3$  and  $\mathbb{P}^2$  get exchanged under mirror symmetry.

However, as previously explained in 2.2.1, discrete symmetries are best understood from a Higgsing perspective in six dimensions. Thus, it is expected that the corresponding multi-sections become apparent in six dimensions. This is indeed, what we will observe.

We would also like to stress that our approach sheds some light on the following observation made in [12] and further confirmed in [34, 35]. That reference has constructed for every two-dimensional reflexive polytope a corresponding Calabi-Yau three-fold which was realized as an elliptic/genus-one fibration, specified by the corresponding two-dimensional polytope, over an arbitrary two-dimensional base. It was furthermore observed that mirror symmetry exchanges fiber ambient spaces that give rise to Calabi-Yau threefolds with non-trivial Tate-Shafarevich groups with fiber ambient spaces that give rise to a three-fold that exhibits non-trivial torsional sections.

Using the Heterotic/F-theory duality, one can explain this phenomenon as follows. It has been shown in [19] that the construction of a discrete gauge symmetry  $D$  in the Heterotic string requires background bundles with structure group of the form of  $G/D$  where  $G$  is a group of ADE type. Groups of the type  $G/D$  require that the Mordell Weil group of the corresponding F-theory compactification has the torsional subgroup  $D$ . Due to the extended conjecture by Berglund and Mayr, the mirror Calabi-Yau manifold should exhibit a discrete symmetry of type  $D$ , where  $D$  is a subgroup of the Tate-Shafarevich group of the mirror Calabi-Yau manifold.

### 3.3 Heterotic field theory perspective

In this section we discuss the conjectured field theory perspective on the Heterotic side which relates the models with U(1) factors to those with discrete gauge symmetries. The approach is conjectured as we did not calculate the charges of the matter fields, responsible for Higgsing of the model with an Abelian gauge symmetry to a discrete one. In particular, we shall focus on the examples where the two U(1)'s factors appear symmetrically. i.e. in our approach we consider the U(1) which arise upon commutation of U(1) background bundles that are embedded symmetrically into the two respective  $E_8$ -bundles.

#### 3.3.1 Stückelberg mechanism

In the Heterotic string theory in six and lower dimensions, a geometric Stückelberg mechanism can render a U(1) gauge field massive [36]. In six dimensions the mass term of U(1), is due to the modified ten-dimensional kinetic term of the Kalb-Ramond field  $B_2$ , which upon dimensional reduction on an elliptically fibered Calabi-Yau threefold  $Z_3$  and a U(1) background bundle, results in a six-dimensional kinetic term for the axions  $\rho_\alpha$ :

$$\mathcal{L}_{\text{Stück.}} = G^{\alpha\beta} (d\rho_\alpha + k_\alpha A_{U(1)}) \wedge \star (d\rho_\beta + k_\beta A_{U(1)}), \quad (3.1)$$

where

$$G^{\alpha\beta} = \int_{Z_3} \omega^\alpha \wedge \star \omega^\beta. \quad (3.2)$$

Here  $\omega^\alpha$ ,  $\alpha = 1, \dots, b_2(Z_3)$ , is a basis of harmonic two-forms in  $H^{(2)}(Z_3)$ , where  $b_2(Z_3)$  is the second Betti number of  $Z_3$ , the axions  $\rho_\alpha$  are associated with the expansion of the Kalb-Ramond field  $B_2$ . and the  $k_\alpha$  are flux quanta associated with the expansion of the U(1) bundle background field strength  $\mathcal{F}$ . Note,  $\mathcal{F} = \frac{1}{2\pi i} c_1(\mathcal{L})$  where  $c_1(\mathcal{L})$  is the first Chern class the corresponding U(1) line bundle  $\mathcal{L}$ .

From (3.1) a single U(1) gauge field will be in general massive if we have a non-trivial first Chern class of the U(1) bundle. However, in the presence of multiple massive U(1) gauge fields appropriate linear combinations of them, which belong to the kernel of the massive matrix, can remain massless U(1) fields.<sup>5</sup>

In the case of the symmetric example with two U(1) factors,  $U(1)_1$  and  $U(1)_2$ , it is evident that a gauge boson associated with a symmetric linear combination,  $U(1)_1 + U(1)_2$ , remains massless, while the orthogonal one, associated with the anti-symmetric linear combination,  $U(1)_1 - U(1)_2$ , becomes massive. Thus, even though the geometry under stable degeneration indicates two U(1) factors in Heterotic theory, the Stückelberg mechanism ensures that only a symmetric combination of two U(1) gauge fields remains massless. It is this latter one, which is identified in the Heterotic/F-theory duality.

#### 3.3.2 Higgsing of symmetric U(1) model

The symmetric Heterotic model with  $\mathbb{Z}_Q$  discrete symmetry can be obtained from the one with symmetric U(1) gauge symmetry factors, by Higgsing the latter model with the matter

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<sup>5</sup>For further details, see, e.g., [21]. For similar computations, see, e.g., [37], where also the case of multiple U(1)'s is systematically discussed.

fields,  $\phi_1$  and  $\phi_2$ . Due to the symmetry structure of the model we conjecture that the  $\phi_1$  and  $\phi_2$  have integer charge  $Q$  under charged under  $U(1)_1$  and  $U(1)_2$ , respectively:

	U(1)	U(1)
$\phi_1$	Q	0
$\phi_2$	0	Q

After having acquired non-zero vacuum expectation values, these field break the symmetry down to  $\mathbb{Z}_Q \times \mathbb{Z}_Q$ .

In particular, the two Higgs fields can be represented as

$$\phi_1 = (V_1 + \beta_1)e^{i\alpha_1 g}, \quad \phi_2 = (V_2 + \beta_2)e^{i\alpha_2 g}, \quad (3.3)$$

where  $V_1$  and  $V_2$  are the respective vacuum expectation values,  $\alpha_1$  and  $\alpha_2$  are real fields and are identified with the Goldstone axions, while  $\beta_1$  and  $\beta_2$  are the Higgs fields.

In this representation the covariant kinetic energy terms for the matter fields  $\phi_1$  and  $\phi_2$  take the form:

$$\mathcal{L}_1 = \sum_{i=1}^2 |V_i + \beta_i|^2 g^2 (\partial_\mu \alpha_i - Q A_i)^2 + \sum_{i=1}^2 (\partial_\mu \beta_i)^2. \quad (3.4)$$

Note that this Lagrangian has a manifest discrete gauge symmetry  $\mathbb{Z}_Q \times \mathbb{Z}_Q$ .

However, due to the Stückelberg mechanism as discussed in 3.3.1, the anti-symmetric combination of the gauge fields  $A_- \equiv \frac{1}{\sqrt{2}}(A_1 - A_2)$  acquires the mass. The contribution (3.1) to the effective Lagrangian can schematically written as:

$$\mathcal{L}_2 = M^2 (\partial_\mu b - A_-)^2, \quad (3.5)$$

where  $b$  is an axion field, associated with the expansion of the Kalb-Ramond field  $B_2$  and  $M$  sets the mass scale. It is this term that manifestly breaks the anti-symmetric combination of discrete symmetry factors, i.e.  $\mathbb{Z}_{Q_-}$ , while the symmetric combination, i.e.  $\mathbb{Z}_{Q_+}$ , is preserved.

By introducing the redefined fields

$$A_{1/2} = \frac{1}{\sqrt{2}}(A_+ \pm A_-), \quad \alpha_{1/2} = \frac{1}{\sqrt{2}}(\alpha_+ \pm \alpha_-) \quad (3.6)$$

and for simplicity setting Higgs fields  $\beta_{1/2} = 0$  the sum of (3.4) and (3.5) reads:

$$\mathcal{L} = \frac{1}{2} V_1^2 (D_\mu \alpha_+ + D_\mu \alpha_-)^2 + \frac{1}{2} V_2^2 (D_\mu \alpha_+ - D_\mu \alpha_-)^2 + M_s^2 (D_\mu b)^2, \quad (3.7)$$

where the covariant derivatives are explicitly given as follows

$$D_\mu \alpha_\pm := \partial_\mu \alpha_\pm - Q A_\pm, \quad D_\mu b = \partial_\mu b - A_-. \quad (3.8)$$

It is evident from (3.7) that the effective theory has only *one* discrete gauge symmetry factor  $\mathbb{Z}_{Q_+}$ , which is a symmetric combination of the two  $\mathbb{Z}_Q \times \mathbb{Z}_Q$ , while the anti-symmetric one is *not* present in the effective theory.

We have therefore demonstrated in the field theory that the Higgsing of a Heterotic model with two symmetric  $U(1)$ 's leads to a Heterotic model with two symmetric  $\mathbb{Z}_Q$ , however due to the Stückelberg mechanism only the symmetric combination of the two  $\mathbb{Z}_Q$  is present in the effective theory. It is this surviving discrete symmetry that is matched via Heterotic/F-theory duality onto the F-theory discrete symmetry. Note that this field theory Higgsing mechanism on the Heterotic side parallels the one on the F-theory side, where only the massless  $U(1)$  is visible and gets broken to  $\mathbb{Z}_Q$  via Higgs mechanism.

This provides an independent verification of why the anti-symmetric combination of the discrete symmetry factors is not present in the effective theory. In addition, we present for the concrete examples in section 4 an additional geometrical argument which has been first given in [38] and is reviewed in detail in subsection 4.2.1. As further elaborated at that place, it explains that the elimination of the anti-symmetric combination of gauge factors in comparable set-ups is due to orbifold singularities of the Heterotic K3 surface.

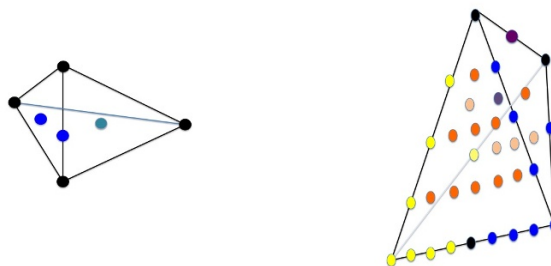
In the case of examples where the two  $U(1)$  factors do not appear in a symmetric way, we expect that one particular linear combination of two  $U(1)$  gauge bosons becomes massive, while the orthogonal one is massive. However, determination of the Stückelberg mass terms of the form (3.1) would require a detailed calculation for specific Heterotic compactification on elliptically fibered Calabi-Yau threefold  $Z_3$  and the calculation of charges of the matter fields responsible for Higgsing to a model with discrete symmetry. Consequently, in this case the origin of the single discrete gauge symmetry factor would be a specific linear combination of the two discrete symmetry factors.

## 4 Examples

In this section, we apply the methods from the previous section to construct two examples which realize the discrete symmetries  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$  within the Heterotic string. For this purpose, we use mirror pairs of K3 surfaces which are elliptically fibered and we promote them to mirror pairs of Calabi-Yau three-folds. In conclusion, we find that the six-dimensional perturbative gauge group associated to one geometry is given as the commutant of the perturbative gauge group of the dual Calabi-Yau within the group  $E_8 \times E_8$ . As has been pointed out in detail in section 3.3.1, it is important to stress that under Heterotic/F-theory duality only massless gauge degrees of freedom are matched. Thus, there appears sometimes just the symmetric combination of certain gauge factors.

For the  $\mathbb{Z}_2$ -model, we start with an elliptic K3 surface  $X_2$  that has  $E_7 \times E_7 \times SU(2) \times SU(2)$  gauge symmetry. Its mirror  $\tilde{X}_2$  is also elliptic and has gauge symmetry  $U(1)$  where the  $U(1)$  originates from an additional section. As a next step, we promote the mirror pair  $X_2, \tilde{X}_2$  of K3 surfaces to a mirror pair of three-folds, named  $X_3, \tilde{X}_3$ . Here,  $\tilde{X}_3$  exhibits a two-section, which originates from the former two sections in  $\tilde{X}_2$ , that have been glued together, and indicates a  $\mathbb{Z}_2$ -symmetry. In addition, there are two non-perturbative  $SU(2)$  factors.

The construction of the  $\mathbb{Z}_3$ -model is quite parallel to the first one, but it does not exhibit any non-perturbative factors in contrast to the former one.



**Figure 1.** The polytope on the right is  $\Delta^\circ$  and provides the ambient space for  $X_2$ , with gauge symmetry  $(E_7 \times E_7 \times SU(2) \times SU(2))/\mathbb{Z}_2$ . The ambient space of the elliptic fiber (the zero plane) is marked by the black points. The polytope on the left shows the dual polytope  $\Delta$  where the symplectic cut is performed, providing the dual polytopes of the half K3 surfaces  $X_2^\pm$ . The yellow and blue points give the affine Dynkin diagram of  $E_7$ . The latter are inherited by the half K3 surfaces  $X_2^\pm$ , respectively. The purple point corresponds to an  $SU(2)$  gauge group which also appears in both half K3 surfaces  $X_2^\pm$  after the stable degeneration limit. Orange points mark inner points of the facets. The dual polytope  $\Delta$  on the left provides the ambient space for the mirror K3  $\tilde{X}_2$ . Note that the semi-stable degeneration of the mirror elliptic K3  $\tilde{X}_2$  is obtained by performing a symplectic cut along the zero plane marked by the black points on the polytope  $\Delta^\circ$  on the right.

#### 4.1 The model with $\mathbb{Z}_2$ gauge symmetry

As outlined above, we start with a K3 surface that exhibits  $(E_7 \times E_7 \times SU(2) \times SU(2))/\mathbb{Z}_2$  gauge symmetry and use its mirror dual to construct a Heterotic compactification with discrete gauge symmetry  $\mathbb{Z}_2$ . We start by constructing a pair of dual polytopes  $(\Delta^\circ, \Delta)$ . Here,  $\Delta^\circ$  is the ambient space of the K3 surface with gauge symmetry  $(E_7 \times E_7 \times SU(2) \times SU(2))/\mathbb{Z}_2$ .

##### 4.1.1 The geometry with gauge symmetry $(E_7 \times E_7 \times SU(2) \times SU(2))/\mathbb{Z}_2$

A very familiar geometry to this one has already been studied in [21] and we refer to that discussion for some of the details. Its polytope  $\Delta^\circ$  is given by the convex hull of the points

$$(-2, 1, 0), (0, 1, 0), (2, -3, 4), (2, -3, -4), \quad (4.1)$$

while the dual polytope  $\Delta$ , leading to the geometry with a bi-section is specified by the points

$$(-2, -1, 0), (0, -1, 0), (1, 1, 0), (0, -1, -1), (0, -1, 1), \quad (4.2)$$

and is shown in figure 1. As a first step, we assign the following coordinates to the points of  $\Delta^\circ$

$$\begin{aligned} (-2, 1, 0) &\mapsto y_1, & (2, -3, 0) &\mapsto y_2, & (0, 1, 0) &\mapsto y_3, \\ (2, -3, 1) &\mapsto \tilde{U}, & (2, -3, -1) &\mapsto \tilde{V}. \end{aligned} \quad (4.3)$$

Here  $\tilde{U}, \tilde{V}$  parameterize the base  $\mathbb{P}^1$  of the fibration, while  $y_1, y_2, y_3$  are coordinates of the fiber ambient space  $\mathbb{P}^{(1,1,2)}/\mathbb{Z}_2$ .

Using equation (2.4), we find that  $X_2$  the general K3 is defined by the equation

$$\chi_{\Delta^\circ} : a_1 y_1^4 + a_2 y_1^2 y_2^2 + a_3 y_2^4 + a_4 y_3^2 + a_5 y_1 y_2 y_3 = 0. \quad (4.4)$$

Here, the coefficients read in terms of  $\tilde{U}, \tilde{V}$  as follows

$$\begin{aligned} a_1 &= a_{11}, \\ a_2 &= a_{21} \tilde{U}^2 \tilde{V}^2, \\ a_3 &= a_{31} \tilde{U}^5 \tilde{V}^3 + a_{32} \tilde{U}^4 \tilde{V}^4 + a_{33} \tilde{U}^3 \tilde{V}^5, \\ a_4 &= a_{41}, \\ a_5 &= a_{51} \tilde{U} \tilde{V}. \end{aligned} \quad (4.5)$$

One observes that there are two sections located at  $y_2 = 0$  and are given by

$$a_{11} y_1^4 + a_{41} y_3^2 = 0. \quad (4.6)$$

Thus we confirm that  $\chi_{\Delta^\circ}$  defines an elliptically fibered K3.

In fact, a similar K3 surface has already been investigated in [21]. To make contact with that description, we transform the constraint (4.4) into a hypersurface within the ambient space  $\mathbb{P}^1 \times \text{Bl}_1 \mathbb{P}^{(1,1,2)}$ . The coordinate transformation

$$y_1 \mapsto (x_1^3 x_2 UV)^{\frac{1}{4}}, \quad y_2 \mapsto (x_1 x_2^3 U^{-3} V^{-3})^{\frac{1}{4}}, \quad y_3 \mapsto x_3 (UV)^{\frac{1}{2}}, \quad (\tilde{U}, \tilde{V}) \mapsto (U, V) \quad (4.7)$$

maps (4.4) onto

$$s_1 x_1^3 x_2 + s_2 x_1^2 x_2^2 + s_3 x_1 x_2^3 + s_4 x_1 x_2 x_3 + s_5 x_3^2 = 0. \quad (4.8)$$

Here, one has

$$s_1 = a_{11} UV, \quad s_2 = a_{21} UV, \quad s_3 = a_{31} U^2 + a_{32} UV + a_{33} V^2, \quad s_4 = a_{51} UV, \quad s_5 = a_{41} UV. \quad (4.9)$$

This geometry also has two sections given by  $[x_1 : x_2 : x_3] = [0 : 1 : 0]$  and  $[x_1 : x_2 : x_3] = [1 : 0 : 0]$ . It should also be stressed that the construction relies on the fact that we have admitted a fiber ambient space different from  $\mathbb{P}^{(1,2,3)}$  given by  $\mathbb{P}^{(1,1,2)}/\mathbb{Z}_2$ . After a transformation into Weierstrass normal form, it becomes transparent that these two sections map onto the zero section and a torsional section of order two, respectively. From the discriminant of the Weierstrass normal form (A.4), one also easily reads off again that there are two  $E_7$  as well as two  $SU(2)$  singularities. In figure 1 we see the affine Dynkin diagrams of the gauge factors of the singular fibers. Note that there is only the non-affine Dynkin diagram visible for the  $SU(2)$  factor, as the additional affine node corresponds to a non-toric divisor. Thus, the full gauge group is given by  $(E_7 \times E_7 \times SU(2) \times SU(2))/\mathbb{Z}_2$ .

As described in detail in [21], the stable degeneration limit decomposes (4.9) into two half K3 surfaces  $X_2^+, X_2^-$ . Both,  $X_2^+, X_2^-$ , inherit an  $SU(2) \times E_7$  singularity, as well as the torsional section of order two.

These findings can be checked by the computation of the rank of the Picard lattice  $\text{rk Pic}(X_{\Delta^\circ})$ . We find:

$$\text{rk Pic}(X_{\Delta^\circ}) = 35 - 3 - 1 - 14 + 1 = 18. \tag{4.10}$$

This accounts for the class of the fiber, the one of the base as well as sixteen exceptional divisors of the corresponding gauge group. This, again, confirms the result that the additional section is torsional.

In summary, one expects that the dual geometry should give rise to a  $\mathbb{Z}_2$  symmetry. This will be the next step of our analysis. As discrete symmetries arising as from a Higgsed  $U(1)$  symmetry are visible only in six dimensions<sup>6</sup> in the F-theory description, one needs to fiber the K3 manifold  $X_2$  over another  $\mathbb{P}^1$ .  $X_3$  is the resulting K3 fiber threefold, where the two  $SU(2)$  singularities are promoted to a curve of  $SU(2)$  singularities resulting in only one  $SU(2)$  factor in six dimensions. As argued in [38] and further elaborated on in subsection 4.2.1, that accounts to an identification of the two  $SU(2)$  factors such that only their diagonal combination survives.

#### 4.1.2 The dual geometry with a fiber ambient space $\mathbb{P}^{(1,1,2)}$

To continue the study of the geometries, we now turn to the analysis of  $\Delta$ . Here we assign the following coordinates to the points (4.2)

$$\begin{aligned} (-2, -1, 0) &\mapsto x_1, & (-1, -1, 0) &\mapsto x_4, & (0, -1, 0) &\mapsto x_2, \\ (1, 1, 0) &\mapsto x_3, & (0, -1, -1) &\mapsto U, & (0, -1, 1) &\mapsto V. \end{aligned} \tag{4.11}$$

Using equation (2.4), we find that  $\tilde{X}_2$  the general K3 is defined by the following equation

$$\begin{aligned} \chi := s_1 x_1^4 x_4^3 + s_2 x_1^3 x_2 x_4^2 + s_3 x_1^2 x_2^2 x_4 + s_4 x_1 x_2^3 + s_5 x_2^4 \\ + s_6 x_1 x_2 x_3 x_4 + s_7 x_1^2 x_3 + s_8 x_2^2 x_3 + s_9 x_3^2 x_4 = 0. \end{aligned} \tag{4.12}$$

Here, the  $s_i$  take explicitly the form

$$\begin{aligned} s_1 &= s_{11}, \\ s_2 &= s_{21}U^2 + s_{22}UV + s_{23}V^2, \\ s_3 &= s_{31}U^4 + s_{32}U^3V + \dots + s_{35}V^4, \\ s_4 &= s_{41}U^6 + s_{42}U^5V + \dots + s_{47}V^6, \\ s_5 &= s_{51}U^8 + s_{52}U^7V + \dots + s_{59}V^8, \\ s_6 &= s_{61}U^2 + s_{62}UV + s_{63}V^2, \\ s_7 &= s_{71}, \\ s_8 &= s_{81}U^4 + s_{82}U^3V + \dots + s_{85}V^4, \\ s_9 &= s_{91}, \end{aligned} \tag{4.13}$$

---

<sup>6</sup>However, it is important to stress that there *are also* discrete symmetries in eight dimensions. In that case, the discrete becomes apparent from the Tate-Shafarevich group of the K3 surface and can be identified by discrete holonomies in a circle reduction to seven dimensions. We would like to thank our referee for illuminating remarks on that point. See also [39–41] for related discussions.

where the  $s_{ij}$  are complex numbers. Thus we confirm that  $\chi_{\Delta^\circ}$  defines an elliptically fibered K3  $\tilde{X}_2$ . One notices that the above geometry has two sections which are given by  $x_1 = 1$ ,  $x_2 = 0$ , leading to the equation

$$s_{11} + s_{71}x_3 + s_{91}x_3^2 = 0, \quad (4.14)$$

which is solvable over  $\mathbb{C}$ . In other words the two sections are located at

$$S_{1/2} = \left[ \sqrt{2s_{91}} : 0 : -s_{71} \pm \sqrt{s_{71}^2 - s_{11}s_{91}} \right]. \quad (4.15)$$

In addition, the study of the Jacobian reveals that there are no further gauge symmetries. This is confirmed by the dimension of the Picard lattice. Indeed, an application of formula (2.5) reveals that

$$\text{rk Pic} = 7 - 3 - 1 - 1 + 1 = 3, \quad (4.16)$$

which accounts for the base and the class of the elliptic fiber and the additional section.

The stable degeneration limit is as in reference [21] defined by a symplectic cut along the  $(x, y, 0)$ -plane within the coordinate system defined by (4.1); it gives two polytopes  $\Delta^+$  and  $\Delta^-$ . These polytopes determine two toric varieties  $(\mathbb{P}_{\Sigma^+}, \mathcal{L}^+)$ ,  $(\mathbb{P}_{\Sigma^-}, \mathcal{L}^-)$  together with line bundles  $\mathcal{L}^\pm$ . A general zero section of the line bundle  $\mathcal{L}^+$  in  $\mathbb{P}_{\Sigma^+}$  defines a rational elliptic surface  $\tilde{X}_2^+$ ; similarly, a general section of the line bundle  $\mathcal{L}^-$  defines a rational elliptic surface  $\tilde{X}_2^-$  in  $\mathbb{P}_{\Sigma^-}$ .

However, it is crucial to note that the appropriate coordinates in the dual fans are

$$\begin{aligned} (-2, -1, 0) &\mapsto x_1, & (-1, -1, 0) &\mapsto x_4, & (0, -1, 0) &\mapsto x_2, \\ (1, 1, 0) &\mapsto x_3, & (0, 0, -1) &\mapsto \lambda^+, & (0, -1, 1) &\mapsto V, \end{aligned} \quad (4.17)$$

for  $\Sigma^+$  and

$$\begin{aligned} (-2, -1, 0) &\mapsto x_1, & (-1, -1, 0) &\mapsto x_4, & (0, -1, 0) &\mapsto x_2, \\ (1, 1, 0) &\mapsto x_3, & (0, 0, +1) &\mapsto \lambda^-, & (0, -1, -1) &\mapsto U, \end{aligned} \quad (4.18)$$

for  $\Sigma^-$ .

Then, if we denote by  $v_j \in N$  the lattice point associated to each  $x_j$ , the general section of the line bundle  $\mathcal{L}^\pm$  in  $\mathbb{P}_{\Sigma^\pm}$  are:

$$\chi^+ = \sum_j \prod_{m \in P^+} a_P x_j^{\langle v_j, P^+ \rangle + 1} V^{\langle (0, -1, 1), P^+ \rangle + 1} (\lambda^-)^{\langle (0, 0, -1), P^+ \rangle} \quad (4.19)$$

and

$$\chi^- = \sum_j \prod_{m \in P^-} a_P x_j^{\langle v_j, P^- \rangle + 1} U^{\langle (0, -1, -1), P^- \rangle + 1} (\lambda^+)^{\langle (0, 0, +1), P^- \rangle}. \quad (4.20)$$

In summary, one obtains two constraints for the two rational elliptic surfaces  $\tilde{X}_2^\pm$  (half K3 surfaces).

$$\begin{aligned} \chi^+ &:= s_1^+ x_1^4 x_4^3 + s_2^+ x_1^3 x_2 x_4^2 + s_3^+ x_1^2 x_2^2 x_4 + s_4^+ x_1 x_2^3 + s_5^+ x_1^2 x_3 x_4^2 \\ &\quad + s_6^+ x_1 x_2 x_3 x_4 + s_7^+ x_1^2 x_3 + s_8^+ x_2^2 x_3 + s_9^+ x_3^2 x_4 = 0, \\ \chi^- &:= s_1^- x_1^4 x_4^3 + s_2^- x_1^3 x_2 x_4^2 + s_3^- x_1^2 x_2^2 x_4 + s_4^- x_1 x_2^3 + s_5^- x_1^2 x_3 x_4^2 \\ &\quad + s_6^- x_1 x_2 x_3 x_4 + s_7^- x_1^2 x_3 + s_8^- x_2^2 x_3 + s_9^- x_3^2 x_4 = 0, \end{aligned} \quad (4.21)$$



where

$$\begin{aligned}
 s_1^+ &= s_{11}, \\
 s_2^+ &= s_{21}U + s_{22}\lambda^+, \\
 s_3^+ &= s_{31}U^2 + s_{32}U\lambda^+ + s_{33}\lambda^{+2}, \\
 s_4^+ &= s_{41}U^3 + s_{42}U^2\lambda^+ + \dots + s_{44}\lambda^{+3}, \\
 s_5^+ &= s_{51}U^4 + s_{52}U^3\lambda^+ + \dots + s_{55}\lambda^{+4}, \\
 s_6^+ &= s_{61}U + s_{62}\lambda^+, \\
 s_7^+ &= s_{71}, \\
 s_8^+ &= s_{81}U^2 + s_{82}U\lambda^+ + s_{83}\lambda^{+2}, \\
 s_9^+ &= s_{91}.
 \end{aligned}
 \tag{4.22}$$

The coefficients for  $\chi^-$  are obtained analogously. In particular, one immediately notices that one obtains the correct number of “layers” (i.e. slices parallel to the  $x$ - $y$ -plane) that are required to obtain an  $E_7$  background bundle according to the results of [20].

We close this section by noting that our geometry under consideration has two sections which lead to an additional  $U(1)$  symmetry. However, from an F-theory point of view, it is expected that discrete symmetries become manifest only in six dimensions. Thus, as a final step of our analysis, we compactify our geometry further to six dimensions.

### 4.2 The six-dimensional geometry

As a next step, we investigate a six-dimensional set-up. We want to construct a Calabi-Yau threefold  $\tilde{X}_3$  which is fibered by the elliptic K3 surface  $\tilde{X}_2$ . For this purpose, we take the direct product of our ambient space (4.2) with another  $\mathbb{P}^1$ . The vertices of the four-dimensional polyhedron  $\Delta_4$  read:

$$(-2, -1, 0, 0), (1, 1, 0, 0), (0, -1, -1, 0), (0, -1, 1, 0), (0, 0, 0, 1), (0, 0, 0, -1). \tag{4.23}$$

In addition, the vertices of the dual four-dimensional polyhedron  $\Delta_4^\circ$

$$(2, -3, -4, 1), (-2, 1, 0, 1), (2, -3, 4, 1), (0, 1, 0, 1), \tag{4.24}$$

$$(2, -3, -4, -1), (-2, 1, 0, -1), (2, -3, 4, -1), (0, 1, 0, -1). \tag{4.25}$$

By comparing to the Kreuzer-Skarke list [42], one finds that  $h^{(1,1)}(X_{\Delta_4})$  is five, where  $\chi_{\Delta_4}$  denotes the three-fold given as a section of the anti-canonical bundle of the toric ambient space specified by  $\Delta_4$ . In particular,  $\chi_{\Delta_4}$  takes the same form as (4.12) and (4.13), where the  $s_{ij}$  now depend on the two additional  $\mathbb{P}^1$  coordinates  $[S : T]$ . Thus,

$$s_{ij} = s_{ij1}S^2 + s_{ij2}ST + s_{ij3}T^2. \tag{4.26}$$

Clearly, the two sections defined by (4.14) are no longer well-defined in six-dimensions and form a bi-section. We can still perform a semi-stable degeneration with the methods of [21] and find the Heterotic K3 dual. In addition, one finds two  $SU(2)$ -singularities located at

$s_{91} = 0$  which becomes apparent from the explicit equation for the determinant (A.4). As these extend along the Heterotic K3, these contributions to the gauge group are non-perturbative in nature. Summarizing, one finds that the six-dimensional gauge group  $G$  is given by

$$G = G_{\text{pert}} \times G_{\text{non-pert}} = \mathbb{Z}_2 \times (\text{SU}(2) \times \text{SU}(2)). \tag{4.27}$$

This matches  $h^{(1,1)}(X_{\Delta_4}) = 5$ , which corresponds to two classes of  $\mathbb{P}^1 \times \mathbb{P}^1$ , the class of the bi-section and the two resolutional divisors of the  $\text{SU}(2)$  singularities. We also note that the location of the two  $\text{SU}(2)$  factors are precisely the points where the two leaves of the bi-section glue together.

In conclusion, we obtain a perfect match of the six-dimensional perturbative gauge groups of the mirror dual geometries in terms of commutants within  $E_8 \times E_8$ . For the explanation, why there is however only one  $\mathbb{Z}_2$ -factor, we refer to the following sub-section.

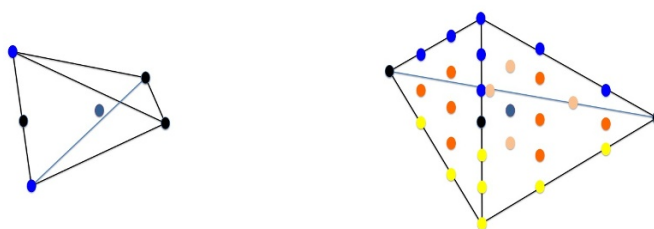
### 4.2.1 Comparing field theory and geometry

Finally, we comment on the appearance of only one  $\mathbb{Z}_2$ -factor in six dimensions although there are two background bundles with structure group  $(E_7 \times \text{SU}(2))/\mathbb{Z}_2$ . Thus, naively, one might expect the appearance of an  $\mathbb{Z}_2 \times \mathbb{Z}_2$  gauge symmetry. From a field theory perspective, this puzzle gets re-solved by viewing the discrete symmetry as a higgsed version of Abelian groups as explained in detail in section 3.3.2. In particular, if one turns off the coefficient  $s_1$ , the fiber ambient space changes from  $\mathbb{P}^{(1,1,2)}$  into  $\text{Bl}_1\mathbb{P}^{(1,1,2)}$ . The latter one gives rise to an additional section, such that there are two  $\text{U}(1)$ -background bundles which are symmetrically embedded into the two  $E_8$ -factors. As analyzed in [21, 38], upon commutation, these background bundles give rise to two  $\text{U}(1)$  gauge symmetries. One linear combination of these two  $\text{U}(1)$ -factors turns out to be massive, while the orthogonal linear combination of  $\text{U}(1)$ -factors is massless. Only the massless factor is seen on the F-theory side and it is this massless  $\text{U}(1)$  that is higgsed to a discrete symmetry  $\mathbb{Z}_2$ .

This discussion is related to the appearance of only one  $\text{SU}(2)$ -factor in six dimensions of the threefold  $X_3$ , of section 4.1.1, which is in fact the mirror dual geometry 4.2 of the threefold  $\tilde{X}_3$  with  $\mathbb{Z}_2$  symmetry considered above in 4.2. From an algebraic point of view, the two loci of  $\text{SU}(2)$  singularities get promoted to a curve of  $\text{SU}(2)$  singularities in six dimensions. A field theory interpretation of this phenomenon has been given in [38]. Here, the K3 surface acquires orbifold singularities at the locations where it hits the  $\text{SU}(2)$  locus. Thus, in order to give an interpretation of centralizing a group action that corresponds to the gauge group  $E_7 \times E_7 \times \text{SU}(2)$ , one has to include an exchange of the  $\text{SU}(2)$  subgroups of  $E_8 \times E_8$  in the monodromy of the two bundles around the orbifold points [38]. It is expected that a similar argument should also hold true in the  $\mathbb{Z}_2$  case considered above.

### 4.3 The model with $\mathbb{Z}_3$ gauge symmetry

The construction of the example with discrete  $\mathbb{Z}_3$  gauge symmetry parallels the example with  $\mathbb{Z}_2$  gauge symmetry and we therefore keep the discussion brief. This time we start with a geometry that has gauge symmetry  $(E_6 \times E_6 \times \text{SU}(3) \times \text{SU}(3))/\mathbb{Z}_3$ .



**Figure 2.** The polytope on the left shows the ambient space whose associated hypersurface leads to the  $\mathbb{Z}_3$ -geometry. The polytope on the right provides the ambient space with gauge symmetry  $(E_6 \times E_6 \times SU(3) \times SU(3))/\mathbb{Z}_3$ . The zero plane along which the symplectic cut is performed is marked by the black points. The yellow and blue points give the affine Dynkin diagram of  $E_6$ . The latter are inherited by the half K3 surfaces  $\chi^\pm$ , respectively. Beige-coloured points are on the invisible facets of the polytope. In particular, the two points on the invisible edge correspond to the Dynkin diagram of  $SU(3)$  which is inherited by both half K3 surfaces. Finally, orange points mark inner points of the facets and the purple point marks the inner point of the polytope.

### 4.3.1 The geometry with $(E_6 \times E_6 \times SU(3) \times SU(3))/\mathbb{Z}_3$ gauge symmetry

We start again with a pair of dual polytopes  $(\Delta^\circ, \Delta)$ .  $\Delta^\circ$  gives rise to a K3 surface with gauge group  $(E_6 \times E_6 \times SU(3) \times SU(3))/\mathbb{Z}_3$ , while  $\Delta$  gives rise to a K3 surface with fiber ambient space given by  $\mathbb{P}^2$ .  $\Delta^\circ$  is given by the convex hull of

$$(2, -1, 0), (-1, 2, 0), (-1, -1, 3), (-1, -1, -3), \tag{4.28}$$

while  $\Delta$  is given as the convex hull of

$$(-1, -1, 0), (1, 0, 0), (1, 1, 0), (0, 1, 0), (-1, -1, 1), (-1, -1, -1). \tag{4.29}$$

The two polytopes are displayed in figure 2. Assigning coordinates as

$$\begin{aligned} (-2, 1, 0) &\mapsto y_1, & (2, -3, 0) &\mapsto y_2, & (0, 1, 0) &\mapsto y_3, \\ (2, -3, 1) &\mapsto \tilde{U}, & (2, -3, -1) &\mapsto \tilde{V}, \end{aligned} \tag{4.30}$$

an application of formula (2.4) reveals that the hypersurface equation for  $\chi_{\Delta^\circ}$  is given as

$$\chi_{\Delta^\circ} : a_1 y_1^4 + a_2 y_1^2 y_2^2 + a_3 y_2^4 + a_4 y_3^2 = 0. \tag{4.31}$$

The coefficients read as follows

$$\begin{aligned} a_1 &= a_{11}, \\ a_2 &= a_{21} \tilde{U}^2 \tilde{V}^2, \\ a_3 &= a_{31} \tilde{U}^5 \tilde{V}^3 + a_{32} \tilde{U}^4 \tilde{V}^4 + a_{33} \tilde{U}^3 \tilde{V}^5, \\ a_4 &= a_{41}. \end{aligned} \tag{4.32}$$

The rank of the Picard lattice is found to be  $h^{(1,1)}(X_{\Delta^\circ}) = 18$  which accounts for two  $E_6$  singularities (to be more precise, its exceptional divisors), two  $SU(3)$  singularities, the class

of the fiber as well as the base. In addition, the Mordell Weil group equals  $\mathbb{Z}_3$ . Thus, the full gauge group is given by  $(E_6 \times E_6 \times SU(3) \times SU(3))/\mathbb{Z}_3$ .

Again, after the compactification to six dimensions, the two  $SU(3)$  singularities merge into a curve of  $SU(3)$  singularities. From the field theory perspective, it is again the symmetric combination of the two  $SU(3)$  factors which survives in this limit.

### 4.3.2 The dual geometry with fiber ambient space $\mathbb{P}^2$

We analyse the dual geometry by assigning the following coordinates to the points (4.2)

$$\begin{aligned} (-1, -1, 0) &\mapsto x_1, & (1, 0, 0) &\mapsto x_2, & (0, 1, 0) &\mapsto x_3, \\ (-1, -1, 1) &\mapsto U, & (-1, -1, -1) &\mapsto V. \end{aligned} \tag{4.33}$$

In this way, one obtains the following hypersurface constraint:

$$\chi := s_1 x_1^3 + s_2 x_1^2 x_2 + s_3 x_1 x_2^2 + s_4 x_2^3 + s_5 x_1^2 x_3 + s_6 x_1 x_2 x_3 + s_7 x_2^2 x_3 + s_8 x_1 x_3^2 + s_9 x_2 x_3^2 + s_{10} x_3^3 = 0. \tag{4.34}$$

Here, the  $s_i$  take explicitly the form

$$\begin{aligned} s_1 &= s_{11} U^6 + s_{12} U^5 V + \dots + s_{17} V^6, \\ s_2 &= s_{21} U^4 + s_{22} U^3 V + \dots + s_{25} V^4, \\ s_3 &= s_{31} U^2 + s_{32} UV + s_{33} V^2, \\ s_4 &= s_{41}, \\ s_5 &= s_{51} U^4 + s_{52} U^3 V + \dots + s_{55} V^4, \\ s_6 &= s_{61} U^2 + s_{62} UV + s_{63} V^2, \\ s_7 &= s_{71}, \\ s_8 &= s_{81} U^2 + s_{82} UV + s_{83} V^2, \\ s_9 &= s_{91}, \\ s_{10} &= s_{10}. \end{aligned} \tag{4.35}$$

A closer inspection of this geometry reveals that there are apart from the zero section two further linear independent sections, which is confirmed by the computation of  $h^{(1,1)}(X_2)$ . In fact, these three sections will glue into a tri-section, once one compactifies further down to six dimensions. We demonstrate that using the Hirzebruch surface  $\mathbb{F}_0$  as the base of the fibration. To be more concrete, the vertices of the four-dimensional polyhedron are given by

$$(-1, -1, 1, 0), (-1, -1, -1, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 1), (0, 0, 0, -1). \tag{4.36}$$

In contrast to the other example, there are no further non-perturbative enhancements, such that the gauge group is given by  $G = \mathbb{Z}_3$ .

## 5 Concluding remarks

In this note we have presented core steps in the understanding of discrete symmetries within the Heterotic/F-theory duality. We propose that for mirror pairs of Calabi-Yau manifolds the gauge group of one geometry is given by the commutant of the gauge group of the dual geometry within  $E_8 \times E_8$ . (However, as explained in detail in section 3.3.1, only massless gauge fields are matched under Heterotic/F-theory duality, such that sometimes only a symmetric combination of certain gauge factors appears.) Our analysis is based on a two-pronged approach. On the one hand, we have proposed concrete constructions of background bundles whose structure group is given as the commutant of a discrete group within  $E_8$ . On the other hand, our analysis relies on the field theory investigation of Higgsing of Heterotic compactifications that exhibit  $U(1)$  symmetries. For the latter ones, we have restricted ourselves to examples where the  $U(1)$  symmetry originates from  $U(1)$  background bundles that are symmetrically embedded into both  $E_8$  factors. In this way there are two  $U(1)$  factors out of which the antisymmetric linear combination is massive due to Stückelberg mechanism in six dimensions, while the symmetric linear combination remains massless. It is the massless  $U(1)$  which is spontaneously broken to the discrete symmetry which is to be identified with the F-theory side.

In addition, we are able to shed light on the conjecture made in [12, 34, 35] concerning general F-theory compactifications. It states that mirror symmetry restricted to the elliptic/genus-one fiber exchanges fibrations with multi-sections and geometries that exhibit torsional sections. This is expected from the dual Heterotic side as follows. The commutant of a discrete symmetry  $D$  within  $E_8$  takes the form  $G/D$ , where  $G$  is a group of ADE type. Using our proposal, which relies on the interpretation of mirror symmetry for K3 surfaces in terms of orthogonal lattices [20], we can translate this phenomenon to the exchange of gauge group and structure group on the Heterotic side. In these terms, it is natural that a multi-section of order  $|D|$  gets exchanged with a torsional factor of the Mordell-Weil group of type  $D$ .

There are many interesting directions this work should be extended to. It would be desirable to be able to compute charges of matter fields and masses of gauge fields explicitly in the six-dimensional effective theory on the Heterotic side. This would require a deeper understanding of the mathematical structure of the appearing bundles and the computation of their Chern classes. Furthermore, it would be interesting to prove the mirror symmetry conjecture, which could probably be done building on the results of Belcastro and Dolgachev [43, 44].

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## A Weierstrass normal forms

In this appendix, we summarize the Weierstrass normal form of the most general quartic in  $\mathbb{P}^{(1,1,2)}$  as well as that of the most general cubic in  $\mathbb{P}^2$ . We use the following convention for the Weierstrass normal form

$$y^2 = x^3 + fx + g, \quad (\text{A.1})$$

where the discriminant is given by

$$\Delta = 4f^3 + 27g^2. \quad (\text{A.2})$$

### A.1 The quartic

The most general quartic with coordinates  $[x_1 : x_2 : x_3] \in \mathbb{P}^{(1,1,2)}$  is given by

$$\chi : s_1x_1^4 + s_2x_1^3x_2 + s_3x_1^2x_2^2 + s_4x_1x_2^3 + s_5x_2^4 + s_6x_1^2x_3 + s_7x_1x_2x_3 + s_8x_2^2x_3 + s_9x_3^2 = 0. \quad (\text{A.3})$$

The Weierstrass normal form has been determined in, e.g., [12, 45] and is given as

$$\begin{aligned} f &= \frac{1}{48} \left( -s_7^4 + 8s_6s_7^2s_8 - 16s_6^2s_8^2 + 48s_5s_6^2s_9 - 24s_4s_6s_7s_9 + 8s_3s_7^2s_9 + 16s_3s_6s_8s_9 \right. \\ &\quad \left. - 24s_2s_7s_8s_9 + 48s_1s_8^2s_9 - 16s_3^2s_9^2 + 48s_2s_4s_9^2 - 192s_1s_5s_9^2 \right), \\ g &= \frac{1}{864} \left( s_7^6 - 12s_6s_7^4s_8 + 48s_6^2s_7^2s_8^2 - 64s_6^3s_8^3 - 72s_5s_6^2s_7^2s_9 + 36s_4s_6s_7^3s_9 - 12s_3s_7^4s_9 \right. \\ &\quad + 288s_5s_6^3s_8s_9 - 144s_4s_6^2s_7s_8s_9 + 24s_3s_6s_7^2s_8s_9 + 36s_2s_7^3s_8s_9 + 96s_3s_6^2s_8^2s_9 \\ &\quad - 144s_2s_6s_7s_8^2s_9 - 72s_1s_7^2s_8^2s_9 + 288s_1s_6s_8^3s_9 + 216s_4^2s_6^2s_9^2 - 576s_3s_5s_6^2s_9^2 \\ &\quad - 144s_3s_4s_6s_7s_9^2 + 864s_2s_5s_6s_7s_9^2 + 48s_3^2s_7^2s_9^2 - 72s_2s_4s_7^2s_9^2 - 576s_1s_5s_7^2s_9^2 \\ &\quad + 96s_3^2s_6s_8s_9^2 - 144s_2s_4s_6s_8s_9^2 - 1152s_1s_5s_6s_8s_9^2 - 144s_2s_3s_7s_8s_9^2 + 864s_1s_4s_7s_8s_9^2 \\ &\quad + 216s_2^2s_8^2s_9^2 - 576s_1s_3s_8^2s_9^2 - 64s_3^3s_9^3 + 288s_2s_3s_4s_9^3 - 864s_1s_4^2s_9^3 - 864s_2^2s_5s_9^3 \\ &\quad \left. + 2304s_1s_3s_5s_9^3 \right), \\ \Delta &= -\frac{1}{16}s_9^2(\dots). \end{aligned} \quad (\text{A.4})$$

We note that there is a factor of  $s_9^2$  that splits off the remaining polynomial. This is the origin of the non-perturbative  $\text{SU}(2)^2$  factor as discussed in 4.1.2.

### A.2 The cubic

The most general cubic with coordinates  $[x : y : z] \in \mathbb{P}^{(2)}$  is given by

$$a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 + a_5x^2y + a_6xyz + a_7y^2z + a_8xz^2 + a_9yz^2 + a_{10}z^3 = 0. \quad (\text{A.5})$$

Its Weierstrass normal form reads explicitly

$$\begin{aligned}
 f = & \left( -a_6^4 + 24a_{10}(2a_3^2a_5 - 6a_2a_4a_5 + 9a_1a_4a_6 + 2a_2^2a_7 - a_3(a_2a_6 + 6a_1a_7)) \right. \\
 & + 8a_6^2(a_5a_7 + a_3a_8 + a_2a_9) - 24a_6(a_4a_5a_8 + a_2a_7a_8 + a_3a_5a_9 + a_1a_7a_9) \\
 & + 16 \left( -a_3^2a_8^2 + 3a_2a_4a_8^2 + a_2a_3a_8a_9 - a_2^2a_9^2 + a_5a_7(a_3a_8 + a_2a_9) \right. \\
 & \quad \left. - a_5^2(a_7^2 - 3a_4a_9) + 3a_1(a_7^2a_8 - 3a_4a_8a_9 + a_3a_9^2) \right) \Big), \\
 g = & \left( 216a_{10}^2 \left( -a_2^2a_3^2 + 4a_2^3a_4 - 18a_1a_2a_3a_4 + a_1(4a_3^3 + 27a_1a_4^2) \right) - a_6^6 \right. \\
 & + 12a_6^4(a_5a_7 + a_3a_8 + a_2a_9) - 36a_6^3(a_4a_5a_8 + a_2a_7a_8 + a_3a_5a_9 + a_1a_7a_9) \\
 & - 24a_6^2(2a_3^2a_8^2 - 3a_2a_4a_8^2 + a_2a_3a_8a_9 + 2a_2^2a_9^2 + a_5a_7(a_3a_8 + a_2a_9) + a_5^2(2a_7^2 - 3a_4a_9) \\
 & \quad - 3a_1(a_7^2a_8 + 9a_4a_8a_9 + a_3a_9^2)) + 144a_6 \left( a_2^2a_7a_8a_9 + (a_1a_5a_7^2 + a_3^2a_5a_8 \right. \\
 & \quad + a_3a_7(a_5^2 - 5a_1a_8))a_9 + a_2(a_5a_7^2a_8 + a_3a_7a_8^2 + a_3a_5a_9^2 + a_1a_7a_9^2) \\
 & \quad \left. + a_4(a_5^2a_7a_8 - 6a_1a_7a_8^2 + a_5(a_3a_8^2 - 5a_2a_8a_9 - 6a_1a_9^2)) \right) \\
 & + 8 \left( 72a_1a_3a_7^2a_8^2 + 8a_3^3a_8^3 + 108a_1a_4^2a_8^3 - 108a_1a_3a_4a_8^2a_9 - 27a_1^2a_7^2a_9^2 + 72a_1a_3^2a_8a_9^2 \right. \\
 & \quad + 8a_2^3a_9^3 + 108a_1^2a_4a_9^3 + 4a_5^3(2a_7^3 - 9a_4a_7a_9) - 3a_5^2(4a_3a_7^2a_8 + 9a_4^2a_8^2 + 4a_2a_7^2a_9 \\
 & \quad - 6a_3a_4a_8a_9 + 9a_3^2a_9^2 - 24a_2a_4a_9^2) - 3a_2^2a_8(9a_7^2a_8 + 4a_9(-6a_4a_8 + a_3a_9)) \\
 & \quad - 6a_2(2a_3^2a_8^2a_9 + 3a_1a_8a_9(-a_7^2 + 6a_4a_9) + 6a_3(a_4a_8^3 + a_1a_9^3)) - 6a_5a_7(2a_3^2a_8^2 \\
 & \quad \left. + a_2a_3a_8a_9 + a_2(-3a_4a_8^2 + 2a_2a_9^2) + 3a_1(2a_7^2a_8 - a_9(9a_4a_8 + a_3a_9)) \right) \Big) \\
 & + 36a_{10} \left( -8a_3^3a_5a_8 + 12a_4^2(2a_5^3 - 9a_1a_5a_8) + 2a_7(-12a_1a_2a_6a_7 + 12a_1^2a_7^2 \right. \\
 & \quad + a_2^2(a_6^2 + 8a_5a_7) - 4a_2^3a_9) + a_3(6a_1a_7(3a_6^2 - 4a_5a_7) + 4a_2^2(a_7a_8 + a_6a_9) \\
 & \quad - a_2(a_6^3 + 20a_5a_6a_7 - 36a_1a_7a_9)) - 3a_4 \left( 8a_2^2(a_6a_8 + a_5a_9) + a_1(5a_6^3 - 12a_5a_6a_7 \right. \\
 & \quad + 36a_1a_7a_9) + 4a_3(2a_5^2a_6 - 3a_1a_6a_8 - 3a_5(a_2a_8 + a_1a_9)) + 2a_2(-3a_5a_6^2 + 4a_5^2a_7 \\
 & \quad \left. - 6a_1(a_7a_8 + a_6a_9)) \right) + 2a_3^2 \left( 8a_5^2a_7 + a_5(a_6^2 + 2a_2a_9) + 2(a_2a_6a_8 - 6a_1(a_7a_8 + a_6a_9)) \right) \Big) \Big).
 \end{aligned} \tag{A.6}$$

Using these explicit expressions for  $f$  and  $g$ , one can construct the discriminant (A.2), however, in contrast to the quartic, it does not enjoy any particular factorization properties.

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