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# Sequence Semantics for Norms and Obligations

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**Abstract**

This paper presents a new version of the sequence semantics presented at DEON 2014. This new version allows us for a capturing the distinction between logic of obligations and logic of norms. Several axiom schemata are discussed, while soundness and completeness results are proved.

*Keywords:* Deontic systems, Neighbourhood semantics, Logic of norms.

## 1 Introduction

Most of the work in deontic logic has focused on the study of the concepts of obligation, permission, prohibition and related notions, but little attention has been dedicated on how these prescriptions are generated within a normative system. [1] The general idea of norms is that they describe conditions under which some behaviours are deemed as 'legal'. In the simplest case, a behaviour can be described by an obligation (or a prohibition, or a permission), but often norms additionally specify what are the consequences of not complying with them, and what sanctions follow from violations and whether such sanctions compensate for the violations.

To address the above issues, Governatori and Rotolo [12] presented a Gentzen style sequent system to describe a non classical operator ($\otimes$) which models chains of obligations and compensatory obligations. The interpretation of a chain like $a \otimes b \otimes c$ is that $a$ is obligatory, but if it is violated (i.e., $\neg a$ holds), then $b$ is the new obligation (and $b$ compensates for the violation of $a$); again, if the obligation of $b$ is violated as well, then $c$ is obligatory (and so on).

As we argued in [12, 8], the logic of $\otimes$ offers a proof-theoretic approach to normative reasoning (and in particular, CTD reasoning), which, as done by [18, 17] in the context of Input/Output Logic, follows the principle "no logic of norms without attention to the normative systems in which they occur" [16].

[1] A normative system can be understood as a, possibly hierarchically structured, set of norms and mechanisms that systematically interplay for deriving deontic prescriptions in force in a given situation.

This idea draws inspiration from the pioneering works in [20] and [1], and focuses on the fact that normative conclusions derive form of norms as interplaying together in normative systems. Indeed, it is essential in this perspective to distinguish prescriptive and permissive norms from obligations and permissions [3, 10]: the latter ones are merely the effects of the application of norms.

While Input/Output approach mainly works by imposing some constraints on the manipulation of conditional norms, the $\otimes$-logic uses $\otimes$-chains to express the logical structures (norms) that generate actual obligations and permissions. In [4], we proposed a model-theoretic semantics (called *sequence semantics*) for the $\otimes$-logic, that addresses the problem identified in [7] that affects most of the existing approaches for the representation of norms, in particular compensatory obligations, using 'standard' possible world semantics. A compensatory obligation is a sub-class of a contrary-to-duty obligation, where the violation of the primary obligation is compensated by the fulfilment of the secondary obligation. Compensatory obligations can be modelled by $\otimes$-chains. As we have already discussed, an expression like $a \otimes b$ means that $a$ is obligatory, but its violation is compensated by $b$ or, in other terms, it is obligatory to do $b$ to compensate the violation of the obligation of $a$. Thus, a situation where $a$ does not hold (or $\neg a$ holds) and $b$ holds is still deemed as a 'legal' situation. Accordingly, when we use a 'standard' possible world semantics, there is a deontically accessible world where $\neg a$ holds, but this implies, according to the usual evaluation conditions for permission (something is permitted, if there is a deontically accessible world where it holds), that $\neg a$ is permitted. However, we have the norm modelling the compensatory obligation that states that $a$ is obligatory (and if it were not, then there would be no need for $b$ to compensate for such a violation since, there would be no violation of the obligation of $a$ to begin with). The sequence semantics solves this problem by establishing that to have an obligation, we must have a norm generating the obligation itself (where a norm is represented by an $\otimes$-chain), and not simply that something is obligatory because it holds in all the deontically accessible worlds.

The work of the present paper completes the picture in three points.

- We extend sequence semantics and split the treatment of $\otimes$-chains and obligations; the intuition is that chains are the generators of obligations and permissions, we hence semantically separate structures interpreting norms from those interpreting obligations and permissions.
- We add $\oplus$-sequences to express ordering among explicit permissions [8]; as for $\otimes$, given the chain $a \oplus b$, we can proceed through the $\oplus$-chain to obtain the derivation of $\mathsf{P}b$. However, permissions cannot be violated. Consequently, it does not make sense to obtain $\mathsf{P}b$ from $a \oplus b$ and $\neg a$. Here, the reason to proceed in the chain is rather that the normative system allows us to prove $\mathsf{O}\neg a$;
- We systematically study several options for the axiomatisation of $\otimes$ and $\oplus$.

The layout of the paper is as follows: in Section 2 we introduce the language of our logics. In Section 3 we progressively introduce axioms for the deontic operators to axiomatise more expressive deontic logics with and without interaction between the operators, and we discuss some intuition behind the axiomatisation. In Section 4 we provide the definitions of sequence semantics to cover the case of weak and strong permission. Soundness and completeness of the various deontic logic with the novel semantics are proved in Section 5. Finally, a short discussion of related work and further work (Section 6) concludes the paper.

## 2 Language

The language consists of a countable set of atomic formulae. Well-formed-formulae are then defined using the typical Boolean connectives, the $n$-ary connectives $\otimes$ and $\oplus$, and the modal (deontic) operators $\mathsf{O}$ for obligation and $\mathsf{P}$ for permission. The intended reading of $\otimes$ is that it encodes a sequence of obligations where each obligation is meant to compensate the violation of the previous obligation. The intuition behind $\oplus$ is instead meant to model ordered lists of permissions, i.e., a preference order among different permissions [8].

Let $\mathcal{L}$ be a language consisting of a countable set of propositional letters $Prop = \{p_1, p_2, \ldots\}$, the propositional constant $\bot$, round brackets, the boolean connective $\rightarrow$, the unary operators $\mathsf{O}$ and $\mathsf{P}$, the set of $n$-ary operators $\otimes^n$ for $n \in \mathbb{N}^+$ and the set of $n$-ary operators $\oplus^n$ for $n \in \mathbb{N}^+$. We shall refer to the language where $\oplus$ does not occur as $\mathcal{L}^{\otimes}$, and the language where $\otimes$ does not occur as $\mathcal{L}^{\oplus}$. There is no technical difficulty in avoiding that $\otimes$ and $\oplus$ be binary operators: the reason why we define them as $n$-ary ones is mainly conceptual and is meant to exclude the nesting of $\otimes$- and $\oplus$-expressions. Consider $a \otimes \neg(b \otimes c) \otimes d$. The expression $\neg(b \otimes c)$ means either that $b$ is not obligatory or that it is so but $c$ does not compensate the violation of $\mathsf{O}b$. What does it mean this as a compensation of the violation of $\mathsf{O}a$? Also, what is the meaning of $a \otimes (b \oplus c) \otimes d$?

**Definition 2.1** [Well Formed Formulae] Well formed formulae (wffs) are defined as follows:

- Any propositional letter $p \in Prop$ and $\bot$ are wffs;
- If $a$ and $b$ are wffs, then $a \rightarrow b$ is a wff;
- If $a$ is a wff and no operator $\otimes^m$, $\oplus^m$, $\mathsf{O}$ and $\mathsf{P}$ occurs in $a$, then $\mathsf{O}a$ and $\mathsf{P}a$ are a wff;
- If $a_1, \ldots, a_n$ are wffs and no operator $\otimes^m$, $\oplus^m$, $\mathsf{O}$ and $\mathsf{P}$ occurs in any of them, then $a_1 \otimes^n \cdots \otimes^n a_n$ and $a_1 \oplus^n \cdots \oplus^n a_n$ are a wff, where $n \in \mathbb{N}^+$; [2]
- Nothing else is a wff.

We use $WFF$ to denote the set of well formed formulae.

Other Boolean operators are defined in the standard way, in particular $\neg a =_{def} a \rightarrow \bot$ and $\top =_{def} \bot \rightarrow \bot$.

We use $\odot$ to refer to either $\otimes$ or $\oplus$. Accordingly, we say that any formula $a_1 \odot \cdots \odot a_n$ is an $\odot$-*chain*; also the negation of an $\odot$-chain is an $\odot$-chain. The formation rules allow us to have $\odot$-chains of any (finite) length, and the arity of

[2] We use the prefix forms $\otimes^1 a$ and $\oplus^1 a$ for the case of $n = 1$.

the operator is equal to number of elements in the chain; we thus drop the index $m$ from $\odot^m$. Moreover, we use the prefix notation $\bigodot_{i=1}^n a_i$ for $a_1 \odot \cdots \odot a_n$.

## 3 Logics for $\otimes$ and $\oplus$

The aim of this section is to discuss the intuitions behind some principles governing the behaviour and the interactions of the various deontic operators. These principles are captured by axioms or inference rules.

### 3.1 Basic Axiomatisation

In this paper, we assume classical propositional logic, CPC, as the underlying logic on which all the deontic logics we examine are based.

The first principle is that of syntax independence or, in other terms, that the deontic operators are closed under logical equivalence. To this end, all the logics have the following inference rules:

$$\frac{a \equiv b}{\mathsf{O}a \equiv \mathsf{O}b}\mathsf{O}\text{-RE} \qquad \frac{a \equiv b}{\mathsf{P}a \equiv \mathsf{P}b}\mathsf{P}\text{-RE}$$

$$\frac{\bigwedge_{i=1}^n (a_i \equiv b_i)}{\bigotimes_{i=1}^n a_i \equiv \bigotimes_{i=1}^n b_i}\otimes\text{-RE} \qquad \frac{\bigwedge_{i=1}^n (a_i \equiv b_i)}{\bigoplus_{i=1}^n a_i \equiv \bigoplus_{i=1}^n b_i}$$

Consider the $\odot$ chain $a \odot b \odot a \odot c$. If $\odot$ is $\otimes$, the meaning of the chain above is that $a$ is obligatory, but if $a$ is violated (meaning that $\neg a$ holds) then $b$ is obligatory. If also $b$ is violated, then $a$ becomes obligatory. But we already know that we will incur in the violation of it, since $\neg a$ holds. Accordingly, we have the obligation of $c$. However, this is the meaning of the $\otimes$-chain: $a \otimes b \otimes c$.

If $\odot$ is $\oplus$, the intuitive reading of $a \odot b \odot a \odot c$ is that $a$ should be permitted unless (for other reasons) $a$ is forbidden; in such a case $b$ is permitted. However, if also $b$ is forbidden, then $a$ is permitted. Nevertheless, we have already established that this is not possible, since $a$ is forbidden, we thus have the permission of $c$. Again, this is what is encoded by the $\oplus$-chain $a \oplus b \oplus c$.

The above example shows that duplications of formulae in $\odot$-chains do not contribute to the meaning of the chains themselves. This motivates us to adopt the following axioms to remove (resp., introduce) an element from (to) a chain if an equivalent formula occurs on the left of it.

$$\bigotimes_{i=1}^{n} a_i \equiv \bigotimes_{i=1}^{k-1} a_i \otimes \bigotimes_{i=k+1}^{n} a_i \text{ where } a_j \equiv a_k,\ j < k \qquad (\otimes\textbf{-contraction})$$

$$\bigoplus_{i=1}^{n} a_i \equiv \bigoplus_{i=1}^{k-1} a_i \oplus \bigoplus_{i=k+1}^{n} a_i \text{ where } a_j \equiv a_k,\ j < k \qquad (\oplus\textbf{-contraction})$$

The minimal logics resulting from the above axioms and inference rules are $\mathsf{E}^{\otimes}$ when the language is restricted to $\mathcal{L}^{\otimes}$, $\mathsf{E}^{\oplus}$ for $\mathcal{L}^{\oplus}$, and $\mathsf{E}^{\otimes\oplus}$ for $\mathcal{L}$.

### 3.2 Deontic Axioms

The logics presented in the previous section are minimal, and besides the intended deontic reading of the operators, they do not not provide any 'genuine' deontic principle. In the present section, we introduce axioms to model the relationships between $\mathsf{O}$ and $\mathsf{P}$; specifically, the axioms lay down the conditions under which the various operators are consistent.

The first axiom defines the duality of obligation and permission.

$$\mathsf{P}a \equiv \neg\mathsf{O}\neg a \qquad (\mathsf{OP}\text{-}\mathbf{duality})$$

This axiom implies the reading of permission as weak permission, i.e., the lack of the obligation of the contrary.

$$\mathsf{O}a \rightarrow \mathsf{P}a \qquad (\mathsf{O}\text{-}\mathsf{P})$$

Axiom O-P, is the standard **D** axiom of modal/deontic logic. This axiom can have different meanings depending on whether O and P are the dual of each other. If they are, the axiom is trivially equivalent to the following one:

$$\mathsf{O}a \rightarrow \neg\mathsf{O}\neg a \qquad (\mathbf{D}\text{-}\mathsf{O})$$

The axiom states the external consistency of a normative system: a normative system is externally consistent if no formula is obligatory and forbidden at the same time. If O and P are independent modalities, then Axiom O-P establishes the consistency between obligations and permissions, while Axiom **D**-O must be assumed to guarantee the external consistency of obligations.

Internal consistency of obligation is the property that no obligation is self-inconsistent; this is expressed by:

$$\neg\mathsf{O}\bot \qquad (\overline{\mathbf{P}}\text{-}\mathsf{O})$$

Finally, when obligation and permission are not dual, while the consistency between obligation and permission is covered by Axiom O-P, we have yet to cover the consistency between prohibition and permission. To this end, we can use one direction of the duality, namely:

$$\mathsf{O}a \rightarrow \neg\mathsf{P}\neg a \qquad (\mathsf{O}\neg\mathsf{P})$$

The axioms we consider hitherto focus on consistency principles for O and P. The next axioms provide consistency principles for $\odot$-chains.

Given that we use classical propositional logic as the underlying logic, it is not possible that an $\odot$-chain and its negation hold at the same time. What about when $\odot$-chains like $a \odot b \odot c$ and $\neg(a \odot b)$ hold. In case $\odot$ is $\otimes$, the first chain states that $a$ is obligatory and its violation is compensated by $b$, which in turn is itself obligatory and it is compensated by $c$. The second expression states that 'either it is not the case that $a$ is obligatory, but if it is so, then its violation is not compensated by $b$'. Accordingly, the combination of the two expressions should result in a contradiction (a similar argument can be made for $\oplus$-chains). To ensure this, we must assume the following axioms that allow us to derive, given a chain, all its sub-chains with the same initial element(s).

$$a_1 \otimes \cdots \otimes a_n \rightarrow a_1 \otimes \cdots \otimes a_{n-1},\ n \geq 2 \qquad (\otimes\text{-}\mathbf{shortening})$$
$$a_1 \oplus \cdots \oplus a_n \rightarrow a_1 \oplus \cdots \oplus a_{n-1},\ n \geq 2 \qquad (\oplus\text{-}\mathbf{shortening})$$

While any combination of the axioms presented in this section can be added to any of the minimal logics of the previous section, we focus on two options that we believe are meaningful for the representation of norms. For the first option, we call the resulting logic $\mathsf{D}^{\otimes}$, we consider $\mathsf{O}$ and $\mathsf{P}$ as dual, and it extends $\mathsf{E}^{\otimes}$ with $\mathsf{OP}$-**duality**, $\overline{\mathbf{P}}$-$\mathsf{O}$ and $\otimes$-**shortening**. For the second option, we reject the duality of $\mathsf{O}$ and $\mathsf{P}$, essentially taking the strong permission stance, and we assume $\mathsf{E}^{\otimes\oplus}$ plus all axioms presented in this section with the exclusion of $\mathsf{OP}$-**duality**. We use $\mathsf{D}^{\otimes\oplus}$ for the resulting logic.

### 3.3 Axioms for $\otimes$ and $\mathsf{O}$

In this section, we address the relationships between $\otimes$ and $\mathsf{O}$; we thus focus on axioms for extending $\mathsf{D}^{\otimes}$ (though the axioms are suitable for extensions of $\mathsf{D}^{\otimes\oplus}$). As we have repeatedly argued, $\otimes$-chains are meant to generate obligations. In particular, we have seen that the first element of an $\otimes$-chain is obligatory. This is formalised by the following axiom:

$$a_1 \otimes \cdots \otimes a_n \to \mathsf{O}a_1. \qquad (\otimes\text{-}\mathsf{O})$$

Furthermore, we say that if the negation of the first element does not hold, we can infer the obligation of the second element. Formally

$$a_1 \otimes \cdots \otimes a_n \wedge \neg a_1 \to \mathsf{O}a_2. \qquad (1)$$

Moreover, we argued that we can repeat the same procedure. This leads us to generalise (1) for the axiom that expresses the detachment principle for $\otimes$-chains and factual statements about the opposites of the first $k$ elements of an $\otimes$-chain.

$$a_1 \otimes \cdots \otimes a_n \wedge \bigwedge_{i=1}^{k<n} \neg a_i \to \mathsf{O}a_{k+1} \qquad (\mathsf{O}\text{-}\mathbf{detachment})$$

A possible intuition behind this schema is that it can be used to determine which are the obligations that can be complied with. For example, since $\neg a_1$ holds, then we know that it is no longer possible to comply with the obligation of $a_1$. In a similar way, we could ask what are the parts of norms which are effective in a particular situation. In this case, instead of detaching an obligation we could detach an $\otimes$-chain. Accordingly, we formulate the following axiom:

$$a_1 \otimes \cdots \otimes a_n \wedge \neg a_1 \to a_2 \otimes \cdots \otimes a_n \qquad (\otimes\text{-}\mathbf{detachment})$$

where $a_2 \otimes \cdots \otimes a_n$ does non contain $a_1$ or formulae equivalent to it.

Notice that, contrary to what we did for (1), there is no need to generalise $\otimes$-**detachment** to a version where we consider the negation of the first $k$ elements of the $\otimes$-chain since

$$a_1 \otimes \cdots \otimes a_n \wedge \bigwedge_{i=1}^{k<n} \neg a_i \to a_{k+1} \otimes \cdots \otimes a_n \qquad (2)$$

is derivable from $k$ applications of $\otimes$-**detachment**; hence, there is no need to take (2) as an axiom. Furthermore, in case Axiom $\otimes$-**detachment** holds, it is possible to use (1) to detach $\mathsf{O}$ from an $\otimes$-chain instead of $\mathsf{O}$-**detachment** which would then be derivable from $\otimes$-**detachment** and (1).

The attentive reader will not fail to observe that the above detachment axioms do not explicitly mention that the negations of the first $k$ elements of an $\otimes$-chain are violations. The next few axioms address this aspect:

$$a_1 \otimes \cdots \otimes a_n \wedge \bigwedge_{i=1}^{k<n} (\mathsf{O}a_i \wedge \neg a_i) \to \mathsf{O}a_{k+1} \qquad (\mathsf{O}\textbf{-violation-detachment})$$

$$a_1 \otimes \cdots \otimes a_n \wedge \mathsf{O}a_1 \wedge \neg a_1 \to a_2 \otimes \cdots \otimes a_n \qquad (\otimes\textbf{-violation-detachment})$$

Axioms O-**violation-detachment** and $\otimes$-**violation-detachment** are the immediate counterpart of Axioms O-**detachment** and $\otimes$-**detachment** just including the violation condition in the their antecedent (and we can repeat the argument about the possible axiom combination for their counterparts).

The question is now what are the differences between the cases with or without the explicit violations. Suppose, we have the $\otimes$-chains

$$a \otimes b \qquad\qquad \neg a \otimes c$$

Applying $\otimes$-O and **D**-O results in a contradiction. Suppose that a normative system is equipped with some mechanisms (as it is the case of real life normative systems) to resolve conflicts like this (maybe, using some form of preferences over norms).[3] Also, for the sake of the example, the resolution prefers the first $\otimes$-chain to the second one, and that the first norm has been complied with, that is $a$ holds. Then, we can ask what the obligations in force are.

On the one hand, one can argue that the norm prescribing the second $\otimes$-chain is still effective and thus it is able to generate obligations, but since the first option ($\neg$) would produce a violation, then we can settled for the second option, and we can hence derive $\mathsf{O}c$ from it. If one subscribes to this interpretation, then Axioms O-**detachment** and, eventually, $\otimes$-**detachment** are to be assumed. On the other hand, it is possible to argue that when a norm overrides other norms, then the norms that are overridden are no longer effective. Accordingly, in the case under analysis, $a$ is not a violation of the second $\otimes$-chain, and then there is no ground to proceed with the derivation of $\mathsf{O}c$. But, if $\neg a$ holds instead of $a$, then we have a violation of the first $\otimes$-chain: we can apply $\otimes$-O to conclude $\mathsf{O}a$, and then O-**violation-detachment** to obtain $\mathsf{O}b$. Hence, the axioms suitable for modelling this intuition are O-**violation-detachment** and, eventually, $\otimes$-**violation-detachment** in case one wants to derive which sub-chains are effective after violations.

Notice that the logic of $\otimes$ was devised to grasp the ideas of violation and compensation: for this reason, we do *not* commit to any reading in which, given $a \otimes b$, the fact $\neg a$ prevents the derivation of $\mathsf{O}a$. If this were the case, we would not have any violation at all. On the contrary, $\mathsf{O}b$ is precisely meant to compensate for the effects of the non legal situation described by $\mathsf{O}a \wedge \neg a$. To further illustrate the idea behind compensatory obligations, consider a situation where $\neg a$ and $b$ hold. Suppose, that you have the norm $a \otimes b$. Here, we can derive the obligations $\mathsf{O}a$ and $\mathsf{O}b$, the first of which is violated, and such a

[3] It is beyond the scope of the present paper to discuss mechanisms to resolve conflicts, the focus of the paper is to propose which combinations of formulae result in conflicts, the reader interested in some solutions using the $\otimes\oplus$-logic can consult [8].

violation triggers the second obligation, i.e., $\mathsf{O}b$, whose fulfilment compensates the violation. Accordingly, the situation, while not ideal, can be still considered compliant with the norm. Suppose that instead of $a \otimes b$ we have two norms $\otimes^1 a$, and $\otimes^1 b$. Similarly, we derive the obligations $\mathsf{O}a$ and $\mathsf{O}b$. However, $\mathsf{O}b$ does not depend on having the violation of $\mathsf{O}a$, nor does it compensate for that violation. Thus, in the last case, $\mathsf{O}a$ is an obligation that cannot be compensated for, and $\mathsf{O}b$ is in force even when we comply with $\mathsf{O}a$.

### 3.4 Axioms for $\otimes$, $\oplus$, $\mathsf{O}$, $\mathsf{P}$

We now turn our attention to the study of the relationships between $\oplus$-chains and permissions. The basic Axiom $\oplus$-$\mathsf{P}$ states that the first element of a permissive chain is a permission.

$$a_1 \oplus \cdots \oplus a_n \to \mathsf{P}a_1 \qquad (\oplus\text{-}\mathsf{P})$$

As we have seen, the intuitive reading of $a \oplus b \oplus c$ is that $a$ should be permitted, but if it is not, then $b$ should be permitted and, if even $b$ is not permitted, then, finally, $c$ should be permitted. Consequently, we formulate the following axioms for detaching a permission from a permissive chain, and for detaching a permissive sub-chain.

$$a_1 \oplus \cdots \oplus a_n \wedge \bigwedge_{i=1}^{k<n} \neg \mathsf{P}a_i \to \mathsf{P}a_{k+1} \qquad (\mathsf{P}\textbf{-detachment})$$

$$a_1 \oplus \cdots \oplus a_n \wedge \neg \mathsf{P}a_1 \to a_2 \oplus \cdots \oplus a_n \qquad (\oplus\textbf{-detachment})$$

The considerations we made about the choice of axioms for $\otimes$ and $\mathsf{O}$ apply for the axioms relating $\mathsf{P}$ and $\oplus$ as well.

If we assume the obligation-permission and prohibition-permission consistency principles, i.e., Axioms $\mathsf{O}$-$\mathsf{P}$ and $\mathsf{O}\neg\mathsf{P}$, then the axioms in the previous section and the axioms above suffice to describe the relationships among the various deontic operators. In absence of such axioms, several variations of the axioms are possible to maintain consistency between obligations and permissions.

$$a_1 \otimes \cdots \otimes a_n \wedge \neg \mathsf{P}\neg a_1 \to \mathsf{O}a_1 \qquad (3)$$

$$a_1 \oplus \cdots \oplus a_n \wedge \neg \mathsf{O}\neg a_1 \to \mathsf{P}a_1. \qquad (4)$$

In the situation where a norm holds while the permission of contrary of the first element (of the chain) does not, (3) allows us to determine that the first element is mandatory. Symmetrically, (4) derives the first element of a permissive chain as a permission whereas its contrary is not mandatory. Similar combinations can be used for the detachment axioms we have proposed. For instance, we can integrate the obligation-permission consistency in Axiom **O-violation-detachment** to obtain

$$a_1 \otimes \cdots \otimes a_n \wedge \bigwedge_{i=1}^{k<n} \neg a_i \wedge \neg \mathsf{P}\neg a_{k+1} \to \mathsf{O}a_{k+1} \qquad (5)$$

or we integrate the prohibition-permission in (4) resulting in

$$a_1 \oplus \cdots \oplus a_n \wedge \mathsf{O}\neg a_1 \to a_2 \oplus \cdots \oplus a_n. \qquad (6)$$

Notice that (3)–(6) (and similar extensions of the various detachment axioms) are derived when Axioms O-P and O¬P as well as the corresponding detachment axioms hold.

### 3.5 Logics

In this paper, we shall prove completeness results for three groups of systems, as outlined in the table below.

| | |
|---|---|
| **Basic Systems** | |
| $\mathsf{E}^{\otimes}$ | CPC + O-RE + ⊗-RE + ⊗-**contraction** |
| $\mathsf{E}^{\oplus}$ | CPC + P-RE + ⊕-RE + ⊕-**contraction** |
| $\mathsf{E}^{\otimes\oplus}$ | $\mathsf{E}^{\otimes}$ + $\mathsf{E}^{\oplus}$ |
| **Basic Deontic Systems** | |
| $\mathsf{D}^{\otimes}$ | $\mathsf{E}^{\otimes}$ + OP-**duality** + O-P + $\overline{\mathbf{P}}$-O + ⊗-**shortening** |
| $\mathsf{D}^{\otimes\oplus}$ | $\mathsf{E}^{\otimes\oplus}$ + O-P + $\overline{\mathbf{P}}$-O + **D**-O + O¬P + ⊗-**shortening** + ⊕-**shortening** |
| $\mathsf{D}^{\mathsf{O}\otimes}$ | $\mathsf{D}^{\otimes}$ + ⊗-O |
| **Basic Full Deontic System** | |
| $\mathsf{D}^{\mathsf{OP}\otimes\oplus}$ | $\mathsf{D}^{\otimes\oplus}$ + ⊗-O |

Besides these systems, in Section 5 we shall also analyse systems extending $\mathsf{D}^{\mathsf{OP}\otimes\oplus}$ with combinations of detachments axioms (including ⊕-P).

## 4 Sequence Semantics

Sequence semantics is an extension of neighbourhood semantics. The extension is twofold: (1) we introduce a second neighbourhood-like function, and (2) the new function generates a set of sequences of sets of possible worlds instead of set of sets of possible worlds. This extension allows us to provide a clean semantic representation of ⊙-chains.

Before introducing the semantics, we provide some technical definitions for the operation of *s-zipping*, i.e., the removal of repetitions or redundancies occurring in sequences of sets of worlds. This operation is required to capture the intuition described for the ⊙-shortening axioms.

**Definition 4.1** Let $X = \langle X_1, \ldots, X_n \rangle$ be such that $X_i \in 2^W$ $(1 \leq i \leq n)$. $Y$ is *s-zipped from* $X$ iff $Y$ is obtained from $X$ by applying the following operation: for $1 \leq k \leq n$, if $X_j = X_k$ and $j < k$, delete $X_k$ from the sequence.

**Definition 4.2** A set $S$ of sequences of sets of possible worlds is closed under s-zipping iff if $X \in S$, then (i) for all $Y$ such that $X$ is s-zipped from $Y$, $Y \in S$; and (ii) for all $Z$ such that $Z$ is s-zipped from $X$, $Z \in S$.

Closure under s-zipping essentially determines classes of equivalences for ⊙-chain based on Axioms ⊗-**shortening** and ⊕-**shortening**.

The next three definitions provide the basic scaffolding for sequence semantics: frame, valuation, and model.

**Definition 4.3** A *sequence frame* is a structure $\mathcal{F} = \langle W, \mathcal{C}, \mathcal{N} \rangle$, where

- $W$ is a non empty set of possible worlds,
- $\mathcal{C}$ is a function with signature $W \to 2^{(2^W)^n}$ such that for every world $w$, every $X \in \mathcal{C}_w$ is closed under s-zipping.
- $\mathcal{N}$ is a function with signature $W \to 2^{2^W}$.

**Definition 4.4** A *sequence model* is a structure $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where
- $\mathcal{F}$ is a sequence frame, and
- $V$ is a valuation function, $V\colon \mathit{Prop} \to 2^W$.

**Definition 4.5** The valuation function for a sequence model is a follows:
- usual for atoms and boolean conditions,
- $w \models \odot_{i=1}^n a_i$ iff $\langle \|a_1\|_V, \ldots, \|a_n\|_V \rangle \in \mathcal{C}_w$,
- $w \models \Box a$ iff $\|a\|_V \in \mathcal{N}_w$.

Sequence models are meant to be used for the combination of a deontic operator (in this paper $\Box$ ranges over $\mathsf{O}$ and $\mathsf{P}$) and the corresponding $\odot$-chain operator ($\otimes$ and $\oplus$, respectively). We are going to use sequence models for the logics where we consider only $\otimes$ and $\mathsf{O}$, and $\mathsf{P}$ is defined as the dual of $\mathsf{O}$.

The next three definitions extend sequences semantics to the case of two sets of independent combinations of $\odot$ and the corresponding unary deontic operator.

**Definition 4.6** A *bi-sequence frame* is a structure $\mathcal{F} = \langle W, \mathcal{C}^{\mathsf{O}}, \mathcal{C}^{\mathsf{P}}, \mathcal{N}^{\mathsf{O}}, \mathcal{N}^{\mathsf{P}} \rangle$, where
- $W$ is a non empty set of possible worlds;
- $\mathcal{C}^{\mathsf{O}}$ and $\mathcal{C}^{\mathsf{P}}$ are two functions with signature $W \to 2^{(2^W)^n}$, such that for every world $w \in W$, for every $X \in \mathcal{C}^{\mathsf{O}}_w$ and $Y \in \mathcal{C}^{\mathsf{P}}_w$, $X$ and $Y$ are closed under s-zipping;
- $\mathcal{N}^{\mathsf{O}}$ and $\mathcal{N}^{\mathsf{P}}$ are two functions with signature $W \to 2^{2^W}$.

**Definition 4.7** A *bi-sequence model* is a structure $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where
- $\mathcal{F}$ is a bi-sequence frame, and
- $V$ is a valuation function, $V\colon \mathit{Prop} \to 2^W$.

**Definition 4.8** The valuation function for a bi-sequence model is as follows:
- usual for atoms and boolean conditions,
- $w \models a_1 \otimes \cdots \otimes a_n$ iff $\langle \|a_1\|_V, \ldots, \|a_n\|_V \rangle \in \mathcal{C}^{\mathsf{O}}_w$,
- $w \models a_1 \oplus \cdots \oplus a_n$ iff $\langle \|a_1\|_V, \ldots, \|a_n\|_V \rangle \in \mathcal{C}^{\mathsf{P}}_w$,
- $w \models \mathsf{O}a$ iff $\|a\|_V \in \mathcal{N}^{\mathsf{O}}_w$,
- $w \models \mathsf{P}a$ iff $\|a\|_V \in \mathcal{N}^{\mathsf{P}}_w$.

## 5 Soundness and Completeness

In this section we study the soundness and completeness of the logics defined in Section 3.5. Completeness is based on adaptation of the standard Lindenbaum's construction for modal (deontic) neighbourhood semantics.

**Definition 5.1** [$\mathcal{L}$-maximality] A set $w$ is $\mathcal{L}$-maximal iff for any formula $a$ of $\mathcal{L}$, either $a \in w$, or $\neg a \in w$.

**Lemma 5.2 (Lindenbaum's Lemma)** *Any consistent set $w$ of formulae in the language $\mathcal{L}$ can be extended to a consistent $\mathcal{L}$-maximal set $w^+$.*

**Proof.** Let $a_1, a_2, \ldots$ be an enumeration of all the possible formulae in $\mathcal{L}$.

- $w_0 := w$;
- $w_{n+1} := w_n \cup \{a_n\}$ if its closure under the axioms and rules of $\mathsf{S}$ is consistent, $w \cup \{\neg a_n\}$ otherwise;
- $w^+ := \bigcup_{n \geq 0} w_n$. □

### 5.1 Basic classical systems: $\mathsf{E}^{\otimes}$, $\mathsf{E}^{\oplus}$

The construction of a sequence canonical model is as follows.

3 [$\mathsf{E}^{\otimes}$-Canonical Models] A sequence canonical model $\mathcal{M}$ = ɔr a system $\mathsf{S}$ in the language $\mathcal{L}^{\otimes}$ (where $\mathsf{S} \supseteq \mathsf{E}^{\otimes}$) is defined as

1. $W$ is the set of all the $\mathcal{L}^{\otimes}$-maximal consistent sets.
2. For any propositional letter $p \in Prop$, $\|p\|_V := |p|_{\mathsf{S}}$, where $|p|_{\mathsf{S}} := \{w \in W \mid p \in w\}$.
3. Let $\mathcal{C} := \bigcup_{w \in W} \mathcal{C}_w$, where, for each $w \in W$, $\mathcal{C}_w := \{\langle \|a_1\|_V, \ldots, \|a_n\|_V \rangle \mid \bigotimes_{i=1}^{n} a_i \in w\}$, where each $a_i$ is a meta-variable for a Boolean formula.
4. Let $\mathcal{N} := \bigcup_{w \in W} \mathcal{N}_w$ where for each world $w$, $\mathcal{N}_w := \{\|a_i\|_V \mid \mathsf{O}a_i \in w\}$.

Any canonical model for a logic extending $\mathsf{E}^{\oplus}$, on the other hand, would be exactly the same, but for condition (3), to be changed as to read: Let $\mathcal{C} := \bigcup_{w \in W} \mathcal{C}_w$, where, for each $w \in W$, $\mathcal{C}_w := \{\langle \|a_1\|_V, \ldots, \|a_n\|_V \rangle \mid \bigoplus_{i=1}^{n} a_i \in w\}$, where each $a_i$ is a meta-variable for a Boolean formula.

**Lemma 5.4 (Truth Lemma for Canonical Sequence Models)** *If $\mathcal{M} = \langle W, \mathcal{C}, \mathcal{N}, V \rangle$ is canonical for $\mathsf{S}$, where $\mathsf{S} \supseteq \mathsf{E}^{\otimes}$ or $\mathsf{S} \supseteq \mathsf{E}^{\oplus}$, then for any $w \in W$ and for any formula $A$, $A \in w$ iff $w \models A$.*

**Proof.** Given the construction of the canonical model, this proof is easy and can be given by induction on the length of an expression $A$. We consider only some relevant cases.

Assume $A$ has the form $a_1 \otimes \cdots \otimes a_n$. If $A \in w$, by definition of the canonical model, then there is a sequence $\langle \|a_1\|_V, \ldots, \|a_n\|_V \rangle \in \mathcal{C}_w$. Following from the semantic clauses given to evaluate $\otimes$-formulae, it holds that $w \models a_1 \otimes \cdots \otimes a_n$. For the opposite direction, assume that $w \models a_1 \otimes \cdots \otimes a_n$. By definition, there is $\mathcal{C}_w$ which contains an ordered $n$-tuple $\langle \|a_1\|_V, \ldots, \|a_n\|_V \rangle$ and by construction $a_1 \otimes \cdots \otimes a_n \in w$. Clearly the same argument holds in the case of operator $\oplus$.

If, on the other hand, $A$ has the form $\mathsf{O}b$ and $\mathsf{O}b \in w$, then $\|b\|_V \in \mathcal{N}_w$ by construction, and by definition $w \models \mathsf{O}b$. Conversely, if $w \models \mathsf{O}b$, then $\|b\|_V \in \mathcal{N}_w$ and, by construction of $\mathcal{N}$, $\mathsf{O}b \in w$. □

It is easy to verify that the canonical model exists, it is not empty, and it is a sequence semantics model. Consider any formula $A \notin \mathsf{S}$ such that $\mathsf{S} \supseteq \mathsf{E}^{\otimes}, \mathsf{S} \supseteq \mathsf{E}^{\oplus}$; $\{\neg A\}$ is consistent and it can be extended to a maximal set $w$ such that for some canonical model, $w \in W$. By Lemma 5.4, $w \not\models A$. That $\mathcal{C}_w$ is closed under zipping follows immediately from the Lindembaum construction.

**Corollary 5.5 (Completeness of $\mathsf{E}^{\otimes}$ and $\mathsf{E}^{\oplus}$)** *The systems $\mathsf{E}^{\otimes}$ and $\mathsf{E}^{\oplus}$ are sound and complete with respect to the class of sequence frames.*

**Definition 5.6** [Bi-sequence Canonical Models] A bi-sequence canonical model $\mathcal{M} = \langle W, \mathcal{C}^{\mathsf{O}}, \mathcal{C}^{\mathsf{P}}, \mathcal{N}^{\mathsf{O}}, \mathcal{N}^{\mathsf{P}}, V\rangle$ for a system $\mathsf{S}$ in $\mathcal{L}^{\otimes\oplus}$ (where $\mathsf{S} \supseteq \mathsf{E}^{\otimes\oplus}$) is defined as follows:

1. $W$ is the set of all the $\mathcal{L}^{\otimes\oplus}$-maximal consistent sets.
2. For any propositional letter $p \in Prop$, $\|p\|_V := |p|_{\mathsf{S}}$, where $|p|_{\mathsf{S}} := \{w \in W \mid p \in w\}$.
3. Let $\mathcal{C}^{\mathsf{O}} := \bigcup_{w\in W} \mathcal{C}^{\mathsf{O}}_w$, where for each $w \in W$, $\mathcal{C}^{\mathsf{O}}_w := \{\langle \|a_1\|_V, \dots, \|a_n\|_V\rangle \mid \bigotimes_{i=1}^{n} a_i \in w\}$, where each $a_i$ is a meta-variable for a Boolean formula.
4. Let $\mathcal{C}^{\mathsf{P}} := \bigcup_{w\in W} \mathcal{C}^{\mathsf{P}}_w$, where for each $w \in W$, $\mathcal{C}^{\mathsf{P}}_w := \{\langle \|a_1\|_V, \dots, \|a_n\|_V\rangle \mid \bigoplus_{i=1}^{n} a_i\}$, where each $a_i$ is a meta-variable for a Boolean formula.
5. Let $\mathcal{N}^{\mathsf{O}} := \bigcup_{w\in W} \mathcal{N}^{\mathsf{O}}_w$ where for each world $w$, $\mathcal{N}^{\mathsf{O}}_w := \{\|a_i\|_V \mid \mathsf{O}a_i \in w\}$.
6. Let $\mathcal{N}^{\mathsf{P}} := \bigcup_{w\in W} \mathcal{N}^{\mathsf{P}}_w$ where for each world $w$, $\mathcal{N}^{\mathsf{P}}_w := \{\|a_i\|_V \mid \mathsf{P}a_i \in w\}$.

**Lemma 5.7 (Truth Lemma for Canonical Bi-sequence Models)** *If $\mathcal{M} = \langle W, \mathcal{C}^{\mathsf{O}}, \mathcal{C}^{\mathsf{P}}, \mathcal{N}^{\mathsf{O}}, \mathcal{N}^{\mathsf{P}}, V\rangle$ is canonical for $\mathsf{S}$, where $\mathsf{S} \supseteq \mathsf{E}^{\otimes\oplus}$, then for any $w \in W$ and for any formula $A$, $A \in w$ iff $w \models A$.*

Since the modal operators do not interact with each other, we can state:

**Corollary 5.8 (Completeness of $\mathsf{E}^{\otimes\oplus}$)** *The system $\mathsf{E}^{\otimes\oplus}$ is sound and complete with respect to the class of bi-sequence frames.*

### 5.2 Deontic Systems

**Theorem 5.9 (Completeness of $\mathsf{D}^{\otimes}$)** *The frame of a canonical model for $\mathsf{D}^{\otimes}$, as defined in Definition 5.3, has the following properties. For any$w \in W$,*

*1. $X \in \mathcal{N}_w$ if and only if $-X \notin \mathcal{N}_w$. (see* $\mathsf{OP}$-**duality**, $\mathsf{O}$-$\mathsf{P}$ *and* **D**-$\mathsf{O}$)

*2. $\emptyset \notin \mathcal{N}_w$ (see* $\overline{\mathbf{P}}$-$\mathsf{O}$)

*3. $\langle X_1, \dots, X_n\rangle \in \mathcal{C}_w$ for $n \geq 2$ then $\langle X_1, \dots, X_{n-1}\rangle \in \mathcal{C}_w$ (see* $\otimes$-**shortening**)

**Proof.**

1. $X \in \mathcal{N}_w$ iff $X = \|a\|_V$ for some $\mathsf{O}a \in w$, i.e., iff $\neg\mathsf{O}\neg a \in w$, $\mathsf{O}\neg a \notin w$, $-\|a\|_V \notin \mathcal{N}_w$.
2. Assume by reductio that $\emptyset \in \mathcal{N}_w$. Then $w \models \mathsf{O}\bot$, $\mathsf{O}\bot \in w$, reaching a contradiction.
3. Assume $\langle \|a\|_1, \dots, \|a_n\|\rangle \in \mathcal{C}_w$. By construction it means that $\bigotimes_{i=1}^{n} a_i \in w$ and by $\otimes$-**shortening**, $\bigotimes_{i=1}^{n-1} a_i \in w$, thus $\langle \|a\|_1, \dots, \|a_{n-1}\|\rangle \in \mathcal{C}_w$. □

**Theorem 5.10 (Completeness of $\mathsf{D}^{\mathsf{O}\otimes}$)** *The frame of a canonical model for $\mathsf{D}^{\mathsf{O}\otimes}$ (Definition 5.3) has the properties expressed in Theorem 5.9 and the following: For any world $w$, if $\langle X_1, \dots, X_n\rangle \in \mathcal{C}_w$ then $X_1 \in \mathcal{N}_w$ (see* $\otimes$-$\mathsf{O}$)

**Proof.** If $\langle X_1, \dots, X_n\rangle \in \mathcal{C}_w$ , then there are $n$ formulae such that for $1 \leq i \leq n$, $X_i = \|a_i\|_V$ and $a_1 \otimes \dots \otimes a_n \in w$. By Axiom $\otimes$-$\mathsf{O}$, $\mathsf{O}a_1 \in w$ and hence $\|a_1\|_V \in \mathcal{N}_w$. □

**Theorem 5.11 (Completeness of $\mathsf{D}^{\otimes\oplus}$)** *The frame of a canonical model for $\mathsf{D}^{\otimes\oplus}$, as defined in Definition 5.6, has the following properties. For any $w \in W$,*
1. $\mathcal{N}_w^{\mathsf{P}} \supseteq \mathcal{N}_w^{\mathsf{O}}$ *(see* $\mathsf{O}$-$\mathsf{P}$*)*
2. $X \in \mathcal{N}_w^{\mathsf{O}}$ *implies* $-X \notin \mathcal{N}_w^{\mathsf{O}}$ *(see* **D**-$\mathsf{O}$*)*
3. $\emptyset \notin \mathcal{N}_w^{\mathsf{O}}$ *(see* $\overline{\mathbf{P}}$-$\mathsf{O}$*)*
4. $X \in \mathcal{N}_w^{\mathsf{O}}$ *implies* $-X \notin \mathcal{N}_w^{\mathsf{P}}$ *(see* $\mathsf{O}\neg\mathsf{P}$*)*
5. $\langle X_1, \ldots, X_n\rangle \in \mathcal{C}_w^{\mathsf{O}}$ *for* $n \geq 2$ *then* $\langle X_1, \ldots, X_{n-1}\rangle \in \mathcal{C}_w^{\mathsf{O}}$ *(see* $\otimes$-**shortening***)*
6. $\langle X_1, \ldots, X_n\rangle \in \mathcal{C}_w^{\mathsf{P}}$ *for* $n \geq 2$ *then* $\langle X_1, \ldots, X_{n-1}\rangle \in \mathcal{C}_w^{\mathsf{P}}$ *(see* $\oplus$-**shortening***)*

**Proof.** Recall that $\mathsf{D}^{\otimes\oplus} = \mathsf{E}^{\otimes\oplus} + \mathsf{O}$-$\mathsf{P} + \overline{\mathbf{P}}$-$\mathsf{O} + \mathbf{D}$-$\mathsf{O} + \mathsf{O}\neg\mathsf{P} + \otimes$-**shortening** $+$ $\oplus$-**shortening**; remember that the operator $\mathsf{P}$ is not defined as a dual of $\mathsf{O}$.
1. Assume $\|a\|_V \in \mathcal{N}_w^{\mathsf{O}}$, then $\mathsf{O}a \in w$ and, by $\mathsf{O}$-$\mathsf{P}$, $\mathsf{P}a \in w$. Hence $\|a\|_V \in \mathcal{N}_w^{\mathsf{P}}$.
2. Assume $X \in \mathcal{N}_w^{\mathsf{O}}$ for some $w \in W$, then, by construction, there is some formula $\mathsf{O}a \in w$ and $X = \|a\|_V$. By **D**-$\mathsf{O}$ and **MP**, $\neg\mathsf{O}\neg a \in w$, i.e., $\mathsf{O}\neg a \notin w$, $\not\models_w^V \mathsf{O}\neg a$, $\|\neg a\|_V \notin \mathcal{N}_w^{\mathsf{O}}$, hence $-\|a\|_V \notin \mathcal{N}_w^{\mathsf{O}}$.
3. See the proof of Theorem 5.9.
4. Assume $X \in \mathcal{N}_w^{\mathsf{O}}$; by Definition 5.6 $X = \|a\|_V$ for some $a$ such that $\mathsf{O}a \in w$. Then, by $\mathsf{O}\neg\mathsf{P}$, $\neg\mathsf{P}\neg a \in w$, $\mathsf{P}\neg a \notin w$, hence $\|a\|_V \notin \mathcal{N}_w^{\mathsf{P}}$.
5. See the proof of Theorem 5.9.
6. See the proof of Theorem 5.9. □

### 5.3 Extended Deontic Systems

In what follows we shall prove completeness results for various systems by adding 6 detachment schemata that combine the modal operators introduced.

**Theorem 5.12 (Completeness of $\mathsf{D}^{\mathsf{OP}\otimes\oplus}$)** *The canonical frame* (*see Definition 5.6*) *for the logic $\mathsf{D}^{\mathsf{OP}\otimes\oplus}$ has the properties stated in Theorem 5.11 plus: For any world $w$ if $\langle X_1, \ldots, X_n\rangle \in \mathcal{C}_w^{\mathsf{O}}$ then $X_1 \in \mathcal{N}_w^{\mathsf{O}}$ (see $\otimes$-$\mathsf{O}$).*

**Proof.** See the proof of Theorem 5.10. □

**Theorem 5.13** *Let $\mathsf{S}$ be a system such that $\mathsf{S} \supseteq \mathsf{D}^{\mathsf{OP}\otimes\oplus}$. If $\mathsf{S}$ contains any of the axioms listed below, the canonical frame enjoys the corresponding property: For any world $w$*
1. $\mathsf{O}$-**detachment***:*
   *If $\langle X_1, \ldots, X_n\rangle \in \mathcal{C}_w^{\mathsf{O}}$ and $w \notin X_i$ for $1 \leq i \leq k$ and $k < n$, then $X_{k+1} \in \mathcal{N}_w^{\mathsf{O}}$.*
2. $\otimes$-**detachment***:*
   *If $\langle X_1, \ldots, X_n\rangle \in \mathcal{C}_w^{\mathsf{O}}$ and $w \notin X_1$, then $\langle X_2, \ldots, X_n\rangle \in \mathcal{C}_w^{\mathsf{O}}$.*
3. $\mathsf{O}$-**violation-detachment***:*
   *If $\langle X_1, \ldots, X_n\rangle \in \mathcal{C}_w^{\mathsf{O}}$ and, for $1 \leq i \leq k$ and $k < n$, $w \notin X_i$ and $X_i \in \mathcal{N}_w^{\mathsf{O}}$, then $X_{k+1} \in \mathcal{N}_w^{\mathsf{O}}$.*
4. $\otimes$-**violation-detachment***:*
   *If $\langle X_1, \ldots, X_n\rangle \in \mathcal{C}_w^{\mathsf{O}}$ and $X_1 \in \mathcal{N}_w^{\mathsf{O}}$ and $w \notin X_1$, then $\langle X_2, \ldots, X_n\rangle \in \mathcal{C}_w^{\mathsf{O}}$.*
5. $\oplus$-$\mathsf{P}$*:*
   *If $\langle X_1, \ldots, X_n\rangle \in \mathcal{C}_w^{\mathsf{P}}$ then $X_1 \in \mathcal{N}_w^{\mathsf{P}}$.*
6. $\mathsf{P}$-**detachment***:*

*If* $\langle X_1, \ldots, X_n \rangle \in \mathcal{C}_w^{\mathsf{P}}$ *and* $X_i \notin \mathcal{N}_w^{\mathsf{P}}$ *for* $1 \leq i \leq k < n$*, then* $X_{k+1} \in \mathcal{N}_w^{\mathsf{P}}$.

7. $\oplus$-**detachment***:*
   *If* $\langle X_1, \ldots, X_n \rangle \in \mathcal{C}_w^{\mathsf{P}}$ *and* $X_i \notin \mathcal{N}_w^{\mathsf{P}}$ *for* $1 \leq i \leq k$ *and* $k < n$*, then* $\langle X_{k+1}, \ldots, X_n \rangle \in \mathcal{C}_w^{\mathsf{P}}$.

**Proof.** Again, the proof is very straightforward and it follows closely the syntactical structure of the schemata. Notice that the fact $\langle X_1, \ldots, X_n \rangle \in \mathcal{C}_w^{\mathsf{O}}$ always implies that for $1 \leq i \leq n$ formulae $X_i = \|a_i\|_V$.

1. If $\langle X_1, \ldots, X_n \rangle \in \mathcal{C}_w^{\mathsf{O}}$ and $w \notin X_i$ for $1 \leq i \leq k$ with $k < n$, then for $1 \leq i \leq n$ formulae it holds that $X_i = \|a_i\|_V$, $a_1 \otimes \cdots \otimes a_n \in w$, $a_i \notin w$ for $1 \leq i \leq k$, hence $\bigwedge_{i=1}^{k} \neg a_i \in w$. Thus, by $\mathsf{O}$-**detachment**, $\mathsf{O}a_{k+1} \in w$ and $\|a_{k+1}\|_V \in \mathcal{N}_w^{\mathsf{O}}$.
2. If $\langle \|a_1\|_V, \ldots, \|a_n\|_V \rangle \in \mathcal{C}_w^{\mathsf{O}}$ and $w \notin \|a_1\|_V$, then $a_1 \otimes \cdots \otimes a_n \in w$ and $\neg a_1 \in w$, thus, by $\otimes$-**detachment**, $a_2 \otimes \cdots \otimes a_n \in w$ and $\langle \|a_2\|_V, \ldots, \|a_n\|_V \rangle \in \mathcal{C}_w^{\mathsf{O}}$.
3. Assume $\langle \|a_1\|_V, \ldots, \|a_n\|_V \rangle \in \mathcal{C}_w^{\mathsf{O}}$ and, for $1 \leq i \leq k$ with $k < n$, $w \notin \|a_i\|_V$ and $\|a_i\|_V \in \mathcal{N}_w^{\mathsf{O}}$. Then $a_1 \otimes \cdots \otimes a_n \in w$, $\bigwedge_{i=1}^{k} \neg a_i \in w$, and $\bigwedge_{i=1}^{k} \mathsf{O}a_i \in w$. By classical propositional logic $\bigwedge_{i=1}^{k} (\mathsf{O}a_i \wedge \neg a_i) \in w$ and, by $\mathsf{O}$-**violation-detachment**, $\mathsf{O}a_{k+1} \in w$ and $\|a_{k+1}\|_V \in \mathcal{N}_w^{\mathsf{O}}$.
4. Assume $\langle \|a_1\|_V, \ldots, \|a_n\|_V \rangle \in \mathcal{C}_w^{\mathsf{O}}$, $w \notin \|a_1\|_V$, and $\|a_1\|_V \in \mathcal{N}_w^{\mathsf{O}}$. Then $a_1 \otimes \cdots \otimes a_n \in w$, $\neg a_1 \in w$ and $\mathsf{O}a_1 \in w$ and, by $\otimes$-**violation-detachment**, $a_2 \otimes \cdots \otimes a_n \in w$ and $\langle \|a_2\|_V, \ldots, \|a_n\|_V \rangle \in \mathcal{C}_w^{\mathsf{O}}$.
5. See Theorem 5.10.
6. Assume $\langle \|a_1\|_V, \ldots, \|a_n\|_V \rangle \in \mathcal{C}_w^{\mathsf{P}}$ and, for $1 \leq i \leq k$ with $k < n$, $\|a_i\|_V \notin \mathcal{N}_w^{\mathsf{P}}$. Then $a_1 \oplus \cdots \oplus a_n \in w$ and $\bigwedge_{i=1}^{k} \neg \mathsf{P}a_i \in w$ and, by $\mathsf{P}$-**detachment**, $\mathsf{P}a_{k+1} \in w$, implying that $\|a_{k+1}\|_V \in \mathcal{N}_w^{\mathsf{P}}$.
7. Assume $\langle \|a_1\|_V, \ldots, \|a_n\|_V \rangle \in \mathcal{C}_w^{\mathsf{P}}$ and, for $1 \leq i \leq k$ with $k < n$, $\|a_i\|_V \notin \mathcal{N}_w^{\mathsf{P}}$. Then $a_1 \oplus \cdots \oplus a_n \in w$ and $\bigwedge_{i=1}^{k} \neg \mathsf{P}a_i \in w$ and, by $\oplus$-**detachment**, $a_{k+1} \oplus \cdots \oplus a_n \in w$ and hence $\langle \|a_{k+1}\|_V, \ldots, \|a_n\|_V \rangle \in \mathcal{C}_w^{\mathsf{P}}$. □

## 6 Conclusions and Related Work

The deontic logic literature on CTD reasoning is vast. However, two fundamental mainstreams have emerged as particularly interesting.

A first line of inquiry is mainly semantic-based. Moving from well-known studies on dyadic obligations, CTD reasoning is interpreted in settings with ideality or preference orderings on possible worlds or states [15]. The value of this approach is that the semantic structures involved are rather flexible: different deontic logics can thus be obtained. This semantic approach has been fruitfully renewed in the '90s, for instance by [19, 21], and most recently by works such as [14, 2], which have confirmed the vitality of this line of inquiry. However, most of these approaches are based on 'standard' possible world semantics with the risk of being affected by the paradox advanced in [7].

While the original systems for $\otimes$ were mainly motivated by modelling CTD reasoning [12, 4], in this paper we have broadened our analysis by extending chains to permissions and by generically dealing with compensations and violations. Indeed, we accept different types of $\mathsf{O}$-detachment, either allowing for the derivation of all obligations from any $\otimes$-chain, or only the subsequent ones in the chains with respect to the ones that have been violated. Our aim was to provide the semantics analysis for several axioms (principles) for the novel operators $\otimes$ and $\oplus$ and how they can be used to generated obligations and permissions. In this paper, we did not study what combinations of axioms are suitable to model different interpretations for different intuitions for the various deontic notions. This study is left to future investigations.

The second mainstream is mostly proof-theoretic. Examples, among others, are various systems springing from Input/Output Logic [18, 17] and the $\otimes$-logic originally proposed in [12]. The logic for $\otimes$ proved to be flexible for several applied domains, such as in business process modelling [13], normative multi-agent systems [6, 9], temporal deontic reasoning [11], and reasoning about different types of defeasible permission [8].

This paper completes the effort in [4] and offers a systematic semantic study of the $\otimes$ and $\oplus$ operators originally introduced in [12] and [8]. We showed that suitable axiomatisations can be characterised in a class of structures extending neighbourhood frames with sequences of sets of worlds. In this perspective, our contribution may offer useful insights for establishing connections between the proof-theoretic and model theoretic approaches to CTD reasoning. Also, we have shown that the semantic structures can easily keep separate structures interpreting norms from those interpreting obligations and permissions, thus mirroring the difference between $\otimes$ and $\oplus$ operators from $\mathsf{O}$ and $\mathsf{P}$.

A number of open research issues are left for future work. Among others, we plan to explore decidability questions using, for example, the filtration methods. The fact that neighbourhoods contain sequences of sets of worlds instead of sets is not expected to make the task significantly harder than the one in standard neighbourhood semantics for modal logics.

Second, we intend to study richer deontic logic. For example, we could extend rule RM for $\mathsf{O}$ (i.e., $a \to b/\mathsf{O}a \to \mathsf{O}b$), this would allow us to determine that the combination of $a \otimes c$, $b \otimes d$, where $a \to \neg b$, results in a contradiction. In this case, the semantic condition to add is that $\mathcal{N}$ is supplemented. Similarly, we may study what are the $\odot$ counterpart of axioms like $\mathbf{M}$, $\mathbf{C}$ an so on. [5] shows how to provide a generalisation of rule RM to the case of $\otimes$.

Third, [9] investigates how to characterise different degrees and types of goal-like mental attitudes of agents (including obligation) with chain operators. We plan to explore the use of sequence semantics to provide axioms (and corresponding semantic conditions) that correspond to the mechanisms governing the goal-like attitudes and their interactions.

Finally, we expect to enrich the language and to further explore the meaning of the nesting of $\otimes$- and $\oplus$-expressions, thus having formulae like $a \otimes \neg(b \otimes c) \otimes d$. As we have said, the meaning of those formulae is not clear. However, a semantic analysis of them in the sequence semantics can clarify the issue. Indeed, in the current language we can evaluate in any world $w$ formulae like $\neg(a \otimes b)$, which semantically means that there is no sequence $\langle \|a\|_V, \|b\|_V \rangle \in \mathcal{C}^{\mathsf{O}}_w$. Conceptually, this means that there is no norm stating that $a$ is obligatory and that the violation of this primary obligation generates an obligation $b$. Accordingly, the truth at $w$ of $a \otimes \neg(b \otimes c) \otimes d$ means that there exists a norm stating that $a$ is obligatory, but either $b$ does not compensate $a$ or, otherwise, $c$ does not compensate $b$, and $d$ compensates what compensates $a$, whatever it is.

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