



Concrete Operators

Research Article

Nicola Arcozzi* and Matteo Levi

On a class of shift-invariant subspaces of the Drury-Arveson space

<https://doi.org/10.1515/conop-2018-0001>

Received March 11, 2017; accepted February 9, 2018.

Abstract: In the Drury-Arveson space, we consider the subspace of functions whose Taylor coefficients are supported in a set $Y \subset \mathbb{N}^d$ with the property that $\mathbb{N} \setminus X + e_j \subset \mathbb{N} \setminus X$ for all $j = 1, \dots, d$. This is an easy example of shift-invariant subspace, which can be considered as a RKHS in its own right, with a kernel that can be explicitly calculated for specific choices of X . Every such a space can be seen as an intersection of kernels of Hankel operators with explicit symbols. Finally, this is the right space on which Drury's inequality can be optimally adapted to a sub-family of the commuting and contractive operators originally considered by Drury.

Keywords: Drury-Arveson space, Von Neumann's inequality, Hankel operators, Invariant subspaces, Reproducing kernel

MSC: 46E22, 47A15, 47A13, 47A20, 47A60, 47A63

1 Introduction

We begin by fixing some notation and delimiting the framework we work in. Let H be an abstract Hilbert space and for $d \geq 2$ consider a d -tuple of operators $A = (A_1, \dots, A_d) : H \rightarrow H^d$. It is not difficult to see that the formal adjoint operator $A^* : H^d \rightarrow H$ acts as follows

$$A^*k = \sum_j A_j^* k_j, \quad \text{for } k = (k_1, \dots, k_d) \in H^d.$$

Given a polynomial Q in d variables, say $Q(z) = \sum_k c_k z^k$, where $z = (z_1, \dots, z_d)$, $k \in \mathbb{N}^d$ and the sum is finite, we write $Q(A)$ for the operator from H to itself given by

$$Q(A) = \sum_k c_k A^k = \sum_k c_k A_1^{k_1} \dots A_d^{k_d}.$$

Following Drury, we will relate A to an operator acting on a Hilbert space of holomorphic functions of several variables on the unit ball. We write \mathbb{B}^d for the open unit ball $\{z = (z_1, \dots, z_d) \in \mathbb{C}^d : |z| < 1\}$, where $|z|^2 := \sum_{j=1}^d |z_j|^2$. Assuming that multiplication by z_j defines a bounded linear operator (and it does on the spaces we are dealing with), on such a space we can consider a very natural d -tuple of operators, namely the d -shift

$$M_z = (M_1, \dots, M_d) : H \rightarrow H^d,$$

where $M_j : f(z) \mapsto z_j f(z)$.

*Corresponding Author: Nicola Arcozzi: University of Bologna, E-mail: nicola.arcozzi@unibo.it

Matteo Levi: University of Bologna, E-mail: matteo.levi2@unibo.it

Definition 1.1. The Drury-Arveson space is the space H_d of functions $f(z) = \sum_{n \in \mathbb{N}^d} a(n)z^n$ holomorphic on the unit ball $\mathbb{B}^d \subset \mathbb{C}^d$, such that

$$\|f\|_{H_d}^2 := \sum_{n \in \mathbb{N}^d} |a(n)|^2 \beta(n)^{-1} < \infty,$$

where the weight function $\beta : \mathbb{N}^d \rightarrow \mathbb{N}$ is given by $\beta(n) = |n|/n!$.

This space has a reproducing kernel. For $f \in H_d$ and $z \in \mathbb{B}^d$, we have

$$f(z) = \sum_n a_n z^n = \sum_n a_n \frac{z^n}{\beta(n)} \beta(n) = \langle f, k_z \rangle_{H_d},$$

with $k_z(w) = \sum_n \beta(n) \bar{z}^n w^n$ for $w \in \mathbb{D}$.

The series can be explicitly calculated and we get

$$k_z(w) = \sum_{n \in \mathbb{N}} \beta(n) \bar{z}^n w^n = \sum_{k \geq 0} \sum_{|n|=k} \binom{k}{n} \bar{z}^n w^n = \sum_{k \geq 0} \left(\sum_{j=1}^d \bar{z}_j w_j \right)^k = \sum_{k \geq 0} (\bar{z} \cdot w)^k = \frac{1}{1 - \bar{z} \cdot w}.$$

This function space was first introduced by Drury in [3], then further developed in [1]. See also [7]. It naturally arises as the right space to consider when trying to generalize to tuples of commuting operators a notable result by Von Neumann, saying that for any linear contraction A on a Hilbert space and any complex polynomial Q , it holds

$$\|Q(A)\| \leq \|Q\|_{\mathcal{M}(H^2)},$$

where $\mathcal{M}(H^2) = H^\infty$ denotes the multiplier space of the Hardy space of the unit disc H^2 .

In fact, Drury shows that for a d -tuples of operators $A = (A_1, \dots, A_d) : H \rightarrow H^d$, $d \geq 2$, such that $[A_i, A_j] = 0$ and $\|A\| \leq 1$, it holds

$$\|Q(A)\| \leq \|Q\|_{\mathcal{M}(H_d)}.$$

The map T given by

$$(Tg)(z) := \sum_{n \in \mathbb{N}^d} g(n) \beta(n) z^n,$$

defines an isometric isomorphism from $\ell^2(\mathbb{N}^d, \beta)$ to H_d . This correspondence in particular tells us that the shift operator on $\ell^2(\mathbb{N}^d, \beta)$, given by

$$S_j g(n) = \chi_{\mathbb{N}^d + e_j}(n) g(n - e_j) \beta(n - e_j) \beta(n)^{-1},$$

and the multiplication operator M_j on H_d are unitarily equivalent, i.e. it turns out that $M_j T = T S_j$ for all $j = 1, \dots, d$.

2 A class of shift invariant subspaces of H_d

We are interested in considering subspaces of H_d of functions having Taylor coefficients with a prescribed support. Given some subset X of \mathbb{N}^d , we write $\ell^2(X, \beta)$ for the closed subspace of $\ell^2(\mathbb{N}^d, \beta)$ of functions supported in X . We say that a set $X \subseteq \mathbb{N}^d$ is *monotone*, if its complement in \mathbb{N}^d is shift invariant, namely

$$\mathbb{N}^d \setminus X + e_j \subset \mathbb{N}^d \setminus X \quad \text{for all } j = 1, \dots, d, \quad (1)$$

where $\mathbb{N}^d \setminus X$ is the complement of X in \mathbb{N}^d . In all what follows we always consider X to be a monotone set.

Given $g \in \ell^2(\mathbb{N}^d \setminus X, \beta)$, for $n \in X$ we have $S_j g(n) = 0$ since $n - e_j \in X$ as well (or it has a negative component). Therefore $\ell^2(\mathbb{N}^d \setminus X, \beta)$ is a shift-invariant subspace of $\ell^2(\mathbb{N}^d, \beta)$. To any such a set X , we can associate the space $H_d(X)$ of functions of H_d whose Taylor coefficients vanish on $\mathbb{N}^d \setminus X$. Since $M_j T \ell^2(\mathbb{N}^d \setminus X, \beta) = T S_j \ell^2(\mathbb{N}^d \setminus X, \beta)$, it follows that $H_d(\mathbb{N}^d \setminus X)$ is a shift-invariant subspace of H_d .

We can construct compressions of tuples of operators to the subspaces associated to the monotone set X .

In particular, let $B_j = S_j^*$ denote the backwards shift operator on $\ell^2(\mathbb{N}^d, \beta)$, given by $B_j g(n) = g(n + e_j)$. We consider the d -tuple of operators

$$B^X = (B_1^X, \dots, B_d^X) : \ell^2(X, \beta) \rightarrow \ell^2(X, \beta)^d,$$

where for each $j = 1, \dots, d$,

$$B_j^X = P_X B_j|_{\ell^2(X, \beta)},$$

being P_X the orthogonal projection of $\ell^2(\mathbb{N}^d, \beta)$ onto $\ell^2(X, \beta)$. In other words, B_j^X is the compression of the standard j^{th} -backwards shift operator B_j to $\ell^2(X, \beta)$.

Observe that the adjoint of B^X is a *row contraction* from $\ell^2(X, \beta)^d$ to $\ell^2(X, \beta)$,

$$(B^X)^*(g_1, \dots, g_d) = \sum_j (B_j^X)^* g_j.$$

In the same way, we write M_z^X for the compressed d -tuple (M_1^X, \dots, M_d^X) , where

$$M_j^X = P_X M_j|_{H_d(X)},$$

P_X being in this context the orthogonal projection from H_d onto $H_d(X)$.

3 Hankel operators and shift invariant subspaces

Shift-invariant subspaces for the Drury-Arveson space are characterized in [5], where it is shown that they can be represented as intersections of countably many kernels of Hankel operators, to be defined shortly. See also the PhD thesis [8].

Consider a Hilbert space \mathcal{H} of holomorphic functions on the unit ball \mathbb{B}^d , such that functions holomorphic on $\overline{\mathbb{B}^d}$ are dense in it. The function $b \in \mathcal{H}$ is a symbol if there exists $C > 0$ such that

$$|\langle fg, b \rangle_{\mathcal{H}}| \leq C \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}} \quad \text{for all } f, g \in \text{Hol}(\overline{\mathbb{B}^d}).$$

Endowing the space $\overline{\mathcal{H}} := \{\bar{f} : f \in \mathcal{H}\}$ with the inner product $\langle \bar{f}, \bar{g} \rangle_{\overline{\mathcal{H}}} := \langle g, f \rangle_{\mathcal{H}}$, we say that $H_b : \mathcal{H} \rightarrow \overline{\mathcal{H}}$ is a Hankel operator with symbol $b \in \mathcal{H}$ if there exists $C > 0$ such that

$$\langle H_b f, \bar{g} \rangle_{\overline{\mathcal{H}}} = \langle fg, b \rangle_{\mathcal{H}} \quad \text{for } f, g \in \text{Hol}(\overline{\mathbb{B}^d}).$$

On H_d , consider the Hankel operator with symbol $b(z) = z^m$, for some $m \in \mathbb{N}^d$. We have $f \in \ker H_b$ iff $\langle fg, b \rangle = 0$ for all $g \in \text{Hol}(\overline{\mathbb{B}^d})$. Since,

$$\langle fg, b \rangle_{H_d} = \widehat{fg}(m) \beta(m) = \left(\sum_{n,k} \widehat{f}(k) \widehat{g}(n) z^{n+k} \right)^{\wedge}(m) \beta(m) = \beta(m) \sum_k \widehat{f}(k) \widehat{g}(m-k),$$

it follows that $f \in \ker H_b$ iff $\widehat{f}(k) = 0$ for $k \leq m$, i.e. $\widehat{f} \equiv 0$ on the rectangle $R_m = \{n \in \mathbb{N}^d : n_j \leq m_j \forall j\}$. Hence, $f \in H_d(\mathbb{N}^d \setminus X)$ with $X = R_m$. This is the easiest example of shift-invariant subspace of the Drury-Arveson space with explicit symbol.

Actually, each set X satisfying (1) can be associated to a collection of Hankel symbols. Observe that X is bounded if and only if for all j there exists $n \in \mathbb{N}^d \setminus X$ such that $n \in \mathbb{N}e_j$. In such a case, X is a finite union of rectangles, $X = \bigcup_{k=1, \dots, K} R_{m_k}$ and hence,

$$H_d(\mathbb{N}^d \setminus X) = \bigcap_{k=1, \dots, K} \ker H_{z^{m_k}}.$$

If X is unbounded, then for every j such that $\mathbb{N}^d \setminus X \cap \mathbb{N}e_j = \emptyset$, we have an increasing sequence of rectangles covering the strip unbounded in the j -th direction. Summing up, it follows that

$$H_d(\mathbb{N}^d \setminus X) = \bigcap_{k=1}^{\infty} \ker H_{z^{m_k}}.$$

4 Drury type inequality

In the introduction we have defined polynomials valued on operators, $Q(A)$. The concept of operators being variables of functions can be properly extended. Following Nagy and Foias [9], given a contraction A on a Hilbert space H one can define the holomorphic functional calculus

$$\varphi(A) := \sum_k c_k A^k,$$

whenever $\varphi \in \mathcal{A} := \{a(z) = \sum_k c_k z^k : a \in \text{Hol}(\mathbb{D}), a \text{ continuous on } \overline{\mathbb{D}}, (c_k) \in \ell^\infty\}$.

Now, for any $\varphi \in \text{Hol}(\mathbb{D})$, the function $\varphi_r(\cdot) := \varphi(r \cdot)$ is in the class \mathcal{A} for $r \in (0, 1)$. Moreover, if $\varphi \in H^\infty$, we have the uniform bound $|\varphi_r(z)| \leq \|\varphi\|_\infty$, for $z \in \mathbb{D}$, $0 < r < 1$. Hence, for every $\varphi \in H^\infty$ it can be defined the functional calculus

$$\varphi(A) = \lim_{r \rightarrow 1^-} \varphi_r(A),$$

whenever the above limit exists in the strong operator topology, which is always the case when A is a completely non-unitary contraction (see [9]).

In particular, for $\varphi \in \mathcal{M}(H_d) \subset H^\infty$ and $A = M_z$, we can define the operator of multiplication by φ via the functional calculus

$$M_\varphi = \varphi(M_z) = \lim_{r \rightarrow 1^-} \varphi_r(M_z). \quad (2)$$

This defines a bounded operator from H_d to itself, and its adjoint is clearly given by $(M_\varphi)^* = \lim_{r \rightarrow 1^-} (\varphi_r(M_z))^*$.

We have the following version of Drury's inequality.

Theorem 4.1. *Let H be an abstract Hilbert space and $A = (A_1, \dots, A_d) : H \rightarrow H^d$, $d \geq 2$ a d -tuple of operators such that*

- (i) $A_i A_j = A_j A_i$ for $i, j = 1, \dots, d$.
- (ii) $\|Ah\|_{H^d} \leq \|h\|_H$ for all $h \in H$.

Let X be the complement in \mathbb{N}^d of the set $N := \{n \in \mathbb{N}^d : A^n = 0\}$. Then for every complex polynomial Q of d variables, we have

$$\|Q(A)\| \leq \|Q(B^X)\| \leq \inf\{\|\varphi\|_{\mathcal{M}(H_d)} : \varphi \in \mathcal{M}(H_d), \varphi(M_z^X) = Q(M_z^X)\}. \quad (3)$$

Proof. For $N = \emptyset$ we have $X = \mathbb{N}^d$ and this is just Drury's theorem, while for $N = \mathbb{N}^d \setminus \{0\}$, A reduces to a d -tuple of zeros (we set 0^0 to be the identity). So, suppose that $\{0\} \subsetneq X \subsetneq \mathbb{N}^d$.

It is enough to show that the theorem is true when (ii) is replaced by the stronger condition

$$(ii)^* \quad \|Ah\|_{H^d} \leq r \|h\|_H \quad \text{for all } h \in H,$$

where $r \in (0, 1)$.

We write $\tilde{H}(X)$ for the space $\ell^2(X, \tilde{H}, \beta)$, where \tilde{H} has the same underlying space as H but a different norm, $\|h\|_{\tilde{H}} = \|Dh\|_H$, where D is the defect operator of A , $D = \sqrt{I - A^*A}$, (see [3] for the details). Drury constructs an injective isometry $\theta : H \rightarrow \tilde{H}(\mathbb{N}^d)$, $\theta h(n) := A^n h$, and shows that $\tilde{B}^m \theta = \theta A^m$ for all $m \in \mathbb{N}^d$ (here \tilde{B} is the d -tuple of backshifts on $\tilde{H}(\mathbb{N}^d)$).

We rephrase this in our setting. Let π_X be the orthogonal projection of $\tilde{H}(\mathbb{N}^d)$ onto $\tilde{H}(X)$, $\tilde{B}_j^X := \pi_X \tilde{B}_j|_{\tilde{H}(X)}$ and $\psi := \pi_X \circ \theta$.

Since θ is an isometry, it is easy to see that that

$$\psi \text{ is an isometry} \iff \theta h = 0 \text{ on } \mathbb{N}^d \setminus X \iff A^n = 0 \text{ for } n \in \mathbb{N}^d \setminus X. \quad (4)$$

We have

$$\psi A_j = \pi_X \tilde{B}_j \theta, \quad \text{and} \quad \tilde{B}_j^X \psi = \pi_X \tilde{B}_j|_{\tilde{H}(X)} \pi_X \theta = \pi_X \tilde{B}_j \pi_X \theta.$$

For $n \in X$ and $h \in H$,

$$(\tilde{B}_j - \tilde{B}_j \pi_X) \theta h(n) = \theta h(n + e_j) - \pi_X \theta h(n + e_j) = \begin{cases} 0 & n + e_j \in X \\ \theta h(n + e_j) & n + e_j \notin X \end{cases}$$

which equals zero by (4). It follows that

$$\psi A^m = (\tilde{B}^X)^m \psi \quad \text{for all } m \in \mathbb{N}.$$

At this point, it is standard (for example follow [3]) that for every complex polynomial Q we have,

$$\|Q(A)\| \leq \|Q(B^X)\| = \|Q(M_z^X)\|. \tag{5}$$

The equality above follows from the intertwining relation $M_j^X T = T(B_j^X)^*$, where the operator T in our case is the isometric isomorphism from $\ell^2(X, \beta)$ to $H_d(X)$ given by $(Tg)(z) := \sum_{n \in X} g(n) \beta(n) z^n$. For $f(z) = \sum_n a_n z^n \in H_d(X)$, we have $M_j^X f(z) = \sum_{n \in X \cap X + e_j} a_{n - e_j} z^n$, and so

$$\|M_j^X f\|_{H_d}^2 = \sum_{n \in X \cap X - e_j} |a_n|^2 \beta(n + e_j)^{-1} \leq \sum_{n \in X \cap X - e_j} |a_n|^2 \beta(n)^{-1} \leq \|f\|_{H_d}^2.$$

Then, all polynomials are multipliers for H_d and

$$\|Q(M_z^X)\| = \|P_X Q(M_z)\| \leq \|Q(M_z)\| = \|Q\|_{\mathcal{M}(H_d)}. \tag{6}$$

Of course, there are in general many functions φ such that $P_X \varphi(M_z) = P_X Q(M_z)$. In particular, let φ be a multiplier of H_d such that $\widehat{\varphi(n)} = \widehat{Q(n)}$ for $n \in X$. Then, for any $g \in H_d$ we have,

$$\begin{aligned} \|P_X Q(M_z)g - P_X \varphi(M_z)g\|_{H_d} &\leq \|P_X(Q(M_z) - \varphi_r(M_z))g\|_{H_d} + \|P_X(\varphi(M_z) - \varphi_r(M_z))g\|_{H_d} \\ &\leq \left\| \sum_X \widehat{\varphi(n)}(1 - r^{|n|}) M_z^n g \right\|_{H_d} + \|\varphi(M_z)g - \varphi_r(M_z)g\|_{H_d} \\ &\leq \sum_X (1 - r^{|n|}) \widehat{\varphi(n)} \|M_z^n g\|_{H_d} + \|\varphi(M_z)g - \varphi_r(M_z)g\|_{H_d}. \end{aligned}$$

The term on the right goes to zero as $r \rightarrow 1^-$, so it follows $P_X Q(M_z) = P_X \varphi(M_z)$. Then, (6) can be generalized as follows

$$\|Q(M_z^X)\| = \|P_X \varphi(M_z)\| \leq \|\varphi(M_z)\| = \|\varphi\|_{\mathcal{M}(H_d)},$$

for any $\varphi \in \mathcal{M}(H_d)$ such that $\widehat{\varphi(n)} = \widehat{Q(n)}$. We have then proved that,

$$\|Q(A)\| \leq \|Q(B^X)\| \leq \inf\{\|\varphi\|_{\mathcal{M}(H_d)} : \varphi \in \mathcal{M}(H_d), \varphi(M_z^X) = Q(M_z^X)\}. \quad \square$$

Remark 4.2. Observe that the first inequality in the theorem is optimal if the backshift d -tuple B^X satisfies (i) and (ii) and if $\{n \in \mathbb{N}^d : (B^X)^n = 0\}$ equals N . It is clear that condition (ii) holds for B^X , for every choice of X . Also, $n \in N$ if and only if $n + m \in N$ for all $m \in \mathbb{N}^d$ and since $N = \mathbb{N}^d \setminus X$ this is equivalent as asking $f(n + m) = 0$ for all $m \in \mathbb{N}^d$, $f \in \ell^2(X, \beta)$. But $f(n + m) = (B^X)^n f(m)$ and so $\{n \in \mathbb{N}^d : (B^X)^n = 0\} = N$.

On the other hand, the commuting property (i) is not fulfilled on most sets X . Of course, if X is chosen such that $\ell^2(X, \beta)$ is backshift-invariant, then $B^X = B|_{\ell^2(X, \beta)}$ and (i) and (ii) hold, see [3]. More in general, doing standard calculations it is not hard to see that B^X satisfies (i) if and only if

$$n, n + e_i + e_j, n + e_i \in X \implies n + e_j \in X, \quad \text{for } i, j = 1, \dots, d. \tag{7}$$

This is a shape-condition on the set X , saying that it cannot have any subset with one of the following configurations

Fig. 1. Fat dots are elements of not permitted subsets of X .



It is clear that $X = \mathbb{N}^d$ satisfies (7), since $n + e_j \in \mathbb{N}^d \setminus X$ for some j would imply $n + e_j + e_i \in \mathbb{N}^d \setminus X$ for all $i = 1, \dots, d$. It follows that the inequality in the theorem is optimal.

5 Further considerations

We want to look closer at the second inequality in (3). In particular, we are interested in understanding if it is an equality indeed. The reason to be optimistic in this sense comes from a theorem proved by Sarason in [6] (see also [4, Theorem 3.1]) in the one-dimensional case, i.e. for the Hardy space. Let K be a closed backshift-invariant subspace of the Hardy space H^2 , and write S_K for the compression of the shift operator to this subspace. Sarason proved the following.

Theorem 5.1. *Let T be an operator commuting with S_K . Then there exists a function $\varphi \in H^\infty$ such that $T = \varphi(S_K)$ and $\|T\| = \|\varphi\|_{H^\infty}$.*

Now, on $H_1 = H^2$ the operator $T = Q(M_z^X)$ clearly commutes with M_z^X , so there exists a function $\varphi \in \mathcal{M}(H^2) = H^\infty$, possibly different from the polynomial Q , such that $T = \varphi(M_z^X)$ and $\|T\| = \|\varphi\|_{\mathcal{M}(H_1)}$. Then, (3) would become

$$\|Q(A)\| \leq \|Q(B^X)\| = \|Q(M^X)\| = \|\varphi\|_{\mathcal{M}(H_1)}.$$

So we have equality in the case $d = 1$. For higher dimensions, we have the following generalized *commutant lifting theorem* (see [2, Theorem 5.1]).

Theorem 5.2. *Let $k(z, w)$ be a nondegenerate positive kernel on a domain Ω such that $1/k$ has 1 positive square. Let $H(k)$ be the associated RKHS. Suppose that $W \subset H(k)$ is a \star -invariant subspace and that T is a bounded linear contraction from W to itself such that*

$$T^* M_\varphi^*|_W = M_\varphi^* T^*, \quad (8)$$

for all $\varphi \in \mathcal{M}(H(k))$. Then, there exists a multiplier $\psi \in \mathcal{M}(H(k))$ such that $\|(M_\psi)\| \leq 1$ and $(M_\psi)^*|_W = T^*$.

Asking that $1/k$ has 1 positive square means that the self adjoint matrix $\{1/k(z_i, z_j)\}_{i,j=1}^N$ has exactly one positive eigenvalue, counted with multiplicity, for every finite set of disjoint points $\{z_1, \dots, z_N\} \subset \mathbb{B}^d$. It is well known that the Drury-Arveson kernel has this property.

So, in order to apply the theorem, take H_d as the RKHS and let $W = H^d(X)$. We have to show that $H_d(X)$ is \star -invariant, i.e. that for every $\varphi \in \mathcal{M}(H_d)$ it holds $M_\varphi^* H^d(X) \subset H^d(X)$. Suppose that the multiplier function φ has the power series expansion $\varphi(z) = \sum_n a_n z^n$. Then $\varphi_r(z) = \sum_n a_n(r) z^n$, where $a_n(r) = a_n r^{|n|}$. Using the fact that $(M_j^{n_j})^* = (M_j^*)^{n_j}$ and the uniform absolute convergence of the series, we get

$$(\varphi_r(M_z))^* = \left(\sum_n a_n(r) M_z^n \right)^* = \sum_n \overline{a_n(r)} (M_z^n)^* = \sum_n \overline{a_n(r)} ((M_1^*)^{n_1}, \dots, (M_d^*)^{n_d}). \quad (9)$$

To prove the \star -invariance, thanks to (2) it is enough to show that $(\varphi_r(M_z))^*$ maps $H_d(X)$ in itself for all r , but this is immediate by (9), since M_j^* does.

The operator $T = Q(M_z^X) = P_X M_Q$ maps continuously $H_d(X)$ to itself. Moreover we have,

$$T^* M_\varphi^*|_{H_d(X)} = M_Q^* P_X M_\varphi^*|_{H_d(X)} = M_Q^* M_\varphi^*|_{H_d(X)}.$$

It follows that for $f \in H_d(X)$ it holds

$$T^* M_\varphi^* f = M_Q^* M_\varphi^* f = (M_\varphi M_Q)^* f = (M_Q M_\varphi)^* f = M_\varphi^* M_Q^* f = M_\varphi^* M_Q^* P_X f = M_\varphi^* T^* f.$$

Therefore, we have $T^* M_\varphi^*|_{H_d(X)} = M_\varphi^* T^*$.

Hence Theorem 5.2 applies, and there exists a multiplier $\psi \in \mathcal{M}(H_d)$ such that $(M_\psi)^*|_W = (Q(M_z^X))^*$. In particular, it follows

$$\|Q(M_z^X)\| = \|M_\psi|_{H^d(X)}\|. \quad (10)$$

Question. Does this help in proving that equality holds in place of the second inequality in (3) for any dimension $d > 1$?

6 A closed formula for the reproducing kernel on slabs

Let X be some subset of \mathbb{N}^d satisfying (1). Clearly, the space $H_d(X)$ has a reproducing kernel $k^X(w, z)$ which is given by the orthogonal projection of the Drury-Arveson kernel onto $H_d(X)$, in the sense that

$$k^X(w, z) = P_X k(w, z) = P_X k_z(w) = \sum_{n \in X} \beta(n) \bar{z}^n w^n. \quad (11)$$

For some special choices of the set X we are able to get a closed formula for the reproducing kernel in (11). In particular, this can be done when X is what we call a *slab*, $\mathcal{S}_1 = \{n \in \mathbb{N}^d : n_1 = 0, \dots, N_1\}$.

Proposition 6.1. For $X = \mathcal{S}_1$ it holds

$$k^{\mathcal{S}_1}(w, z) = \frac{1}{1 - \bar{z} \cdot w} \left(1 - \frac{\bar{z}_1 w_1}{1 - \bar{z} \cdot w + \bar{z}_1 w_1} \right)^{N_1}. \quad (12)$$

Proof. Set $t = \bar{z}w$. As a first step, suppose that $d = 2$. Using the fact that for $j, k \in \mathbb{N}$ it holds

$$\sum_{j=0}^{\infty} \binom{j+k}{j} x^j = \frac{1}{(1-x)^{k+1}},$$

we get

$$\begin{aligned} k^X(w, z) &= \sum_{n \in X} \beta(n) \bar{z}^n w^n = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{\infty} \binom{n_1+n_2}{n_2} t_1^{n_1} t_2^{n_2} = \sum_{n_1=0}^{N_1} t_1^{n_1} \frac{1}{(1-t_2)^{n_1+1}} \\ &= \frac{1}{1-t_2} \sum_{n_1=0}^{N_1} \left(\frac{t_1}{1-t_2} \right)^{n_1} = \frac{1}{1-t_2} \frac{1 - \left(\frac{t_1}{1-t_2} \right)^{N_1}}{1 - \frac{t_1}{1-t_2}} \\ &= \frac{1 - \left(\frac{t_1}{1-t_2} \right)^{N_1}}{1 - t_1 - t_2} = \frac{1}{1 - \bar{z} \cdot w} \left(1 - \frac{\bar{z}_1 w_1}{1 - \bar{z} \cdot w + \bar{z}_1 w_1} \right)^{N_1}. \end{aligned}$$

Now, suppose that (12) holds on \mathbb{N}^{d-1} . Again, suppose to re-order the basis e_1, \dots, e_d so that $j = 1$. On \mathbb{N}^d we have

$$\begin{aligned} k^X(w, z) &= \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{\infty} \dots \sum_{n_d=0}^{\infty} \binom{(|n| - n_d) + n_d}{n_d} \frac{(n_1 + \dots + n_{d-1})!}{n_1! \dots n_{d-1}!} t_1^{n_1} \dots t_d^{n_d} \\ &= \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{\infty} \dots \sum_{n_{d-1}=0}^{\infty} \frac{(n_1 + \dots + n_{d-1})!}{n_1! \dots n_{d-1}!} t_1^{n_1} \dots t_{d-1}^{n_{d-1}} \frac{1}{(1-t_d)^{n_1 + \dots + n_{d-1} + 1}} \\ &= \frac{1}{1-t_d} \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{\infty} \dots \sum_{n_{d-1}=0}^{\infty} \frac{(n_1 + \dots + n_{d-1})!}{n_1! \dots n_{d-1}!} \left(\frac{t_1}{(1-t_d)} \right)^{n_1} + \dots + \left(\frac{t_{d-1}}{(1-t_{d-1})} \right)^{n_{d-1}} \\ &= \frac{1}{1-t_d} \frac{1}{1 - \sum_{i=1}^{d-1} \frac{t_i}{(1-t_d)}} \left(1 - \frac{\frac{t_1}{1-t_d}}{1 - \sum_{i=2}^{d-1} \frac{t_i}{(1-t_d)}} \right)^{N_1} \\ &= \frac{1}{1 - \sum_{i=1}^d t_i} \left(1 - \frac{1}{1 - \sum_{i=2}^d t_i} \right)^{N_1} = \frac{1}{1 - \bar{z} \cdot w} \left(1 - \frac{\bar{z}_j w_j}{1 - \bar{z} \cdot w + \bar{z}_1 w_1} \right)^{N_1}. \quad \square \end{aligned}$$

Acknowledgement: We would like to thank Orr Shalit and Jaydeb Sarkar for some useful conversations.

This research was partially supported by the project PRIN 2015 "Varietà reali e complesse: geometria, topologia e analisi armonica".

References

- [1] William Arveson. Subalgebras of C^* -algebras. III. Multivariable operator theory. *Acta Math.*, 181(2):159–228, 1998.
- [2] Joseph A. Ball, Tavan T. Trent, and Victor Vinnikov. *Interpolation and Commutant Lifting for Multipliers on Reproducing Kernel Hilbert Spaces*, pages 89–138. Birkhäuser Basel, Basel, 2001.
- [3] S. W. Drury. A generalization of von Neumann’s inequality to the complex ball. *Proc. Amer. Math. Soc.*, 68(3):300–304, 1978.
- [4] Vladimir V Peller. An excursion into the theory Hankel operators. *Holomorphic spaces (Berkeley, CA, 1995)*, *Math. Sci. Res. Inst. Publ*, 33:65–120, 1998.
- [5] Stefan Richter and James Sunkes. Hankel operators, invariant subspaces, and cyclic vectors in the Drury-Arveson space. *Proc. Amer. Math. Soc.*, 144, 2016.
- [6] Donald Sarason. Generalized interpolation in H^∞ . *Transactions of the American Mathematical Society*, 127(2):179–203, 1967.
- [7] Orr Shalit. Operator theory and function theory in Drury-Arveson space and its quotients. *Operator Theory*, pages 1125–1180, 2015.
- [8] James Allen Sunkes III. Hankel operators on the Drury-Arveson space. *PhD diss., University of Tennessee*, 2016.
- [9] Béla Sz.-Nagy and Ciprian Foias. *Harmonic analysis of operators on Hilbert space*. Translated from the French and revised. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest, 1970.