Necessary Conditions for Output Regulation with Exosystem Modelled by Differential Inclusions ⋆

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Abstract The problem of output regulation has always been tackled in frameworks in which the references to be tracked and the disturbances to be rejected are generated by an autonomous differential equation, referred to as the exosystem. This assumption, that is routinely used in the design of asymptotic regulators, plays also a fundamental role in the formulation of the regulation problem and in the definition of the basic concepts such as the steady state and the zero dynamics of nonlinear systems. In this paper we show that the concepts of steady state, zero dynamics and the output regulation problem can be equivalently defined in a framework in which the exosystem is generated by a differential inclusion.

Keywords: Output Regulation, Nonlinear Systems, Differential Inclusion

1. INTRODUCTION

Output regulation refers to the class of control problems in which some outputs of the plant must be asymptotically driven to a given reference, in presence of exogenous disturbances and parameters uncertainties. Although under the assumption of full knowledge of the references and the exogenous disturbances the problem of output regulation reduces to a standard stabilisation problem, no existing designs are known to obtain asymptotic regulation in absence of any knowledge on the exogenous signals. The output regulation problem for general multivariable linear systems has been solved in the 70s in the seminal works of Francis and Wonham (1975); Davison (1976); Francis and Wonham (1976), under an assumption on the exogenous signal that lies in the middle between the case in which the exogenous signals are known and those in which no information is available. They assumed that all the exogenous signals are *unknown* trajectories of an autonomous system (referred to as the *exosystem*) of known structure. In that framework, the structure of the exosystem permitted to properly define the regulation problem by yielding the celebrated *internal model principle*, stating that if asymptotic regulation is sought then the closed-loop system must necessarily embed an *internal model* of the exosystem (see Francis and Wonham (1975, 1976)).

The exosystem has been a fundamental brick also in the formulation of the output regulation problem for nonlinear systems, at first in a local framework (see Isidori and Byrnes (1990); Huang and Rugh (1990); Huang and Lin (1994); Byrnes et al. (1997a)) and then (semi)global settings (see Byrnes and Isidori (2003, 2004); Marconi et al. (2007); Marconi and Praly (2008)). In that frameworks, the exosystem dynamics is defined by a nonlinear ordinary differential equation, whose outputs represent the disturbances and the references acting on the plant.

At the *analysis* level, the exosystem is needed to define fundamental notions such as the *steady state*, the *zero dynamics* of a nonlinear system. Those are indeed the main elements that are needed to formulate necessary conditions (such as the internal model principle) for the problem of output regulation (see Byrnes and Isidori (2003)). Under a *synthesis* point of view, the knowledge of the exosystem is usually exploited to identify an ideal steady state in which the regulated variables vanish, with the regulator that is designed to stabilise the closed-loop system to that steady-state (see for instance Khalil (1994); Byrnes et al. (1997b); Ding (2003); Isidori et al. (2012) for what concern designs with linear exosystem and Byrnes and Isidori (2004); Marconi et al. (2007); Marconi and Praly (2008); Bin and Marconi (2017a) for regulators dealing with nonlinear exosystems). In all these designs, the structure of the exosystem enters explicitly in the definition of the regulator, and generally only a perfect knowledge of the exosystem dynamics can guarantee asymptotic tracking.

Nevertheless, some recent papers (see Forte et al. (2016); Bin and Marconi (2017b)) in the field of adaptive regulation proposed a framework in which the ideal control law is estimated at runtime, by means of possibly *black-box* identification techniques. Although in these designs some a-priori knowledge of the exogenous signals is still needed at a conceptual level to set up a meaningful identification problem, the dependency between the exosystem knowledge and the regulator design results to be consistently weakened. This, in particular, raised the question whether this kind of regulators may be able to solve regulation problems in which the exogenous signals belong to more general classes of signals than the set of solutions of a differential equation. As a first step in this direction, in this

paper we investigate the possibility of defining the output regulation problem in a framework in which the exogenous signals are modelled as solutions to a nonlinear *differential inclusion* (see Aubin and Cellina (1984)) rather than an ordinary differential equation. Namely, we suppose that the references and the disturbances acting on the plant are solutions of a system of the form

$$\dot{w} \in S(w), \qquad w \in \mathbb{R}^s \qquad (1)$$

with $S : \mathbb{R}^s \rightrightarrows \mathbb{R}^s$ a set-valued map. The solutions to (1) are *absolute continuous* functions that need not to be differentiable. Hence, the system (1) can generate a consistently larger multitude of signals than an ordinary differential equation. For instance, it is worth noting that each solution to (1) might have a different (distributional) derivative and thus (1) can model also the cases of differential equations with *uncertain* time-varying parameters (in this case $S(\cdot)$ is a *parametrized map*) as well as exosystem with variable structure. As a first preliminary work, by following the line of Byrnes and Isidori (2003), in this paper we show that concepts like limit sets, steady state and zero dynamics can be equivalently defined also when the exosystem has the form (1). We also give necessary conditions for the solvability of the output regulation problem and we extend the notion of zero dynamics and efficient controllers as given in Byrnes and Isidori (2003).

Notation: We let $\mathbb{N}^* := \mathbb{N}/\{0\}$ and $\mathbb{R}_+ := [0, \infty)$. With $\dot{x} \in F(x)$ a differential inclusion defined on a vector space $\mathcal{X}$ we denote by $\mathcal{S}_x(X)$ the set of all maximal solutions starting from a set $X \subset \mathcal{X}$. If X is a singleton, i.e. $X = \{x\}$, we let $\mathcal{S}_x(x) = \mathcal{S}_x(\{x\})$. With $\dot{x} = f(x, u)$ a differential equation with input, where $x \in \mathcal{X}$ and $u \in \mathcal{U}$, being $\mathcal{X}$ and $\mathcal{U}$ vector spaces, we denote by $(t, x, u) \mapsto \phi_x(t, x, u)$ the value of the solution originating in $x \in \mathcal{X}$ at time $t = 0$ with input $u(\cdot)$. Moreover, for all fixed $u(\cdot)$ and all $X \subset \mathcal{X}$, we let $\mathcal{S}_x(X, u) = \{\phi_x(\cdot, x, u) : x \in X\}$ be the set of all the solutions starting in X driven by the input $u(\cdot)$. With $\mathcal{B}$ a set of functions from $\mathbb{R}_+$ into $\mathbb{R}^d$, $d \in \mathbb{N}^*$ and with $t \geq 0$, we denote by $\mathcal{B}|_t$ the set of all functions obtained by restricting an element of $\mathcal{B}$ to the interval $[0, t]$. We say that a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class-$\mathcal{K}$ if it is continuous, increasing and $\gamma(0) = 0$. If $\gamma(s) \to \infty$ as $s \to \infty$ we say that γ is of class-$\mathcal{K}_\infty$. We say that a continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class-$\mathcal{K}\mathcal{L}$ if for each $t \in \mathbb{R}_+$, $\beta(\cdot, t)$ is of class-$\mathcal{K}$ and for each $s \in \mathbb{R}^+$, $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$. We denote by $AC(\mathcal{X}, \mathcal{Y})$ the set of all the absolute continuous functions from $\mathcal{X}$ to $\mathcal{Y}$. We denote by $(t_n)_{n \in \mathbb{N}} \nearrow$ a sequence of $t_n \in \mathbb{R}_+$ that are strictly increasing and $\lim_{n \to \infty} t_n = \infty$. With $r > 0$, we denote by $B_n(0, r)$ the closed ball in $\mathbb{R}^n$ with radius r. Let $(w, \xi) \in \mathbb{R}^s \times \mathbb{R}^q$ we denote by $p_w : \mathbb{R}^s \times \mathbb{R}^q \to \mathbb{R}^s$ the projection of (w, x) on $\mathbb{R}^s$, namely $p_w(w, \xi) = w$. For a set $X \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we denote by $T_X(x)$ the tangent space to X at point x and by $d(x, X) := \inf_{s \in X} |x - s|$ the distance from x to X.

2. PRELIMINARIES

In this section we consider the following interconnection

$$\dot{w} \in S(w) \qquad (2)$$
$$\dot{\xi} = \psi(w, \xi) \qquad (3)$$

in which an autonomous differential inclusion with state $w \in \mathbb{R}^s$, $s \in \mathbb{N}^*$, drives a nonlinear system with state

$\xi \in \mathbb{R}^q$, $q \in \mathbb{N}^*$. We suppose that the initial conditions of (2), (3) range in a compact subset $W \times \Xi \subset \mathbb{R}^s \times \mathbb{R}^q$. We assume that $\psi : W \times \mathbb{R}^q \to \mathbb{R}^q$ is locally Lipschitz and $S : W \rightrightarrows \mathbb{R}^s$ is Lipschitz on W and has non-empty and compact values at each $w \in W$. In what follows we introduce the essential preliminary concepts that are instrumental for the forthcoming analysis.

2.1 Basics concepts

With $N, M > 0$, we define the set of *admissible solutions* to (2) as

$$\mathcal{L}^w(w_0) = \Big\{ \varphi_w \in \mathcal{S}_w(w_0) : ||\varphi_w||_\infty \leq M \text{ and }$$

$$\forall \varphi_\xi \in \mathcal{S}_\xi(\Xi, \varphi_w), ||\varphi_\xi||_\infty \leq N \Big\}$$

which is the set of all the bounded solutions to (2) which produce bounded solutions to (3). For ease of notation, with $Z \subset \mathbb{R}^s \times \mathbb{R}^q$, we define the set

$$\mathcal{A}(Z) = \Big\{ (\varphi_\xi, \varphi_w) \in AC(\mathbb{R}_+, \mathbb{R}^s \times \mathbb{R}^q) :$$

$$\varphi_w \in \mathcal{L}^w(w_0), \; \varphi_\xi \in \mathcal{S}_\xi(\xi_0, \varphi_w), \; (\xi_0, w_0) \in Z \Big\}.$$

With $\mathcal{B}$ a set of functions from $\mathbb{R}_+$ into $\mathbb{R}^d$, $d \in \mathbb{N}^*$, we define the *flow* of $\mathcal{B}$ as the set-valued map $\Phi_\mathcal{B} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ as

$$\Phi_\mathcal{B}(t, b_0) = \Big\{ b \in \mathbb{R}^d : \exists \varphi \in \mathcal{B}, \varphi(t) = b, \; \varphi(0) = b_0 \Big\}.$$

With $\varphi \in \mathcal{B}$ we define the *ω-limit set* of φ as the set

$$\omega(\varphi) = \Big\{ b \in \mathbb{R}^d : \exists (t_n)_{n \in \mathbb{N}} \nearrow, \varphi(t_n) \to_{n \to +\infty} b \Big\}$$

or equivalently $\omega(\varphi) = \cap_{t \geq 0} \overline{\Phi_\varphi(t, \varphi(0))}$. Furthermore, we define the *Ω-limit set* of $\mathcal{B}$ as

$$\Omega(\mathcal{B}) = \Big\{ b \in \mathbb{R}^d : \exists (t_n)_{n \in \mathbb{N}} \nearrow,$$

$$\exists (\varphi^n)_{n \in \mathbb{N}} \in \mathcal{B}^\mathbb{N}, \varphi^n(t_n) \to_{n \to +\infty} b \Big\}.$$

Let A be a set such that $A = \cup_{\varphi \in \mathcal{B}} \{\varphi(0)\}$. We shall say that the set A is *Poisson Stable* if $A = \omega(\mathcal{B}) = \cup_{\varphi \in \mathcal{B}} \omega(\varphi)$. With $A \subset \mathbb{R}^d$, we say A *uniformly attracts* $\mathcal{B}$ if

$$\forall \varepsilon > 0, \; \exists T > 0, \; \forall \varphi \in \mathcal{B}, \; \forall t > T, \; d(\varphi(t), A) \leq \varepsilon.$$

We say A is *invariant for* $\mathcal{B}$ if

$$\forall t \in \mathbb{R}_+, \; \forall \varphi \in \mathcal{B}, \; \varphi(t) \in A.$$

If the set $\mathcal{B}$ is clear from the context, we omit to mention it. When invariance or attractiveness refer to the solutions to a differential equation (or inclusion), we always refer to the set of complete solutions if not other set is mentioned.

2.2 Properties of Limit Sets

In this section we study the asymptotic behaviour of the *admissible* solutions to the interconnection (2), (3). Under mild existence and regularity assumptions, we show that the Ω-limit set of $\mathcal{A}(W \times \Xi)$ is a well-defined compact set that uniformly attracts $\mathcal{A}(W \times \Xi)$. This results are instrumental for the forthcoming analysis in the context of output regulation. From now on, we assume W to be *invariant* for (2) and we fix $N > 0$ in the definition of $\mathcal{L}^w(W)$. We denote with Φ the flow of the set $\mathcal{A}(W \times \Xi)$. Finally, with slight abuse of notation, we let $\Omega := \Omega(\mathcal{A}(W \times \Xi))$.

Standing Assumptions:

We make the following existence and admissibility assumptions:

A1) For all $w_0 \in W$, $\mathcal{L}^w(w_0)$ is non empty.
A2) $\mathcal{S}_w(W) = \mathcal{L}^w(W)$

A relevant case in which Assumptions A1-A2 hold is when the system (3) is input-to-state-stable. In this case, indeed, there exist $\gamma, \rho \in \mathcal{K}_\infty$ such that (see Sontag (1995))

$$\forall \varphi_w \in \mathcal{S}_w(W), \quad \|\varphi_\xi(t)\| \le \gamma(\|\varphi_\xi(0)\|) + \rho(\|\varphi_w\|_\infty)$$

for all $\varphi_\xi \in \mathcal{S}_\xi(\Xi)$ and for all $t \in \mathbb{R}_+$. Therefore, every pair (φ_w, φ_ξ) is in $\mathcal{A}(W \times \Xi)$ with $M := \max_{w \in W} \|w\|$ and $N := \rho(M) + \max_{\xi \in \Xi} \gamma(\|\xi\|)$.

Properties of the Ω-limit set:

With the next theorem we show that under assumptions A1-A2 the Ω-limit set of $\mathcal{A}(W \times \Xi)$ is a well-defined compact attractor for the admissible solutions to (2), (3).
Theorem 1. *Suppose A1. Then Ω is non-empty, compact, and uniformly attracts $\mathcal{A}(W \times \Xi)$. Moreover, Ω is the graph of an upper semicontinuous set-valued map and, if in addition A2 holds, then Ω is invariant for $\mathcal{A}(\Omega)$.*

The proof of Theorem 1 is postponed in appendix A. The claim of Theorem 1 stating that Ω is the graph of a map means that asymptotically, when the transitory elapses, the trajectories of (3) are algebraically (even if in a set-valued sense) defined by the value of $w(t)$. In this sense, the set Ω is the *steady-state locus* of (2), (3). Moreover, the next proposition states that Ω is also the smallest closed set that uniformly attracts $\mathcal{A}(W \times \Xi)$.
Proposition 1. *Assume Ω is not empty. Then Ω is the smallest closed set (in the sense of inclusion) which uniformly attracts $\mathcal{A}(W \times \Xi)$.*

3. NECESSARY CONDITIONS FOR OUTPUT REGULATION

In this section, we show how the asymptotic characterisation of the interconnections of the kind (2), (3) presented so far can be used to deduce necessary conditions for the output regulation problem to be solved. In doing this we follows the line of development of Byrnes and Isidori (2003).

3.1 The Framework

We consider a system of the kind

$$\begin{aligned} \dot{x} &= f(w, x, u) \\ y &= h_y(w, x) \\ e &= h_e(w, x) \end{aligned} \tag{4}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, output $(y, e) \in \mathbb{R}^\nu \times \mathbb{R}^p$ and with $w \in \mathbb{R}^s$ that is an exogenous signal that represents disturbances acting on the system (4) as well as references to be tracked or parameters uncertainties. We assume that f and h are locally Lipschitz. The output $e := h_e(w, x) \in \mathbb{R}^p$ is referred to as the *regulation errors* and represents the system outputs that need to asymptotically vanish, while $y_a := h_a(w, x) \in \mathbb{R}^{\nu - p}$ is the set of measured outputs that might be needed for stabilisation purposes but that are not required to vanish at the steady state.

We suppose w to be generated by an *exosystem* of the form

$$\dot{w} \in S(w) \tag{5}$$

with initial conditions that range in a compact invariant set $W \subset \mathbb{R}^s$. Furthermore, we assume $S : \mathbb{R}^s \rightrightarrows \mathbb{R}^s$ to be Lipschitz on W with non-empty compact values at each $w \in W$.

In this framework the problem of semiglobal *output regulation* reads as follows: given a compact set $X \subset \mathbb{R}^n$ of initial conditions for (4), find a controller of the form

$$\begin{aligned} \dot{\eta} &= g(\eta, y) \\ u &= \gamma(\eta, y) \end{aligned} \tag{6}$$

with state $\eta \in \mathbb{R}^\ell$ and with $g : \mathbb{R}^\ell \times \mathbb{R}^{\nu - p} \to \mathbb{R}^\ell$, $\gamma : \mathbb{R}^\ell \times \mathbb{R}^{\nu - p} \to \mathbb{R}^m$ locally Lipschitz, and a compact set $H \subset \mathbb{R}^\ell$, such that, with $K := h_e^{-1}(0)$, the closed-loop system

$$\begin{aligned} \dot{w} &\in S(w) \\ \dot{x} &= f(w, x, \gamma(\eta, h_y(w, x))) \\ \dot{\eta} &= \varphi(\eta, h_y(w, x)) \end{aligned} \tag{7}$$

satisfy the following:

(1) All the solutions to (7) originating in $W \times X \times H$ are admissible.
(2) K uniformly attracts $\mathcal{A}(W \times X \times H)$.

With $\xi := \operatorname{col}(x, \eta)$ and $\Xi := X \times H$, the first requirement can be equivalently expressed as $\mathcal{S}_w(W) = \mathcal{L}^w(W)$. The second one, instead, requires the regulation error e to vanish at the steady state.

3.2 Necessary Conditions

As a consequence of Theorem 1 and Proposition 1 we obtain the following necessary conditions for the output regulation problem to admit a solution.
Proposition 2. *Suppose the problem of output regulation is solvable on $W \times X$. Then there exists an upper semi-continuous set valued map*

$$\pi : \operatorname{dom} \pi \subset W \rightrightarrows \mathbb{R}^n$$

with compact graph, such that

a) $\operatorname{graph} \pi \subset K$.
b) *For each $(w, x) \in \operatorname{graph} \pi$, the set of all input functions $u \in \mathbb{R}^m$ such that*

$$S(w) \times \{f(x, w, u)\} \subset T_{\operatorname{graph} \pi}(w, x),$$

is non empty.

Proof. Assume that the problem of output regulation is solved, i.e. points 1 and 2 of the definition hold. Point 1 implies assumption A1-A2 and hence Ω is well defined and Theorem 1 holds. Let $E := \{w \in \mathbb{R}^s : (w, x, \eta) \in \Omega\}$. From the invariance of W it follows that $E \subset W$. For each $w \in E$, let

$$\pi(w) = \left\{ x \in \mathbb{R}^n : (w, x, \eta) \in \Omega \right\}.$$

Then, $\pi : E \to \mathbb{R}^n$ is well-defined and has compact graph. Upper semicontinuity follows from Theorem 1. Point a) follows from Proposition 1. By considering the restrictions of S, f and g to any open neighbourhood Ω and applying (Aubin, 1991, Thm. 5.3.4) we obtain that $S(w) \times \{f(w, x, \gamma(\eta, h_y(w, x)))\} \times \{g(\eta, h_y(w, x))\} \in T_\Omega(w, x, \eta)$. Hence, with $E(w, x) := \{\eta \in \mathbb{R}^\ell : (w, x, \eta) \in \Omega\}$, the

set $\mathcal{U}(w,x) := \{u \in \mathbb{R}^m : u = \gamma(\eta, h_y(w,x)), \eta \in E(w,x), (w,x) \in \operatorname{graph} \pi\}$ is precisely the set of inputs for which point b) holds.   □

Remark 1. If in addition we assume W to be *Poisson Stable* for $\mathcal{A}(W \times X \times H)$ then $\operatorname{dom} \pi = W$, as in the case in Byrnes and Isidori (2003).

4. OUTPUT REGULATION AND ZERO DYNAMICS

4.1 Zero Dynamics

In this section we extend the concept of zero dynamics as given in Byrnes and Isidori (2003) to the case of exosystems given by a differential inclusion. Consider the system

$$
\begin{aligned}
\dot{z} &\in F(z,u) \\
y &= H(z)
\end{aligned}
\tag{8}
$$

with $z \in \mathbb{R}^\mu$ and $y \in \mathbb{R}^\nu$. We say that $Z \subset \mathbb{R}^\mu$ is a *viability domain* for (8) if, for all $z \in Z$, there exists u such that $F(z,u) \subset T_Z(z)$. We define the *regulation map $r : Z \rightrightarrows \mathbb{R}^\mu$* as

$$
r(z) = \{u \in \mathbb{R}^m : F(z,u) \subset T_Z(z)\} .
$$

If the regulation map $r(\cdot)$ admits a selection $\alpha : \mathbb{R}^\mu \to \mathbb{R}^m$ such that the map $z \mapsto F(z, \alpha(z))$ is sufficiently regular, then from from (Aubin, 1991, Thm, 5.3.4) we obtain that Z is invariant for (8).

We say (8) possesses a well-defined *zero dynamics* if there exists a non-empty closed subset Z in $\mathbb{R}^n$ such that

(1) $Z \subset H^{-1}(0)$
(2) Z is a viability domain and the regulation map possesses a continuous selection α such that $z \mapsto F(z, \alpha(z))$ is Lipschitz and has compact values.
(3) If $z_0 \in \mathbb{R}^n$, $u_0 \in \mathcal{C}(\mathbb{R}_+)$ and $z \in \mathcal{S}_z(z_0, u_0)$ are defined on an interval I and such that

$$
\forall t \in I, H(z(t)) = 0
$$

then

$$
z_0 \in Z \text{ and } u_0(t) = \alpha(z(t)) \text{ a.e. on } I
$$

The above definition generalizes zero dynamics notion appearing in Byrnes and Isidori (2003) and, in particular, the third condition ensures the uniqueness of the selection α.

Consider now the system

$$
\begin{aligned}
\dot{w} &\in S(w) \\
\dot{x} &= f(w,x,u) \\
e &= h_e(w,x)
\end{aligned}
\tag{9}
$$

and assume that (9) possesses well-defined zero dynamics. Let Z_e denote the zero dynamics kernel of (9) and let $\alpha : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^m$ be the unique continuous selection of the regulation map associated to Z_e. Let $A \subset W \times \mathbb{R}^n$ be the (possibly empty) set for which all the solutions to the system

$$
\begin{aligned}
\dot{w} &\in S(w) \\
\dot{x} &= f(w,x,\alpha(w,x))
\end{aligned}
\tag{10}
$$

originating in A are admissible. Then the following result holds.

Proposition 3. *Assume* (6) *solves the problem of output regulation for* (4), (5) *on* $W \times X$ *and suppose that* (9) *possesses a well defined zero dynamics. Then,*

(1) $p_{(w,x)}(\Omega) \subset A$
(2) *Any trajectory in $\mathcal{A}(\Omega)$ is p_ξ-related to a trajectory of the zero dynamics system.*
(3) *For any $(x_0, w_0) \in p_\xi(\Omega)$ there exists ξ_0 such that the response u_δ of the system*

$$
\begin{aligned}
\dot{w} &\in S(w) \\
\dot{x} &= f(x, w, \alpha(x,w)) \\
\dot{\xi} &= \varphi(\xi, k(x,w)) \\
u_\delta &= \alpha(x,w) - \gamma(\xi, k(x,w))
\end{aligned}
\tag{11}
$$

is define for all $t \geq 0$ and identically zero.

Proof. The result directly follows from the properties of Ω with the same arguments used in (Byrnes and Isidori, 2003, Prop. 6.1).   □

5. EFFICIENT CONTROLLERS

In this section we restrict the focus on the case in which the map π introduced in Proposition 2 is single valued. By borrowing the terminology of Byrnes and Isidori (2003), we say that (6) is an *efficient controller* if it solves the problem of output regulation and there exist two single-valued maps $\varpi : p_w(\Omega(W \times X \times H)) \to \mathbb{R}^n$ and $\rho : p_w(\Omega(W \times X \times H)) \to \mathbb{R}^\ell$ such that

$$
\begin{aligned}
\Omega(W \times X \times H) = \{(w,x,\eta) \in W \times \mathbb{R}^n \times \mathbb{R}^\ell : \\
x = \varpi(w) \text{ and } \eta = \rho(w)\}.
\end{aligned}
$$

The following results hold.

Proposition 4. *If* (6) *is an efficient controller, then ϖ and ρ are continuous.*

Proposition 5. *Suppose a controller of the form* (6) *is efficient and* (4) *possesses a well-defined zero dynamics. Then there exist $\varpi : p_w(\Omega(W \times X \times H)) \to \mathbb{R}^n$ and $\rho : p_w(\Omega(W \times X \times H)) \to \mathbb{R}^\ell$ such that*

(1) $\operatorname{graph} \varpi$ *is invariant for the set of solutions of* (10) *starting in $W \times X$. Moreover $h_e(\varpi(w), w) = 0$.*
(2) *For any $(w_0, \eta_0) \in \{(w, \eta) \in \mathbb{R}^s \times \mathbb{R}^\ell : \eta = \rho(w)\}$ the response of the system*

$$
\begin{aligned}
\dot{w} &\in S(w) \\
\dot{x} &= \varphi(\xi, k(\nu(w), w)) \\
u_\delta &= \alpha(\rho(w), w) - \gamma(\rho(w), k(\nu(w), w))
\end{aligned}
\tag{12}
$$

is defined for all $t \geq 0$ and is identically equal to zero.

6. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper we extended the notions of limit sets, steady state and zero dynamics to the case of exosystems modelled by differential inclusions. We have given necessary conditions for the output regulation problem to be solved and we have shown that the concepts of zero dynamics given in Byrnes and Isidori (2003) extend in a natural way. With these results we have thus built the fundamental background in which future research toward designs of regulators able to deal with more general regulation problems can be carried out. Future directions concern the design of regulators that need less a-priori knowledge of the exogenous signals and that can solve the output regulation problem in more general settings. A prominent way to design such controllers is throughout adaptive schemes, that permit do decouple the regulator synthesis with the strict knowledge of the dynamics of the exogenous signals.

Appendix A. PROOF OF THEOREM 1

Before proving Theorem 1, we state the next result.
Lemma 1. *For all $t \in \mathbb{R}_+$ and any two solutions $\varphi_w^0 \in \mathcal{S}_w(W)$ and $\varphi_w^1 \in \mathcal{S}_w(\varphi_w^0(t))$, let $\oplus_t$ be the concatenation operator*

$$\varphi_w^0 \oplus_t \varphi_w^1(s) := \mathbf{1}_{[0,t]}(s)\varphi_w^0(s) + \mathbf{1}_{]t,+\infty[}(s)\varphi_w^1(s-t)$$

Then $\varphi_w^0 \oplus_t \varphi_w^1 \in \mathcal{S}_w(W)$.

Proof. Since, by definition, φ_w^i, $i = 0, 1$, is absolutely continuous then it admits derivative in L_{loc}^1 such that

$$\varphi_w^i(s) = \varphi_w^i(0) + \int_0^s (\varphi_w^i)'(u)du$$

and $(\varphi_w^i)'(s) \in S(\varphi_w^i(s))$ a.e., $i = 0, 1$. Then

$$\varphi_w^0 \oplus_t \varphi_w^1(s) = \varphi_w^0(0) + \int_0^s \mathbf{1}_{[0,t]}(u)(\varphi_w^0)'(u)du \\ + \mathbf{1}_{]t,+\infty[}(u)(\varphi_w^1)'(u-t)du$$

and then
$$(\varphi_w^0 \oplus_t \varphi_w^1)'(s) = \mathbf{1}_{[0,t]}(s)(\varphi_w^0)'(s) + \mathbf{1}_{]t,+\infty[}(s)(\varphi_w^1)'(s-t)$$
which proves that $\varphi_w^0 \oplus_t \varphi_w^1$ is absolutely continuous and satisfies $(\varphi_w^0 \oplus_t \varphi_w^1)'(s) \in S(\varphi_w^0 \oplus_t \varphi_w^1)$ a.e. on $\mathbb{R}_+$. $\qquad\square$

Proof of Theorem 1. First we prove that Ω is compact. Boundedness follows from the definition of $\mathcal{L}^w(W)$, hence it suffices to prove it is closed. Let $(w_n, \xi_n)_{n \in \mathbb{N}}$ be a sequence in Ω converging to (w, ξ). By definition of Ω, for all $n \in \mathbb{N}$,

$$\exists (t_k^n)_{k \in \mathbb{N}} \nearrow, \exists (\varphi_\xi^{n,k}, \varphi_w^{n,k})_{k \in \mathbb{N}} \in \mathcal{A}(W \times \Xi)^{\mathbb{N}}, \\ (\varphi_\xi^{n,k}(t_k^n), \varphi_w^{n,k}(t_k^n)) \to_{k \to +\infty} (\xi_n, w_n).$$

We can index k on n to obtain for all $n \in \mathbb{N}$
$$\left\| (\varphi_\xi^{n,k}(t_k^n), \varphi_w^{n,k}(t_k^n)) - (\xi_n, w_n) \right\| \leq 2^{-n}$$
that in turn implies
$$\left\| (\varphi_\xi^{n,k}(t_k^n), \varphi_w^{n,k}(t_k^n)) - (\xi, w) \right\| \leq 2^{-n} + \|(\xi_n, w_n) - (\xi, w)\|.$$
This shows that $(\xi, w) \in \Omega$ and thus Ω is closed, hence compact.

We now show uniform attractiveness of Ω for $\mathcal{A}(W \times \Xi)$. By contradiction, assume

$$\exists \varepsilon > 0, \forall T > 0, \exists t > T, \exists (\varphi_\xi, \varphi_w) \in \mathcal{A}(W \times \Xi), \\ d(\varphi_\xi(t), \varphi_x(t)) > \varepsilon.$$

Then there exist a sequence $(t_n)_{n \in \mathbb{N}} \nearrow$ and a sequence $(\varphi_\xi^n, \varphi_w^n)_{n \in N}$ in $\mathcal{A}(W \times \Xi)$ such that

$$d((\varphi_\xi^n(t_n), \varphi_w^n(t_n)), \Omega) > \varepsilon.$$

By definition of $\mathcal{A}(W \times \Xi)$, $(\varphi_\xi^n(t_n), \varphi_w^n(t_n))_{n \in \mathbb{N}}$ lives in a compact set and thus there exists a subsequence which converges to (ξ, w), which is in Ω by definition. As this is a contradiction and we claim the uniformly attractiveness of Ω.

We now show invariance of Ω for $\mathcal{A}(\Omega)$. Pick arbitrarily $(w_0, \xi_0) \in \Omega$ and let $\varphi_w \in \mathcal{S}_w(w_0)$. We now show a element $\varphi_\xi \in \mathcal{S}_\xi(\xi_0, \varphi_w)$ is defined at least on $[0, T]$, where $T > 0$ does not depend on (w_0, ξ_0) picked in Ω. Consider $\varphi_\xi^{ref} \in \mathcal{S}_\xi(\Xi, \varphi_w)$, so as, by definition, $\|\varphi_\xi^{ref}\|_\infty \leq N$. Let $\eta > 0$ be such that $\Omega \subset \text{int } B_{s+q}(0, \eta)$. By the fact ψ is

locally Lipschitz there exists $T > 0$ such that φ_ξ is defined on $[0, T]$ and $\varphi_\xi(t) \leq \eta$ for all $t \in [0, T]$. In fact Gronwall lemma gives us :

$$\forall t \in [0, T], \left\| \varphi_\xi^{ref}(t) - \varphi_\xi(t) \right\| \leq \left\| \varphi_\xi^{ref}(0) - \varphi_\xi(0) \right\| e^{Lt} \tag{A.1}$$

where L is the Lipschitz constant of ψ on $B_{s+q}(0, \eta)$. With $\mu > \max \left\{ \left\| \varphi_\xi^{ref}(0) - \xi \right\| : \xi \in p_\xi(\Omega) \right\}$, let

$$T^* = \min_{\xi \in p_w(\Omega)} \left\{ \frac{1}{L} \ln \left(\frac{\mu}{\left\| \varphi_\xi^{ref}(0) - \xi \right\|} \right) \right\},$$

which is non-negative by the choice of μ. Let us take η big enough to have

$$\forall \xi \in \mathbb{R}^q, \forall t \in \mathbb{R}_+, \left\| \varphi_\xi^{ref}(t) - \xi \right\| < \mu \Rightarrow \|\xi\| < \eta,$$

which is possible as φ_ξ^{ref} has a compact positive orbit.

In view of (A.1), if $T < T^*$, then $\left\| \varphi_\xi^{ref}(T) - \varphi_\xi(T) \right\| < \mu$ and $\|\varphi_\xi(T)\| < \eta$. T^* is independent of (w_0, ξ_0) picked in Ω. From now on we considered only maximal solutions that, in view of the previous analysis are defined for $T > T^*$.

By definition of Ω, there exit $(t_n)_{n \in \mathbb{N}} \nearrow$ and $(\varphi_w^n, \varphi_\xi^n) \in \mathcal{A}(W \times \Xi)^{\mathbb{N}}$ such that

$$(\varphi_w^n(t_n), \varphi_\xi^n(t_n)) \to_{n \to +\infty} (w_0, \xi_0).$$

By the hypotheses on S (see (Aubin and Cellina, 1984, Thm. 1, ch. 2.4)), for all $n \in \mathbb{N}$, there exists $\bar\varphi_w^n \in \mathcal{S}_w(\varphi_w^n(t_n))$ such that

$$\sup_{t \in [0, T^*]} \|\bar\varphi_w^n(t) - \varphi_w(t)\| \leq \|\bar\varphi_w^n(t_n) - \varphi_w(0)\| e^{\alpha T^*}.$$

By lemma 1, $\varphi_w^n \oplus_{t_n} \bar\varphi_w^n \in \mathcal{S}_w(W)$ and, from the last estimate, we obtain

$$\forall t \in [0, T^*], \ \varphi_w^n \oplus_{t_n} \bar\varphi_w^n(t_n + t) \to_{n \to +\infty} \varphi_w(t).$$

We now have to prove the same kind of result for the variable ξ. For all $n \in \mathbb{N}$, consider the solution $\bar\varphi_\xi^n$ of (3) with initial condition $\varphi_\xi^n(0)$ and input $\bar\varphi_w^n$. Since $\varphi_w^n \oplus_{t_n} \bar\varphi_w^n \in \mathcal{S}_w(W)$ then, by using the fact that $\mathcal{S}_w(W) = \mathcal{L}^w(W)$ and from the definition of $\bar\varphi_\xi^n$, we deduce that $\bar\varphi_\xi^n$ is defined on $\mathbb{R}_+$ and bounded by N. From the Gronwall lemma we get

$$\left\| \phi_\xi(t, \varphi_\xi^n(0), \bar\varphi_w^n) - \phi_\xi(t, \xi_0, \varphi_w) \right\| \\ \leq \left(\left\| \varphi_\xi^n(0) - \xi_0 \right\| + LT^* \sup_{s \in [0, T^*]} \|\bar\varphi_w^n(s) - \varphi_w(s)\| \right) e^{Lt}$$

for all $t \in [0, T^*]$. Hence, for all $t \in [0, T^*]$,
$$(\varphi_w^n \oplus_{t_n} \bar\varphi_w^n(t_n+t), \varphi_\xi^n \oplus_{t_n} \bar\varphi_\xi^n(t_n+t)) \to_{n \to +\infty} (\varphi_w(t), \varphi_\xi(t)) \tag{A.2}$$

and then $(\varphi_w(t), \varphi_\xi(t)) \in \Omega$. Since $(\varphi_w(t), \varphi_\xi(t)) \in \Omega$ and T^* does not depends of the element chosen in it, invariance is obtained by induction.

Finally, to prove Ω is the graph of an upper semi continuous map. Define

$$\pi(w) = \left\{ \xi \in \mathbb{R}^q : (w, \xi) \in \Omega \right\}$$

then π is well-defined and upper-semicontinuity follows from (Aubin and Cellina, 1984, Thm. 1, ch. 1). $\qquad\square$

Appendix B. PROOF OF PROPOSITION 1

Proof. Assume a closed set K uniformly attracts $\mathcal{A}(W \times \Xi)$. Pick a point $\omega \in \Omega$, it suffices to prove that $\omega \in K$. By definition there exist $(t_n)_{n \in \mathbb{N}} \nearrow$ and $(\varphi_\xi^n, \varphi_w^n)_{n \in \mathbb{N}} \in \mathcal{A}(W \times \Xi)$ such that

$$(\varphi_w^n(t_n), \varphi_\xi^n(t_n)) \to_{n \to +\infty} \omega \qquad (B.1)$$

and, from uniform attractiveness of K,

$$\forall \varepsilon > 0, \exists T > 0, \forall t \geq T, \forall(\varphi_\xi, \varphi_w) \in \mathcal{A}(W \times \Xi),$$

$$d((\varphi_w(t), \varphi_\xi(t)), K) \leq \frac{\varepsilon}{2}.$$

Fix $\varepsilon > 0$. Then, for n big enough,

$$\begin{aligned}
d(\omega, K) &\leq \left\| (\varphi_w^n(t_n), \varphi_\xi^n(t_n)) - \omega \right\| \\
&\quad + d((\varphi_w^n(t_n), \varphi_\xi^n(t_n)), K) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

As it holds for any $\varepsilon > 0$, we claim $d(\omega, K) = 0$ hence $\omega \in K$ as K is closed. $\qquad\square$

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