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**Research Article**

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# Some Hilbert spaces related with the Dirichlet space

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**Abstract:** We study the reproducing kernel Hilbert space with kernel $k^d$, where $d$ is a positive integer and $k$ is the reproducing kernel of the analytic Dirichlet space.

**Keywords:** Dirichlet space, Complete Nevanlinna Property, Hilbert-Schmidt operators, Carleson measures

**MSC:** 30H25, 47B35

## 1 Introduction

Consider the Dirichlet space $\mathcal{D}$ on the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ of the complex plane. It can be defined as the Reproducing Kernel Hilbert Space (RKHS) having kernel

$$k_z(w) = k(w, z) = \frac{1}{\bar{z}w} \log \frac{1}{1 - \bar{z}w} = \sum_{n=0}^{\infty} \frac{(\bar{z}w)^n}{n+1}.$$

We are interested in the spaces $\mathcal{D}_d$ having kernel $k^d$, with $d \in \mathbb{N}$. $\mathcal{D}_d$ can be thought of in terms of function spaces on polydiscs, following ideas of Aronszajn [4]. To explain this point of view, note that the tensor $d$-power $\mathcal{D}^{\otimes d}$ of the Dirichlet space has reproducing kernel $k_d(z_1, \cdots, z_d; w_1, \ldots, w_d) = \Pi_{j=1}^d k(z_j, w_j)$. Hence, the space of restrictions of functions in $\mathcal{D}^{\otimes d}$ to the diagonal $z_1 = \cdots = z_d$ has the reproducing kernel $k^d$, and therefore coincides with $\mathcal{D}_d$.

We will provide several equivalent norms for the spaces $\mathcal{D}_d$ and their dual spaces in Theorem 1.1. Then we will discuss the properties of these spaces. More precisely, we will investigate:

– $\mathcal{D}_d$ and its dual space $HS_d$ in connection with Hankel operators of Hilbert-Schmidt class on the Dirichlet space $\mathcal{D}$;
– the complete Nevanlinna-Pick property for $\mathcal{D}_d$;
– the Carleson measures for these spaces.

Concerning the first item, the connection with Hilbert-Schmidt Hankel operators served as our original motivation for studying the spaces $\mathcal{D}_d$.

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Note that the spaces $\mathcal{D}_d$ live infinitely close to $\mathcal{D}$ in the scale of weighted Dirichlet spaces $\tilde{\mathcal{D}}_s$, defined by the norms

$$\|\varphi\|^2_{\tilde{\mathcal{D}}_s} = \int_{-\pi}^{+\pi} \left|\varphi(e^{it})\right|^2 \frac{dt}{2\pi} + \int_{|z|<1} \left|\varphi'(z)\right|^2 (1-|z|^2)^s \frac{dA(z)}{\pi}, \; 0 \le s < 1,$$

where $\frac{dA(z)}{\pi}$ is normalized area measure on the unit disc.

**Notation:** We use multiindex notation. If $n = (n_1, \dots, n_d)$ belongs to $\mathbb{N}^d$, then $|n| = n_1 + \dots + n_d$. We write $A \approx B$ if $A$ and $B$ are quantities that depend on a certain family of variables, and there exist independent constants $0 < c < C$ such that $cA \le B \le CA$.

## Equivalent norms for the spaces $\mathcal{D}_d$ and their dual spaces $HS_d$

**Theorem 1.1.** *Let $d$ be a positive integer and let*

$$a_d(k) = \sum_{|n|=k} \frac{1}{(n_1+1)\dots(n_d+1)}.$$

*Then the norm of a function $\varphi(z) = \sum_{k=0}^{\infty} \widehat{\varphi}(k) z^k$ in $\mathcal{D}_d$ is*

$$\|\varphi\|_{\mathcal{D}_d} = \left(\sum_{k=0}^{\infty} a_d(k)^{-1} |\widehat{\varphi}(k)|^2\right)^{1/2} \approx [\varphi]_d, \tag{1}$$

*where*

$$[\varphi]_d = \left(\sum_{k=0}^{\infty} \frac{k+1}{\log^{d-1}(k+2)} |\widehat{\varphi}(k)|^2\right)^{1/2}. \tag{2}$$

*An equivalent Hilbert norm $|[\varphi]|_d \approx [\varphi]_d$ for $\varphi$ in terms of the values of $\varphi$ is given by*

$$|[\varphi]|_d = |\varphi(0)|^2 + \left(\int_{\mathbb{D}} |\varphi'(z)|^2 \frac{1}{\log^{d-1}\left(\frac{1}{1-|z|^2}\right)} \frac{dA(z)}{\pi}\right)^{1/2}. \tag{3}$$

*Define now the holomorphic space $HS_d$ by the norm:*

$$\|\psi\|_{HS_d} = \left(\sum_{k=0}^{\infty} (k+1)^2 a_d(k) \left|\widehat{\psi}(k)\right|^2\right)^{1/2}. \tag{4}$$

*Then, $HS_d \equiv (\mathcal{D}_d)^*$ is the dual space of $\mathcal{D}_d$ under the duality pairing of $\mathcal{D}$. Moreover,*

$$\|\psi\|_{HS_d} \approx [\psi]_{HS_d} := \left(\sum_{k=0}^{\infty} (k+1) \log^{d-1}(k+2) \left|\widehat{\psi}(k)\right|^2\right)^{1/2} \approx$$
$$|[\psi]|_{HS_d} := \left(|\psi(0)|^2 + \int_{\mathbb{D}} |\psi'(z)|^2 \log^{d-1}\left(\frac{1}{1-|z|^2}\right) \frac{dA(z)}{\pi}\right)^{1/2}. \tag{5}$$

*Furthermore, the norm can be written as*

$$\|\psi\|^2_{HS_d} = \sum_{(n_1,\dots,n_d)} |\langle e_{n_1} \dots e_{n_d}, \psi\rangle_{\mathcal{D}}|^2, \tag{6}$$

*where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis of $\mathcal{D}$, $e_n(z) = \frac{z^n}{\sqrt{n+1}}$.*

The remainder of this section is devoted to the proof of Theorem 1.1. The expression for $\|\varphi\|_{\mathcal{D}_d}$ in (1) follows by expanding $(k_z)^d$ as a power series. The equivalence $\|\varphi\|_{\mathcal{D}_d} \approx [\varphi]_d$, as well as $\|\varphi\|_{HS_d} \approx [\varphi]_{HS_d}$, are consequences of the following lemma. We denote by $c, C$ positive constants which are allowed to depend on $d$ only, whose precise value can change from line to line.

**Lemma 1.2.** *For each $d \in \mathbb{N}$ there are constants $c, C > 0$ such that for all $k \geq 0$ we have*

$$c a_d(k) \leq \frac{\log^{d-1}(k+2)}{k+1} \leq C a_d(k).$$

*Consequently, if $t \in (0,1)$, then*

$$c\left(\frac{1}{t}\log\frac{1}{1-t}\right)^d \leq \sum_{k=0}^{\infty} \frac{\log^{d-1}(k+2)}{k+1} t^k \leq C\left(\frac{1}{t}\log\frac{1}{1-t}\right)^d.$$

*Proof of Lemma 1.2.* We will prove the Lemma by induction on $d \in \mathbb{N}$. It is obvious for $d = 1$. Thus let $d \geq 2$ and suppose the lemma is true for $d-1$. Also we observe that there is a constant $c > 0$ such that for all $k \geq 0$ and $0 \leq n \leq k$ we have

$$c \log^{d-2}(k+2) \leq \log^{d-2}(n+2) + \log^{d-2}(k-n+2) \leq 2\log^{d-2}(k+2).$$

Then for $k \geq 0$

$$\begin{aligned}
a_d(k) &= \sum_{n_1+\dots+n_d=k} \frac{1}{(n_1+1)\dots(n_d+1)} \\
&= \sum_{n=0}^{k} \frac{1}{n+1} \sum_{n_2+\dots+n_d=k-n} \frac{1}{(n_2+1)\dots(n_d+1)} \\
&\approx \sum_{n=0}^{k} \frac{1}{n+1} \frac{\log^{d-2}(k-n+2)}{k-n+1} \quad \text{by the inductive assumption} \\
&= \frac{1}{2}\sum_{n=0}^{k} \frac{\log^{d-2}(n+2)+\log^{d-2}(k-n+2)}{(n+1)(k-n+1)} \\
&\approx \log^{d-2}(k+2) \sum_{n=0}^{k} \frac{1}{(n+1)(k-n+1)} \quad \text{by the earlier observation} \\
&= \frac{\log^{d-2}(k+2)}{k+2} \sum_{n=0}^{k} \frac{1}{n+1} + \frac{1}{k-n+1} \\
&\approx \frac{\log^{d-1}(k+2)}{k+1}.
\end{aligned}$$

□

Next, we prove the equivalence $[\varphi]_{HS_d} \approx |[\varphi]|_{HS_d}$ which appears in (5).

**Lemma 1.3.** *Let $d \in \mathbb{N}$. Then*

$$\int_0^1 t^k \left(\frac{1}{t}\log\frac{1}{1-t}\right)^{d-1} dt \approx \frac{\log^{d-1}(k+2)}{k+1}, \quad k \geq d.$$

Given the Lemma, we expand

$$\begin{aligned}
|[\psi]|^2_{HS_d} &= |\widehat{\psi}(0)|^2 + \int_{\mathbb{D}} \left|\sum_{k=1}^{\infty} \widehat{\psi}(k) k z^{k-1}\right|^2 \log^{d-1}\frac{1}{1-|z|^2}\frac{dA(z)}{\pi} \\
&= |\widehat{\psi}(0)|^2 + \sum_{k=1}^{\infty} k^2 \left|\widehat{\psi}(k)\right|^2 \int_0^1 \log^{d-1}\frac{1}{1-t} t^{k-1} dt
\end{aligned}$$

$$\approx |\widehat{\psi}(0)|^2 + \sum_{k=1}^{\infty} k^2 \left|\widehat{\psi}(k)\right|^2 \frac{\log^{d-1}(k+2)}{k+1}$$
$$\approx [\psi]^2_{HS_d},$$

obtaining the desired conclusion.

*Proof of Lemma 1.3.* The case $d = 1$ is obvious, leaving us to consider $d \geq 2$. We will also assume that $k \geq 2$. Then by Lemma 1.2 we have

$$\int_0^1 t^k \left(\frac{1}{t}\log\frac{1}{1-t}\right)^{d-1} dt \approx \int_0^1 t^k \sum_{n=0}^{\infty} \frac{\log^{d-2}(n+2)}{n+1} t^n dt = \sum_{n=0}^{\infty} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} = S_1 + S_2,$$

where

$$S_1 = \sum_{n=0}^{k-1} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} \approx \frac{1}{k+1}\sum_{n=0}^{k-1} \frac{\log^{d-2}(n+2)}{n+1} \approx \frac{1}{k+1}\int_1^{k+2} \frac{\log^{d-2}(t)}{t} dt$$
$$= \frac{1}{d-1}\frac{\log^{d-1}(k+2)}{k+1}$$

and

$$S_2 = \sum_{n=k}^{\infty} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} \leq \sum_{n=k+1}^{\infty} \frac{\log^{d-2}(n+1)}{n^2} \leq \sum_{j=1}^{\infty}\sum_{n=k^j}^{k^{j+1}-1} \frac{\log^{d-2}(n+1)}{n^2}$$
$$\leq \sum_{j=1}^{\infty} (j+1)^{d-2} \log^{d-2} k \sum_{n=k^j}^{k^{j+1}-1} \frac{1}{n^2} \leq \log^{d-2}(k+2) \sum_{j=1}^{\infty} (j+1)^{d-2} \int_{k^j-1}^{\infty} \frac{1}{x^2} dx$$
$$= \frac{\log^{d-2}(k+2)}{k+1} \sum_{j=1}^{\infty} (j+1)^{d-2} \frac{k+1}{k^j-1} \leq \frac{\log^{d-2}(k+2)}{k+1} \sum_{j=1}^{\infty} (j+1)^{d-2} \frac{k+1}{(k-1)k^{j-1}}$$
$$\leq \frac{\log^{d-2}(k+2)}{k+1} \sum_{j=1}^{\infty} (j+1)^{d-2} \frac{3}{2^{j-1}} = o\left(\frac{\log^{d-1}(k+2)}{k+1}\right). \qquad \square$$

Now, the duality between $\mathcal{D}_d$ and $HS_d$ under the duality pairing given by the inner product of $\mathcal{D}$ is easily seen by considering $[\cdot]_d$ and $[\cdot]_{HS_d}$. They are weighted $\ell^2$ norms and duality is established by means of the Cauchy-Schwarz inequality.

Next we will prove that $[\varphi]_d \approx \|[\varphi]\|_d$. This is equivalent to proving that the dual space of $HS_d$, with respect to the Dirichlet inner product $\langle \cdot, \cdot \rangle_{\mathcal{D}}$, is the Hilbert space with the norm $\|[\cdot]\|_d$.

Let $d \in \mathbb{N}$ and set, for $0 < t < 1$, $w_d(t) = \left(\frac{1}{t}\log\frac{1}{1-t}\right)^d$ and, for $0 < |z| < 1$, $W_d(z) = w_d(|z|^2)$ and $W_d(0) = 1$.

**Lemma 1.4.** *Let $d \in \mathbb{N}$. Then*

$$\int_{1-\varepsilon}^1 w_d(t)dt \cdot \int_{1-\varepsilon}^1 \frac{1}{w_d(t)} dt \approx \varepsilon^2 \quad \text{as } \varepsilon \to 0.$$

*Proof.* Write $\tilde{w}(t) = (\log\frac{1}{1-t})^d$, and note that it suffices to establish the lemma for $\tilde{w}$ in place of $w_d$. Let $\varepsilon > 0$. Then $\tilde{w}$ is increasing in $(0, 1)$ and $\tilde{w}(1-\varepsilon^{k+1}) = (k+1)^d (\log\frac{1}{\varepsilon})^d$, hence

$$\int_{1-\varepsilon}^1 \tilde{w}(t)dt = \sum_{k=1}^{\infty} \int_{1-\varepsilon^k}^{1-\varepsilon^{k+1}} \tilde{w}(t)dt \leq \sum_{k=1}^{\infty} \tilde{w}(1-\varepsilon^{k+1})(\varepsilon^k - \varepsilon^{k+1})$$

$$= \sum_{k=1}^{\infty} (k+1)^d (\log \frac{1}{\varepsilon})^d \varepsilon^k (1-\varepsilon) \approx \varepsilon (\log \frac{1}{\varepsilon})^d \frac{1}{(1-\varepsilon)^d}$$

For $1/\tilde{w}$ we just notice that it is decreasing and hence

$$\int_{1-\varepsilon}^{1} \frac{1}{\tilde{w}(t)} dt \leq \frac{1}{\tilde{w}(1-\varepsilon)} \varepsilon = \frac{\varepsilon}{(\log \frac{1}{\varepsilon})^d}$$

Thus as $\varepsilon \to 0$ we have

$$\varepsilon^2 \leq \int_{1-\varepsilon}^{1} \tilde{w}(t) dt \int_{1-\varepsilon}^{1} \frac{1}{\tilde{w}(t)} dt = O(\varepsilon^2).$$

□

For $0 < h < 1$ and $s \in [-\pi, \pi)$ let $S_h(e^{is})$ be the Carleson square at $e^{is}$, i.e.

$$S_h(e^{is}) = \{re^{it} : 1-h < r < 1, |t-s| < h\}.$$

A positive function $W$ on the unit disc is said to satisfy the Bekollé-Bonami condition (B2) if there exists $c > 0$ such that

$$\int_{S_h(e^{is})} W dA \cdot \int_{S_h(e^{is})} \frac{1}{W} dA \leq ch^4$$

for every Carleson square $S_h(e^{is})$. If $d \in \mathbb{N}$ and if $W_d(z)$ is defined as above, then

$$\int_{S_h(e^{is})} W_d dA \cdot \int_{S_h(e^{is})} \frac{1}{W_d} dA = h^2 \int_{1-h}^{1} w_d(t) dt \cdot \int_{1-h}^{1} \frac{1}{w_d(t)} dt \approx h^4$$

by Lemma 1.4, at least if $0 < h < 1/2$. Observe that both $W_d$ and $1/W_d$ are positive and integrable in the unit disc, hence it follows that the estimate holds for all $0 < h \leq 1$.

Thus $W_d$ satisfies the condition (B2). Furthermore, note that if $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ is analytic in the open unit disc, then

$$\int_{|z|<1} |f(z)|^2 w_d(|z|^2) \frac{dA(z)}{\pi} = \sum_{k=0}^{\infty} w_k |\hat{f}(k)|^2,$$

where $w_k = \int_0^1 t^k w_d(t) dt \approx \frac{\log^d(k+2)}{k+1}$.

A special case of Theorem 2.1 of Luecking's paper [7] says that if $W$ satisfies the condition (B2) by Bekollé and Bonami [5], then one has a duality between the spaces $L_a^2(W dA)$ and $L_a^2(\frac{1}{W} dA)$ with respect to the pairing given by $\int_{|z|<1} f \overline{g} dA$. Thus, we have

$$\int_{|z|<1} |g(z)|^2 \frac{1}{W_d(z)} dA \approx \sup_{f \neq 0} \frac{\left| \int_{|z|<1} g(z) \overline{f(z)} \frac{dA(z)}{\pi} \right|^2}{\int_{|z|<1} |f(z)|^2 W_d(z) dA} = \sup_{f \neq 0} \frac{\left| \sum_{k=0}^{\infty} \frac{\hat{g}(k)}{(k+1)\sqrt{w_k}} \sqrt{w_k} \overline{\hat{f}(k)} \right|^2}{\sum_{k=0}^{\infty} w_k |\hat{f}(k)|^2}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)^2 w_k} |\hat{g}(k)|^2$$

This finishes the proof of (5). It remains to demonstrate (6). We defer its proof to the next section.

By Theorem 1.1 we have the following chain of inclusions:

$$\ldots \hookrightarrow HS_{d+1} \hookrightarrow HS_d \hookrightarrow \ldots \hookrightarrow HS_2 \hookrightarrow HS_1 = \mathcal{D} = \mathcal{D}_1 \hookrightarrow \mathcal{D}_2 \hookrightarrow \ldots \hookrightarrow \mathcal{D}_d \hookrightarrow \mathcal{D}_{d+1} \hookrightarrow \ldots$$

with duality w.r.t. $\mathcal{D}$ linking spaces with the same index. It might be interesting to compare this sequence with the sequence of Banach spaces related to the Dirichlet spaces studied in [3]. Note that for $d \geq 3$ the reproducing kernel of $HS_d$ is continuous up to the boundary. Hence functions in $HS_d$ extend continuously to the closure of the unit disc, for $d \geq 3$.

## Hilbert-Schmidt norms of Hankel-type operators

Let $\{e_n\}$ be the canonical orthonormal basis of $\mathcal{D}$, $e_n(z) = \frac{z^n}{\sqrt{n+1}}$. Equation (6) follows from the computation

$$\sum_{k=0}^{\infty}\sum_{|n|=k}|\langle e_{n_1}...e_{n_d},\psi\rangle|^2 = \sum_{k=0}^{\infty}\sum_{|n|=k}\frac{1}{(n_1+1)\cdot\ldots\cdot(n_d+1)}|\langle z^{n_1}...z^{n_d},\psi\rangle|^2$$
$$= \sum_{k=0}^{\infty}\sum_{|n|=k}\frac{1}{(n_1+1)\cdot\ldots\cdot(n_d+1)}|\langle z^k,\psi\rangle|^2 = \sum_{k=0}^{\infty}\sum_{|n|=k}\frac{(k+1)^2}{(n_1+1)\cdot\ldots\cdot(n_d+1)}|\hat{\psi}(k)|^2$$
$$= \sum_{k=0}^{\infty}(k+1)a_d(k)|\hat{\psi}(k)|^2 \approx \sum_{k=0}^{\infty}\frac{\log^{d-1}(k+2)}{k+1}|\hat{\psi}(k)|^2.$$

Polarizing this expression for $\|\cdot\|_{HS_d}$, the inner product of $HS_d$ can be written

$$\langle\psi_1,\psi_2\rangle_{HS_d} = \sum_{(n_1,\ldots,n_d)}\langle\psi_1, e_{n_1}\ldots e_{n_d}\rangle_{\mathcal{D}}\langle e_{n_1}\ldots e_{n_d},\psi_2\rangle_{\mathcal{D}}.$$

Hence, for any $\lambda,\zeta\in\mathbb{D}$,

$$\langle k_\lambda,k_\zeta\rangle_{HS_d} = \sum_{n\in\mathbb{N}^d}\langle k_\lambda, e_{n_1}\ldots e_{n_d}\rangle_{\mathcal{D}}\langle e_{n_1}\ldots e_{n_d},k_\zeta\rangle_{\mathcal{D}} = \sum_{n\in\mathbb{N}^d}\overline{e_{n_1}(\lambda)\ldots e_{n_d}(\lambda)}e_{n_1}(\zeta)\ldots e_{n_d}(\zeta)$$
$$= \left(\sum_{i=0}^{\infty}\overline{e_i(\lambda)}e_i(\zeta)\right)^d = k_\lambda(\zeta)^d = \langle k_\lambda^d,k_\zeta^d\rangle_{\mathcal{D}_d}.$$

That is,

**Proposition 1.5.** *The map $U: k_\lambda\mapsto k_\lambda^d$ extends to a unitary map $HS_d\to\mathcal{D}_d$.*

When $d = 2$, $HS_2$ contains those functions $b$ for which the Hankel operator $H_b:\mathcal{D}\to\overline{\mathcal{D}}$, defined by $\langle H_b e_j,\overline{e_k}\rangle_{\overline{\mathcal{D}}} = \langle e_j e_k, b\rangle_{\mathcal{D}}$, belongs to the Hilbert-Schmidt class.

Analogous interpretations can be given for $d\geq 3$, but then function spaces on polydiscs are involved. We consider the case $d=3$, which is representative. Consider first the operator $T_b:\mathcal{D}\to\overline{\mathcal{D}}\otimes\overline{\mathcal{D}}$ defined by

$$\left\langle T_b f,\overline{g}\otimes\overline{h}\right\rangle_{\overline{\mathcal{D}}\otimes\overline{\mathcal{D}}} = \langle fgh,b\rangle_{\mathcal{D}}.$$

The formula uniquely defines an operator, whose action is

$$\begin{aligned}T_b f(z,w) &= \langle T_b f,\overline{k}_z\overline{k}_w\rangle_{\overline{\mathcal{D}}\otimes\overline{\mathcal{D}}}\\ &= \langle f k_z k_w,b\rangle_{\mathcal{D}}\\ &= \sum_{n,m,j}\hat{f}(j)\frac{\overline{z}^n}{n+1}\frac{\overline{w}^m}{m+1}\langle\zeta^{n+m+j},b\rangle_{\mathcal{D}}\\ &= \sum_{n,m,j}\hat{f}(j)\overline{\hat{b}(n+m+j)}\frac{n+m+j+1}{(n+1)(m+1)}\overline{z}^n\overline{w}^m\end{aligned}$$

Then, the Hilbert-Schmidt norm of $T_b$ is:

$$\sum_{l,m,n}\left|\langle T_b e_l, e_m e_n\rangle_{\overline{\mathcal{D}}\otimes\overline{\mathcal{D}}}\right|^2 = \sum_{l,m,n}|\langle e_l e_m e_n,b\rangle_{\mathcal{D}}|^2 = \|b\|_{HS_3}^2.$$

Similarly, we can consider $U_b:\mathcal{D}\otimes\mathcal{D}\to\overline{\mathcal{D}}$ defined by

$$\left\langle U_b(f\otimes g),\overline{h}\right\rangle_{\overline{\mathcal{D}}} = \langle fgh,b\rangle_{\mathcal{D}}.$$

The action of this operator is given by

$$U_b(f\otimes g)(\overline{z}) = \sum_{l,m,n=0}^{\infty} \widehat{f}(l)\widehat{g}(m)\frac{(l+m+n+1)\overline{\widehat{b}(l+m+n)}}{n+1}\overline{z}^n.$$

The Hilbert-Schmidt norm of $U_b$ is still $\|b\|_{HS_3}$.

## Carleson measures for the spaces $\mathcal{D}_d$ and $HS_d$

The (B2) condition allows us to characterize the Carleson measures for the spaces $\mathcal{D}_d$ and $HS_d$. Recall that a nonnegative Borel measure $\mu$ on the open unit disc is Carleson for the Hilbert function space $H$ if the inequality

$$\int_{|z|<1} |f|^2 d\mu \le C(\mu)\|f\|_H^2$$

holds with a constant $C(\mu)$ which is independent of $f$. The characterization [2] shows that, since the (B2) condition holds, then

**Theorem 1.6.** *For $d \in \mathbb{N}$, a measure $\mu \ge 0$ on $\{|z| < 1\}$ is Carleson for $\mathcal{D}_d$ if and only if for $|a| < 1$ we have:*

$$\int_{\tilde{S}(a)} \log^{d-1}\left(\frac{1}{1-|z|^2}\right)(1-|z|^2)\mu(S(z)\cap S(a))^2\frac{dxdy}{(1-|z|^2)^2} \le C_1(\mu)\mu(S(a)),$$

*where $S(a) = \{z : 0 < 1-|z| < 1-|a|,\ |\arg(z\overline{a})| < 1-|a|\}$ is the Carleson box with vertex $a$ and $\tilde{S}(a) = \{z : 0 < 1-|z| < 2(1-|a|),\ |\arg(z\overline{a})| < 2(1-|a|)\}$ is its “dilation”.*

The characterization extends to $HS_2$, with the weight $\log^{-1}\left(\frac{1}{1-|z|^2}\right)$. Since functions in $HS_d$ are continuous for $d \ge 3$, all finite measures are Carleson measures for these spaces. Once we know the Carleson measures, we can characterize the multipliers for $\mathcal{D}_d$ in a standard way.

## The complete Nevanlinna-Pick property for $\mathcal{D}_d$

Next, we prove that the spaces $\mathcal{D}_d$ have the Complete Nevanlinna-Pick (CNP) Property. Much research has been done on kernels with the CNP property in the past twenty years, following seminal work of Sarason and Agler. See the monograph [1] for a comprehensive and very readable introduction to this topic. We give here a definition which is simple to state, although perhaps not the most conceptual. An irreducible kernel $k : X \times X \to \mathbb{C}$ has the CNP property if there is a positive definite function $F : X \to \mathbb{D}$ and a nowhere vanishing function $\delta : X \to \mathbb{C}$ such that:

$$k(x,y) = \frac{\overline{\delta}(x)\delta(y)}{1-F(x,y)}$$

whenever $x, y$ lie in $X$. The CNP property is a property of the kernel, not of the Hilbert space itself.

**Theorem 1.7.** *There are norms on $\mathcal{D}_d$ such that the CNP property holds.*

*Proof.* A kernel $k : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ of the form $k(w,z) = \sum_{k=0}^{\infty} a_k(\overline{z}w)^k$ has the CNP property if $a_0 = 1$ and the sequence $\{a_n\}_{n=0}^{\infty}$ is positive and log-convex:

$$\frac{a_{n-1}}{a_n} \le \frac{a_n}{a_{n+1}}.$$

See [1], Theorem 7.33 and Lemma 7.38. Consider $\eta(x) = \alpha \log\log(x) - \log(x)$, with real $\alpha$. Then, $\eta''(x) = \frac{\log^2(x)-\alpha\log(x)-\alpha}{x^2\log^2(x)}$, which is positive for $x \geq M_\alpha$, depending on $\alpha$. Let now

$$a_n = \frac{\log^{d-1}(M_d(n+1))}{\log(M_d)\cdot(n+1)} \approx \frac{1}{n+1} + \frac{\log^{d-1}(n+1)}{n+1} \tag{7}$$

Then, the sequence $\{a_n\}_{n=0}^{\infty}$ provides the coefficients for a kernel with the CNP property for the space $\mathcal{D}_d$. □

The CNP property has a number of consequences. For instance, we have that the space $\mathcal{D}_d$ and its multiplier algebra $M(\mathcal{D}_d)$ have the same interpolating sequences. Recall that a sequence $Z = \{z_n\}_{n=0}^{\infty}$ is *interpolating* for a RKHS $H$ with reproducing kernel $k^H$ if the weighted restriction map $R : \varphi \mapsto \left\{\frac{\varphi(z_n)}{k^H(z_n,z_n)^{1/2}}\right\}_{n=0}^{\infty}$ maps $H$ boundedly onto $\ell^2$; while $Z$ is interpolating for the multiplier algebra $M(H)$ if $Q : \psi \mapsto \{\psi(z_n)\}_{n=0}^{\infty}$ maps $M(H)$ boundedly onto $\ell^\infty$. The reader is referred to [1] and to the second chapter of [8] for more on this topic.

It is a reasonable guess that the *universal interpolating sequences* for $\mathcal{D}_d$ and for its multiplier space $M(\mathcal{D}_d)$ are characterized by a Carleson condition and a separation condition, as described in [8] (see the Conjecture at p. 33). See also [6], which contains the best known result on interpolation in general RKHS spaces with the CNP property. Unfortunately we do not have an answer for the spaces $\mathcal{D}_d$.

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