Research Article

Generating q-Commutator Identities and the q-BCH Formula

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Motivated by the physical applications of q-calculus and of q-deformations, the aim of this paper is twofold. Firstly, we prove the q-deformed analogue of the celebrated theorem by Baker, Campbell, and Hausdorff for the product of two exponentials. We deal with the q-exponential function $\exp_q(x) = \sum_{n=0}^{\infty}(x^n/[n]_q!)$, where $[n]_q = 1 + q + \cdots + q^{n-1}$ denotes, as usual, the nth q-integer. We prove that if x and y are any noncommuting indeterminates, then $\exp_q(x)\exp_q(y) = \exp_q(x + y + \sum_{n=2}^{\infty} Q_n(x, y))$, where $Q_n(x, y)$ is a sum of iterated q-commutators of x and y (on the right and on the left, possibly), where the q-commutator $[y, x]_q := yx - qxy$ has always the innermost position. When $[y, x]_q = 0$, this expansion is consistent with the known result by Schützenberger-Cigler: $\exp_q(x)\exp_q(y) = \exp_q(x + y)$. Our result improves and clarifies some existing results in the literature. Secondly, we provide an algorithmic procedure for obtaining identities between iterated q-commutators (of any length) of x and y. These results can be used to obtain simplified presentation for the summands of the q-deformed Baker-Campbell-Hausdorff Formula.

1. Introduction

The celebrated Baker-Campbell-Hausdorff (BCH, for short in the sequel) Theorem allows the representation of the product of two exponentials in terms of a single exponential (see [1] for a comprehensive investigation of this result). The applications of the BCH Theorem range over many areas of mathematics and physics, including theoretical physics, quantum statistical mechanics, perturbation and transformation theory, the representation of time-evolution in quantum mechanics in terms of the exponential of the Hamiltonian, the study of nonclassical (i.e., coherent, squeezed) states of light, group theory, control theory, the exponentiation of Lie algebras into Lie groups, linear subelliptic PDEs, and geometric integration in numerical analysis. A quite extensive review of exponential operators and their many roles in physics was presented by Wilcox [2]. In order to motivate the main topics of the present paper (i.e., the q-deformed BCH Formula, and an algorithm for generating q-commutator identities), we first review what is known so far as the q-analogue (or q-deformation) of the BCH Theorem, along

with motivations for the physical interest in this subject; see also [3] by the authors with Achilles.

The idea of q-deformation goes back to Euler in the mid eighteenth century and to Gauss in the early nineteenth century. If one defines the nth q-integer as $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and, accordingly, if the q-factorial is defined as $[n]_q! = [n-1]_q! \cdot [n]_q$ (where $[0]_q! = 1$), then the q-exponential is

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}. \tag{1}$$

It is well known that Jackson's q-derivative, defined by the ratio

$$D_q f(x) = \frac{f(qx) - f(x)}{x(q-1)} \tag{2}$$

satisfies $D_q \exp_q(x) = \exp_q(x)$ (see, e.g., the monograph [4] for an introduction to these topics). The reader is referred to the recent monograph [5] for an in-depth analysis and

a comprehensive historical presentation of q-calculus. The advent of quantum groups, some thirty years ago [6], gave rise to vigorous renewed interest in q-deformations, although the symmetric q-integers introduced in that context, namely, $\{n\}_q := (q^n - q^{-n})/(q - q^{-1})$ (hence the corresponding q-exponential), differ from those defined above.

The earliest introduction of q-deformations into physics is probably due to Arik and Coon [7], who studied the q-deformed oscillator, whose creation and annihilation operators, $a^\dagger$ and a, satisfy $[a, a^\dagger]_q := aa^\dagger - qa^\dagger a = 1$. The conventional boson operators $[a, a^\dagger]_1 = 1$ satisfy a special case of the BCH identity:

$$\exp\left(\alpha a^\dagger + \beta a\right) = \exp\left(\alpha a^\dagger\right) \exp\left(\beta a\right) \exp\left(\frac{\alpha\beta}{2}\right), \quad (3)$$

which allows transformation from normal to symmetric ordering of the boson operators. The failure to obtain q-analogue of this relation was referred to in [8] as an instance of the *"q-Campbell-Baker-Hausdorff enigma."* In connection with this problem, in a very recent note [3] we addressed this "enigma," and we announced the q-deformed BCH Theorem (which we prove here), providing improvements of existing partial results previously given in [8, 9].

The q-deformation of the BCH identity has also been considered, from very different points of view, in [10–13]; as a related topic to q-exponentiation, see [14–17] for the q-analogue of the Zassenhaus formula; as another related topic, see also the very recent works [18–20] for the q-analogue of the so-called pre-Lie Magnus expansion. q-exponentials (and their products) naturally appear in the study of the group-like elements in Hopf algebras; see [21, 22]; for the relation to Hall algebras and quantum groups, see also [23–25].

One mathematical motivation for the study of q-deformations concerns the fact that they allow a more refined view of the features of the systems involved. Thus, the characterization of the irreducible representations of the special unitary group SU(n) requires the specification of the eigenvalues of all the $n - 1$ Casimir operators, whereas the fundamental Casimir operator is sufficient for characterizing the irreducible representations of the corresponding quantum groups [26]. This is a consequence of the fact that the eigenvalues of the latter are polynomials in q rather than integers. An encyclopedic treatment of q-deformed special functions is provided by Vilenkin and Klimyk [27].

In physical applications one can identify two types of motivations for the study of q-deformations. On the one hand, q-deformations are invoked as generalizations of fundamental theories (see Wachter [28]). On the other hand, they are proposed as models or approximations of more complicated Hamiltonians. Thus, the Arik-Coon q-oscillator mentioned above has been used to mimic anharmonicities in molecular vibrations [29]. Composite bosons and quasi-bosonic elementary excitations such as excitons have also been modelled by q-deformed boson operators [30] that were shown to exhibit Bose-Einstein condensation in two dimensions [31].

Nowadays, as described in the recent monograph by Ernst [5], the interest in q-calculus deserves no further motivation,

due to its wide applications, in addition to the physical contexts described above, to many branches of pure mathematics as well: from analytic number theory to noncommutative geometry, from combinatorics to hypergeometric function theory.

After this short introduction on the physical interest in q-deformation and q-exponentiation, we now focus on the q-analogue of the BCH Theorem and on the identities between q-commutators. Let us denote by $[a, b]_q := ab - qba$ the so-called q-commutator (also abbreviated as *q-mutator*) of a and b. If x and y are any symbols, by an *iterated q-commutator centered at* $[y, x]_q$ (see also Definition 2) we mean any arbitrarily long polynomial in x, y of the form

$$\beta_{z_1} \circ \beta_{z_2} \circ \cdots \circ \beta_{z_n} \left([y, x]_q\right), \quad (4)$$

where $z_1, \ldots, z_n$ may be any of x and y and where β_z denotes indifferently a left or a right q-commutator operator, that is, any of the maps $\mathrm{ad}_q z$ and $\mathrm{ad}_q^* z$ defined by

$$a \longmapsto \mathrm{ad}_q z\,(a) := [z, a]_q,$$
$$\text{or } a \longmapsto \mathrm{ad}_q^* z\,(a) := [a, z]_q. \quad (5)$$

For example, $[[[y, [x, [y, x]_q]_q]_q, y]_q, x]_q$ is an iterated q-commutator centered at $[y, x]_q$, whereas $[x, y]_q$ or $[y, [x, x]_q]_q$ are not centered at $[y, x]_q$.

In a very recent note [3], we announced the following result: if $\exp_q$ is as in (1), then

$$\exp_q (x) \exp_q (y) = \exp_q \left(z_q (x, y)\right), \quad (6)$$

where the formal power series $z_q(x, y)$ can be represented by infinitely many summands, each of which is an iterated q-commutator *centered at* $[y, x]_q$. In [3] we also provided, without proof, an explicit expression for these summands; we prove these results in the present paper (see Theorem 1).

Our formula (6) is consistent with the 1953 result by Schützenberger [32] (see also Cigler [33]), ensuring that

$$\exp_q (x) \exp_q (y) = \exp_q (x + y) \quad \text{iff } [y, x]_q = 0. \quad (7)$$

Whereas from (7) it follows that the series $z_q(x, y)$ satisfying (6) is of the form $z_q(x, y) = x + y + z_q'(x, y)$, where $z_q'(x, y)$ is an infinite sum of *polynomials* each containing the factor $[y, x]_q$, it does not follow the fact that $z_q'(x, y)$ is a series of *iterated q-commutators centered at* $[y, x]_q$. About this question, which we now answer, we recall that

(a) it was posed and (only) partially solved in the 1995 work [8] by the second-named author and Duchamp;

(b) in 1983, Reiner [9] showed that $z_q'(x, y)$ is a series of right-nested q-commutators of x and y: by the latter we mean any expression of the form (see also (5))

$$\mathrm{ad}_{z_1} \circ \mathrm{ad}_{z_2} \circ \cdots \circ \mathrm{ad}_{z_n} (z_0), \quad (8)$$

where $z_0, z_1, \ldots, z_n$ may be any of x or y;

(c) a crucial tool in our arguments is provided by the following identity:

$$ab = \frac{1}{1-q^2}\left([a,b]_q + q\,[b,a]_q\right), \tag{9}$$

transforming products into q-mutators.

According to Reiner's result recalled in (b), the innermost q-commutator may be (and often will be) any of

$$[x,y]_q,$$
$$[y,x]_q,$$
$$[x,x]_q, \tag{10}$$
$$[y,y]_q,$$

but this expansion does not imply Schützenberger's result, where only $[y,x]_q$ is expected. To this extent our result improves Reiner's result. Broadly speaking, we renounce Reiner's (left- or right-) nested presentation, in favor of presentation with centered $[y,x]_q$ and with iterated commutators (in the above sense), consistently with Schützenberger [32]. Incidentally, due to its relevance in this context, we provide a result playing a role analogous to that of the Dynkin-Specht-Wever Lemma, characterizing the q-commutators centered at $[y,x]_q$.

An expansion of the series $z'_q(x,y)$ up to fourth order in terms of nested q-commutators that depend both on $[y,x]_q$ and $[x,y]_q$ was obtained by the second-named author and Solomon [34], and it was claimed that the dependence on $[x,y]_q$ can be eliminated (consistently with Schützenberger [32]) by means of some (unspecified) operator identities. In the present paper *we determine these operator identities*.

As a byproduct, we exhibit the expansion of $z_q(x,y)$ up to degree 4 in terms of $[y,x]_q$-centered q-mutators only:

$$z_q(x,y) = x + y - \frac{1}{[2]_q}[y,x]_q - \frac{q}{[3]_q!\,[2]_q}$$
$$\cdot\left([x,[y,x]_q]_q + [[y,x]_q,y]_q\right) - \frac{q^2}{[4]_q!\,[2]_q}$$
$$\cdot\left([[[y,x]_q,y]_q,y]_q + [y,[[y,x]_q,y]_q]_q\right.$$
$$\left. + [[x,[y,x]_q]_q,x]_q + [x,[x,[y,x]_q]_q]_q\right)$$
$$+ \frac{1}{[4]_q!\,[2]_q^3}\cdot\left(q\,[5]_q\,[y,[x,[y,x]_q]_q]_q\right.$$
$$\left. - q^2\,[3]_q\,[x,[y,[y,x]_q]_q]_q\right.$$

$$\left. - q^3\,[y,[[y,x]_q,x]_q]_q\right.$$
$$\left. - q^2\,(q^2+3q+1)\,[x,[[y,x]_q,y]_q]_q\right)$$
$$+ \{\text{summands of length} \geq 5 \text{ in } x, y\}. \tag{11}$$

Incidentally, we observe a striking novelty of the q-BCH series compared to the classical undeformed BCH series: the latter does not contain summands with three x and one y or three y and one x (this is due to the properties of the Bernoulli numbers; see [1]), whereas the q-BCH series does. In the formal limit as $q \to 1$ in the above expansion, these summands disappear, due to the skew symmetry of the classical commutator. In a forthcoming study, we shall investigate higher degrees, and we shall also consider computational issues and implementation using the computer algebra system REDUCE [35].

The ultimate goal (to which we shall devote future investigations) will be the analysis of the formal limit as $q \to 1$ of our expansion, which would eventually provide a brand new proof of the classical undeformed BCH Theorem, a problem which seems highly nontrivial, since it is interlaced with the identities holding true among q-commutators. Finally, we hope that an understanding of the q-BCH Formula will shed light on q-Zassenhaus and q-Magnus expansions as well.

As an application (see the Appendix), we show that our explicit q-mutator expansion is convergent in any Banach algebra (when $|q| < 1$); see Theorem A.1: in particular, this is true in any matrix algebra or (more generally) in any finite dimensional associative algebra. This parallels the classical undeformed case, where it is possible to use the Dynkin expansion [36], to give a domain of convergence for the BCH series. Furthermore, we hope that this convergence result may be useful to shed light on the q-analogue of the classical undeformed passage from the Lie algebra to the Lie group multiplication.

Although we crucially use the underlying (free) associative structure of $\mathbb{K}(q)\langle\!\langle x,y\rangle\!\rangle$ (the algebra of the formal power series in x,y with coefficients in $\mathbb{K}(q)$) to obtain a closed formula for the q-BCH series, the proof of the convergence of the latter is obtained only by using the estimate

$$\left\|[a,b]_q\right\| \leq M\,\|a\|\cdot\|b\|, \tag{12}$$

for some constant $M > 0$, and this suggests that our presentation of the q-BCH series in terms of q-mutators may be of relevance for the study of other contexts, with nontrivial commutation identities. Despite the lack of nontrivial relations in $\mathbb{K}(q)\langle\!\langle x,y\rangle\!\rangle$, the analysis in the free associative setting is intended as a first step towards a future comprehension of structures with nontrivial relations, like quantum groups or, more generally, Hopf algebras. To the best of our knowledge, even in the free associative setting, the analysis of the q-commutator-form of the q-deformed BCH series, along with its local convergence in Banach algebras, appears here for the first time.

As happening for the classical BCH Formula (starting, e.g., from Dynkin's expansion [36]), in order to get the above simplified expansion starting from our general formula for $z_q(x, y)$ in Theorem 1, one has to take into account the linear dependency relations among the q-mutators of the same bidegree in x, y. *In this paper we furnish an algorithm to obtain these identities for any bidegree.*

Our procedure for generating q-commutator identities is fully described in Section 3; here we anticipate the main tools: along with identity (9) (which transforms the left and right multiplications into q-commutations), we shall combine the following identities:

$$\left[[a, b]_q, a\right]_q = \left[a, [b, a]_q\right]_q,$$

$$\left[a, [c, b]_q\right]_q + \left[b, [c, a]_q\right]_q = \left[[b, c]_q, a\right]_q$$
$$+ \left[[a, c]_q, b\right]_q, \tag{13}$$

$$\left[a, [c, b]_q\right]_q - \left[[a, c]_q, b\right]_q = \frac{q}{q^2 + 1}\left(\left[[c, a]_q, b\right]_q\right.$$

$$\left. - \left[[c, b]_q, a\right]_q + \left[b, [a, c]_q\right]_q - \left[a, [b, c]_q\right]_q\right),$$

holding true for any a, b, c. Clearly, if we repeatedly apply to these identities any choice of

$$\mathrm{ad}_q x,$$
$$\mathrm{ad}_q y,$$
$$\mathrm{ad}_q^* x, \tag{14}$$
$$\mathrm{ad}_q^* y$$

(see the notation in (5)), we obtain new identities between q-commutators. In Section 3 we shall investigate an algorithm to obtain the *smallest* number of independent (linear) identities existing among the generators (4) of the $[y, x]_q$-centered q-mutators of a fixed bidegree (i, j) in x and y.

In order to show the efficiency of our procedure for generating q-commutator identities (and the nontriviality of the dependency relations among q-mutators of a fixed bidegree), we close the introduction by showing, as an example, the set of 15 independent identities obtained with our algorithm for the 24 generators of the $[y, x]_q$-centered q-mutators of bidegree $(2, 3)$ in x, y (i.e., of degree 2 in x and degree 3 in y): denoting the generators by

$$b_1 = \mathrm{ad}_q x \circ \mathrm{ad}_q y \circ \mathrm{ad}_q y \left([y, x]_q\right),$$

$$b_2 = \mathrm{ad}_q y \circ \mathrm{ad}_q x \circ \mathrm{ad}_q y \left([y, x]_q\right),$$

$$b_3 = \mathrm{ad}_q y \circ \mathrm{ad}_q y \circ \mathrm{ad}_q x \left([y, x]_q\right),$$

$$b_4 = \mathrm{ad}_q^* x \circ \mathrm{ad}_q y \circ \mathrm{ad}_q y \left([y, x]_q\right),$$

$$b_5 = \mathrm{ad}_q^* y \circ \mathrm{ad}_q x \circ \mathrm{ad}_q y \left([y, x]_q\right),$$

$$b_6 = \mathrm{ad}_q^* y \circ \mathrm{ad}_q y \circ \mathrm{ad}_q x \left([y, x]_q\right),$$

$$b_7 = \mathrm{ad}_q x \circ \mathrm{ad}_q^* y \circ \mathrm{ad}_q y \left([y, x]_q\right),$$

$$b_8 = \mathrm{ad}_q y \circ \mathrm{ad}_q^* x \circ \mathrm{ad}_q y \left([y, x]_q\right),$$

$$b_9 = \mathrm{ad}_q y \circ \mathrm{ad}_q^* y \circ \mathrm{ad}_q x \left([y, x]_q\right),$$

$$b_{10} = \mathrm{ad}_q^* x \circ \mathrm{ad}_q^* y \circ \mathrm{ad}_q y \left([y, x]_q\right),$$

$$b_{11} = \mathrm{ad}_q^* y \circ \mathrm{ad}_q^* x \circ \mathrm{ad}_q y \left([y, x]_q\right),$$

$$b_{12} = \mathrm{ad}_q^* y \circ \mathrm{ad}_q^* y \circ \mathrm{ad}_q x \left([y, x]_q\right),$$

$$b_{13} = \mathrm{ad}_q x \circ \mathrm{ad}_q y \circ \mathrm{ad}_q^* y \left([y, x]_q\right),$$

$$b_{14} = \mathrm{ad}_q y \circ \mathrm{ad}_q x \circ \mathrm{ad}_q^* y \left([y, x]_q\right),$$

$$b_{15} = \mathrm{ad}_q y \circ \mathrm{ad}_q y \circ \mathrm{ad}_q^* x \left([y, x]_q\right),$$

$$b_{16} = \mathrm{ad}_q^* x \circ \mathrm{ad}_q y \circ \mathrm{ad}_q^* y \left([y, x]_q\right),$$

$$b_{17} = \mathrm{ad}_q^* y \circ \mathrm{ad}_q x \circ \mathrm{ad}_q^* y \left([y, x]_q\right),$$

$$b_{18} = \mathrm{ad}_q^* y \circ \mathrm{ad}_q y \circ \mathrm{ad}_q^* x \left([y, x]_q\right),$$

$$b_{19} = \mathrm{ad}_q x \circ \mathrm{ad}_q^* y \circ \mathrm{ad}_q^* y \left([y, x]_q\right),$$

$$b_{20} = \mathrm{ad}_q y \circ \mathrm{ad}_q^* x \circ \mathrm{ad}_q^* y \left([y, x]_q\right),$$

$$b_{21} = \mathrm{ad}_q y \circ \mathrm{ad}_q^* y \circ \mathrm{ad}_q^* x \left([y, x]_q\right),$$

$$b_{22} = \mathrm{ad}_q^* x \circ \mathrm{ad}_q^* y \circ \mathrm{ad}_q^* y \left([y, x]_q\right),$$

$$b_{23} = \mathrm{ad}_q^* y \circ \mathrm{ad}_q^* x \circ \mathrm{ad}_q^* y \left([y, x]_q\right),$$

$$b_{24} = \mathrm{ad}_q^* y \circ \mathrm{ad}_q^* y \circ \mathrm{ad}_q^* x \left([y, x]_q\right),$$

$$\tag{15}$$

we have the following 15 independent relations among them:

$$b_6 = b_9,$$

$$b_{18} = b_{21},$$

$$b_7 = b_{13},$$

$$b_{10} = b_{16},$$

$$b_7 + b_8 = b_4 + b_5,$$

$$b_{19} + b_{20} = b_{16} + b_{17},$$

$$b_{14} + b_{15} = b_8 + b_9,$$

$$b_{17} + b_{18} = b_{11} + b_{12},$$

$$\left(q^2 + 1\right)\left(b_{14} - b_9\right) = q\left(b_{21} - b_{20} + b_3 - b_2\right),$$

$$\left(q^2 + 1\right)\left(b_{17} - b_{12}\right) = q\left(b_{24} - b_{23} + b_6 - b_5\right),$$

$$\left(q^2 + 1\right)\left(b_7 - b_5\right) = q\left(b_{11} - b_{10} + b_2 - b_1\right),$$

$$\left(q^2 + 1\right)\left(b_{19} - b_{17}\right) = q\left(b_{23} - b_{19} + b_{14} - b_{13}\right),$$

$$\left(1 - q + q^2\right)\left(b_{20} - b_3\right)$$

$$= q\left(b_9 + b_{15} + b_{21} - b_{14} - b_8 - b_2\right),$$

$$\left(1 - q + q^2\right)\left(b_{23} - b_6\right)$$

$$= q\left(b_{12} + b_{18} + b_{24} - b_{17} - b_{11} - b_5\right),$$

$$\left(1 + q^4\right)b_{22} + q\left(1 + q^2\right)\left(b_{19} + b_{16} + b_{10} + b_1\right)$$

$$+ 2q^2\left(b_{13} + b_7 + b_4\right) = \left(1 + q^4\right)b_2$$

$$+ q\left(1 + q^2\right)\left(b_{23} + b_{14} + b_8 + b_5\right)$$

$$+ 2q^2\left(b_{20} + b_{17} + b_{11}\right).$$

$$(16)$$

2. Method: The q-Deformed BCH Formula

Notation. We fix the algebraic setting we work in: $A = \mathbb{K}(q)\langle\!\langle x, y \rangle\!\rangle$ will denote the associative algebra of the formal power series in two noncommuting indeterminates x and y, with coefficients in $\mathbb{K}(q)$, which is the field of the rational functions in the symbol q over a field $\mathbb{K}$ of characteristic 0. (We recall that whereas $\mathbb{K}[q]$ usually denotes the ring of the polynomials in the indeterminate q, by $\mathbb{K}(q)$ one identifies the field of the quotients of the ring $\mathbb{K}[q]$.) The associative multiplication in A is the usual Cauchy product of formal power series. The notation $\mathbb{K}(q)\langle x, y \rangle$ will stand for the associative algebra of the polynomials in x and y over $\mathbb{K}(q)$. From now on, we introduce on A the bilinear map

$$[a, b]_q := ab - qba, \quad a, b \in A. \tag{17}$$

We say that $[a, b]_q$ is the *q-mutator* (shortcut of "q-commutator") of a and b.

Given $i, j \in \mathbb{N} \cup \{0\}$, $A_{i,j}$ denotes the set of the homogeneous polynomials in A with degree i with respect to x and degree j with respect to y. We say that any element of $A_{i,j}$ has bidegree (i, j) (with respect to x, y, resp.). We also set, for any $n \in \mathbb{N} \cup \{0\}$,

$$A_n := \text{span}\left\{A_{i,j} : i + j = n\right\}. \tag{18}$$

Thus, for example,

$$A_0 = \mathbb{K}(q),$$

$$A_1 = \text{span}_{\mathbb{K}(q)}\left\{x, y\right\}, \tag{19}$$

$$A_2 = \text{span}_{\mathbb{K}(q)}\left\{x^2, xy, yx, y^2\right\}.$$

Obviously, we have the direct sum/product decompositions

$$A_n = \bigoplus_{i+j=n} A_{i,j}, \quad \forall n \geq 0, \ A = \prod_{i,j\geq 0} A_{i,j}, \ A = \prod_{n\geq 0} A_n. \tag{20}$$

The typical element of A is therefore

$$p = p_0 + p_1 + \cdots =: \sum_{n=0}^{\infty} p_n \quad \text{where } p_n \in A_n. \tag{21}$$

Finally, we denote by I the two-sided ideal in A generated by $[y, x]_q$ and we set

$$I_{i,j} := I \cap A_{i,j}. \tag{22}$$

We can consider the quotient of A modulo I, denoted as usual by

$$\frac{A}{I} = \left\{[a]_I : a \in A\right\}. \tag{23}$$

We also use the standard notation for the equivalence $a \sim b$ modulo I:

$$a \equiv b \,(\text{mod } I)$$

$$\text{or } a \equiv_I b. \tag{24}$$

A/I is an associative algebra with the obvious operations. As I is a two-sided ideal generated by a homogeneous polynomial of bidegree $(1, 1)$, then $I = \prod_{i,j\geq 0} I_{i,j}$. Notice that, obviously,

$$I_{i,0} = \{0\} = I_{0,j} \quad \text{for every } i, j \in \mathbb{N} \cup \{0\}. \tag{25}$$

Since the lowest order term in the q-exponential series (1) is 1, there exists the inverse map of $\exp_q$, say $\log_q$ (called the q-logarithm), defined on the set $1 + A_+$, where A_+ is the set of the formal power series in x, y whose zero-degree term is null. We use the notation

$$\log_q\left(1 + p\right) = \sum_{n=1}^{\infty} \lambda_n p^n \quad \left(p \in A_+\right), \tag{26}$$

where the coefficients $\lambda_n \in \mathbb{K}(q)$ are given by the recurrence formula:

$$\lambda_1 = 1,$$

$$\lambda_n = -\sum_{i=1}^{n-1} \lambda_i \sum_{j_1+\cdots+j_i=n} \frac{1}{[j_1]_q!} \cdots \frac{1}{[j_i]_q!} \quad (n \geq 2). \tag{27}$$

Thus, the unique series $z_q(x, y)$ closing the identity (6) is

$$z_q\left(x, y\right) := \log_q\left(\exp_q\left(x\right)\exp_q\left(y\right)\right), \tag{28}$$

referred to as the *q-Baker-Campbell-Hausdorff series*, shortly, *q-BCH series*. Therefore, an explicit expression for $z_q(x, y)$ in terms of polynomials is

$$z_q\left(x, y\right)$$

$$= \sum_{n=1}^{\infty} \lambda_n \sum_{(i_1,j_1),\ldots,(i_n,j_n)\neq(0,0)} \frac{x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}}{[i_1]_q! [j_1]_q! \cdots [i_n]_q! [j_n]_q!}, \tag{29}$$

where λ_n's are as in (26).

Starting from the (tautological) identity $yx = qxy + [y,x]_q$, any monomial $x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}$ in (29) can be rewritten, modulo I, by moving any x on the left and any y on the right. Namely, one has

$$x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}$$
$$= q^{i_2 j_1 + \cdots + i_n(j_1 + j_2 + \cdots + j_{n-1})} x^{i_1 + \cdots + i_n} y^{j_1 + \cdots + j_n} \qquad (30)$$
$$+ \{\text{an element of } I\}.$$

Since, as we prove in Section 4 (starting from (7)), we have $z_q(x,y) = x + y + z'_q(x,y)$ with $z'_q(x,y) \in I$, by means of (30) one can write $z_q(x,y)$ as a series whose summands (other than $x+y$) are polynomials in I; that is, they contain the factor $[y,x]_q$.

Now, by means of the rearranging identity (9), we can write any element of I in terms of iterated q-mutators of x and y centered at $[y,x]_q$. An explicit example will clarify this: from (30) we have $xyxy^3 = qx^2 y^4 + \{\text{element of } I\}$; explicitly (by applying four times (9) in the last four equalities)

$$xyxy^3 - qx^2 y^4 = x\left([y,x]_q + qxy\right) y^3 - qx^2 y^4$$

$$= x[y,x]_q y^3 = \frac{1}{1-q^2}\left([x,[y,x]_q]_q\right.$$

$$+ q\left[[y,x]_q,x\right]_q\right) \cdot y^3$$

$$= \frac{1}{(1-q^2)^2}\left(\left[[x,[y,x]_q]_q,y\right]_q\right.$$

$$+ q\left[y,[x,[y,x]_q]_q\right]_q + q\left[[[y,x]_q,x]_q,y\right]_q$$

$$+ q^2\left[y,[[y,x]_q,x]_q\right]_q\right) \cdot y^2$$

$$= \frac{1}{(1-q^2)^3}\left(\left[\left[[x,[y,x]_q]_q,y\right]_q,y\right]_q\right.$$

$$+ q\left[y,\left[[x,[y,x]_q]_q,y\right]_q\right]_q$$

$$+ q\left[\left[y,[x,[y,x]_q]_q\right]_q,y\right]_q$$

$$+ q^2\left[y,\left[y,[x,[y,x]_q]_q\right]_q\right]_q$$

$$+ q\left[\left[[[y,x]_q,x]_q,y\right]_q,y\right]_q$$

$$+ q^2\left[y,\left[[[y,x]_q,x]_q,y\right]_q\right]_q$$

$$+ q^2\left[\left[y,[[y,x]_q,x]_q\right]_q,y\right]_q$$

$$+ q^3\left[y,\left[y,[[y,x]_q,x]_q\right]_q\right]_q\right) \cdot y$$

$$= \frac{1}{(1-q^2)^4}\left(\left[\left[\left[[x,[y,x]_q]_q,y\right]_q,y\right]_q,y\right]_q\right.$$

$$+ q\left[y,\left[\left[[x,[y,x]_q]_q,y\right]_q,y\right]_q\right]_q$$

$$+ q\left[\left[y,\left[[x,[y,x]_q]_q,y\right]_q\right]_q,y\right]_q$$

$$+ q^2\left[y,\left[y,\left[[x,[y,x]_q]_q,y\right]_q\right]_q\right]_q$$

$$+ q\left[\left[\left[y,[x,[y,x]_q]_q\right]_q,y\right]_q,y\right]_q$$

$$+ q^2\left[y,\left[\left[y,[x,[y,x]_q]_q\right]_q,y\right]_q\right]_q$$

$$+ q^2\left[\left[y,\left[y,[x,[y,x]_q]_q\right]_q\right]_q,y\right]_q$$

$$+ q^3\left[y,\left[y,\left[y,[x,[y,x]_q]_q\right]_q\right]_q\right]_q$$

$$+ q\left[\left[\left[[[y,x]_q,x]_q,y\right]_q,y\right]_q,y\right]_q$$

$$+ q^2\left[y,\left[\left[[[y,x]_q,x]_q,y\right]_q,y\right]_q\right]_q$$

$$+ q^2\left[\left[y,\left[[[y,x]_q,x]_q,y\right]_q\right]_q,y\right]_q$$

$$+ q^3\left[y,\left[y,\left[[[y,x]_q,x]_q,y\right]_q\right]_q\right]_q$$

$$+ q^2\left[\left[\left[y,[[y,x]_q,x]_q\right]_q,y\right]_q,y\right]_q$$

$$+ q^3\left[y,\left[\left[y,[[y,x]_q,x]_q\right]_q,y\right]_q\right]_q$$

$$+ q^3\left[\left[y,\left[y,[[y,x]_q,x]_q\right]_q\right]_q,y\right]_q$$

$$+ q^4\left[y,\left[y,\left[y,[[y,x]_q,x]_q\right]_q\right]_q\right]_q\right).$$

$$(31)$$

This methodology can be applied to any summand in (29). For example, if we use the bidegree notation

$$z_q(x, y) = x + y + \sum_{i,j \geq 1} Q_{i,j}(x, y), \tag{32}$$

where $Q_{i,j}(x, y)$ has degree i with respect to x and degree j with respect to y, we can readily obtain the associative presentation of $Q_{1,3}(x, y)$:

$$Q_{1,3}(x, y) = \frac{1}{[4]_q! \, [2]_q} \cdot \left(q^3 (-q + 1) \cdot xy^3 \right.$$

$$+ q^3 (-q + 1) \cdot y^3 x + q^2 \left(q^3 - 2q^2 + 2q - 1 \right) \tag{33}$$

$$\left. \cdot y^2 xy + q^2 \left(q^3 - 2q^2 + 2q - 1 \right) \cdot yxy^2 \right).$$

By using the cited identity $yx = qxy + [y, x]_q$ on each summand of $Q_{1,3}(x, y)$ (other than xy^3), we get

$$y^3 x = y^2 [y, x]_q + qy [y, x]_q y + q^2 [y, x]_q y^2$$

$$+ q^3 xy^3$$

$$y^2 xy = y [y, x]_q y + q [y, x]_q y^2 + q^2 xy^3 \tag{34}$$

$$yxy^2 = [y, x]_q y^2 + qxy^3.$$

Inserting these identities in the expansion of $Q_{1,3}(x, y)$ we get

$$[4]_q! \, [2]_q \cdot Q_{1,3}(x, y) = [\cdots] = \left\{ \left(-q^4 + q^3 \right) \right.$$

$$+ \left(q^3 - q^4 \right) q^3 + \left(q^5 - 2q^4 + 2q^3 - q^2 \right) q^2$$

$$+ \left(q^5 - 2q^4 + 2q^3 - q^2 \right) q \right\} \cdot xy^3 + \left(q^3 - q^4 \right)$$

$$\cdot y^2 [y, x]_q + \left(\left(q^3 - q^4 \right) q \right.$$

$$+ \left(q^5 - 2q^4 + 2q^3 - q^2 \right) \right) y [y, x]_q y \tag{35}$$

$$+ \left(\left(q^3 - q^4 \right) q^2 + \left(q^5 - 2q^4 + 2q^3 - q^2 \right) q \right.$$

$$+ \left(q^5 - 2q^4 + 2q^3 - q^2 \right) \right) [y, x]_q y^2 = \{0\} \cdot xy^3$$

$$- q^3 (q - 1) y^2 [y, x]_q - q^2 (q - 1)^2 y [y, x]_q y$$

$$+ q^2 (q - 1) [y, x]_q y^2.$$

Obviously, the cancelation of the summand xy^3 (see the curly braces) is not sheer chance, but it derives from the fact that $Q_{1,3}$ belongs to I. We next apply the technique exemplified above, based on (9), thus obtaining the presentation

$$Q_{1,3}(x, y) = -\frac{q^2}{[4]_q! \, [2]_q} \left(\left[\left[[y, x]_q, y \right]_q, y \right]_q \right.$$

$$\left. + \left[y, \left[[y, x]_q, y \right]_q \right]_q \right). \tag{36}$$

Analogously one gets

$$Q_{3,1}(x, y) = -\frac{q^2}{[4]_q! \, [2]_q} \left(\left[x, \left[x, [y, x]_q \right]_q \right]_q \right.$$

$$\left. + \left[\left[x, [y, x]_q \right]_q, x \right]_q \right). \tag{37}$$

We explicitly remark that in order to get simplifications for $Q_{1,3}(x, y)$ and $Q_{3,1}(x, y)$ one also needs to take into account the fact that

$$\left[\left[y, [y, x]_q \right]_q, y \right]_q = \left[y, \left[[y, x]_q, y \right]_q \right]_q, \tag{38}$$

which is a particular case of an identity which will take a crucial role in the sequel:

$$\left[[a, b]_q, a \right]_q = \left[a, [b, a]_q \right]_q \quad \text{for every } a, b. \tag{39}$$

No more relations intervene among the q-mutators

$$\left[y, \left[y, [y, x]_q \right]_q \right]_q,$$

$$\left[\left[[y, x]_q, y \right]_q, y \right]_q, \tag{40}$$

$$\left[y, \left[[y, x]_q, y \right]_q \right]_q,$$

which are linearly independent; it is a striking fact, however, that $Q_{1,3}(x, y)$ can be written by means of the last two only. With the same techniques we obtained the fourth-degree expansion in (11).

The above methodology in attacking the study of the q-BCH Formula shows that it is of relevance to study the following issues:

(1) to obtain an explicit expression of the q-BCH summands in (32) in terms of iterated q-mutators centered at $[y, x]_q$;

(2) to obtain identities among the iterated q-mutators centered at $[y, x]_q$, allowing simplifying the presentation of $Q_{i,j}(x, y)$'s and studying bases/dependence-relations in the spaces of the q-mutators centered at $[y, x]_q$.

The answer to the first issue is given by the following theorem which we prove in Section 4; the second problem is investigated in the next section.

Theorem 1. *The q-BCH series has form (32), where any homogeneous summand $Q_{i,j}(x, y)$ is given by the following formula, as a linear combination of iterated $[y, x]_q$-centered q-mutators:*

$$Q_{i,j}(x, y) = \sum_{n=1}^{i+j} \lambda_n$$

$$\cdot \sum_{\substack{(i_1, j_1), \ldots, (i_n, j_n) \neq (0,0) \\ i_1 + \cdots + i_n = i, \, j_1 + \cdots + j_n = j}} \frac{1}{[i_1]_q! \, [j_1]_q! \cdots [i_n]_q! \, [j_n]_q!}$$

$$\cdot \frac{1}{\left(1-q^2\right)^{i-1}} \sum_{r=1}^{n-1} \frac{q^{i_2 j_1 + \cdots + i_r(j_1 + \cdots + j_{r-1})}}{\left(1-q^2\right)^{j_{r+1}+j_{r+2}+\cdots+j_n}}$$

$$\cdot \sum_{k=0}^{j_1+\cdots+j_r-1} \frac{q^k}{\left(1-q^2\right)^k} \binom{j_1+\cdots+j_r}{k} \sum_{h=0}^{i_{r+1}-1} q^{h(j_1+\cdots+j_r)}$$

$$\cdot \left(\mathrm{ad}_q^* y + q\mathrm{ad}_q y\right)^{j_n} \cdots \left(\mathrm{ad}_q^* x + q\mathrm{ad}_q x\right)^{i_{r+2}}$$

$$\cdot \left(\mathrm{ad}_q^* y + q\mathrm{ad}_q y\right)^{j_{r+1}} \left(\mathrm{ad}_q^* x + q\mathrm{ad}_q x\right)^{i_{r+1}-h-1}$$

$$\cdot \left(\mathrm{ad}_q^* y + q\mathrm{ad}_q y\right)^{k} \left(\mathrm{ad}_q x + q\mathrm{ad}_q^* x\right)^{h+i_1+\cdots+i_r}$$

$$\cdot \left(\mathrm{ad}_q y\right)^{j_1+\cdots+j_r-k} (x).$$

$$(41)$$

Here the numbers λ_n are the coefficients of the expansion of the q-logarithm in (27). Finally, any power of $\mathrm{ad}_q z + q\mathrm{ad}_q^* z$ and $\mathrm{ad}_q^* z + q\mathrm{ad}_q z$ (with $z = x$ or y) can be further expanded by Newton's binomial, since $\mathrm{ad}_q z$ and $\mathrm{ad}_q^* z$ commute for every z (as it derives from (39); see also (5) for the meaning of ad_q, ad_q^*).

3. The Identity-Generating Technique

In this section we provide one of our two main results: an algorithm for the generation of identities between iterated q-mutators. We fix the definitions used in the sequel (see also the notations for A, $A_{i,j}$, I, $I_{i,j}$ introduced at the beginning of Section 2).

Definition 2. One gives the following three definitions.

(i) Fixing $b \in A$, one sets

$$R_b : A \longrightarrow A, \quad R_b(a) := ab, \qquad (42)$$

$$L_b : A \longrightarrow A, \quad L_b(a) := ba. \qquad (43)$$

In other words, R_b and L_b are, respectively, the right and left multiplications in the associative algebra A.

(ii) Let one use notation (5) for the right/left q-adjoint operators ad_q^* and ad_q; given $z \in A$, any q-mutator of the form

$$\left(\mathrm{ad}_q^* x\right)^{h_1} \left(\mathrm{ad}_q^* y\right)^{k_1} \left(\mathrm{ad}_q x\right)^{h_1'} \left(\mathrm{ad}_q y\right)^{k_1'} \cdots \left(\mathrm{ad}_q^* x\right)^{h_n}$$

$$\cdot \left(\mathrm{ad}_q^* y\right)^{k_n} \left(\mathrm{ad}_q x\right)^{h_n'} \left(\mathrm{ad}_q y\right)^{k_n'} (z) \qquad (44)$$

(with $n \in \mathbb{N}$ and $h_1, k_1, h_1', k_1', \ldots, h_n, k_n, h_n', k_n' \in \mathbb{N} \cup \{0\}$) will be called an *iterated q-mutator* (of x and y) *centered at z* (or z-*centered*). For one's aims, one shall be interested in $[y, x]_q$-centered q-mutators only.

(iii) With the above notation, for every $i, j \in \mathbb{N}$ one denotes by $S_{i,j}$ the subspace of $I_{i,j}$ spanned by the $[y, x]_q$-centered q-mutators (44) additionally satisfying

$$h_1 + h_1' + \cdots + h_n + h_n' = i - 1,$$
$$k_1 + k_1' + \cdots + k_n + k_n' = j - 1. \qquad (45)$$

If $i = 0$ or $j = 0$, one sets $S_{i,j} := \{0\}$. Furthermore one sets

$$S := \sum_{i,j \in \mathbb{N}\cup\{0\}} S_{i,j}, \qquad (46)$$

to denote the formal power series in A with summands in the sets $S_{i,j}$'s.

The letter "S" has been chosen to remind us of Schützenberger's result (7). For example, $x^3 y[y, x]_q y$ belongs to $I_{4,3}$, while $[x, [x, [y, [[[y, x]_q, x]_q, y]_q]_q]_q]_q$ belongs to $S_{4,3}$. A priori, whereas it is trivial that $S_{i,j} \subseteq I_{i,j}$, it is not at all obvious that $I_{i,j} = S_{i,j}$, which is stated in the next result.

Lemma 3. *With the notation in Definition 2, one has*

$$I_{i,j} = S_{i,j} \quad \text{for every } i, j \geq 0. \qquad (47)$$

The proof of this result is contained in Proposition 14.

3.1. Some Dimensions. We next take into account the space S of the $[y, x]_q$-centered polynomials of bidegree (i, j). For example, we have (all spans are understood over the field $\mathbb{K}(q)$)

$$S_{1,1} = \mathrm{span}\left\{[y, x]_q\right\}$$

$$S_{2,1} = \mathrm{span}\left\{\left[x, [y, x]_q\right]_q, \left[[y, x]_q, x\right]_q\right\}$$

$$S_{1,2} = \mathrm{span}\left\{\left[y, [y, x]_q\right]_q, \left[[y, x]_q, y\right]_q\right\}$$

$$S_{3,1} = \mathrm{span}\left\{\left[x, \left[x, [y, x]_q\right]_q\right]_q, \left[\left[x, [y, x]_q\right]_q, x\right]_q, \right.$$
$$\left. \left[x, \left[[y, x]_q, x\right]_q\right]_q, \left[\left[[y, x]_q, x\right]_q, x\right]_q\right\}$$

$$S_{2,2} = \mathrm{span}\left\{\left[y, \left[x, [y, x]_q\right]_q\right]_q, \left[\left[x, [y, x]_q\right]_q, y\right]_q, \right.$$
$$\left[y, \left[[y, x]_q, x\right]_q\right]_q, \left[\left[[y, x]_q, x\right]_q, y\right]_q,$$
$$\left[x, \left[y, [y, x]_q\right]_q\right]_q, \left[\left[y, [y, x]_q\right]_q, x\right]_q,$$
$$\left. \left[x, \left[[y, x]_q, y\right]_q\right]_q, \left[\left[[y, x]_q, y\right]_q, x\right]_q\right\}$$

$$S_{1,3} = \text{span} \left\{ \left[y, \left[y, [y,x]_q \right]_q \right]_q, \left[\left[y, [y,x]_q \right]_q, y \right]_q, \right.$$
$$\left. \left[y, \left[[y,x]_q, y \right]_q \right]_q, \left[\left[[y,x]_q, y \right]_q, y \right]_q \right\}.$$

$$(48)$$

The above spaces are expressed in terms of generators, not all of which may be linearly independent (nor different!). For example one has (due to (39))

$$\left[\left[x, [y,x]_q \right]_q, x \right]_q = \left[x, \left[[y,x]_q, x \right]_q \right]_q,$$
$$\left[\left[y, [y,x]_q \right]_q, y \right]_q = \left[y, \left[[y,x]_q, y \right]_q \right]_q,$$

$$(49)$$

and it can be proved that no other dependency relations hold among the generators of $S_{3,1}$ or the generators of $S_{1,3}$, so that $\dim(S_{3,1}) = \dim(S_{1,3}) = 3$.

The problem of determining the dimension of $S_{i,j}$ is rather simple (see Proposition 4), whereas the problem of discovering the dependency relations among the generators of a given $S_{i,j}$ is much more difficult: here we determine the pertinent number of relations and we propose an algorithm for discovering all of them.

For example, we consider the case of total degree $i + j = 4$: one can prove that $\dim(S_{2,2}) = 5$ and that the dependency relations among the 8 generators of $S_{2,2}$ are the following three:

$$\left[\left[x, [y,x]_q \right]_q, y \right]_q - \left[x, \left[[y,x]_q, y \right]_q \right]_q$$
$$= \left[y, \left[[y,x]_q, x \right]_q \right]_q - \left[\left[y, [y,x]_q \right]_q, x \right]_q$$
$$= \left[y, \left[x, [y,x]_q \right]_q \right]_q - \left[\left[[y,x]_q, y \right]_q, x \right]_q$$
$$= \frac{q}{[3]_q} \left(\left[x, \left[y, [y,x]_q \right]_q \right]_q - \left[\left[[y,x]_q, x \right]_q, y \right]_q \right).$$

$$(50)$$

These identities may obviously produce infinitely many others; for example (as we shall see by a very general procedure for obtaining identities), hidden in the above identities one has

$$\left[\left[[y,x]_q, x \right]_q, y \right]_q - \left[\left[[y,x]_q, y \right]_q, x \right]_q$$
$$+ \left[y, \left[x, [y,x]_q \right]_q \right]_q - \left[x, \left[y, [y,x]_q \right]_q \right]_q$$
$$= \left(q + \frac{1}{q} \right)$$
$$\cdot \left(\left[\left[y, [y,x]_q \right]_q, x \right]_q - \left[y, \left[[y,x]_q, x \right]_q \right]_q \right).$$

$$(51)$$

In the next result it is understood that the field underlying all vector space structures is $\mathbb{K}(q)$. Along with other dimensional facts, we aim to count the following set of generators of $S_{i,j}$: these are the iterated q-mutators of the form

$$\beta_1 \circ \beta_2 \circ \cdots \circ \beta_{i+j-2} \left([y,x]_q \right), \qquad (52)$$

where $\beta_1, \ldots, \beta_{i+j-2}$ all belong to the set of maps $\{\text{ad}_q^* x, \text{ad}_q^* y, \text{ad}_q x, \text{ad}_q y\}$ in such a way that x appears exactly $i-1$ times and y appears exactly $j-1$ times (if $i = 1$ or $j = 1$ it is understood that these maps are not counted).

Proposition 4 (dimensions). *Let $i, j \in \mathbb{N}$. Let the vector space $S_{i,j} \subseteq A_{i,j}$ be as in Definition 2. Then one has the following:*

(i) $\dim(A_{i,j}) = \binom{i+j}{i}$;

(ii) $d(i, j) := \dim(S_{i,j}) = \binom{i+j}{i} - 1$;

(iii) *the number of possible $[y,x]_q$-centered q-mutators writable as in (52) defining $S_{i,j}$ is*

$$n(i, j) := \binom{i + j - 2}{i - 1} \cdot 2^{i+j-2}; \qquad (53)$$

(iv) *the number of the linearly independent dependency relations among the list of the q-mutators in part (iii) above is*

$$r(i, j) := n(i, j) - d(i, j)$$
$$= \binom{i + j - 2}{i - 1} \cdot 2^{i+j-2} - \binom{i + j}{i} + 1. \qquad (54)$$

A clarification of point (iii) above is needed: here we are counting separately any of the formal objects in (52) even if, *a posteriori*, some of these q-mutators may be equal. In other words, we count the $(i+j-2)$-tuples $(\beta_1, \beta_2, \ldots, \beta_{i+j-2})$, where $\beta_1, \ldots, \beta_{i+j-2}$ belong to $\{\text{ad}_q^* x, \text{ad}_q^* y, \text{ad}_q x, \text{ad}_q y\}$ in such a way that x appears exactly $i - 1$ times and y appears exactly $j - 1$ times (if $i = 1$ or $j = 1$ it is understood that these maps are not counted).

Proof. We split the proof into four steps.

(i) $\dim(A_{i,j})$ follows from the cardinality of the set

$$\left\{ x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n} : i_1 + \cdots + i_n = i, \ j_1 + \cdots + j_n = j \right\}. \qquad (55)$$

(ii) We claim that $\dim(S_{i,j})$ is one unit less than $\dim(A_{i,j})$. To this aim, we recall that in Section 2 we introduced on $A = \prod_{i,j} A_{i,j}$ the quotient modulo I, the two-sided ideal generated by $[y,x]_q$. Due to homogeneity and degree reasons, we can also consider this quotient on each $A_{i,j}$ separately and we can infer that $A_{i,j}/I$ is isomorphic to $A_{i,j}/I_{i,j}$. We therefore get $\dim(A_{i,j}/I_{i,j}) = \dim(A_{i,j}/I) = 1$. This last identity follows by (30). From Lemma 3 we know that $I_{i,j} = S_{i,j}$, so that $\dim(A_{i,j}/S_{i,j}) = 1$. Since $S_{i,j}$ is a vector subspace of $A_{i,j}$, this gives the claimed $\dim(S_{i,j}) = \dim(A_{i,j}) - 1$.

(iii) If $i = j = 1$ the only object in (52) is $[y, x]_q$ and (53) is correct, providing $n(1, 1) = 1$. If $i \neq 1$ or $j \neq 1$, we need to count the $(i + j - 2)$-tuples $(\beta_1, \beta_2, \ldots, \beta_{i+j-2})$, where $\beta_1, \ldots, \beta_{i+j-2} \in \{\mathrm{ad}_q^* x, \mathrm{ad}_q^* y, \mathrm{ad}_q x, \mathrm{ad}_q y\}$, and x, y, respectively, appear $i - 1$ and $j - 1$ times. These $(i + j - 2)$-tuples are in bijective relation with the formal monomials

$$X^{a_1} Y^{b_1} \cdots X^{a_n} Y^{b_n} \quad \text{with} \quad \begin{cases} a_1 + \cdots + a_n = i - 1 \\ b_1 + \cdots + b_n = j - 1, \end{cases} \tag{56}$$

where $X \in \{\mathrm{ad}_q^* x, \mathrm{ad}_q x\}$ and $Y \in \{\mathrm{ad}_q^* y, \mathrm{ad}_q y\}$, via the identification

$$\left(\beta_1, \ldots, \beta_{i+j-2} \right) \longleftrightarrow$$
$$\left(\underbrace{X, \ldots, X}_{a_1 \text{ times}}, \underbrace{Y, \ldots, Y}_{b_1 \text{ times}}, \ldots, \underbrace{X, \ldots, X}_{a_n \text{ times}}, \underbrace{Y, \ldots, Y}_{b_n \text{ times}} \right). \tag{57}$$

Now, the number of the monomials in (56) is precisely $\binom{i+j-2}{i-1}$, and for any such a monomial we can choose X's in $2^{a_1 + \cdots + a_n} = 2^{i-1}$ different ways and Y's in $2^{b_1 + \cdots + b_n} = 2^{j-1}$ different ways; this proves (53).

(iv) This follows from (ii) and (iii).

$\square$

Remark 5. Proposition 4 provides us with a very simple basis for $S_{i,j}$ (which is not, however, constituted of iterated q-mutators of x and y as in (52)). Indeed, from (30) we know that

$$x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n} - q^{i_2 j_1 + \cdots + i_n(j_1 + j_2 + \cdots + j_{n-1})} x^i y^j \in S_{i,j}, \tag{58}$$

whenever $i_1 + \cdots + i_n = i$ and $j_1 + \cdots + j_n = j$. Taking into account that $\dim(S_{i,j}) = \dim(A_{i,j}) - 1$, it is then very easy to construct a basis for $S_{i,j}$ by means of this procedure.

An example will clarify this. $A_{2,4}$ is spanned by the 15 monomials

$$x^2 y^4,$$
$$xyxy^3,$$
$$xy^2 xy^2,$$
$$xy^3 xy,$$
$$xy^4 xy^0,$$
$$x^0 yx^2 y^3,$$
$$x^0 yxyxy^2,$$
$$x^0 yxy^2 xy,$$
$$x^0 yxy^3 xy^0,$$

$$x^0 y^2 x^2 y^2,$$
$$x^0 y^2 xyxy,$$
$$x^0 y^2 xy^2 xy^0,$$
$$x^0 y^3 x^2 y,$$
$$x^0 y^3 xyxy^0,$$
$$x^0 y^4 x^2 y^0. \tag{59}$$

We apply the procedure in (58) to all of these monomials except for the first:

$$xyxy^3 - qx^2 y^4,$$
$$xy^2 xy^2 - q^2 x^2 y^4,$$
$$xy^3 xy - q^3 x^2 y^4,$$
$$xy^4 xy^0 - q^4 x^2 y^4,$$
$$yx^2 y^3 - q^2 x^2 y^4,$$
$$yxyxy^2 - q^3 x^2 y^4,$$
$$yxy^2 xy - q^4 x^2 y^4,$$
$$yxy^3 x - q^5 x^2 y^4,$$
$$y^2 x^2 y^2 - q^4 x^2 y^4,$$
$$y^2 xyxy - q^5 x^2 y^4,$$
$$y^2 xy^2 x - q^6 x^2 y^4,$$
$$y^3 x^2 y - q^6 x^2 y^4,$$
$$y^3 xyx - q^7 x^2 y^4,$$
$$y^4 x^2 - q^8 x^2 y^4. \tag{60}$$

Due to (58) these 14 polynomials all belong to $S_{2,4}$ and they are (clearly) linearly independent (as they are obtained from linearly independent vectors by subtracting multiples of a given vector); since $\dim(S_{2,4}) = 14$ by Proposition 4-(ii), they form a basis for $S_{2,4}$. This also means that each of them can be written as a sum of $[y, x]_q$-centered q-mutators: it is not difficult to obtain such a representation for each of them by using the technique described in Section 2.

3.2. Producing General Identities: The Basic Maps. We are ready to provide a general technique which produces q-mutator identities. For later reference, we give for each formula/procedure a one-letter name. In the sequel,

$$[B, C]_\circ = B \circ C - C \circ B \tag{61}$$

denotes the commutator of two operators B, C with respect to the composition $\circ$ of maps (whenever this makes sense).

Here is the list of our procedures for obtaining q-mutator identities:

(T) We say that identity (9) is the *Transformation Rule*; it allows transforming polynomials (under their associative presentation) into a linear combination of iterated q-mutators. With the notation in Definition 2, (9) can be rewritten as

$$R_b = \frac{1}{1-q^2}\left(\mathrm{ad}_q^* b + q \mathrm{ad}_q b\right),$$
$$L_a = \frac{1}{1-q^2}\left(\mathrm{ad}_q a + q \mathrm{ad}_q^* a\right), \tag{62}$$

holding true for every a and b.

(R) The following identity (see also (39)) is implicitly contained in the work [9] by Reiner; we call it *Reiner's identity*:

$$\left[[a,b]_q, a\right]_q = \left[a, [b,a]_q\right]_q \quad \text{for every } a, b. \tag{63}$$

With the formalism in Definition 2, it can be rewritten as the commuting relation

$$\mathrm{ad}_q^* a \circ \mathrm{ad}_q a \equiv \mathrm{ad}_q a \circ \mathrm{ad}_q^* a, \quad \text{for every } a, \tag{64}$$

which is also equivalent to

$$\left[\mathrm{ad}_q^* a, \mathrm{ad}_q a\right]_\circ \equiv 0. \tag{65}$$

(A) We introduce the following identity involving three letters a, b, c:

$$\left[a, [c,b]_q\right]_q + \left[b, [c,a]_q\right]_q$$
$$= \left[[b,c]_q, a\right]_q + \left[[a,c]_q, b\right]_q. \tag{66}$$

It can be written as

$$\mathrm{ad}_q a \circ \mathrm{ad}_q^* b + \mathrm{ad}_q b \circ \mathrm{ad}_q^* a$$
$$\equiv \mathrm{ad}_q^* a \circ \mathrm{ad}_q b + \mathrm{ad}_q^* b \circ \mathrm{ad}_q a, \tag{67}$$

for every a, b, or alternatively as a relation involving the $\circ$-commutators of left and right q-mutator operators:

$$\left[\mathrm{ad}_q a, \mathrm{ad}_q^* b\right]_\circ \equiv \left[\mathrm{ad}_q^* a, \mathrm{ad}_q b\right]_\circ. \tag{68}$$

Identity (67) can be proved starting from identity (64) by the substitution of a with $a + b$ (and then by two cancellations, using (64)). We note that (66) is symmetric with respect to an interchange of a with b. Finally, when $a = b$, (67) gives at once (64). Therefore (R) and (A) are equivalent, but, for our purposes, we shall use them in different ways, so it is more convenient to keep them separated.

(B) We introduce another identity for three letters a, b, c; namely,

$$\left[a, [c,b]_q\right]_q - \left[[a,c]_q, b\right]_q = \frac{q}{q^2+1}\left(\left[[c,a]_q, b\right]_q\right.$$
$$\left. - \left[[c,b]_q, a\right]_q + \left[b, [a,c]_q\right]_q - \left[a, [b,c]_q\right]_q\right). \tag{69}$$

It can be written as

$$\mathrm{ad}_q a \circ \mathrm{ad}_q^* b - \mathrm{ad}_q^* b \circ \mathrm{ad}_q a \equiv \frac{q}{q^2+1}\left(\mathrm{ad}_q^* b \circ \mathrm{ad}_q^* a\right.$$
$$\left. - \mathrm{ad}_q^* a \circ \mathrm{ad}_q^* b + \mathrm{ad}_q b \circ \mathrm{ad}_q a - \mathrm{ad}_q a \circ \mathrm{ad}_q b\right). \tag{70}$$

This gives an alternative way of writing the $\circ$-commutator of left and right q-mutator operators by means of the $\circ$-commutators of two right and two left q-mutator operators:

$$\left[\mathrm{ad}_q a, \mathrm{ad}_q^* b\right]_\circ$$
$$= \frac{q}{q^2+1}\left(\left[\mathrm{ad}_q^* b, \mathrm{ad}_q^* a\right]_\circ + \left[\mathrm{ad}_q b, \mathrm{ad}_q a\right]_\circ\right). \tag{71}$$

The proof of (69) follows by applying twice the Transformation Rule (T) to acb, by writing the latter alternatively as $(a \cdot c) \cdot b$ and $a \cdot (c \cdot b)$ and then using identity (A). If we interchange a and b in (70), the right-hand side changes sign; it then easily follows that (70) implies (67), whence (B) implies (A). Furthermore, if $a = b$, identity (69) reduces to (63).

(C) Let p be any monomial of the form $x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}$. We consider the tautological identity

$$Y \cdot p \cdot (yx - qxy) = (yx - qxy) \cdot p \cdot Y,$$
$$\text{where } Y = [y, x]_q. \tag{72}$$

We repeatedly apply the Transformation Rule (T) to both of its sides, in the following way: we write the left-hand side as

$$Y \cdot \underbrace{x \cdots x}_{i_1 \text{ times}} \underbrace{y \cdots y}_{j_1 \text{ times}} \cdots \underbrace{x \cdots x}_{i_n \text{ times}} \underbrace{y \cdots y}_{j_n \text{ times}} \cdot y \cdot x$$
$$- q\{\ldots \text{analogous} \ldots\}, \tag{73}$$

and we apply (T) *from left to right* (to both summands), without breaking Y into its summands $yx - qxy$, so that Y will always appear in the innermost position of a sum of iterated q-mutators of x and y (ultimately producing a linear combination of $[y, x]_q$-centered polynomials); we do the same on the right-hand side, *starting from right to left*, in order to preserve again Y in innermost positions. See Example 6 for an example of this technique.

(I) Under the name *inherited relations*, we call any iden-
tity which can be directly obtained from lower order
identities in (R), (A), (B), and (C) by applying either

$$
\mathrm{ad}_q^* x,
$$

$$
\mathrm{ad}_q^* y,
$$

$$
\mathrm{ad}_q x \tag{74}
$$

$$
\text{or } \mathrm{ad}_q y
$$

to both sides. Furthermore, starting from total degree
6 (see Table 1) this procedure will also apply on lower
order identities previously obtained by (I) itself.

Example 6. We give an example for the procedure (C), when
$p = x$. We have

$$
Y \cdot x \cdot (yx - qxy) = (yx - qxy) \cdot x \cdot Y,
$$

$$
\text{where } Y = [y, x]_q. \tag{75}
$$

The left-hand side is $Y \cdot x \cdot y \cdot x - qY \cdot x \cdot x \cdot y$. For the first
summand we have

$$
Y \cdot x \cdot y \cdot x = (((Y \cdot x) \cdot y) \cdot x) \overset{(T)}{=} \frac{1}{1 - q^2} \left([Y, x]_q \right.
$$

$$
+ q [x, Y]_q) \cdot y \cdot x \overset{(T)}{=} \frac{1}{(1 - q^2)^2} \left([[Y, x]_q \right.
$$

$$
+ q [x, Y]_q, y]_q + q [y, [Y, x]_q + q [x, Y]_q]_q) \cdot x
$$

$$
\overset{(T)}{=} \frac{1}{(1 - q^2)^3} \left([[[Y, x]_q + q [x, Y]_q, y]_q \right. \tag{76}
$$

$$
+ q [y, [Y, x]_q + q [x, Y]_q]_q, x]_q
$$

$$
+ q [x, [[Y, x]_q + q [x, Y]_q, y]_q
$$

$$
\left. + q [y, [Y, x]_q + q [x, Y]_q]_q]_q \right).
$$

Analogously, the second summand is

$$
- qY \cdot x \cdot x \cdot y = [\cdots]
$$

$$
= -\frac{q}{(1 - q^2)^3} \left([[[Y, x]_q + q [x, Y]_q, x]_q \right.
$$

$$
+ q [x, [Y, x]_q + q [x, Y]_q]_q, y]_q \tag{77}
$$

$$
+ q [y, [[Y, x]_q + q [x, Y]_q, x]_q
$$

$$
\left. + q [x, [Y, x]_q + q [x, Y]_q]_q]_q \right).
$$

The right-hand side is $y \cdot x \cdot x \cdot Y - qx \cdot y \cdot x \cdot Y$. For the first
summand we have

$$
y \cdot x \cdot x \cdot Y = (y \cdot (x \cdot (x \cdot Y))) \overset{(T)}{=} \frac{1}{1 - q^2} y \cdot x
$$

$$
\cdot \left([x, Y]_q + q [Y, x]_q \right) \overset{(T)}{=} \frac{1}{(1 - q^2)^2} y \cdot \left([x, [x, Y]_q \right.
$$

$$
+ q [Y, x]_q]_q + q [[x, Y]_q + q [Y, x]_q, x]_q)
$$

$$
\overset{(T)}{=} \frac{1}{(1 - q^2)^3} \left([y, [x, [x, Y]_q + q [Y, x]_q]_q \right. \tag{78}
$$

$$
+ q [[x, Y]_q + q [Y, x]_q, x]_q]_q
$$

$$
+ q [[x, [x, Y]_q + q [Y, x]_q]_q
$$

$$
\left. + q [[x, Y]_q + q [Y, x]_q, x]_q, y]_q \right).
$$

Analogously, the second summand is

$$
- qx \cdot y \cdot x \cdot Y = [\cdots]
$$

$$
= -\frac{q}{(1 - q^2)^3} \left([x, [y, [x, Y]_q + q [Y, x]_q]_q \right.
$$

$$
+ q [[x, Y]_q + q [Y, x]_q, y]_q]_q \tag{79}
$$

$$
+ q [[y, [x, Y]_q + q [Y, x]_q]_q
$$

$$
\left. + q [[x, Y]_q + q [Y, x]_q, y]_q, x]_q \right).
$$

Putting the pieces together, we obtain an identity for nested
q-mutators in $S_{3,2}$.

3.3. Producing General Identities: Counting the Identities.
Finally, we describe how to obtain identities in each space
$S_{i,j}$ (see Definition 2). Let $i, j \in \mathbb{N}$ be fixed. According to
Proposition 4, we know that the dimension of $S_{i,j}$ is $d(i, j) =$
$\binom{i+j}{i} - 1$, while the total number of the formal $[y, x]_q$-
centered q-mutators spanning $S_{i,j}$ is $n(i, j) = \binom{i+j-2}{i-1} \cdot 2^{i+j-2}$.

We show, inductively, how to construct $n(i, j) - d(i, j)$
*identities among the generators of $S_{i,j}$ by using the procedures
described above.* We here conjecture that the identities that we
are able to obtain are linearly independent, and we shall deal
with the proof of this conjecture in a future investigation.

If $(i, j) = (1, 1)$ there is nothing to prove since $d(1, 1) =$
$n(1, 1) = 1$. The same is true for the bidegrees $(1, 2)$ and $(2, 1)$
since $d(1, 2) = d(2, 1) = n(1, 2) = n(2, 1) = 2$. For the

TABLE 1: A few numbers relative to the analysis of the identities obtained by our procedure, for $[y, x]_q$-centered q-mutators of total degree ≤ 10.

Bidegree	$n(i, j)$	$d(i, j)$	$r(i, j)$	$I(i, j)$	$R(i, j)$	$A(i, j)$	$B(i, j)$	$C(i, j)$
(1, 1)	1	1	0	0	0	0	0	0
(1, 2)	2	2	0	0	0	0	0	0
(1, 3)	4	3	1	0	1	0	0	0
(2, 2)	8	5	3	0	0	1	1	1
(1, 4)	8	4	4	2	2	0	0	0
(2, 3)	24	9	15	8	2	2	2	1
(1, 5)	16	5	11	8	3	0	0	0
(2, 4)	64	14	50	38	5	3	3	1
(3, 3)	96	19	77	60	6	5	5	1
(1, 6)	32	6	26	22	4	0	0	0
(2, 5)	160	20	140	122	9	4	4	1
(3, 4)	320	34	286	254	13	9	9	1
(1, 7)	64	7	57	52	5	0	0	0
(2, 6)	384	27	357	332	14	5	5	1
(3, 5)	960	55	905	852	24	14	14	1
(4, 4)	1280	69	1211	1144	28	19	19	1
(1, 8)	128	8	120	114	6	0	0	0
(2, 7)	896	35	861	828	20	6	6	1
(3, 6)	2688	83	2605	2524	40	20	20	1
(4, 5)	4480	125	4355	4232	54	34	34	1
(1, 9)	256	9	247	240	7	0	0	0
(2, 8)	2048	44	2004	1962	27	7	7	1
(3, 7)	7168	119	7049	6932	62	27	27	1
(4, 6)	14336	209	14127	13920	96	55	55	1
(5, 5)	17920	251	17669	17420	110	69	69	1

For every bidegree (i, j) we have the following.
$n(i, j)$: number of symbolic $[y, x]_q$-centered q-mutators of bidegree (i, j).
$d(i, j)$: dimension of the vector space $S_{i,j}$.
$r(i, j)$: number of relations among the symbolic q-mutators.
$I(i, j)$: number of inherited relations (I).
$R(i, j)$: number of relations according to Reiner's identity (R).
$A(i, j)$: number of relations according to identity (A).
$B(i, j)$: number of relations according to identity (B).
$C(i, j)$: number of relations according to procedure (C).

bidegrees (i, j) with total degree $i + j = 4$ we have the following *scenario*.

(i) For bidegree $(1, 3)$ we have $d(1, 3) = 3$ and $n(1, 3) = 4$; there is one relation from (R); namely (see (63) with $b = [y, x]_q$ and $a = y$),

$$\left[\left[y, [y, x]_q \right]_q, y \right]_q = \left[y, \left[[y, x]_q, y \right]_q \right]_q. \tag{80}$$

Nothing can be obtained from procedure (I) since in previous degree 3 there are no relations; nothing can be obtained as well from identities (A) and (B) since, with the notations in (66) and (69), one has to choose $c = [y, x]_q$ and this forces taking $a = b = y$, but we already know that (A) and (B) reduce to (R) when a and b are equal. Analogous facts hold for bidegree $(3, 1)$.

(ii) For bidegree $(2, 2)$ we have $d(2, 2) = 5$ and $n(2, 2) = 8$; one has to find 3 independent relations. Nothing can be obtained from procedure (I) since in cases $(1, 2)$ and $(2, 1)$ there are no relations; (R) is not useful either, since one is forced to choose $b = [y, x]_q$ in (63), but then a cannot be either x or y, due to bidegree $(2, 2)$. The three needed relations are instead provided by (A), (B), and (C); indeed

  (a) from (A) we get (see (66) with $c = [y, x]_q, a = x,$ and $b = y$)

$$\left[x, \left[[y, x]_q, y \right]_q \right]_q + \left[y, \left[[y, x]_q, x \right]_q \right]_q$$

$$= \left[\left[y, [y, x]_q \right]_q, x \right]_q + \left[\left[x, [y, x]_q \right]_q, y \right]_q; \tag{81}$$

(b) from (B) we get (see (69) with $c = [y, x]_q$, $a = x$, and $b = y$)

$$\left[x, \left[[y, x]_q, y \right]_q \right]_q - \left[\left[x, [y, x]_q \right]_q, y \right]_q$$

$$= \frac{q}{q^2 + 1} \left(\left[\left[[y, x]_q, x \right]_q, y \right]_q \right.$$

$$- \left[\left[[y, x]_q, y \right]_q, x \right]_q + \left[y, \left[x, [y, x]_q \right]_q \right]_q \qquad (82)$$

$$\left. - \left[x, \left[y, [y, x]_q \right]_q \right]_q \right);$$

(c) from (C) we get (use the procedure with $p = 1$)

$$\left[\left[[y, x]_q, y \right]_q + q \left[y, [y, x]_q \right]_q, x \right]_q$$

$$+ q \left[x, \left[[y, x]_q, y \right]_q + q \left[y, [y, x]_q \right]_q \right]_q$$

$$- q \left[\left[[y, x]_q, x \right]_q + q \left[x, [y, x]_q \right]_q, y \right]_q$$

$$- q^2 \left[y, \left[[y, x]_q, x \right]_q + q \left[x, [y, x]_q \right]_q \right]_q$$

$$= \left[y, \left[x, [y, x]_q \right]_q + q \left[[y, x]_q, x \right]_q \right]_q \qquad (83)$$

$$+ q \left[\left[x, [y, x]_q \right]_q + q \left[[y, x]_q, x \right]_q, y \right]_q$$

$$- q \left[x, \left[y, [y, x]_q \right]_q + q \left[[y, x]_q, y \right]_q \right]_q$$

$$- q^2 \left[\left[y, [y, x]_q \right]_q + q \left[[y, x]_q, y \right]_q, x \right]_q.$$

It can be proved with some tedious linear algebra computations that these three identities are independent of each other.

After warming up with low degrees (which also serve for starting the induction), we are ready to take into account the general bidegree (i, j). In the sequel we can suppose that $i + j > 4$ and we count the number of expected relations deriving from (I), (R), (A), (B), and (C), provided that we know these numbers for degrees strictly less than $i + j$.

(i) *Number of Relations from Procedure (I)*. Procedure (I) requires application of the following.

(a) $\mathrm{ad}_q^* x$ or $\mathrm{ad}_q x$: if $i = 1$ there is nothing to do; otherwise we apply any of these maps *to the relations of bidegree* $(i - 1, j)$; there are precisely $r(i - 1, j)$ of these relations.

(b) $\mathrm{ad}_q^* y$ or $\mathrm{ad}_q y$: if $j = 1$ there is nothing to do; otherwise we apply any of these maps *to the relations of bidegree* $(i, j - 1)$; there are precisely $r(i, j - 1)$ of these relations.

Summing up, after some simple computation on binomials, the total number of expected identities from procedure (I) is (see also (54))

$$\mathrm{I}(i, j) := 2r(i - 1, j) + 2r(i, j - 1)$$

$$= \begin{cases} 0 & \text{if } i = j = 1 \\ \binom{i + j - 2}{i - 1} 2^{i+j-2} - 2 \binom{i + j}{i} + 4 & \text{if } i \geq 2 \text{ or } j \geq 2. \end{cases} \qquad (84)$$

(ii) *Number of Relations from Identity (R)*. We apply identity (63) with $a = x$ or with $a = y$. In the first case we choose as b any of the members of a basis of the set $S_{i-2,j}$; in the second case we choose as b any of the members of a basis of $S_{i,j-2}$. Since the iterated q-mutator b must contain $[y, x]_q$ in the innermost position, the first case occurs only if $i \geq 3$ and the second case only if $j \geq 3$. Summing up, the expected relevant number of relations from identity (R) is (see also Proposition 4-(ii)):

$$\mathrm{R}(i, j)$$

$$:= \begin{cases} 0 & \text{if } i \leq 2, \ j \leq 2 \\ \binom{i + j - 2}{i} - 1 & \text{if } i \leq 2, \ j \geq 3 \\ \binom{i + j - 2}{j} - 1 & \text{if } i \geq 3, \ j \leq 2 \\ \binom{i + j - 2}{i} + \binom{i + j - 2}{j} - 2 & \text{if } i \geq 3, \ j \geq 3. \end{cases} \qquad (85)$$

(iii) *Number of Relations from Identity (A)*. As we already remarked, identity (66) boils down to (R) when $a = b$; hence we can take $a \neq b$. Also, we do not get new information if a and b are interchanged; thus *we can always choose $a = x$ and $b = y$*, so that (A) can be applied only when $i \geq 2$ and $j \geq 2$ (since c must contain $[y, x]_q$). *In place of c we can then take any member of a basis of $S_{i-1,j-1}$*. Summing up, taking into account the formula for $d(i - 1, j - 1)$, the expected number of relations from identity (A) is

$$\mathrm{A}(i, j) := \begin{cases} 0 & \text{if } i = 1 \text{ or } j = 1 \\ \binom{i + j - 2}{i - 1} - 1 & \text{if } i \geq 2, \ j \geq 2. \end{cases} \qquad (86)$$

(iv) *Number of Relations from Identity (B)*. An argument similar to the above one applies for (B): *we take $a = x$ and $b = y$ and c is any member of a basis of $S_{i-1,j-1}$*. Thus, the expected number of relations from identity (B) is

$$\mathrm{B}(i, j) := \begin{cases} 0 & \text{if } i = 1 \text{ or } j = 1 \\ \binom{i + j - 2}{i - 1} - 1 & \text{if } i \geq 2, \ j \geq 2. \end{cases} \qquad (87)$$

(v) *Number of Relations from Identity (C)*. With the notation in the description of procedure (C), since x and y appear at least twice (remember that $Y = [y, x]_q$), procedure (C) is inapplicable if $i = 1$ or $j = 1$. Our conjecture states that, denoting by $C(i, j)$ the number of relations deriving from (C), there suffices one and only one such relation, when $i \geq 2$ and $j \geq 2$; thus we define

$$C(i, j) := \begin{cases} 0 & \text{if } i = 1 \text{ or } j = 1 \\ 1 & \text{if } i \geq 2, \ j \geq 2. \end{cases} \tag{88}$$

(It is expected that this can be obtained by taking $p = x^{i-2} y^{j-2}$.)

Remark 7 (consistency of the number of identities). In order to support our conjecture on the linear independence of the identities obtained via (I), (R), (A), (B), and (C) (and the minimal application of the latter), we verify that the sum of the numbers of the relations obtained above fill the number of the needed independent relations; in other words, for every $i, j \in \mathbb{N}$,

$$I(i, j) + R(i, j) + A(i, j) + B(i, j) + C(i, j)$$
$$= n(i, j) - d(i, j). \tag{89}$$

This is a simple verification, which we omit, based on the Pascal rule for binomials and on formulas (84) to (88).

See Table 1 for the computation of the above numbers in (84) to (88), up to degree 10. Up to degree 10 it has been verified, with the help of the computer algebra system REDUCE [35], that our conjecture is true.

4. The q-Deformed CBH Theorem

It is understood for the rest of the paper that the notations of Section 2 are fixed. To make our study of the q-BCH Formula precise, we need to endow A with a metric structure. Indeed, as in [1, Theorem 2.58, p. 94], A can be equipped with a *metric space structure* by the distance

$$d(a, b) := \exp(-\text{md}(a - b)), \quad a, b \in A, \tag{90}$$

where $\text{md}(0) := \infty$ and if $p = \sum_n p_n \neq 0$ (see the notation in (21)), we set

$$\text{md}(p) := \min\{n \geq 0 : p_n \neq 0\}, \tag{91}$$

with the convention $\exp(-\infty) = 0$.

The metric space (A, d) is complete and it is an isometric completion of $\mathbb{K}(q)\langle x, y \rangle$ (as a metric subspace); moreover it is ultrametric; that is,

$$d(a, b) \leq \max\{d(a, c), d(c, b)\}, \quad a, b, c \in A. \tag{92}$$

Remark 8. As a consequence of these facts and of the invariant property $d(a, b) = d(a + c, b + c)$, any series $\sum_n a_n$ in A (where $a_n \in A$ for any n) is convergent *if and only if*

$\lim_{n \to \infty} a_n = 0$ in A, that is, if and only if $\lim_{n \to \infty} \text{md}(a_n) = \infty$. (See [1, Section 2.3.1] for all the details.) In the sequel we shall tacitly use the well-behaved properties of the topology of (A, d) allowing us to easily perform any passage to the limit or limit/series interchange.

With this topology, the series in (21) not only is a formal expression but also becomes a genuine convergent series in (A, d), since $\text{md}(p_n) \geq n \to \infty$ as $n \to \infty$ (because $p_n \in A_n$).

Since, for any $p \in A_+$, one has $\text{md}(p^n) \to \infty$ as $n \to \infty$, Remark 8 ensures that the following maps are well posed, as convergent series in the metric space A:

$$\exp : A_+ \longrightarrow 1 + A_+, \quad \exp(p) = \sum_{n=0}^{\infty} \frac{p^n}{n!},$$

$$\exp_q : A_+ \longrightarrow 1 + A_+, \quad \exp_q(p) = \sum_{n=0}^{\infty} \frac{p^n}{[n]_q!}. \tag{93}$$

We have the following results, whose simple proofs are omitted.

(i) Each of the maps $\exp$ and $\exp_q$ in (93) admits an inverse function, which we, respectively, denote by $\log$ and $\log_q$, from $1 + A_+$ to A_+. We say that $\log_q$ is the q-logarithm.

(ii) There exists a map $\phi_q : A_+ \to A_+$ such that

$$\exp\left(\phi_q(p)\right) = \exp_q(p),$$

$$\exp_q\left(\phi_q^{-1}(p)\right) = \exp(p) \tag{94}$$

$$\text{for every } p \in A_+.$$

It is known that ϕ_q has the explicit expansion (see [16, 37])

$$\phi_q(p) = \sum_{n=1}^{\infty} \frac{(1 - q)^{n-1}}{n[n]_q} p^n, \quad p \in A_+. \tag{95}$$

Then we infer that in the associative algebra A there exists one and only one formal power series $z_q(x, y)$ such that (6) is valid, and this is defined as in (28). The series $z_q(x, y)$ is referred to as the *q-Baker-Campbell-Hausdorff series* (shortly, the *q-BCH series*), and it will be also denoted by $x \diamond_q y$. Its associative presentation is (29). Grouping together the summands of the same degree, we use the notation

$$x \diamond_q y = \sum_{n=1}^{\infty} Q_n(x, y), \tag{96}$$

$$\text{with } Q_n(x, y) \in A_n \text{ for every } n \geq 1.$$

The operation $\diamond_q$ gives the q-deformation of the classical BCH Formula

$$x \diamond y = x + y + \frac{1}{2}[x, y]$$

$$+ \frac{1}{12}\left([x, [x, y]] + [y, [y, x]]\right) \tag{97}$$

$$- \frac{1}{24}[x, [y, [x, y]]] + \cdots.$$

Remark 9 (intertwining of the deformed/undeformed BCH series). By using the map ϕ_q in (95) we can obtain a representation for $x \diamond_q y$ deriving from the following argument:

$$
\begin{aligned}
\exp_q (x \diamond_q y) &\stackrel{(6)}{=} \exp_q (x) \exp_q (y) \\
&\stackrel{(94)}{=} \exp \left(\phi_q (x) \right) \exp \left(\phi_q (y) \right) \\
&\stackrel{(97)}{=} \exp \left(\phi_q (x) \diamond \phi_q (y) \right) \\
&\stackrel{(94)}{=} \exp_q \left(\phi_q^{-1} \left(\phi_q (x) \diamond \phi_q (y) \right) \right).
\end{aligned}
\tag{98}
$$

Since $\exp_q$ is injective we get the identity

$$
x \diamond_q y = \phi_q^{-1} \left(\phi_q (x) \diamond \phi_q (y) \right),
\tag{99}
$$

intertwining the undeformed and the q-deformed BCH series. Due to explicit form (95) of ϕ_q, identity (99) can be used for computational issues to derive explicit summands of $\diamond_q$ starting from those of $\diamond$; unfortunately, computations become cumbersome very rapidly.

As it happens for the undeformed case of the classical Campbell-Baker-Hausdorff-Dynkin series, the most natural problem is the study of the distinguished algebraic properties of the polynomials $Q_n(x, y)$ in (96). We next claim that, apart from $Q_1(x, y) = x + y$, any $Q_n(x, y)$ (with $n \geq 2$) is a sum of polynomials containing $yx - qxy$ as a factor. This fact can be seen as a consequence of Schützenberger's result (7) recalled in the Introduction.

In order to prove the above claim, we first observe that, from $yx - qxy \in I$, one gets

$$
[y]_I [x]_I = q [x]_I [y]_I.
\tag{100}
$$

By arguing inductively one obtains

$$
[y]_I^j [x]_I^i = q^{ij} [x]_I^i [y]_I^j, \quad \text{for every } i, j \geq 1.
\tag{101}
$$

By the aid of this identity, we can provide a basis for $A_{i,j}/I$ made of one single element; for instance,

$$
\frac{A_{i,j}}{I} = \operatorname{span}_{\mathbb{K}(q)} \left\{ [x]_I^i [y]_I^j \right\}, \quad \text{for every } i, j \geq 0.
\tag{102}
$$

Clearly, $[x]_I^i [y]_I^j$ is nonvanishing, since $x^i y^j \notin I$ for any $i, j \geq 0$, whence the dimension of $A_{i,j}/I$ is 1 precisely. Identity (101) has another remarkable consequence, resemblant to Newton's binomial formula, namely (see Schützenberger [32]; see also Cigler [33]),

$$
\left([x]_I + [y]_I \right)^n = \sum_{k=0}^{n} \binom{n}{k}_q [x]_I^k [y]_I^{n-k},
\tag{103}
$$

$$
\text{for every } n \geq 1,
$$

where

$$
\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}, \quad 0 \leq k \leq n.
\tag{104}
$$

Indeed, one can easily prove (103) by induction on n, using (101) and the well-known q-Pascal rule (see, e.g., [4])

$$
\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q, \quad 1 \leq k \leq n.
\tag{105}
$$

By the aid of identity (103) solely, one can prove the next result. We name it after Schützenberger, even if its original formulation was in terms of q-commuting variables (see (7)).

Lemma 10 (Schützenberger [32]). *The q-BCH series has the decomposition*

$$
x \diamond_q y = \sum_{i,j \geq 0} Q_{i,j}(x, y), \quad \left(\text{with } Q_{i,j}(x, y) \in A_{i,j} \right)
\tag{106}
$$

with the following properties:

(1) $Q_{0,0}(x, y) = 0$, $Q_{(1,0)}(x, y) = x$, $Q_{(0,1)}(x, y) = y$;

(2) $Q_{(i,0)}(x, y) = Q_{(0,j)}(x, y) = 0$ *for every $i, j \geq 2$;*

(3) *for any i, j with $i, j \geq 1$, one has $Q_{i,j}(x, y) \in I_{i,j}$.*

For example (see also [38]), one has

$$
\begin{aligned}
Q_{1,1}(x, y) &= -\frac{1}{[2]_q} [y, x]_q, \\
Q_{2,1}(x, y) &= -\frac{q}{[2]_q [3]_q!} \left[x, [y, x]_q \right]_q, \\
Q_{1,2}(x, y) &= -\frac{q}{[2]_q [3]_q!} \left[[y, x]_q, y \right]_q.
\end{aligned}
\tag{107}
$$

Proof. We first equip A/I with the structures of a topological algebra and of a complete ultrametric space, by imitating the corresponding structures on A. Then one can define a q-exponential on A/I as well, denoted by $\widetilde{\exp}_q$. Since the projection $[\cdot]_I$ is continuous morphism of the underlying algebras one obtains

$$
\left[\exp_q (x) \exp_q (y) \right]_I = \widetilde{\exp}_q \left([x \diamond_q y]_I \right).
\tag{108}
$$

On the other hand, (103) gives

$$
\begin{aligned}
\widetilde{\exp}_q \left([x]_I + [y]_I \right) &= \sum_{n=0}^{\infty} \frac{1}{[n]_q!} \sum_{k=0}^{n} \binom{n}{k}_q [x]_I^k [y]_I^{n-k} \\
&= \left[\sum_{k,j \geq 0} \frac{x^k y^j}{[k]_q! [j]_q!} \right]_I \\
&= \left[\exp_q (x) \exp_q (y) \right]_I.
\end{aligned}
\tag{109}
$$

The injectivity of $\widetilde{\exp}_q$ implies that $[x \diamond_q y]_I = [x]_I + [y]_I$ or equivalently $x \diamond_q y - (x + y) \in I$, which immediately proves the theorem.   □

Next we describe another feature of the q-BCH series.

Definition 11 (nested q-mutators). One says that any element of the form

$$\left[z_k, \cdots \left[z_3, \left[z_2, z_1 \right]_q \right]_q \cdots \right]_q, \tag{110}$$

with $k \geq 2$ and $z_1, z_2, \ldots, z_k \in \{x, y\}$, is a *right-nested q-mutator of x and y of length k*. Analogously, one says that any element of the form

$$\left[\cdots \left[\left[z_1, z_2 \right]_q, z_3 \right]_q \cdots, z_k \right]_q \tag{111}$$

is a *left-nested q-mutator of x and y of length k*. For $k = 1$, one qualifies x and y as the right-nested (and the left-nested) q-mutators of length 1.

We use the following result.

Theorem 12 (Reiner [9]). *Let $i, j \in \mathbb{N} \cup \{0\}$. Let one construct the set $\mathscr{R}_{i,j}$ of the polynomials in $A_{i,j}$ consisting of the left-nested q-mutators*

$$\left[\left[\left[\left[\left[\left[\left[\underbrace{\left[x, x \right]_q \cdots x}_{i_1 \; times} \right]_q \right. \right. \right. \right. \right. \right. \right. \tag{112}$$

$$\left. \left. \left. \left. \left. \left. \left. \cdot \underbrace{y \right]_q \cdots y}_{j_1 \; times} \right]_q \cdots \underbrace{x \right]_q \cdots x}_{i_n \; times} \right]_q \underbrace{y \right]_q \cdots y}_{j_n \; times} \right]_q,$$

with $i_1 + \cdots + i_n = i$ and $j_1 + \cdots + j_n = j$.
Then $\mathscr{R}_{i,j}$ is a basis of $A_{i,j}$ as well.

Clearly one can obtain an analogous result with right-nested q-mutators. For brevity, we shall refer to $\mathscr{R}_{i,j}$ as the *left-nested Reiner basis of $A_{i,j}$* (the choice of the notation "$\mathscr{R}$" refers to "Reiner"). Since we have an associative presentation of $x \diamond_q y$, we immediately get, from Reiner's Theorem 12, the following result.

Corollary 13. *With the notation in Lemma 10, any $Q_{i,j}(x, y)$ can be expressed in a unique way as a linear combination of elements of the left-nested Reiner basis $\mathscr{R}_{i,j}$ of $A_{i,j}$. Therefore, the q-BCH series $x \diamond_q y$ admits a presentation as a series of left-nested q-mutators of x and y as in (112) (with coefficients in $\mathbb{K}(q)$).*
An analogous result holds for right-nested q-mutators.

The disadvantage of the above nested presentation of the q-BCH series, based on Reiner's Theorem 12, is that it (necessarily) allows for summands of the form

$$\left[[x, x]_q, \cdots \right]_q,$$

$$\left[[x, y]_q, \cdots \right]_q,$$

$$\left[[y, x]_q, \cdots \right]_q, \tag{113}$$

$$\left[[y, y]_q, \cdots \right]_q,$$

which are not manifestly consistent with what is known from Schützenberger's result (7) (encoded in Lemma 10); for example, already in degree three the nested presentation differs from (107) in that

$$Q_{2,1}(x, y) = -\frac{q}{[2]_q [3]_q!} \left[[x, y]_q, x \right]_q. \tag{114}$$

Compared to our expression (107) for $Q_{2,1}$, we get general identity (39) (implicitly contained in [9]). Our main task is to compound Lemma 10 and Corollary 13 and prove that $x \diamond_q y$ admits a presentation with q-mutators (not necessarily nested) *where the innermost q-mutator is $[y, x]_q$*, consistently with the mentioned Schützenberger's result. This is not obvious since, for example, $[[x, x]_q, y]_q$ (which is *a priori* legitimate in Reiner's presentation) is not a linear combination of $[[y, x]_q, x]_q$ and $[x, [y, x]_q]_q$.

Proposition 14. *For any $i, j \geq 1$, the summand $Q_{i,j}(x, y)$ in q-BCH series (106) belongs to $S_{i,j}$ (see Definition 2); that is, $Q_{i,j}(x, y)$ can be expressed as a linear combination (with coefficients in $\mathbb{K}(q)$) of $[y, x]_q$-centered q-mutators.*
More generally, for every $i, j \in \mathbb{N} \cup \{0\}$ we have $I_{i,j} = S_{i,j}$ and $I = S$.

Proposition 14 improves the result provided in [34], where $Q_1, \ldots, Q_4$ are given in terms of nested q-mutators centered at $[y, x]_q$ or centered at $[x, y]_q$.

Proof. We prove that $I_{i,j} = S_{i,j}$ for every $i, j \in \mathbb{N} \cup \{0\}$. Since $S_{i,j} \subseteq I_{i,j}$, we are left to prove the reverse inclusion. From (25) (and the definition of $S_{i,j}$ when i or j vanishes) we have $I_{i,j} = \{0\} = S_{i,j}$ whenever i or j is 0. We can therefore suppose that $i, j \geq 1$. If $i = j = 1$ we trivially have $I_{1,1} = \text{span}\{[y, x]_q\} = S_{1,1}$.

We can thus suppose that $i + j \geq 3$. Any element of $I_{i,j}$ is, by definition, a linear combination (with coefficients in $\mathbb{K}(q)$) of

$$B_1 \circ \cdots \circ B_{i+j-2} \left([y, x]_q \right), \tag{115}$$

where $B_1, B_2, \ldots, B_{i+j-2}$ all belong to the set of maps $\{R_x, R_y, L_x, L_y\}$ (see Definition 2), in such a way that x appears exactly $i - 1$ times and y appears exactly $j - 1$ times. By (9), any of the maps $B_1, B_2, \ldots, B_{i+j-2}$ in (115) is a linear combination of suitable maps belonging to $\{\text{ad}_q^* x, \text{ad}_q^* y, \text{ad}_q x, \text{ad}_q y\}$ (preserving the total number of x's and y's).

Therefore (115) is a $[y, x]_q$-centered polynomial in $A_{i,j}$. This shows that $I_{i,j} \subseteq S_{i,j}$. In particular, since $Q_{i,j}(x, y) \in I_{i,j}$ (see Lemma 10-(3)), we get $Q_{i,j}(x, y) \in S_{i,j}$. □

5. A Criterion for $[y, x]_q$-Centered q-Mutators

Due to our interest in $[y, x]_q$-centered q-mutators, we introduce a criterion for characterizing the elements of $I_{i,j}$ (or equivalently, of $S_{i,j}$).

Any nonvanishing monomial in A can be written *in a unique way* as a scalar multiple of the following basis[1] monomials (we agree that $x^0 = y^0 = 1_{\mathbb{K}}$, the identity of $\mathbb{K}$):

$$x^{i_1} y^{j_1} \quad \text{with } i_1, j_1 \geq 0, \tag{116a}$$

$$x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n} \tag{116b}$$

$$\text{with } n \geq 2, \ i_1, j_n \geq 0, \ j_1, i_2, \ldots, j_{n-1}, i_n \geq 1.$$

We denote by m any of the above monomials. In the sequel, we also agree that any monomial in x, y has been written in the above unique way.

Definition 15. Let $\mathcal{M} = \{m\}$ denote the collection of the monomials in (116a)-(116b). We set

$$H : \mathcal{M} \longrightarrow \mathbb{N} \cup \{0\}$$

$$H(m)$$

$$:= \begin{cases} 0 & \text{if } m \text{ is as in } (116a) \\ i_2 j_1 + \cdots + i_n (j_1 + j_2 + \cdots + j_{n-1}) & \text{if } m \text{ is as in } (116b). \end{cases} \tag{117}$$

By an abuse of notation, we agree that the map H is also defined on the multi-indices appearing in (116a)-(116b), so that we also write $H(i_1, j_1) = 0$ if the indexes are as in (116a), and

$$H(i_1, j_1, \ldots, i_n, j_n) = i_2 j_1 + \cdots + i_n (j_1 + \cdots + j_{n-1}), \tag{118}$$

if the indexes are as in (116b).

Starting from (101), which can be rewritten as $y^n x^m \equiv_I q^{nm} x^m y^n$, by an inductive argument one gets (30); namely,

$$x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}$$

$$\equiv_I q^{i_2 j_1 + \cdots + i_n (j_1 + j_2 + \cdots + j_{n-1})} x^{i_1 + \cdots + i_n} y^{j_1 + \cdots + j_n}. \tag{119}$$

The following map plays, in a certain sense, the same role played by the Dynkin-Specht-Wever map (see, e.g., [1, Lemma 3.26]) in detecting the Lie-polynomials.

Lemma 16 (criterion for $[y, x]_q$-centrality). *With the notation in Definition 15, we consider the unique (continuous) $\mathbb{K}(q)$-linear map $\varphi : A \to A$ defined on monomials as follows:*

$$\varphi(m)$$

$$:= \begin{cases} m - q^{H(m)} x^{i_1} y^{j_1} = 0 & \text{if } m \text{ is as in } (116a) \\ m - q^{H(m)} x^{i_1 + \cdots + i_n} y^{j_1 + \cdots + j_n} & \text{if } m \text{ is as in } (116b). \end{cases} \tag{120}$$

Then, given $p \in A$, one has $p \in I$ (or, equivalently, $p \in S$) if and only if $\varphi(p) = p$. Moreover, φ is valued in $I = S$ so that φ is a projection onto S.

By using the abused notation following Definition 15, a homogeneous polynomial p belonging to $A_{i,j}$, say

$$p = \sum_{\substack{i_1 + \cdots + i_n = i \\ j_1 + \cdots + j_n = j}} c(i_1, j_1, \ldots, i_n, j_n) x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}, \tag{121}$$

belongs to S if and only if

$$\sum_{\substack{i_1 + \cdots + i_n = i \\ j_1 + \cdots + j_n = j}} c(i_1, j_1, \ldots, i_n, j_n) q^{H(i_1, j_1, \ldots, i_n, j_n)} = 0. \tag{122}$$

Note that the latter is simply an identity in $\mathbb{K}(q)$.

Example 17. We consider the polynomial p in $A_{2,2}$ defined by

$$p = (1 + q^2) xyxy - qx^2 y^2 + (q^2 - q) yx^2 y$$
$$- (q^3 + q) yxyx + q^2 y^2 x^2. \tag{123}$$

The associated scalar as in (122) is

$$(1 + q^2) q - q q^0 + (q^2 - q) q^2 - (q^3 + q) q^3 + q^2 q^4. \tag{124}$$

Since this is evidently null (as one can check upon expansion), we can infer that $p \in S$. Actually one can verify that p is equal to $[[x, [y, x]_q]_q, y]_q \in S_{2,2}$.

Proof of Lemma 16. We split the proof into five steps.

(I) First we have $\varphi(A) \subseteq I$ thanks to (119): indeed, for any $m \in \mathcal{M}$ we have either $\varphi(m) = m - q^{H(m)} x^{i_1} y^{j_1} = 0 \in I$ (in case (116a)) or

$$\varphi(m) = m - q^{H(m)} x^{i_1 + \cdots + i_n} y^{j_1 + \cdots + j_n} \equiv_I 0 \tag{125}$$

in case (116b) (i.e., $\varphi(m) \in I$). By linearity (and continuity) this gives $\varphi(a) \in I$ for any $a \in A$.

(II) If $p \in A$ is such that $\varphi(p) = p$, then, by part (I), we infer that $p = \varphi(p) \in \varphi(A) \subseteq I$, whence $p \in I$.

(III) Conversely, suppose that $p \in I$; we need to show that $\varphi(p) = p$, or equivalently we have to prove that $\varphi|_I$ is the identity on I. To this aim, it suffices to show that φ is the identity on any $I_{i,j}$. To this end, let $p \in I_{i,j}$; like any polynomial, p can be uniquely written in the basis (116a)-(116b) as

$$p = \sum_{m \in \mathcal{F}_p} c_p(m) m, \tag{126}$$

where $\mathcal{F}_p$ is a finite family of basis monomials in $\mathcal{M}$ and $c_p(m) \in \mathbb{K}(q)$ for any $m \in \mathcal{F}_p$. Then, by recalling that the (nonzero) monomials which span $I_{i,j}$ have bidegree (i, j), we infer

$$\varphi(p) = \sum_{m \in \mathcal{F}_p} c_p(m) \left(m - q^{H(m)} x^i y^j \right)$$

$$= \sum_{m \in \mathcal{F}_p} c_p(m) m - \left(\sum_{m \in \mathcal{F}_p} c_p(m) q^{H(m)} \right) x^i y^j \tag{127}$$

$$= p - \left(\sum_{m \in \mathcal{F}_p} c_p(m) q^{H(m)} \right) x^i y^j.$$

Moving terms around we get $(\sum_{m \in \mathcal{F}_p} c_p(m) q^{H(m)}) x^i y^j = \varphi(p) - p$. Now, the right-hand term belongs to I since φ

is I-valued (see part (I) of the proof) and since $p \in I$ by assumption; so the same is true of the left-hand side, but a scalar multiple of $x^i y^j$ can belong to I iff the scalar factor is null. Hence, from (127) we get $\varphi(p) = p$.

(IV) The surjectivity of φ is a trivial consequence of part (III).

(V) We have to prove the last assertion of the theorem. On the one hand, let $p \in S$ (this part of the proof does not require $p \in A_{i,j}$). By part (III) of the proof we know that $p = \varphi(p)$; hence, if we write p in the basis monomials as

$$p = \sum c(i_1, j_1, \ldots, i_n, j_n) x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}, \tag{128}$$

we get

$$
\begin{aligned}
p = \varphi(p) &= \sum c(i_1, j_1, \ldots, i_n, j_n) \\
&\quad \cdot \varphi\left(x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}\right) = \sum c(i_1, j_1, \ldots, i_n, j_n) \\
&\quad \cdot \left(x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}\right. \\
&\quad \left. - q^{H(i_1, j_1, \ldots, i_n, j_n)} x^{i_1 + \cdots + i_n} y^{j_1 + \cdots + j_n}\right) = p \\
&\quad - \sum c(i_1, j_1, \ldots, i_n, j_n) \\
&\quad \cdot q^{H(i_1, j_1, \ldots, i_n, j_n)} x^{i_1 + \cdots + i_n} y^{j_1 + \cdots + j_n}.
\end{aligned}
\tag{129}
$$

After canceling p and grouping terms of the same bidegree we get

$$0 = \sum_{\substack{i_1 + \cdots + i_n = i \\ j_1 + \cdots + j_n = j}} c(i_1, j_1, \ldots, i_n, j_n) q^{H(i_1, j_1, \ldots, i_n, j_n)}, \tag{130}$$

$$\text{for every } i, j \geq 0,$$

as needed (this gives precisely (122) when $p \in A_{i,j}$).

Conversely, let $p \in A_{i,j}$ and suppose that after we have written p as

$$p = \sum c(i_1, j_1, \ldots, i_n, j_n) x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}, \tag{131}$$

it is known that (122) holds true. In the preceding computations we proved that

$$
\begin{aligned}
\varphi(p) &= p - \sum c(i_1, j_1, \ldots, i_n, j_n) \\
&\quad \cdot q^{H(i_1, j_1, \ldots, i_n, j_n)} x^{i_1 + \cdots + i_n} y^{j_1 + \cdots + j_n}.
\end{aligned}
\tag{132}
$$

If $p \in A_{i,j}$ this is just

$$
\begin{aligned}
\varphi(p) &= p \\
&\quad - \left(\sum_{\substack{i_1 + \cdots + i_n = i \\ j_1 + \cdots + j_n = j}} c(i_1, j_1, \ldots, i_n, j_n) q^{H(i_1, j_1, \ldots, i_n, j_n)} \right) \\
&\quad \cdot x^i y^j.
\end{aligned}
\tag{133}
$$

By assumed (122), the term in the parenthesis is null, whence $\varphi(p) = p$. By part (II) of the proof we therefore get $p \in S$. □

Remark 18. Without any specification on the exponents $i_1, j_1, \ldots, i_n, j_n$, different-looking monomials $p = x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}$ can produce the same monomial m as in (116a)-(116b): for example,

$$
\begin{aligned}
x^1 y^0 x^2 y^0 x^1 y^1 x^0 y^0 x^0 y^2 x y^2 &= x^4 y^3 x y^2 \\
&= x^4 y^3 x^1 y^0 x^0 y^1 x^0 y^1.
\end{aligned}
\tag{134}
$$

However, it can be easily checked that the definition of H is unambiguous for any monomial p and it leads to the same result; that is,

$$
\begin{aligned}
H\left(x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}\right) &:= i_2 j_1 + \cdots \\
&\quad + i_n \left(j_1 + j_2 + \cdots + j_{n-1}\right),
\end{aligned}
\tag{135}
$$

with the convention (which we tacitly assume in the sequel) that the sum is 0 if $n = 1$. Accordingly, the map φ is well posed for every monomial:

$$
\begin{aligned}
\varphi\left(x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}\right) \\
= x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n} \\
- q^{i_2 j_1 + \cdots + i_n(j_1 + j_2 + \cdots + j_{n-1})} x^{i_1 + \cdots + i_n} y^{j_1 + \cdots + j_n}.
\end{aligned}
\tag{136}
$$

The next section provides a closed formula for the terms $Q_{i,j}$ in the q-BCH series, only depending on the coefficients of the q-logarithm. The main tool is Lemma 16.

6. An Explicit Formula for the q-BCH Series

We already showed the basic associative presentation of $x \diamond_q y$ in (29), where the coefficients λ_n (from the expansion of $\log_q$) are used: they can be derived, for example, by the recursion formula (27). In this section we provide an explicit formula for $Q_{i,j}$ in terms of iterated q-mutators. The procedure is quite technical, so that the reader may first want to consult an example, describing the idea behind our formula for the q-BCH series with an example: this is given in Section 7. For obtaining an explicit formula for $Q_{i,j}$ in terms of iterated q-mutators, we first need some lemmas whose proofs (mainly, some inductive arguments) are omitted.

Lemma 19. *For any $j \geq 1$ one has*

$$y^j x = q^j x y^j + \sum_{k=0}^{j-1} q^k \binom{j}{k} \left(\mathrm{ad}_q y\right)^{j-k}(x) y^k, \tag{137}$$

where $\binom{j}{k}$ denotes the classical (undeformed) binomial coefficient, and the notation ad_q denotes the left q-commutation map in (42).

Note that the sum over k in the right-hand side of (137) is an element of I, the bilateral ideal generated by $[y, x]_q$. A direct application of formula (62) proves the following result, starting from (137).

Lemma 20. *For any $j \geq 1$ one has*

$$y^j x = q^j x y^j + \sum_{k=0}^{j-1} \left(\frac{q}{1-q^2} \right)^k \binom{j}{k} \left(\mathrm{ad}_q^* y + q \mathrm{ad}_q y \right)^k$$
$$\cdot \left(\mathrm{ad}_q y \right)^{j-k} (x). \tag{138}$$

Remark 21. Formula (138) could be written in an even more explicit form: indeed, *the operators* $\mathrm{ad}_q^* a$ *and* $q \mathrm{ad}_q a$ *commute*, for every $a \in A$, as identity (39) proves. Hence one can apply Newton's binomial formula to obtain

$$\left(\mathrm{ad}_q^* a + q \mathrm{ad}_q a \right)^k = \sum_{h=0}^{k} q^h \binom{k}{h} \left(\mathrm{ad}_q^* a \right)^{k-h} \left(\mathrm{ad}_q a \right)^h, \tag{139}$$
$$\forall a \in A.$$

Moreover, since any right multiplication R_b commutes with any left multiplication L_a (by associativity), we infer that

any map $\mathrm{ad}_q^* b$

$+ q \mathrm{ad}_q b$ commutes with any map $\mathrm{ad}_q a$ $\qquad$ (140)

$+ q \mathrm{ad}_q^* a$.

We now obtain a formula, generalizing the above lemma, which also expresses in a "quantitative" way the congruence $y^j x^i \equiv q^{ji} x^i y^j \pmod I$.

Lemma 22. *For any $i, j \geq 1$ one has*

$$y^j x^i - q^{ji} x^i y^j = \frac{1}{(1-q^2)^{i-1}} \sum_{k=0}^{j-1} \sum_{h=0}^{i-1} \frac{q^{k+hj}}{(1-q^2)^k}$$
$$\cdot \binom{j}{k} \left(\mathrm{ad}_q^* x + q \mathrm{ad}_q x \right)^{i-1-h}$$
$$\cdot \left(\mathrm{ad}_q x + q \mathrm{ad}_q^* x \right)^h \tag{141}$$
$$\cdot \left(\mathrm{ad}_q^* y + q \mathrm{ad}_q y \right)^k \left(\mathrm{ad}_q y \right)^{j-k} (x).$$

As stated in Remark 21, this formula can be made even more explicit by unraveling the powers of $\mathrm{ad}_q^* x + q \mathrm{ad}_q x$, $\mathrm{ad}_q x + q \mathrm{ad}_q^* x$, and $\mathrm{ad}_q^* y + q \mathrm{ad}_q y$ by means of (139).

Note that the right-hand side of identity (141) is a linear combination of iterated q-mutators, centered at $[y, x]_q$, since $j - k > 0$. In other words it is an element of $S = I$. Our final prerequisite is to find an explicit form (in terms of iterated $[y, x]_q$-centered q-mutators) for the projection φ defined in Lemma 16, when it acts on a generic monomial $x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_m}$. This is given in the next result.

Lemma 23. *For any $n \geq 2$ and any $i_1, j_1, \ldots, i_n, j_n \geq 0$ one has*

$$x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}$$
$$- q^{i_2 j_1 + i_3(j_1+j_2) + \cdots + i_n(j_1 + \cdots + j_{n-1})} x^{i_1 + \cdots + i_n} y^{j_1 + \cdots + j_n}$$

$$= \frac{1}{(1-q^2)^{i_1 + \cdots + i_n - 1}} \sum_{r=1}^{n-1} \frac{q^{i_2 j_1 + \cdots + i_r(j_1 + \cdots + j_{r-1})}}{(1-q^2)^{j_{r+1} + j_{r+2} + \cdots + j_n}}$$
$$\cdot \sum_{k=0}^{j_1 + \cdots + j_r - 1} \frac{q^k}{(1-q^2)^k}$$
$$\cdot \binom{j_1 + \cdots + j_r}{k} \sum_{h=0}^{i_{r+1}-1} q^{h(j_1 + \cdots + j_r)}$$
$$\cdot \left(\mathrm{ad}_q^* y + q \mathrm{ad}_q y \right)^{j_n} \cdots \left(\mathrm{ad}_q^* x + q \mathrm{ad}_q x \right)^{i_{r+2}}$$
$$\cdot \left(\mathrm{ad}_q^* y + q \mathrm{ad}_q y \right)^{j_{r+1}} \left(\mathrm{ad}_q^* x + q \mathrm{ad}_q x \right)^{i_{r+1}-h-1}$$
$$\cdot \left(\mathrm{ad}_q^* y + q \mathrm{ad}_q y \right)^{k} \left(\mathrm{ad}_q x + q \mathrm{ad}_q^* x \right)^{h+i_1 + \cdots + i_r}$$
$$\cdot \left(\mathrm{ad}_q y \right)^{j_1 + \cdots + j_r - k} (x). \tag{142}$$

We agree that, when $r = 1$, the exponent $i_2 j_1 + \cdots + i_r(j_1 + \cdots + j_{r-1})$ has to be considered 0; furthermore we agree that summations over an empty set of indices are to be omitted.

Again (see Remark 21) the above formula can be made more explicit (although more cumbersome) by unraveling the powers of $\mathrm{ad}_q^* z + q \mathrm{ad}_q z$ and of $\mathrm{ad}_q z + q \mathrm{ad}_q^* z$ (with $z = x, y$), by means of (139). We are ready for the proof of Theorem 1.

Proof of Theorem 1. Let $i, j \geq 1$ be fixed. From Proposition 14 we know that $Q_{i,j}(x, y)$ belongs to $S_{i,j} \subset S$. From Lemma 16 we derive that $\varphi(Q_{i,j}(x, y)) = Q_{i,j}(x, y)$, since φ is a projection onto S. From the associative presentation (29) of $x \diamond_q y$ we infer the explicit formula

$$Q_{i,j}(x, y) = \sum_{n=1}^{i+j} \lambda_n$$
$$\cdot \sum_{\substack{(i_1,j_1),\ldots,(i_n,j_n) \neq (0,0) \\ i_1 + \cdots + i_n = i, j_1 + \cdots + j_n = j}} \frac{x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}}{[i_1]_q! [j_1]_q! \cdots [i_n]_q! [j_n]_q!}, \tag{143}$$

where the λ_n are as in (27). We observe that, from (29), we have isolated the summands with $1 \leq n \leq i + j$ since $(i_1, j_1), \ldots, (i_n, j_n) \neq (0, 0)$, so that

$$i + j = (i_1 + j_1) + \cdots + (i_n + j_n) \geq n. \tag{144}$$

Taking into account the above facts (and the definition of φ), if we apply φ to both sides of (143) we get (see also Remark 18)

$$Q_{i,j}(x, y) = \sum_{n=1}^{i+j} \lambda_n$$
$$\cdot \sum_{\substack{(i_1,j_1),\ldots,(i_n,j_n) \neq (0,0) \\ i_1 + \cdots + i_n = i, j_1 + \cdots + j_n = j}} \frac{1}{[i_1]_q! [j_1]_q! \cdots [i_n]_q! [j_n]_q!}$$

$$\cdot \left(x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n} \right.$$
$$\left. - q^{i_2 j_1 + \cdots + i_n (j_1 + j_2 + \cdots + j_{n-1})} x^{i_1 + \cdots + i_n} y^{j_1 + \cdots + j_n} \right).$$

$$(145)$$

Finally, the polynomial in the above parentheses has been explicitly written as a $[y, x]_q$-centered q-mutator in Lemma 23: this proves the theorem. $\square$

7. An Example of the Rearrangement Technique

We know from (101) that $xyxy^3$ is congruent to $qx^2 y^4$ modulo I. By means of a repeated application of the trivial identity

$$yx = [y, x]_q + qxy, \tag{146}$$

we can write $xyxy^3 - qx^2 y^4$ as an element of I (i.e., as a sum of polynomials factorizing $[y, x]_q$); subsequently, we can apply basic procedure (9) to write it as an element of S (i.e., as a linear combination of $[y, x]_q$-centered q-mutators). This is done in the next computation:

$$xyxy^3 - qx^2 y^4 = x \left([y, x]_q + qxy \right) y^3 - qx^2 y^4$$

$$= x [y, x]_q y^3 \overset{(9)}{=} \frac{1}{1 - q^2} \left(\left[x, [y, x]_q \right]_q \right.$$

$$\left. + q \left[[y, x]_q , x \right]_q \right) \cdot y^3$$

$$\overset{(9)}{=} \frac{1}{(1 - q^2)^2} \left(\left[\left[x, [y, x]_q \right]_q , y \right]_q \right.$$

$$+ q \left[y, \left[x, [y, x]_q \right]_q \right]_q + q \left[\left[[y, x]_q , x \right]_q , y \right]_q$$

$$\left. + q^2 \left[y, \left[[y, x]_q , x \right]_q \right]_q \right) \cdot y^2$$

$$\overset{(9)}{=} \frac{1}{(1 - q^2)^3} \left(\left[\left[\left[x, [y, x]_q \right]_q , y \right]_q , y \right]_q \right.$$

$$+ q \left[y, \left[\left[x, [y, x]_q \right]_q , y \right]_q \right]_q$$

$$+ q \left[\left[y, \left[x, [y, x]_q \right]_q \right]_q , y \right]_q$$

$$+ q^2 \left[y, \left[y, \left[x, [y, x]_q \right]_q \right]_q \right]_q$$

$$+ q \left[\left[\left[[y, x]_q , x \right]_q , y \right]_q , y \right]_q$$

$$+ q^2 \left[y, \left[\left[[y, x]_q , x \right]_q , y \right]_q \right]_q$$

$$+ q^2 \left[\left[y, \left[[y, x]_q , x \right]_q \right]_q , y \right]_q$$

$$+ q^3 \left[y, \left[y, \left[[y, x]_q , x \right]_q \right]_q \right]_q \right) \cdot y$$

$$\overset{(9)}{=} \frac{1}{(1 - q^2)^4} \left(\left[\left[\left[\left[x, [y, x]_q \right]_q , y \right]_q , y \right]_q , y \right]_q \right.$$

$$+ q \left[y, \left[\left[\left[x, [y, x]_q \right]_q , y \right]_q , y \right]_q \right]_q$$

$$+ q \left[\left[y, \left[\left[x, [y, x]_q \right]_q , y \right]_q \right]_q , y \right]_q$$

$$+ q^2 \left[y, \left[y, \left[\left[x, [y, x]_q \right]_q , y \right]_q \right]_q \right]_q$$

$$+ q \left[\left[\left[y, \left[x, [y, x]_q \right]_q \right]_q , y \right]_q , y \right]_q$$

$$+ q^2 \left[y, \left[\left[y, \left[x, [y, x]_q \right]_q \right]_q , y \right]_q \right]_q$$

$$+ q^2 \left[\left[y, \left[y, \left[x, [y, x]_q \right]_q \right]_q \right]_q , y \right]_q$$

$$+ q^3 \left[y, \left[y, \left[y, \left[x, [y, x]_q \right]_q \right]_q \right]_q \right]_q$$

$$+ q \left[\left[\left[\left[[y, x]_q , x \right]_q , y \right]_q , y \right]_q , y \right]_q$$

$$+ q^2 \left[y, \left[\left[\left[[y, x]_q , x \right]_q , y \right]_q , y \right]_q \right]_q$$

$$+ q^2 \left[\left[y, \left[\left[[y, x]_q , x \right]_q , y \right]_q \right]_q , y \right]_q$$

$$+ q^3 \left[y, \left[y, \left[\left[[y, x]_q , x \right]_q , y \right]_q \right]_q \right]_q$$

$$+ q^2 \left[\left[\left[y, \left[[y, x]_q , x \right]_q \right]_q , y \right]_q , y \right]_q$$

$$+ q^3 \left[y, \left[\left[y, \left[[y, x]_q , x \right]_q \right]_q , y \right]_q \right]_q$$

$$+ q^3 \left[\left[y, \left[y, \left[[y, x]_q , x \right]_q \right]_q \right]_q , y \right]_q$$

$$+ q^4 \left[y, \left[y, \left[y, \left[[y, x]_q , x \right]_q \right]_q \right]_q \right]_q \right).$$

$$(147)$$

In Lemma 23 we provided a formula for the above procedure for any monomial

$$x^{i_1} y^{j_1} \cdots x^{i_n} y^{j_n}. \tag{148}$$

This procedure allows us to write any $Q_{i,j}$ as a $[y, x]_q$-centered q-mutator. For example, a direct computation based on (143) and (27) gives out the associative presentation of $Q_{1,3}$:

$$\begin{aligned} Q_{1,3} = \frac{1}{[4]_q! \, [2]_q} \cdot \Big(& q^3 \left(-q+1\right) \cdot xy^3 + q^3 \left(-q+1\right) \\ & \cdot y^3 x + q^2 \left(q^3 - 2q^2 + 2q - 1\right) \cdot y^2 xy \\ & + q^2 \left(q^3 - 2q^2 + 2q - 1\right) \cdot yxy^2 \Big). \end{aligned} \tag{149}$$

We then use the identity $yx = [y, x]_q + qxy$ on each summand of $Q_{1,3}$ (other than xy^3) to eventually produce, modulo I, the monomial xy^3. For instance,

$$\begin{aligned} y^3 x &= y^2 [y, x]_q + qy [y, x]_q \, y + q^2 [y, x]_q \, y^2 \\ &\quad + q^3 xy^3 \\ y^2 xy &= y [y, x]_q \, y + q [y, x]_q \, y^2 + q^2 xy^3 \\ yxy^2 &= [y, x]_q \, y^2 + qxy^3. \end{aligned} \tag{150}$$

(These formulas can be improved by shifting all the factors y on the right as in Lemma 19.) Inserting these identities in the expansion of $Q_{1,3}$ we get

$$\begin{aligned} [4]_q! \, [2]_q \cdot Q_{1,3} = [\cdots] = &\left(\left(-q^4 + q^3\right) + \left(q^3 - q^4\right) q^3 \right. \\ &+ \left(q^5 - 2q^4 + 2q^3 - q^2\right) q^2 \\ &\left. + \left(q^5 - 2q^4 + 2q^3 - q^2\right) q \right) \cdot xy^3 + \left(q^3 - q^4\right) \\ &\cdot y^2 [y, x]_q + \left(\left(q^3 - q^4\right) q \right. \\ &+ \left(q^5 - 2q^4 + 2q^3 - q^2\right)\right) y [y, x]_q \, y \\ &+ \left(\left(q^3 - q^4\right) q^2 + \left(q^5 - 2q^4 + 2q^3 - q^2\right) q \right. \\ &\left. + \left(q^5 - 2q^4 + 2q^3 - q^2\right)\right) [y, x]_q \, y^2 = 0 \cdot xy^3 \\ &- q^3 (q - 1) \, y^2 [y, x]_q - q^2 (q - 1)^2 \, y [y, x]_q \, y \\ &+ q^2 (q - 1) [y, x]_q \, y^2. \end{aligned} \tag{151}$$

Obviously, the cancelation of the summand xy^3 is not sheer chance, but it derives from the fact that $Q_{1,3}$ belongs to I. We next apply the technique exemplified above, based on (9), thus obtaining the presentation[2]

$$\begin{aligned} Q_{1,3} = -\frac{q^2}{[4]_q! \, [2]_q} \Big(& \left[\left[[y, x]_q, y \right]_q, y \right]_q \\ & + \left[y, \left[[y, x]_q, y \right]_q \right]_q \Big). \end{aligned} \tag{152}$$

Analogously one gets

$$\begin{aligned} Q_{3,1} = -\frac{q^2}{[4]_q! \, [2]_q} \Big(& \left[x, \left[x, [y, x]_q \right]_q \right]_q \\ & + \left[\left[x, [y, x]_q \right]_q, x \right]_q \Big). \end{aligned} \tag{153}$$

Appendix

The Convergence of the q-BCH Series Near Zero

To test the applicability of our formula for the q-BCH series, we prove the following result concerning the convergence of the series for Banach algebras; we observe that our arguments can also be applied in wider context than associative algebras, as the proof will show (see also Remark A.2).

We recall that a Banach algebra is triple $(A, *, \| \cdot \|)$ where $(A, *)$ is an associative algebra over $\mathbb{R}$ (or $\mathbb{C}$), $(A, \| \cdot \|)$ is a normed Banach space, and $(a, b) \mapsto a * b$ is a continuous map ($A \times A$ is equipped with the product topology of the space A).

Theorem A.1. *Let $(A, *, \| \cdot \|)$ be a Banach algebra over $\mathbb{R}$. Let $q \in \mathbb{R}$ be such that $|q| < 1$. Then there exists an open neighborhood U_q of $0 \in A$ such that the series $\sum_{i,j \geq 1} Q_{i,j}(x, y)$ converges normally for $x, y \in U_q$.*

Here $Q_{i,j}(x, y)$ is given by formula (41) and $[a, b]_q = a * b - qb * a$ for any $a, b \in A$. Finally the numbers λ_n are defined as in (27) relative to $\mathbb{R}$.[3]

In particular, the q-BCH series is convergent in any (real or complex) matrix algebra and—more generally—in any finite dimensional (real or complex) associative algebra.

Remark A.2. Normal convergence on U_q means, as usual,

$$\sum_{i,j \geq 1} \sup_{x,y \in U_q} \left\| Q_{i,j}(x, y) \right\| < \infty. \tag{A.1}$$

Since A is a Banach space, normal convergence implies uniform and pointwise convergence. Theorem A.1 also holds true if the underlying field is $\mathbb{C}$, as the proof will show. Finally, we do not use explicitly the associative structure of A, but only the basic inequality

$$\left\| [a, b]_q \right\| \leq M_q \|a\| \cdot \|b\|, \quad \forall a, b \in A, \tag{A.2}$$

for some constant $M_q > 0$. Hence the same proof works for more general settings than Banach (associative) algebras.

Proof. Since A is a Banach algebra, there exists a constant $c \geq 1$ such that

$$\|a * b\| \leq c \|a\| \cdot \|b\|, \quad \forall a, b \in A. \tag{A.3}$$

To simplify the notation, by replacing the norm $\| \cdot \|$ with the equivalent norm $c\| \cdot \|$, we can assume that the above inequality holds true with $c = 1$. As a consequence (A.2) holds true

with $M_q := (1 + |q|) \geq 1$. From (A.2) one easily obtains the estimates[4]

$$\left\| \left(\mathrm{ad}_q a \right)^j b \right\| \leq M_q^j \|a\|^j \cdot \|b\|, \tag{A.4}$$

$$\left\| \left(\mathrm{ad}_q a + q \mathrm{ad}_q^* a \right)^j b \right\|, \left\| \left(\mathrm{ad}_q^* a + q \mathrm{ad}_q a \right)^j b \right\|$$
$$\leq M_q^{2j} \|a\|^j \cdot \|b\|, \tag{A.5}$$

for any $a, b \in A$ and any $j \in \mathbb{N}$, where ad_q and ad_q^* have the obvious meanings.

We let $\varepsilon > 0$ be small (it will be conveniently chosen in due course) and we take any $x, y \in A$ such that $\|x\|, \|y\| < \varepsilon$. Then, by the triangle inequality and a repeated application of (A.4)-(A.5), we have

$$\left\| Q_{i,j}(x,y) \right\| \leq \sum_{n=1}^{i+j} |\lambda_n| \sum_{\substack{(i_1,j_1),\dots,(i_n,j_n)\neq(0,0) \\ i_1+\cdots+i_n=i, j_1+\cdots+j_n=j}} \frac{1}{\left|[i_1]_q!\right| \left|[j_1]_q!\right| \cdots \left|[i_n]_q!\right| \left|[j_n]_q!\right|} \frac{1}{|1-q^2|^{i-1}} \sum_{r=1}^{n-1} \frac{|q|^{i_2 j_1 + \cdots + i_r(j_1 + \cdots + j_{r-1})}}{|1-q^2|^{j_{r+1}+j_{r+2}+\cdots+j_n}}$$

$$\cdot \sum_{k=0}^{j_1+\cdots+j_r-1} \frac{|q|^k}{|1-q^2|^k} \binom{j_1 + \cdots + j_r}{k} \sum_{h=0}^{i_{r+1}-1} |q|^{h(j_1+\cdots+j_r)}$$

$$\cdot M_q^{2(j_n+i_n+\cdots+i_{r+2}+j_{r+1}+i_{r+1}-h-1+k+h+i_1+\cdots+i_r)+j_1+\cdots+j_r-k} \|y\|^{j_n+\cdots+j_{r+1}+k+j_1+\cdots+j_r-k} \|x\|^{i_n+\cdots+i_{r+2}+i_{r+1}-h-1+h+i_1+\cdots+i_r+1} \tag{A.6}$$

$$\leq M_q^{2(i+j)} \|x\|^i \|y\|^j \sum_{n=1}^{i+j} |\lambda_n| \sum_{\substack{(i_1,j_1),\dots,(i_n,j_n)\neq(0,0) \\ i_1+\cdots+i_n=i, j_1+\cdots+j_n=j}} \frac{1}{\left|[i_1]_q!\right| \cdots \left|[j_n]_q!\right|} \frac{1}{|1-q^2|^{i-1}} \sum_{r=1}^{n-1} \frac{1}{|1-q^2|^{j_{r+1}+j_{r+2}+\cdots+j_n}} \sum_{k=0}^{j_1+\cdots+j_r-1} M_q^k \frac{|q|^k}{|1-q^2|^k}$$

$$\cdot \binom{j_1 + \cdots + j_r}{k} \sum_{h=0}^{i_{r+1}-1} 1 =: (\star).$$

In the last inequality we also used $|q| < 1$ and $M_q \geq 1$.

In order to estimate $(\star)$ we observe the following facts:

(i) the inner sum on h equals i_{r+1} and is therefore bounded above by $i \leq M_q^i$;

(ii) setting $A = M_q|q|/|1-q^2|$, the sum over k is majorized by

$$\sum_{k=0}^{j_1+\cdots+j_r} A^k \binom{j_1 + \cdots + j_r}{k} = (1 + A)^{j_1+\cdots+j_r}. \tag{A.7}$$

As a consequence we have

$$(\star) \leq \frac{M_q^{3i+2j}}{|1-q^2|^{i-1}} \|x\|^i \|y\|^j \sum_{n=1}^{i+j} |\lambda_n|$$

$$\cdot \sum_{\substack{(i_1,j_1),\dots,(i_n,j_n)\neq(0,0) \\ i_1+\cdots+i_n=i, j_1+\cdots+j_n=j}} \frac{1}{\left|[i_1]_q!\right| \cdots \left|[j_n]_q!\right|} \tag{A.8}$$

$$\cdot \sum_{r=1}^{n-1} \left(1 + \frac{M_q|q|}{|1-q^2|} \right)^{j_1+\cdots+j_r} \left(\frac{1}{|1-q^2|} \right)^{j_{r+1}+\cdots+j_n}$$

$$=: (2\star).$$

In order to estimate $(2\star)$ we observe that $|1 - q^2| < 1$ (since $|q| < 1$ by assumption), so that the inner sum over r is bounded above by

$$\sum_{r=1}^{n} \left(1 + \frac{M_q|q|}{|1-q^2|} \right)^{j_1+\cdots+j_n} \cdot \left(\frac{1}{|1-q^2|} \right)^{j_1+\cdots+j_n}$$
$$= n \left(\frac{|1-q^2| + M_q|q|}{|1-q^2|^2} \right)^j. \tag{A.9}$$

Here we used the fact that the second sum in $(2\star)$ specifies that $j_1 + \cdots + j_n = j$. Furthermore, from the first sum in $(2\star)$ we know that $n \leq i + j$, and the latter is obviously $\leq M_q^{i+j}$ (since $M_q \geq 1$). Therefore we get

$$(2\star) \leq \left(\frac{|1-q^2| + M_q|q|}{|1-q^2|^2} \right)^j \cdot \frac{M_q^{4i+3j}}{|1-q^2|^i} \|x\|^i \|y\|^j$$

$$\cdot \sum_{n=1}^{i+j} |\lambda_n| \tag{A.10}$$

$$\cdot \sum_{\substack{(i_1,j_1),\dots,(i_n,j_n)\neq(0,0) \\ i_1+\cdots+i_n=i, j_1+\cdots+j_n=j}} \frac{1}{\left|[i_1]_q!\right| \left|[j_1]_q!\right| \cdots \left|[i_n]_q!\right| \left|[j_n]_q!\right|}$$

$$=: (3\star).$$

(Again we used $|1 - q^2| < 1$.) We set

$$X = \frac{M_q^4}{|1 - q^2|} \|x\|,$$

$$Y = \left(\frac{|1 - q^2| + M_q |q|}{|1 - q^2|^2} \right) \cdot M_q^3 \|y\|, \quad \text{(A.11)}$$

so that $(3\star)$ is equal to

$$\sum_{n=1}^{i+j} |\lambda_n| \sum_{\substack{(i_1, j_1), \dots, (i_n, j_n) \neq (0,0) \\ i_1 + \dots + i_n = i, j_1 + \dots + j_n = j}} \frac{X^{i_1} \cdots X^{i_n}}{\left| [i_1]_q! \right| \cdots \left| [i_n]_q! \right|} \quad \text{(A.12)}$$

$$\cdot \frac{Y^{j_1} \cdots Y^{j_n}}{\left| [j_1]_q! \right| \cdots \left| [j_n]_q! \right|}.$$

Now we get to the crucial part of the estimate (see also the analogy to the classical case [1, Section 5.2]). Summing over all the indices $i, j \geq 1$ we get

$$\sum_{i,j \geq 1} \left\| Q_{i,j}(x, y) \right\|$$

$$\leq \sum_{i,j \geq 1} \sum_{n=1}^{i+j} |\lambda_n| \sum_{\substack{(i_1, j_1), \dots, (i_n, j_n) \neq (0,0) \\ i_1 + \dots + i_n = i, j_1 + \dots + j_n = j}} \frac{X^{i_1} \cdots X^{i_n}}{\left| [i_1]_q! \right| \cdots \left| [i_n]_q! \right|}$$

$$\cdot \frac{Y^{j_1} \cdots Y^{j_n}}{\left| [j_1]_q! \right| \cdots \left| [j_n]_q! \right|}$$

$$\leq \sum_{n=1}^{\infty} |\lambda_n| \sum_{(i_1, j_1), \dots, (i_n, j_n) \neq (0,0)} \frac{X^{i_1} \cdots X^{i_n}}{\left| [i_1]_q! \right| \cdots \left| [i_n]_q! \right|} \quad \text{(A.13)}$$

$$\cdot \frac{Y^{j_1} \cdots Y^{j_n}}{\left| [j_1]_q! \right| \cdots \left| [j_n]_q! \right|}$$

$$= \sum_{n=1}^{\infty} |\lambda_n| \left(\sum_{(i,j) \neq (0,0)} \frac{X^i}{\left| [i]_q! \right|} \cdot \frac{Y^j}{\left| [j]_q! \right|} \right)^n$$

$$= \sum_{n=1}^{\infty} |\lambda_n| \left(\sum_{i=0}^{\infty} \frac{X^j}{\left| [i]_q! \right|} \cdot \sum_{j=0}^{\infty} \frac{Y^j}{\left| [j]_q! \right|} - 1 \right)^n =: (4\star).$$

To end the proof, we recall that (as is well known, see, e.g., [39]), the complex series $\exp_q(z) = \sum_{i=0}^{\infty} (z^i / [i]_q!)$ has a positive radius of convergence (depending on q), say $\alpha_q \in$ $]0, \infty]$; for $|z| < \alpha_q$ one obviously also has

$$E(z) := \sum_{i=0}^{\infty} \frac{|z|^i}{\left| [i]_q! \right|} < \infty. \quad \text{(A.14)}$$

The inverse function $\log_q(z)$ of $\exp_q(z)$ is convergent for z in some neighborhood of 1; as a consequence the series

$\log_q(1 + z) = \sum_{n=1}^{\infty} \lambda_n z^n$ has a positive radius of convergence, say $\beta_q > 0$. Analogously, for $|z| < \beta_q$ one has

$$L(z) := \sum_{n=1}^{\infty} |\lambda_n| |z|^n < \infty. \quad \text{(A.15)}$$

Summing up, from

$$(4\star) = L(E(X) E(Y) - 1), \quad \text{(A.16)}$$

we infer that $\sum_{i,j \geq 1} \|Q_{i,j}(x, y)\|$ is convergent if the right-hand side of (A.16) is finite, and—in its turn—the latter happens if

$$|E(X) E(Y) - 1| < \beta_q. \quad \text{(A.17)}$$

By continuity (since $E(0)E(0) = 1$), there exists a small $\varepsilon_q > 0$ (with $\varepsilon_q < \alpha_q$) such that $|E(X)E(Y) - 1| < \beta_q$ whenever $|X|, |Y| \leq \varepsilon_q$.

For this choice of ε_q and by (A.11), we deduce that the series $\sum_{i,j \geq 1} \|Q_{i,j}(x, y)\|$ is finite whenever

$$\|x\| \leq \varepsilon_q \frac{|1 - q^2|}{(1 + |q|)^4},$$

$$\text{(A.18)}$$

$$\|y\| \leq \varepsilon_q \frac{1}{(1 + |q|)^3} \cdot \frac{|1 - q^2|^2}{|1 - q^2| + (1 + |q|) |q|}.$$

This completes the proof of the absolute convergence of $\sum_{i,j} Q_{i,j}(x, y)$. The proof of the normal convergence is completely analogous, by refining (A.17): for instance, it suffices to require that ε_q satisfy

$$|E(X) E(Y) - 1| < \frac{\beta_q}{2} \quad \text{if } |X|, |Y| \leq \varepsilon_q. \quad \text{(A.19)}$$

This ends the proof. $\qquad \square$

Competing Interests

The authors declare that they have no competing interests.

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Endnotes

1. Here the word basis refers to the space $\mathbb{K}(q)\langle x, y\rangle$.

2. In order to get simplifications one also needs to take into account the fact that

$$\left[\left[y, [y, x]_q\right]_q, y\right]_q = \left[y, \left[[y, x]_q, y\right]_q\right]_q. \qquad (*)$$

The q-mutators $[y, [y, [y, x]_q]_q]_q$, $[[[y, x]_q, y]_q, y]_q$, and $[y, [[y, x]_q, y]_q]_q$ are linearly independent; it is a striking fact that $Q_{1,3}$ can be written by means of the last two only.

3. λ_n are well posed in $\mathbb{R}$ for any q since $[n]_q = 1 + q + \cdots + q^{n-1} \neq 0$ for any $q \in \mathbb{R} \setminus \{-1\}$ (and for any $q \in \mathbb{C}$ with $|q| < 1$).

4. For obtaining (A.5), one can also use (139).

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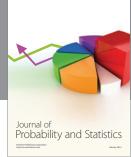
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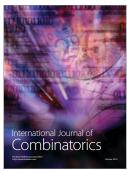[39] H. Exton, q-*Hypergeometric Functions and Applications*, Ellis Horwood, Chichester, UK, 1983.