Stability of the graviton Bose–Einstein condensate in the brane-world

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A B S T R A C T

We consider a solution of the effective four-dimensional Einstein equations, obtained from the general relativistic Schwarzschild metric through the principle of Minimal Geometric Deformation (MGD). Since the brane tension can, in general, introduce new singularities on a relativistic Eötvös brane model in the MGD framework, we require the absence of observed singularities, in order to constrain the brane tension. We then study the corresponding Bose–Einstein condensate (BEC) gravitational system and determine the critical stability region of BEC MGD stellar configurations. Finally, the critical stellar densities are shown to be related with critical points of the information entropy.

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1. Introduction

Several aspects of black hole physics have been recently studied, by considering black holes as Bose–Einstein condensates (BEC) of a large number N of weakly interacting, long-wavelength, gravitons close to a critical point [1–3]. This paradigm has the merit to directly interconnect black hole physics to the study of critical phenomena, where quantum effects are relevant at critical points, even for a macroscopic number N of particles [4]. Although black holes are non-perturbative gravitational objects, the effective quantum field theory of gravitons that describes them can still be weakly coupled, due to large collective effects [3,5,6]. Black hole features that cannot be recovered in a standard semiclassical approach of gravity may then be encoded by the quantum state of the critical BEC [7,8], with the semiclassical regime obtained as a particular limit for $N \to \infty$. Moreover, describing black holes by a condensate of long-wavelength gravitons generates a self-sustained system, whose size equals the standard Schwarzschild radius and the gravitons are maximally packed [1–3,7]. A quantum field-theoretical analysis also clarified the relation between the emerging geometry of spacetime and the quantum theory [9].

Brane-world models are effective five-dimensional (5D) phenomenological realisations of the Hořava–Witten domain wall solutions [10], when moduli effects, engendered from the remain-

ing extra dimensions, may be disregarded [11,12]. The brane self-gravity is identified by the brane tension σ, and the effective four-dimensional (4D) geometry, due to a compact stellar distribution, can be achieved by a Minimal Geometric Deformation (MGD) of the standard Schwarzschild solution in General Relativity (GR) [13–17]. The MGD method ensures that this brane-world effective gravitational solution reduces to the standard Schwarzschild solution, in the limit of infinite brane tension $\sigma^{-1} \to 0$. Therefore the MGD is a framework that provides corrections to GR, controlled by a parameter ζ, that is a function of the stellar distribution effective radius and the brane tension.

Finally, we recall that a harmonic black hole model was recently introduced [18], which can be viewed as an explicit realisation of a BEC of gravitons, with a regular interior. The energy density in this model is obtained from a three-dimensional harmonic potential, "cut" around the horizon size in order to accommodate for the continuum spectrum of scattering modes, and the Hawking radiation. Afterwards, this model was ameliorated by instead considering the Pöschl–Teller potential [19], which naturally contains a continuum spectrum above the bound states, contrary to the harmonic oscillator.

We shall here employ this last model to study a MGD BEC black hole and analyse its critical stable density, from the point of view of the information entropy [20,21], and statistical mechanics [22]. The information entropy has been applied to a variety of settings, and the stability of self-gravitating compact objects was already reported in Refs. [20,23]. In particular, Newtonian polytropes, neutron stars, and boson stars were studied in Ref. [23]. The information entropy is well-known to measure the underlying

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shape complexity of spatially localised configurations [20,21]. The less information involved in the modes that comprise a physical system, the smaller entropic information is required to represent the same physical system. The energy density is the main ingredient to compute the information entropy. In this framework, the critical stable density of a BEC MGD black hole will be here studied, by relating the stellar distribution conditional entropy and its central critical density. In other words, the conditional entropy will be used to study the gravitational stability.

This work is organised as follows: we review the MGD procedure in Section 2, and the BEC description of a MGD black hole is employed to establish a bound for the brane tension of a Eötvös brane-world model in Section 3; Section 4 is devoted to establish the interplay between the critical point in the stellar stability and the critical point of the conditional entropy in a BEC MGD self-gravitating system scenario; finally, we comment on our findings in Section 5.

2. Minimal geometric deformation

The MGD approach is designed to produce brane-world corrections to standard GR solutions, hence it is a suitable method to obtain inhomogeneous, spherically symmetric, stellar distributions that are physically admissible in the brane-world [17,24]. For example, the bound $\sigma \gtrsim 5 \times 10^6$ MeV^4 for the brane tension was obtained from the MGD in Ref. [30]. The MGD was originally applied in order to deform the standard Schwarzschild solution [13, 16,17] and describe the 4D geometry of a brane stellar distribution. Moreover, the MGD paved the way for interesting developments concerning 5D black string solutions of 5D Einstein equations [26] in Eötvös variable brane tension models [27,28].

The method relies on the effective Einstein equations on the brane [29],

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda\, g_{\mu\nu} - \tilde{T}_{\mu\nu} = 0 , \tag{1}$$

where the effective energy–momentum tensor is given by

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + \mathcal{E}_{\mu\nu} + \frac{1}{\sigma}\, S_{\mu\nu} , \tag{2}$$

which contains the usual stress tensor $T_{\mu\nu}$ of brane matter (with four-velocity u^μ), and the (non-local) Weyl and high energy Kaluza–Klein corrections $\mathcal{E}_{\mu\nu}$ and $S_{\mu\nu}$. The Weyl tensor can be further decomposed as

$$\mathcal{E}_{\mu\nu} = \frac{6}{\sigma} \left[\mathcal{U} \left(\frac{1}{3} h_{\mu\nu} + u_\mu u_\nu \right) + \mathcal{P}_{\mu\nu} + \mathcal{Q}_{(\mu} u_{\nu)} \right] , \tag{3}$$

where $h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ denotes the induced spatial metric, $\mathcal{P}_{\mu\nu}$ is the anisotropic stress, $\mathcal{U}$ stands for the Weyl bulk scalar, and $\mathcal{Q}_\mu$ denotes the energy flux field.

One then considers the general spherically symmetric metric,

$$ds^2 = A(r)\, dt^2 - \frac{dr^2}{B(r)} - r^2\, d\Omega^2 , \tag{4}$$

in the effective equations (1). Any deformation of this static metric, with respect to a GR solution, must be caused by 5D bulk effects, in a brane-world scenario. Particularly, the radial component outside a compact stellar distribution, of average radius $r = R$, turns out to be given by [16,17]

$$B_+(r) = 1 - \frac{2 M}{r} + \zeta\, e^{-I} , \tag{5}$$

where

$$I(r) = \int_R^r \left(\frac{A A''}{A'^2} + \frac{A'^2}{A^2} - 1 + \frac{2 A'}{rA} + \frac{1}{r^2} \right) \left(\frac{2}{r} + \frac{A'}{2A} \right)^{-1} d\bar{r} , \tag{6}$$

where primes denote derivatives with respect to r. The parameter ζ describes the deformation induced onto the vacuum by bulk effects, evaluated at the surface of the stellar distribution. Therefore, ζ contains all relevant information of a Weyl fluid on the brane [30]. The matching conditions with the inner star metric then determine the outer metric for $r > R$ [13,26]. In particular, if one considers the standard Schwarzschild metric, the deformed outer metric components read [16]

$$A_+(r) = 1 - \frac{2 M}{r} , \tag{7a}$$

$$B_+(r) = \left(1 - \frac{2 M}{r} \right) \left[1 + \zeta\, \frac{\ell}{r} \left(1 - \frac{3 M}{2 r} \right)^{-1} \right] , \tag{7b}$$

where ℓ is a length given by[1]

$$\ell \equiv R \left(1 - \frac{2 M}{R} \right)^{-1} \left(1 - \frac{3 M}{2 R} \right) . \tag{8}$$

This metric has two event horizons where $B_+ = 0$: one is the usual Schwarzschild horizon, $r_s = 2 M$, and the second horizon is at $r_2 = \frac{3 M}{2} - \zeta\, \ell$. The expression of ζ was previously derived [13,16],

$$\zeta(\sigma, R) \approx - \frac{0.275}{R^2 \sigma} , \tag{9}$$

and the GR limit $\zeta \sim \sigma^{-1} \to 0$ implies that $r_2 < r_s$. One can therefore conclude that the gravitational field around the compact star is weaker than in GR.

3. BEC and MGD: a brane tension bound

In order to study BEC black holes with the MGD methods, let us start from the Klein–Gordon equation for a scalar field Ψ [19]

$$\left\{ \left[i \hbar\, \partial_t - V(\vec{x}) \right]^2 + \hbar^2\, \nabla^2 - \left[\mu + S(\vec{x}) \right]^2 \right\} \Psi(t, \vec{x}) = 0 , \tag{10}$$

where μ denotes the rest mass and one included the time-independent vector and scalar potentials $V(\vec{x})$ and $S(\vec{x})$. Writing $\Psi(t, \vec{x}) = e^{-i \varpi t/\hbar}\, \Psi(\vec{x})$ and assuming $S = V$ yield

$$\left[-\frac{\hbar^2}{2 (\varpi + \mu)} \nabla^2 + V - \frac{1}{2} (\varpi - \mu) \right] \Psi(\vec{x}) = 0 , \tag{11}$$

which is just a Schrödinger equation with $m = \varpi + \mu$, and $E = \frac{1}{2} (\varpi - \mu)$. It represents the relativistic dispersion relation $\varpi^2 = \hbar^2 k^2 + \mu^2$, and we shall in particular consider the spherically symmetric Pöschl–Teller potential [19]

$$V = -\frac{3 \mu}{\cosh(\mu r/\hbar)} , \tag{12}$$

for which one can find explicit solutions for $\Psi = \Psi(r)$ and compute the corresponding energy density. In fact, this graviton BEC can be macroscopically modelled by an anisotropic fluid, with local energy–momentum tensor of the form

$$T^{\mu\nu} = (p_\parallel - p_\perp) v^\mu v^\nu + (\varepsilon + p_\perp) u^\mu u^\nu + p_\perp g^{\mu\nu} , \tag{13}$$

where $u^\mu u_\mu = -1 = -v^\mu v_\mu$, and $u^\mu v_\mu = 0$, ε is the energy density, $p_\perp$ and $p_\parallel$ are the pressures perpendicular and parallel to the

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[1] The deformation around the star surface is negative, in order to prevent a negative pressure for a solid crust [24].

space-like vector v^μ. For the static, spherically symmetric metric in Eq. (4), one has

$$u^\mu = \left(-A^{-1/2}, 0, 0, 0\right)^\mathsf{T}, \quad v^\mu = \left(0, B^{1/2}, 0, 0\right)^\mathsf{T}. \tag{14}$$

One can then introduce the (quasilocal) Misner–Sharp mass function,

$$M(r) = 4\pi \int_0^r \bar{r}^2 \varepsilon(\bar{r}) \, d\bar{r}, \tag{15}$$

which represents the total energy within a stellar distribution of radius r, and find the radial component of the metric [19]

$$B(\rho) = 1 + \frac{1}{\rho} \tanh^3(\nu\rho)\left[3\tanh^2(\nu\rho) - 5\right], \tag{16}$$

where $\rho = r/M = 2r/r_s$ and $\nu = M\mu/\hbar$, with r_s the gravitational radius of the total Misner–Sharp mass M like in the previous Section.

The fluid description by means of Eq. (13) makes it now possible to apply the MGD approach to the above metric. In particular, the graviton BEC model of Ref. [19] considers the equation of state $\varepsilon + p_\parallel = 0$. Upon assuming for the temporal component of the metric the usual Schwarzschild form (7a), the effective energy–momentum tensor $\tilde{T}_{\mu\nu}$ in Eq. (2) yields the effective energy density and pressure [24]

$$\tilde{\varepsilon} = -\frac{(9 - 14cr^2 + c^2r^4)}{7\pi(1 + cr^2)^4}\delta + \frac{c\,(c^2r^4 + 2cr^2 + 9)}{7\pi(1 + cr^2)^3}, \tag{17}$$

$$\tilde{p} = -\frac{4(2c^2r^4 - 9cr^2 + 1)}{7\pi(1 + cr^2)^4}\delta + \frac{2c(c^2r^4 + 7cr^2 + 2)}{7\pi(1 + cr^2)^3}, \tag{18}$$

where $c \simeq 0.275/R^2$. In the next Section, Eqs. (17) and (18) will also be used to compute the effective energy density as the temporal component of the effective energy–momentum tensor (2), namely

$$\tilde{\epsilon}(r) = \tilde{T}^{00}(r). \tag{19}$$

In the above expressions, the brane-world corrections are given by the terms proportional to

$$\delta(\sigma) = \frac{f_R^*}{\sigma\,R^2}\frac{7(1 + cR^2)^2(1 + 9cR^2)}{16\,R^2(7 + 2cR^2)} + \mathcal{O}(\sigma^{-2}), \tag{20}$$

with

$$f_R^* = \frac{4}{49\pi}\left[\frac{80\arctan(y^{1/2})}{(1 + y)^2(3y + 1)\,y^{1/2}} + \frac{3y^4 + 41y^3 + 25y^2 - 589y - 240}{3(1 + y)^4(1 + 3y)}\right], \tag{21}$$

for $y = cR^2$ [24]. Finally, in the limiting case $R = r_s$, which we assume describes the BEC black hole, the deformed radial metric component to leading order in σ^{-1} is given by

$$\frac{B_\nu(\rho)}{B(\rho)} = 1 - \frac{2\,c_0}{\sigma\left[\rho - \frac{3}{4}\tanh^3(\nu\rho)\right]\left[5 - 3\tanh^2(\nu\rho)\right]}, \tag{22}$$

where $c_0 \simeq 0.275$. Fig. 1 shows plots of $B_\nu(\rho)$ for various values of ν. It is clear that, for increasing values of ν, this black hole model rapidly approaches the Schwarzschild black hole. This figure can be compared to Fig. 1 in Ref. [19] for similar parameters.

For any ν, the metric component $B_\nu(\rho)$ has a single local minimum at $\rho_* = a_*/\nu$, where $a_* \approx 1.031$. Writing $B_\nu(\rho_*) = 1 - \nu/\nu_*$, with $\nu_* \approx 0.694$, the condition for the existence of an event horizon is $\nu > \nu_*$. The case $\nu = \nu_*$ is extremal [19].
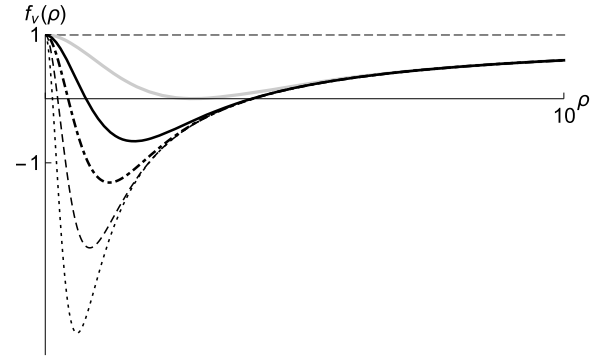

Fig. 1. Plot of $B_\nu(\rho)$ in Eq. (22), for $\nu = 0$ (gray dashed line); $\nu = 0.3$ (thick gray line); $\nu = 0.5$ (thick black line); $\nu = \nu_*$ (black dot-dashed line); $\nu = 1$ (black dashed line); $\nu = 1.4$ (dotted line).

3.1. Variable tension model

A more realistic model can be implemented by considering an Eötvös variable tension brane [31,25]. Essentially, the Eötvös law states that the (fluid) membrane tension depends on the temperature as

$$\sigma = \xi\,(T_\text{crit} - T), \tag{23}$$

where ξ is a positive constant and T_crit a critical temperature that determines the ceasing of the membrane existence. The tension variation is now expressed in terms of the (cosmic) time, instead of the temperature. Indeed, the Universe cools down as it expands. The cosmic microwave background indicates $T \sim a^{-1}$, where $a = a(t)$ denotes the FLRW Universe expansion factor [31,27,28], in agreement with the standard cosmological model. Eq. (23) then yields

$$\sigma(t) = \sigma_0\left(1 - \frac{a_\text{min}}{a(t)}\right), \tag{24}$$

where σ_0 is a constant related to the 4D coupling constants [31], and a_min is the minimum scale factor, below which the brane does cease to exist.

Along each definite phase, in the Universe evolution, brane tension changes are just perceptible across cosmic time scales. Regarding a de Sitter (dS) brane, the variable brane tension reads $\sigma(t) \sim 1 - e^{-\chi t}$, for $\chi > 0$ [32], being admissible from a phenomenological point of view [12,29], predicting a variable Newton coupling constant $G \sim \sigma(t)$. In Fig. 2 and 3, we display a refinement of Fig. 1, along the time scale, which takes into account an Eötvös brane variable tension, according to the law (24).

Since we do not observe extra singularities up to the standard Schwarzschild-like one, the brane tension is constrained by the bound:

$$\sigma \gtrsim 3.18 \times 10^6 \text{ MeV}^4. \tag{25}$$

This limit is much stronger than the cosmological nucleosynthesis constraint, and also better than the one provided by the limits of the MGD parameter ζ [30].

4. Stability analysis of the MGD BEC

Having computed a bound on the brane tension for the MGD BEC, we can now employ the conditional entropy in order to obtain stability bounds of critical points of the MGD BEC stellar distribution density. The entropic information, realised by the conditional entropy, was utilised to scrutinise the stability of physical systems [21], and proved to be based on statistical mechanics grounds
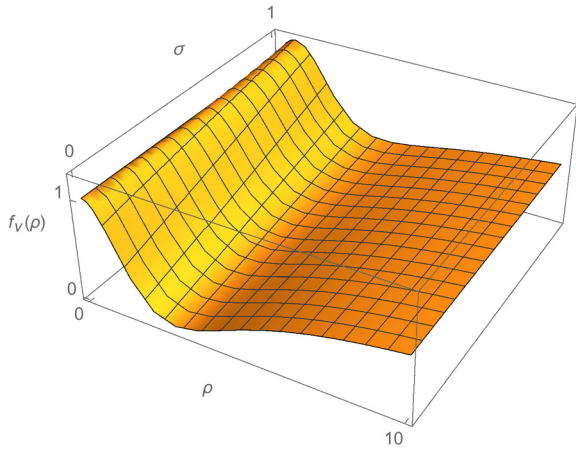
Fig. 2. Plot of $B_\nu(\rho)$ for $\nu = 0.3$, in the era dominated by non-relativistic matter, as a function of the Eötvös brane tension $\sigma(t)$ and the radial coordinate $\rho = r/M$. The cosmological time is normalised according to Eq. (24).
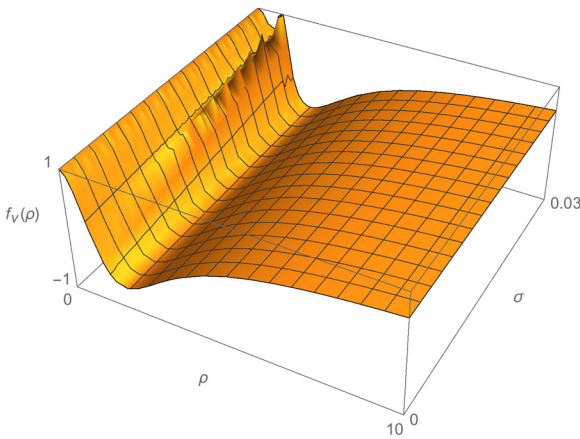

Fig. 3. Plot of $B_\nu(\rho)$ for $\nu = 0.5$, in the era dominated by the cosmological constant, as a function of the Eötvös brane tension $\sigma(t)$ and the radial coordinate $\rho = r/M$. The cosmological time is normalised according to Eq. (24).

in Ref. [22]. Indeed, physical systems have classical field configurations defined as a critical points of the classical action. In a semi-classical approximation, these configurations can then be thought as critical points of the effective action. Critical points of conditional entropy correlated with the most stable configurations, in the context of information entropy [33,34]. Physical systems states, with larger information entropy, either need a larger amount of energy to be produced, or are more scarcely observed – or detected – than their configurationally stable analog states, or even both [22,35]. The conditional entropy is based upon the information entropy and has critical points of stability that can comprise configurations that provide the best compression of informational nature in the system.

In order to compute the conditional entropy for the MGD BEC stellar distribution, let us start by calculating the spatial Fourier transform of the energy density $\tilde{\epsilon} = \tilde{\epsilon}(r)$ in Eq. (19),

$$\epsilon(\omega) = (2\pi)^{-1/2} \lim_{n \to \infty} \int_{-n}^{n} \tilde{\epsilon}(r) \, e^{i\omega r} \, dr \;, \tag{26}$$

where r is again the radial coordinate. Employing the effective energy density appears quite natural, since this quantity effectively describes the spatially localised BEC in the brane-world, also including the physics and boundary conditions that determine the stellar distribution. These Fourier components mimic

the collective coordinates [22]. The structure factor, defined by $s_n = \frac{1}{n} \sum_{k=1}^{n} \langle \epsilon(\omega_k)^* \epsilon(\omega_k) \rangle$, normalises the correlation of collective coordinates, and defines the discrete modal fraction

$$f(\omega_n) = \frac{1}{n \, s_n} \langle \epsilon^*(\omega_n) \epsilon(\omega_n) \rangle \;. \tag{27}$$

The structure factor probes fluctuations in the energy density [22], as the energy density operator fluctuates among system configurations. In the limit $n \to \infty$, the discrete modal fraction [20] can be expressed as the ratio between the correlation of collective coordinates and the structure factor [22],

$$f(\omega) \equiv \lim_{n \to \infty} f(\omega_n) = \frac{\langle |\epsilon(\omega)|^2 \rangle}{\displaystyle \lim_{n \to \infty} \int_{-n}^{n} d\omega \, \langle |\epsilon(\omega)|^2 \rangle} \;. \tag{28}$$

Now, by denoting with $f_{\max}(\omega)$ the maximum modal fraction, define $\mathring{f}(\omega) = f(\omega)/f_{\max}(\omega)$ [20]. The conditional entropy computed in the lattice approach then reads [20,21]

$$S_c[f] = -\lim_{n \to \infty} \int_{-n}^{n} d\omega \, \sigma(\omega) \;, \tag{29}$$

where $\sigma(\omega) = \mathring{f}(\omega) \ln \mathring{f}(\omega)$ denotes the conditional entropy density [23]. Note that, since $\mathring{f}(\omega)$ is not a periodic function, the limits of integration in Eq. (29) are $\omega = \omega_{\min} = \pi/R$ (the lower limit) and $\omega \to \infty$ (the upper limit), as in Ref. [23].

The stability of self-gravitating compact objects by means of the configurational entropy was previously investigated in Refs. [20, 23]. Eq. (29) further measures the configurational, or informational, stability of the system modes. Taking the continuum limit of Shannon entropy is subtle and can miss some features that are demanded for any definition compatible with the expected properties of an entropy. As a matter of fact, the greater the information entropy, the more information, encrypted in the system modes, is compelled to constitute the complexity of a spatially localised system [20–22].

To perform a stability analysis of the MGD BEC distribution, let us recall that most studies of black hole physics are based on the background classical geometry. Therefore, the semi-classical approximation which accounts for small fluctuations about the classical background does not include the quantum effects on the background. In fact, the classical geometry should be an appropriate limit of an effective quantum theory, in which states, with large graviton occupation number, can be thought of as the main constituents. When such states are the ground-state, the gravitational field can be effectively regarded as a BEC.

The equilibrium configurations can be derived in terms of the energy density $\tilde{\epsilon} = \tilde{\epsilon}(r)$ in Eq. (19). The same density $\tilde{\epsilon}$ can then be employed, in order to compute the conditional entropy S_c as a function of the critical central density, $\tilde{\epsilon}(0) \equiv \epsilon_0$, of the compact stellar distribution. The conditional entropy (29) is, in particular, obtained by means of the modal fraction (28), and the critical central density ϵ_0 is found from Eq. (19) to be $\epsilon_0 \approx 0.40925 \, \delta(\sigma) + 0.11254/R^2$, where $\delta(\sigma)$ is given by the expression in Eq. (20). The explicit expression of the conditional entropy turns out to be rather involved, so its numerical estimate is displayed in Fig. 4, where S_c is multiplied by the inverse central density to produce a quantity that scales with dimensions of inverse mass.

The analysis of the conditional entropy for polytropes and neutron stars also makes use of the fiducial value ϵ_c for the critical central density. In Ref. [23], the critical points of the conditional
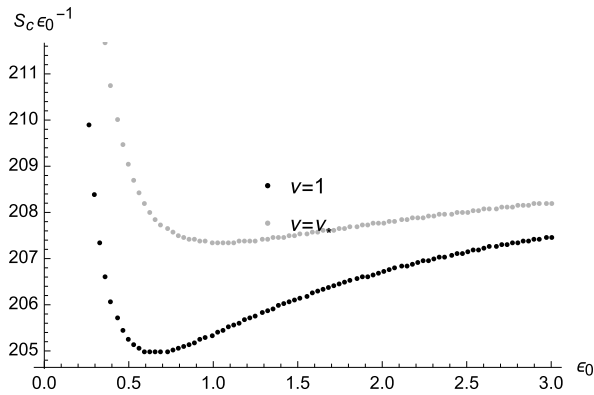
Fig. 4. Conditional entropy times the inverse of the BEC MGD black hole critical central density ϵ_0, with respect to ϵ_0.

entropy were related to ϵ_c and to the Chandrasekhar mass likewise. However, to our knowledge, there is no result in the literature that provides the analogue of the results in Fig. 4. The two-fold reason for it is that the conditional entropy was successfully analysed for boson stars formed by self-interacting scalar fields, and the stability properties of boson stars were explored in their ground state [36,23]. Our results are therefore original and represent a potential reference for further analysis of the stability of boson stars in the graviton BEC scenario.

5. Concluding remarks

Studying a MGD BEC in a Eötvös brane-world scenario provides a strict bound for the brane tension $\sigma \gtrsim 3.18 \times 10^6$ MeV^4, which is stronger than the bound determined by the study of cosmological nucleosynthesis. This bound is also more stringent than the one already obtained in the MGD formalism in Ref. [30]. Moreover, analysing the graviton BEC MGD black hole shows that it is important to take into account quantum effects at the stability critical point, even for a macroscopic number of particles. The MGD BEC has an unlimited number of gapless modes, since it corresponds to a large occupation number N at the stability critical point, coinciding with a maximal packing. The minimum value of the conditional entropy is seen, from Fig. 4, to vary according to the value of the parameter $\nu = M\mu/\hbar$, that was defined as a function of the Klein–Gordon scalar field rest mass μ and the Misner–Sharp mass M. In fact, for $\nu = \nu_*$, the critical conditional entropy occurs at the BEC MGD black hole critical central density $\epsilon_0 \approx 0.62$, whereas the value $\nu = 1$ yields the minimal conditional entropy to be at $\epsilon_0 \approx 1.03$. Based upon previous results in Ref. [23] for polytropes

and neutron stars, a bound on the critical BEC MGD black hole mass can still be obtained, by establishing values of the star density, according to Fig. 4. Nevertheless, the critical point is shifted in the star density, due to the MGD procedure and the graviton BEC as well, to describe the stellar distribution.

Acknowledgements

References

[1] G. Dvali, C. Gomez, Fortschr. Phys. 61 (2013) 742.
[2] G. Dvali, C. Gomez, Phys. Lett. B 716 (2012) 240.
[3] G. Dvali, C. Gomez, Phys. Lett. B 719 (2013) 419.
[4] D. Flassig, A. Pritzel, N. Wintergerst, Phys. Rev. D 87 (2013) 084007.
[5] G. 't Hooft, Nucl. Phys. B 72 (1974) 461.
[6] E. Witten, Nucl. Phys. B 160 (1979) 57.
[7] R. Casadio, A. Giugno, O. Micu, A. Orlandi, Entropy 17 (2015) 6893.
[8] R. Casadio, R.T. Cavalcanti, A. Giugno, J. Mureika, Phys. Lett. B 760 (2016) 36.
[9] S. Hofmann, T. Rug, Nucl. Phys. B 902 (2016) 302.
[10] P. Hořava, E. Witten, Nucl. Phys. B 460 (1996) 506.
[11] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, Phys. Lett. B 436 (1998) 257.
[12] R. Maartens, K. Koyama, Living Rev. Relativ. 13 (2010) 5.
[13] J. Ovalle, Int. J. Mod. Phys. D 18 (2009) 837.
[14] J. Ovalle, Mod. Phys. Lett. A 25 (2010) 3323.
[15] J. Ovalle, F. Linares, A. Pasqua, A. Sotomayor, Class. Quantum Gravity 30 (2013) 175019.
[16] R. Casadio, J. Ovalle, Phys. Lett. B 715 (2012) 251.
[17] R. Casadio, J. Ovalle, Gen. Relativ. Gravit. 46 (2014) 1669.
[18] R. Casadio, A. Orlandi, J. High Energy Phys. 1308 (2013) 025.
[19] W. Mück, G. Pozzo, J. High Energy Phys. 1405 (2014) 128.
[20] M. Gleiser, N. Stamatopoulos, Phys. Lett. B 713 (2012) 304.
[21] M. Gleiser, D. Sowinski, Phys. Lett. B 727 (2013) 272.
[22] A.E. Bernardini, R. da Rocha, Phys. Lett. B 762 (2016) 107.
[23] M. Gleiser, N. Jiang, Phys. Rev. D 92 (2015) 044046.
[24] J. Ovalle, L.A. Gergely, R. Casadio, Class. Quantum Gravity 32 (2015) 045015.
[25] M.C.B. Abdalla, J.M. Hoff da Silva, R. da Rocha, Phys. Rev. D 80 (2009) 046003.
[26] R. Casadio, J. Ovalle, R. da Rocha, Class. Quantum Gravity 31 (2014) 045016.
[27] D. Bazeia, J.M. Hoff da Silva, R. da Rocha, Phys. Lett. B 721 (2013) 306.
[28] D. Bazeia, J.M. Hoff da Silva, R. da Rocha, Phys. Rev. D 90 (2014) 047902.
[29] T. Shiromizu, K. Maeda, M. Sasaki, Phys. Rev. D 62 (2000) 043523.
[30] R. Casadio, J. Ovalle, R. da Rocha, Europhys. Lett. 110 (2015) 40003.
[31] L.A. Gergely, Phys. Rev. D 79 (2009) 086007.
[32] A. Campos, C.F. Sopuerta, Phys. Rev. D 63 (2001) 104012.
[33] R.A.C. Correa, R. da Rocha, A. de Souza Dutra, Ann. Phys. 359 (2015) 198.
[34] R.A.C. Correa, D.M. Dantas, C.A.S. Almeida, R. da Rocha, Phys. Lett. B 755 (2016) 358.
[35] A.E. Bernardini, N.R.F. Braga, R. da Rocha, arXiv:1609.01258 [hep-th].
[36] M. Gleiser, Phys. Rev. D 38 (1988) 2376, Phys. Rev. E 39 (1989) 1257.