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Fault Detection Problems for Switching Linear Systems: a Structural Approach

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***Abstract*—The fault detection and isolation problem is considered in the context of switching linear systems. The problem is tackled by searching for suitable residual signal generators, whose existence is completely characterized in structural terms. Both the situation in which the initial condition of the system subject to possible faults is known and that in which it is not known are considered. The results are compared with those found in the classical linear case for the same problem using a structural geometric approach in order to show consistency and to highlight differences.**

***Index Terms*—Fault detection; Model-based methods; Switching linear systems; Structural methods; Algebraic/geometric methods**

I. Introduction

In the operation of a plant, any fault that may degrade the performances and/or compromise safety needs to be quickly and accurately detected in order to take, if possible, appropriate countermeasures. For this reason, fault detection and isolation (FDI) is an important area of study in the framework of dynamical systems, whose aspects have been investigated by many authors using different approaches.

In particular, model-based (or analytical redundancy-based) FDI methods have been greatly developed for classical linear systems and are by-now widely used (see, e.g., [1]–[3] and the references therein). The characteristic feature of these methods is that the occurrence of faults is detected by comparing a function of the measured output of the (possibly) faulty system with its estimated value, thus obtaining what is called a *residual*. The dynamical system that filters and processes the available information to compute the residual, called *residual generator*, is based on an observer that is designed using a mathematical model of the plant. In that model, faults are represented by specific unknown inputs, while uncertainties, disturbances and noise are taken into account by modelling them as the effects of other, additional unknown inputs.

The existence of observers that generate residuals which are sensitive to the faults and insensitive to the unknown inputs that account for uncertainties and disturbances, or *robust observers* in the terminology of [1], is therefore a key problem to investigate for developing FDI model-based schemes (see [1, Section 5.3], [2, Section 2.3], [3, Section III-B]). A comprehensive survey of the main contributions given in that direction by using the *unknown input observer* approach, in the case of classical linear systems, can be found in [1]–[3].

In the ideal situation, the residual, beside being sensitive to the faults, is independent of (i.e., completely decoupled from) the other unknown inputs. The existence of model-based schemes that achieve this objective is characterized by a set of rank conditions in [4]. An equivalent characterization that better highlights the structural content of the problem is given in [5] by means of a geometric condition which expresses the fact that two specific subspaces of the output space of the plant, the first depending on the faults and the second depending on the other unknown inputs, intersect only at the origin. Indeed, the ideal existence condition is difficult to be satisfied in practice. Nevertheless, also when it is not satisfied, it is useful because it allows a clear identification of the structural obstructions that prevent the residual being completely decoupled from the unknown inputs. This information can be exploited to design the dynamics of the residual generator in such a way to decouple the residual at least from a subset of the set of unknown inputs and to attenuate the effects of the remaining ones. The occurrence of the fault can then be detected with the aid of suitable thresholds.

In this paper, we consider the FDI problem for switching linear systems: i.e., systems whose dynamics varies in a given family according to a known time signal. The main result is a necessary and sufficient condition for the existence of model-based schemes consisting of robust observers (like those considered, in particular, in [1]–[5] for the classical linear case) whose residual is sensitive to the faults and completely decoupled from the other unknown inputs.

From a general point of view, our investigation is motivated by the usefulness and efficacy of the switching system paradigm in modelling complex dynamical behaviors in many situations of interest (in the huge literature on switching linear systems and their applications, the reader can see, e.g., [6]–[10]).

The FDI problem for switching systems and, in particular, the model-based methods for its solution have already received attention by a number of authors. Since qualitative methods for analysis and synthesis of switching linear systems (like those based on norm minimization techniques, Lyapunov functions and LMIs) were, until recently, more developed than structural ones, the construction of FDI schemes where the effects of the unknown inputs on the residual are attenuated, without

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necessarily being annihilated, has been privileged. Indeed, a complete structural characterization of the existence of unknown input observers, which are crucial for constructing model-based FDI schemes that achieve complete decoupling, was given, in the framework of switching linear systems, only lately in [11].

As for the current literature, in [12], a model-based FDI scheme for switching systems affected by disturbances, noise and modeling errors is obtained by approaching the problem as a standard H_{-}/H_{∞} model matching problem. The residual consists of an observer of Luemberger type whose parameters are to be chosen in such a way to attenuate, in terms of H_{∞} performance, the effects of disturbances, noise and modeling errors on the residual, while making it sensitive to the fault according to an H_{-} index. The construction is performed under the somehow limiting assumption that all the single modes of the switching system are observable and that they have the same nominal output matrix. A conceptually similar approach is followed in [13], where the assumption on observability of each mode is maintained and the existence of a suitable observer is formulated as a feasibility LMI problem. In [14], a family of FDI filters with H_{∞} performances for discrete-time switching linear systems is designed via a switched Lyapunov functional approach. Sufficient conditions for the existence of filters/controllers that satisfy H_{-}/H_{∞} performances are given in [15] using LMI techniques. The possible presence of delays is taken into account in [16], where the construction of observers is carried out using Lyapunov-Krasovskii functionals and LMI techniques. Likewise, system inversion, switched Lyapunov functions and LMIs are employed to design fault detection filters in [17]. Model-based schemes for FDI that employ high-order sliding mode observers for switching systems are proposed in [18]–[22]. An approach based on interval observers, whose construction is complicated by the requirement of generating a non-negative estimation error, is developed in [23], [24].

Although the current literature, as illustrated above, considers several ways to design model-based FDI schemes for switching systems, the ideal situation in which a complete decoupling between the residual and the unknown inputs can be achieved has not yet been fully characterized. The specific contribution of this paper consists in giving such characterization in structural terms (that is in terms of dynamical properties and mutual geometric relationships of specific subspaces of the state space of the switching system at issue) in Theorem 1. This provides a sound and complete extension of the results of [5] to the framework of switching systems and it gives a valuable insight into the problem, which cannot be obtained by means of qualitative methods.

The structural characterization given by the existence condition of Theorem 1 is an effective means to define

1) a viable algorithmic procedure for checking the existence of model-based schemes in which the residual is sensitive to a given fault and completely decoupled from the unknown inputs representing other faults or model uncertainties and disturbances;
2) a viable algorithmic procedure for constructing schemes of the above kind, if any exists, by means of unknown input observers.

The solution of the FDI problem is an obvious consequence if the initial condition of the plant is known. Otherwise, if the initial condition of the plant is unknown, asymptotic stability (in a suitable sense, in the framework of switching systems) of the unknown input observers is needed to solve the FDI problem, and the existence of such observers is characterized in Theorem 2.

The conditions of Theorem 1 and Theorem 2 are tight. However, even if they are not satisfied and a complete decoupling of the residual from the unknown inputs is not possible, the structural characterization enables the search of subsets of the set of unknown inputs for which this can be achieved. Then, using the possible degrees of freedom in the construction of the observer, one can try to minimize, according to a suitable norm, the effects of the unknown inputs from which the residual cannot be decoupled.

The paper is organized as follows. In Section II, the class of switching linear systems considered is introduced. In the system's equations, the presence of unknown inputs, other than those representing the faults, serves to take into account uncertainties, disturbances and noise when physical plants are modelled, as is described in [1]–[3], [12]. The problem of generating a residual that, if its generator is suitably initialized, deviates from 0 only at the occurrence of a given fault and is completely decoupled from the other unknown inputs is formally stated in Problem 1. Then, if the initial condition of the possibly faulty plant is known, the solvability of Problem 1 implies the possibility of detecting and isolating the fault, that is the solvability of the FDI problem. The switching behavior makes the construction of unknown input observers that can generate a residual with the desired properties more complex than in the classical linear case. Therefore, specific geometric notions and tools, that are introduced and illustrated in Section III, need to be used to tackle the problem. The results of [11] are instrumentally used in constructing a candidate residual generator as an unknown input observer and, in Section IV, the solvability of Problem 1 is completely characterized in structural terms by Theorem 1. The consequences of the theorem are illustrated in subsequent remarks. In Subsection IV-A, we discuss how to exploit the structural characterization when the existence condition turns out to be not satisfied. In Subsection IV-B, the condition found in the case of switching linear systems is shown to encompass that given in [5] in the case of classical linear systems. However, the formulation given herein is new also when specialized to the classical linear system case. In Section V, an example illustrates the construction of the residual generator and its behavior in the occurrence of faults. The case in which the initial state of the plant is not known is considered in Section VI. In that case, fault detection and isolation can be practically dealt with by means of residual signal generators that enjoy suitable stability properties: namely, whose dynamics is asymptotically stable for sufficiently slow switching. Their existence is characterized in Theorem 2 by modifying the condition of Theorem 1. In Section VII, an example illustrates the construction of a residual generator whose dynamics is globally asymptotically stable for sufficiently slow switching and its behavior in the occurrence of faults (modeled in different ways) in the case where the initial state of the plant is unknown. Finally, Section VIII contains some concluding remarks and a brief description of future work.

Notation: The symbols $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{Z}^+$ are used to denote the sets of real numbers, non negative real numbers and non negative integer numbers, respectively. Real vector spaces and subspaces are denoted by calligraphic letters, like $\mathcal{V}$. The quotient space of a vector space $\mathcal{X}$ over a subspace $\mathcal{V}\subseteq\mathcal{X}$ is denoted by $\mathcal{X}/\mathcal{V}$. The subspace orthogonal to a given subspace $\mathcal{V}$ is denoted by $\mathcal{V}^\perp$. Linear maps between vector spaces and the associated matrices are denoted by the same slanted capital letters, like A. Therefore, the statements $A\in\mathbb{R}^{p\times q}$ and $A:\mathbb{R}^q\to\mathbb{R}^p$ are consistent. The image and the kernel of A are denoted by $\operatorname{Im} A$ and $\operatorname{Ker} A$, respectively. The image of a subspace $\mathcal{V}$ under a map A is simply denoted by $A\mathcal{V}$. The transpose of A is denoted by $A^\top$. The symbols I_n, $0_{m\times n}$ and 0_n are respectively used for the identity matrix of dimension n, for the $m\times n$ zero matrix and for the n-dimensional zero vector (subscripts are omitted if the dimensions are clear from the context).

II. Preliminaries and Problem Statement

A switching linear system Σ_σ is a continuous-time dynamical system described by the equations

$$\Sigma_\sigma \equiv \begin{cases} \dot{x}(t) \,=\, A_{\sigma(t)}\, x(t) + B_{\sigma(t)}\, u(t), \\ y(t) \,=\, C_{\sigma(t)}\, x(t), \end{cases} \tag{1}$$

where $t\in\mathbb{R}^+$ is the time, $x\in\mathcal{X}=\mathbb{R}^n$ the state, $u\in\mathcal{U}=\mathbb{R}^m$ the input, $y\in\mathcal{Y}=\mathbb{R}^p$ the output; letting $I=\{1,\ldots,N\}$ be a finite index set, $\sigma:\mathbb{R}^+\to I$ is a piecewise-constant, left-continuous function that represents the time-driven switching signal; A_i, B_i, C_i, for all $i\in I$, are real matrices of suitable dimensions. The linear systems

$$\Sigma_i \equiv \begin{cases} \dot{x}(t) \,=\, A_i\, x(t) + B_i\, u(t), \\ y(t) \,=\, C_i\, x(t), \end{cases} \quad \text{with } i\in I,$$

are the *modes* of Σ_σ and they form the indexed family $\Sigma=\{\Sigma_i\}_{i\in I}$ of the modes of Σ_σ. The active mode at time t is specified by the value of $\sigma(t)\in I$. In dealing with switching linear systems of the form (1), we assume that $\sigma(t)$ is known at time t for all $t\in\mathbb{R}^+$ and we refer to this assumption by saying that σ is measurable.

The points of discontinuity of the switching signal σ are called the *switching times* and the *dwell time* τ_σ of σ is defined as the lower bound of the set of the lengths of the time intervals between two consecutive switching times. To avoid Zeno phenomena in the behavior of Σ_σ, only the switching signals that have finitely many points of discontinuity in any time interval of finite length are considered to be admissible. Their set is denoted by $\mathscr{S}_0$, while $\mathscr{S}_\alpha\subseteq\mathscr{S}_0$, with $\alpha\in\mathbb{R}^+$, denotes the set of the switching signals such that $\tau_\sigma\geq\alpha$.

We assume that the components of the input signal $u(t)$ may be discontinuous functions (with finitely many points of discontinuity on $[0,+\infty)$) and/or a (linear combination of) Dirac delta functions. Accordingly, the state equation in (1) has to be interpreted in a generalized sense, either by letting it hold almost everywhere (except at the discontinuity points of $u(t)$) and/or considering the derivative $\dot{x}(t)$ in a distributional sense.

Letting $T_\sigma=\{t_0=0,t_1,t_2,\ldots\}$ be the finite or countably infinite ordered set of the discontinuity points of the switching signal $\sigma\in\mathscr{S}_0$, given an initial state $x(0)=x_0\in\mathcal{X}$ and an input $u(t)$ defined over the interval $[0,\bar{t})$, the (generalized) solution $x(t)$ to the state equations of (1) is recursively defined, for any $t\in(t_{k-1},t_k]$ with $t_{k-1},\, t_k\in T_\sigma$, by

$$x(t) = e^{A_{i_k}(t-t_{k-1})}x(t_{k-1}) + \int_{t_{k-1}}^{t} e^{A_{i_k}(t-\tau)}B_{i_k}u(\tau)d\tau$$

where $i_k=\sigma(t)$ for $t\in(t_{k-1},t_k]$. To highlight that $x(t)$ depends on the choice of σ and on t, $x(0)$ and $u(t)$ over $[0,t)$, we will write $x(t)=\phi_\sigma(t,x(0),u(\tau)|_{[0,t)})$.

Note that the notion of switching linear system described above remains consistent also if we let the dimension of the input vector and the dimension of the output vector vary as the system switches from one mode to another. Since it is possible to maintain such dimensions constant by embedding $u(t)$ and $y(t)$, separately, in two larger real vector spaces, we refer, by a slight abuse of notations and without loss of generality, to the representation (1) also in case such variations occur.

The stability of a switching system depends on the stability of its modes as well as on the switching signal. We say that Σ_σ is globally asymptotically stable over $\mathscr{S}_\alpha$ if it is globally asymptotically stable for all $\sigma\in\mathscr{S}_\alpha$, i.e. if the origin of $\mathcal{X}$ is a stable equilibrium point that attracts all the free motions. Asymptotic stability of all the modes does not guarantee global asymptotic stability for arbitrary switching, that is for all $\sigma\in\mathscr{S}_0$ [25]. In general, one is interested in guaranteeing that Σ_σ is globally asymptotically stable over $\mathscr{S}_\alpha$ provided α is sufficiently big or, equivalently, for *(sufficiently) slow switching*. The basic result in this direction is the following.

Lemma 1: [26, Lemma 2] Let all the modes Σ_i, with $i\in I$, of Σ_σ be asymptotically stable. Then, there exists $\alpha\in\mathbb{R}^+$ such that Σ_σ is globally asymptotically stable over $\mathscr{S}_\alpha$.

The proof of Lemma 1 given in [26, Lemma 2] implies that if $\|e^{A_i t}\|\leq e^{(a_i-\lambda_i t)}$, with $\lambda_i>0$, for all $i\in I$ and for all $t\geq 0$, then Σ_σ is globally asymptotically stable over $\mathscr{S}_\alpha$ for $\alpha\geq\max_{i\in I}a_i/\lambda_i$. Since any constant σ belongs to $\mathscr{S}_\alpha$ for any α, asymptotic stability of all the modes is a complete characterization of global asymptotic stability for slow switching. To characterize stability over $\mathscr{S}_\alpha$ for a specific α is, in general, more difficult: sufficient conditions in terms of LMIs can be found in [27] and in the references therein.

Given a switching linear system Σ_σ of the form (1) with $\dim\mathcal{U}=m\geq 2$, let some components of the input u, say, possibly after reordering, $u_1,\ldots,u_{m_1}$, with $m_1\leq m$, model the occurrence of m_1 different faults, while the other components represent control inputs or possible unknown inputs that are not to be regarded as faults. In order to model an abrupt failure (like that due to a sudden component rupture) that occurs at time $\bar{t}\in\mathbb{R}^+$, we can assume, for instance, that $u_j(t)$, with $j\in\{1,\ldots,m_1\}$, is of the form $u_j(t)=a\,\delta(t-\bar{t})$, where $a\in\mathbb{R}$ and $\delta(t)$ denotes the Dirac delta function, or of the form $u_j(t)=a\,H(t-\bar{t})$, where $a\in\mathbb{R}$ and $H(t)$ denotes the Heaviside step function. Otherwise, to model a faulty variation of the performances that initiates less abruptly at time $\bar{t}$, we can assume that $u_j(t)$ is a continuous function such that $u_j(t)=0$ on $[0,\bar{t}]$ and $u_j(t)\neq 0$ on $(\bar{t},\bar{t}+\epsilon)$, for some $\epsilon>0$. The components of the input that model the faults will be generally referred to as the *fault inputs*.

In such situation, the FDI problem consists in recognizing the occurrence of faults, i.e. the occurrence of deviations from 0 of any fault component u_j, with $j\leq m_1$, of the input and in determining which component(s) has (have) deviated by analyzing the output of the system. The most natural way to perform this analysis consists in comparing (a linear combination of the components of) the output of the system with the output of a suitably designed observer, so as to generate a residual signal $r_j(t)$ whose behavior carries the desired information on $u_j(t)$. From this general point of view, since the signals affecting the system behavior through the fault inputs are unknown to the designer, the considered problem shows similarities with the unknown input observation problem studied, for switching linear systems, in [11].

On that basis, if the initial state $x(0)$ of the system is known, one may look for an observer such that $r_j(t)$ remains identically equal to 0 as long as the fault input is equal to zero, while it deviates from zero when the possible fault occurs and the fault input becomes different from zero. To be more precise, this means that, if the fault occurs at time $\bar{t}$, however it is modeled, the residual $r_j(t)$ should be equal to 0 on the interval $[0,\bar{t})$ and different from 0 on the interval $(\bar{t},\bar{t}+\epsilon)$, for some $\epsilon>0$. In that case, the behavior of $r_j(t)$ gives an immediate and sharp indication of the occurrence of a fault.

Otherwise, if $x(0)$ is not known, one may ask that $r_j(t)$ goes asymptotically to 0 for sufficiently slow switching if no faults occur, and that the convergence to 0 is lost or somehow significantly disturbed in the occurrence of faults. However, in this case, it is not immediate and it may not be simple to recognize the occurrence of a fault by analyzing the behavior of $r_j(t)$. We will consider this situation with more detail in Section VI.

In order to state the problem formally, let us assume that the switching linear system Σ_σ, of the form (1), with $m\geq 2$, is the mathematical model of a real, possibly faulty, plant and let us denote by

- $u_j(t)$, with $j\in\{1,\ldots,m\}$, the j-th component of the input vector $u(t)$;
- $u_C(t)$ the subvector of $u(t)$ whose components represent known inputs (like, e.g., control inputs);
- $u^j(t)$ the vector obtained by removing the j-th component $u_j(t)$ and all the components of $u_C(t)$ from $u(t)$.

The inputs $u_j(t)$ and $u^j(t)$ are unknown to the designer and, in particular, $u_j(t)$ is meant to model a specific fault of interest. Some of the components of $u^j(t)$ may model further faults, while the others are used to take into account the presence of model uncertainties and mismatches as well as of disturbances, in the same way as in [1, Eqs. (54)–(55)], [2, Table 4], [3, Eqs. (5)–(6)] in the case of linear systems and in [12, Eqs. (1)–(2)], [13] in the case of switching systems. With these notations, the problem we are interested in is stated as follows.

Problem 1 (Residual Signal Generation Problem): Given a switching linear system Σ_σ of the form (1), with $m\geq 2$, let u_j be a fault input and assume that $\sigma(t)$ is measurable. Then, the j-th Residual Signal Generation Problem (j-RSGP) consists in finding a switching linear system $\Sigma_\sigma^{res_j}$, called the j-th residual signal generator, of the form

$$\Sigma_\sigma^{res_j} \equiv \left\{ \begin{array}{rcl} \dot{z}(t) &=& A_{O\sigma(t)}\,z(t)+B_{OC\sigma(t)}\,u_C(t)+B_{O\sigma(t)}\,y(t), \\ r_j(t) &=& C_{O\sigma(t)}\,z(t)+D_{O\sigma(t)}\,y(t), \end{array}\right. \tag{2}$$

with state space $\mathcal{Z}=\mathbb{R}^q$, together with an exact initialization map $P:\mathcal{X}\to\mathcal{Z}$, with $\operatorname{Im}P=\mathcal{Z}$, such that, for all $\sigma\in\mathscr{S}_0$,

$\mathcal{R}1$. the residual signal $r_j(t)$ is independent of (i.e., completely decoupled from) both $u_C(t)$ and $u^j(t)$

and, if $\Sigma_\sigma^{res_j}$ is initialized exactly at $z(0)=P\,x(0)$,

$\mathcal{R}2$. $r_j(t)=0$ for $t\in[0,\bar{t})$ if and only if $u_j(t)=0$ for $t\in[0,\bar{t})$.

Clearly, if $x(0)$ is known, solvability of Problem 1 implies the possibility of detecting and isolating, with respect to the effect of any other fault, unknown or control input, the fault modeled by the input $u_j(t)$. Hence, we can say that the solution of the FDI problem obtained in this way is robust with respect to the model uncertainties and to the disturbances that, together with the other possible faults, are modeled by means of the unknown input $u^j(t)$. In the case in which the j-RSGP is solved by a j-th residual signal generator $\Sigma_\sigma^{res_j}$ but $x(0)$ is not known, the asymptotic behavior of $r_j(t)$ depends on the global stability properties of $\Sigma_\sigma^{res_j}$ as well as on the characteristics of the fault signal $u_j(t)$. We will consider this situation in Section VI.

Remark 1: Following the approach of [5] more closely, it would be possible to formally state the FDI problem in a more general way than that of Problem 1. More precisely, assuming that the set of the, say, m_1 fault inputs is partitioned in disjoint subsets $\bar{u}_1,\ldots,\bar{u}_{m_2}$, with $m_2\leq m_1$, one could consider a residual generator Σ_σ^{res} of the form

$$\Sigma_\sigma^{res} \equiv \left\{ \begin{array}{rcl} \dot{z}(t) &=& A_{O\sigma(t)}\,z(t)+B_{OC\sigma(t)}\,u_C(t)+B_{O\sigma(t)}\,y(t), \\ r(t) &=& C_{O\sigma(t)}\,z(t)+D_{O\sigma(t)}\,y(t), \end{array}\right. \tag{3}$$

and reformulate Requirement $\mathcal{R}1$ by asking that, for all $\sigma\in\mathscr{S}_0$, the output trajectories of Σ_σ^{res} generated by each subset of fault inputs $\bar{u}_j$ are confined, for each mode Σ_i^{res}, to the subspaces of the output space of that mode which are mutually independent – i.e., such that each of them does not intersect the sum of all the others. Note that this condition is stronger than asking that the output trajectories of each mode Σ_i^{res} generated by each subset of inputs $\bar{u}_j$ are confined to independent subspaces of the output space of the mode. If there exists a residual generator for which such condition is satisfied, the occurrence of faults modeled by different subsets of fault inputs affects components of $r(t)$ that are contained, for all $i \in I$, in independent subspaces of the output space and then it can be identified by looking at the projection of $r(t)$ onto each one of those subspaces. Since, in general, the interest is in detecting each single fault input and in isolating it with respect to all the others (and to all the possible non-fault inputs), in stating Problem 1 we concentrate on the situation in which the set of fault inputs is partitioned into two subsets: one consisting of u_j alone and the other consisting of the remaining fault inputs, viewed as a subset of components of $u^j(t)$. Beside simplifying the notation and the investigation in the switching framework, this choice gives us the possibility of incorporating, in the construction of the residual generator, the projection on the independent subspace of the output that is affected by the fault, thus reducing the dimension of the residual generator. In practice, assuming that m_1 fault inputs are present and that one is interested in detecting and isolating each of them with respect to all the others, the formulation of the problem (called the Beard and Jones Detection Filter Problem, or BJDFP) given in [5] in the classical linear case applies and it leads to look for one residual generator and a bank of m_1 projection operators. By considering the j-RSGP stated here for $j = 1, \ldots, m_1$, we have a different formulation of the same problem and we are led, in the same situation, to look for a bank of m_1 residual generators of lower dimension. Moreover, focusing on a single fault input at a time, as it happens in the classical linear case, we can give conditions to assure that the residual actually differs from 0 if the fault occurs (see the discussion at the end of [5, Section IV]).

Remark 2: Note that the FDI problem concerns the input/output behavior and, therefore, in the classical linear framework, one can assume, without loss of generality, that the system at issue is observable. On the other hand, in a switching linear system, unobservable modes can be present together with observable ones and, in any case, it may be impossible to get rid of their unobservable subsystems without destroying the structure of the overall switching system. This adds a nontrivial complication to the problem and, together with the fact that the independency of the effects of the various fault inputs must be maintained in the switching from one mode to another, it establishes the main technical difference between the classical case and the one considered herein (see also Remark 7). It is worthwhile mentioning that the observability problem has been circumvented in [12], [13] simply by assuming that all the modes of the considered switching system are observable. Our approach is more general, since no observability assumption on the single modes is made.

III. Structural Geometric Tools

The fundamental geometric tool in the investigation of solvability of the j-RSGP is the notion of robust conditioned invariant subspace, which was first introduced and studied in the framework of switching linear systems in [28], [29] and which was exploited to characterize solvability of the unknown input observation problem in [11]. The definitions of robust invariant subspace and of robust conditioned invariant subspace for a switching linear system are recalled below.

Definition 1: Given a switching linear system Σ_σ of the form (1), a subspace $\mathcal{S}_R \subseteq \mathcal{X}$ is said to be:

1) a *(robust) invariant subspace for* Σ_σ if it is an invariant subspace for all the modes of Σ_σ, i.e. if $A_i\,\mathcal{S}_R \subseteq \mathcal{S}_R$ for all $i \in I$;
2) a *(robust) conditioned invariant subspace for* Σ_σ if it is a conditioned invariant subspace for all the modes of Σ_σ, i.e. if $A_i\,(\mathcal{S}_R \cap \operatorname{Ker} C_i) \subseteq \mathcal{S}_R$ for all $i \in I$.

To simplify the notation, when speaking of invariant or conditioned invariant subspaces for Σ_σ, we will drop the adjective *robust*, although we will keep the subscript $_R$.

Let us now recall a number of results from [11], [28], [29], which are instrumental in the subsequent developments.

Proposition 1: Given a subspace $\mathcal{S}_R \subseteq \mathcal{X}$ and a matrix $P^\top$ whose columns form a basis of $\mathcal{S}_R^\perp$, $\mathcal{S}_R$ is a conditioned invariant subspace for Σ_σ if and only if one of the following equivalent conditions holds:

i) there exists an indexed family of linear maps $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ such that

$$(A_i + G_i\,C_i)\,\mathcal{S}_R \subseteq \mathcal{S}_R \quad \text{for all } i \in I; \tag{4}$$

ii) there exists an indexed family of pairs of matrices with real entries $\{(L_i, M_i)\}_{i \in I}$ such that

$$A_i^\top P^\top = P^\top L_i^\top + C_i^\top M_i^\top \quad \text{for all } i \in I. \tag{5}$$

Moreover, $\{(L_i, M_i)\}_{i \in I}$ is an indexed family of pairs of real matrices that satisfy (5) if and only if there exists an indexed family $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ of linear maps that satisfy (4) with

$$\begin{cases} P\,G_i = -\,M_i \\ P\,(A_i + G_i\,C_i) = L_i\,P \end{cases} \quad \text{for all } i \in I. \tag{6}$$

The set of all conditioned invariant subspaces for Σ_σ containing a subspace $\mathcal{W} \subseteq \mathcal{X}$ has a minimal element, which is denoted by $\mathcal{S}_R^*(\mathcal{W})$. The sequence of subspaces $\mathcal{S}_{Rk}$, with $k \in \mathbb{Z}^+$, generated by the recursive algorithm

$$\begin{cases} \mathcal{S}_{R0} = \mathcal{W} \\ \mathcal{S}_{R(k+1)} = \mathcal{S}_{Rk} + \sum_{i \in I} A_i\,(\mathcal{S}_{Rk} \cap \operatorname{Ker} C_i) \end{cases} \tag{7}$$

converges to $\mathcal{S}_R^*(\mathcal{W})$ in n steps at most.

Any indexed family $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ that satisfies Condition i) of Proposition 1 is called a *friend* of $\mathcal{S}_R$. Any friend $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ of a conditioned invariant subspace $\mathcal{S}_R$ for Σ_σ defines a family of output injections which, applied to the corresponding modes of Σ_σ, yield a new switching linear system $\Sigma_\sigma^{\mathscr{G}}$, whose modes are described by the equations

$$\Sigma_i^{\mathscr{G}} \equiv \begin{cases} \dot{x}(t) \,=\, (A_i + G_i\,C_i)\,x(t) + B_i\,u(t), \\ y(t) \,=\, C_i\,x(t), \end{cases} \quad \text{with } i \in I.$$

By (4), the subspace $\mathcal{S}_R$ is invariant for the switching linear system $\Sigma_\sigma^{\mathscr{G}}$. Thus, $\Sigma_\sigma^{\mathscr{G}}$ induces a switching linear dynamics on the subspace $\mathcal{S}_R$ and a switching linear dynamics on the quotient space $\mathcal{X}/\mathcal{S}_R$. The former will be denoted by $\Sigma_\sigma^{\mathscr{G}}|_{\mathcal{S}_R}$ and the latter will be denoted by $\Sigma_\sigma^{\mathscr{G}}|_{\mathcal{X}/\mathcal{S}_R}$.

Proposition 2: Given a conditioned invariant subspace $\mathcal{S}_R \subseteq \mathcal{X}$ for Σ_σ, an $n \times q$ matrix $P^\top$ whose columns form a basis of $\mathcal{S}_R^\perp$ and a friend $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ of $\mathcal{S}_R$, the switching linear dynamics induced by $\Sigma_\sigma^{\mathscr{G}}$ on $\mathcal{X}/\mathcal{S}_R$ is described, up to a change of basis, by the indexed family of matrices $\{L_i\}_{i \in I}$, where $\{(L_i, M_i)\}_{i \in I}$ is a family of pairs which satisfy (5) with $P\,G_i = -M_i$ and $P\,(A_i + G_i\,C_i) = L_i\,P$ for all $i \in I$.

To gain more insight into the situation described by Proposition 2, let us apply the change of basis $x = T\,\xi = [P^\top\; S]\,\xi$, where $P^\top$ is a matrix whose columns form a basis of $\mathcal{S}_R^\perp$ and S is a matrix whose columns form a basis of $\mathcal{S}_R$. Then, the equations that describe the modes $\Sigma_i^{\mathscr{G}}$ of $\Sigma_\sigma^{\mathscr{G}}$ take the form

$$\Sigma_i^{\mathscr{G}} \equiv \left\{ \begin{array}{rcl} \dot{\xi}(t) &=& A'_i\,\xi(t) + B'_i\,u(t), \\ y(t) &=& C'_i\,\xi(t), \end{array} \right. \quad \text{with } i \in I,$$

where, in particular,

$$A'_i = T^{-1}(A_i + G_i\,C_i)\,T = \left[\begin{array}{cc} A'_{11i} & 0 \\ A'_{21i} & A'_{22i} \end{array} \right] \text{ for all } i \in I.$$

Since

$$T^{-1} = \left[\begin{array}{c} (P\,P^\top)^{-1}P \\ (S^\top S)^{-1}S^\top \end{array} \right],$$

in the ξ-coordinates, a basis of $\mathcal{S}_R$ is given by the columns of the matrix $(0_{(n-q)\times q}\; I_{(n-q)})^\top$ and the canonical projection P_c from $\mathcal{X}$ onto $\mathcal{X}/\mathcal{S}_R$ is given by $(I_q\; 0_{q\times(n-q)}) : \mathcal{X} \to \mathcal{X}/\mathcal{S}_R$. Thanks to the lower block-triangular form of A'_i, this shows that, for all $i \in I$, the matrix A'_{11i} describes the dynamics induced by $\Sigma_i^{\mathscr{G}}$ on $\mathcal{X}/\mathcal{S}_R$, while A'_{22i} describes that induced on $\mathcal{S}_R$. Moreover, the diagram below

$$\begin{tikzcd}
\mathcal{X} \arrow[rrr, "A_i + G_i C_i"] \arrow[dr, "T^{-1}"] \arrow[dd, "(PP^\top)^{-1}P"'] & & & \mathcal{X} \arrow[dl, "T^{-1}"'] \arrow[dd, "(PP^\top)^{-1}P"] \\
& \mathcal{X} \arrow[r, "A'_i"] \arrow[dl, "P_c"'] & \mathcal{X} \arrow[dr, "P_c"] & \\
\mathcal{X}/\mathcal{S}_R \arrow[rrr, "A'_{11i}"] & & & \mathcal{X}/\mathcal{S}_R
\end{tikzcd}$$

is commutative, and hence we have

$$\begin{array}{rcl} A'_{11i}\,(PP^\top)^{-1}P &=& A'_{11i}\,(I_q \quad 0_{q\times(n-q)})\,T^{-1} \\ &=& (I_q \quad 0_{q\times(n-q)})\,A'_i\,T^{-1} \\ &=& (I_q \quad 0_{q\times(n-q)})\,T^{-1}\,(A_i + G_iC_i) \\ &=& (PP^\top)^{-1}P\,(A_i + G_iC_i). \end{array}$$

Hence, since P is a full row rank matrix, by the second equation in (6), it follows that $(PP^\top)\,A'_{11i}\,(PP^\top)^{-1} = L_i$, where $\{(L_i, M_i)\}_{i \in I}$, with $P\,G_i = -\,M_i$ for all $i \in I$, is a family of pairs of matrices which satisfy (5). By choosing an orthonormal matrix $P^\top$ to represent a basis of $\mathcal{S}_R^\perp$, we have $A'_{11i} = L_i$ for all $i \in I$.

IV. Solution of the Problem

Given a switching linear system Σ_σ of the form (1), with $m \geq 2$, letting u_j be a fault input and assuming that σ is measurable, we can construct a candidate residual signal generator that solves the j-RSGP. Then, let us denote by

- $b^j_{\sigma(t)}$ the column of the input matrix $B_{\sigma(t)}$ that multiplies $u_j(t)$;
- $B_{C\sigma(t)}$ the submatrix of the input matrix $B_{\sigma(t)}$ that multiplies $u_C(t)$ (recall that $u_C(t)$ is the subvector of $u(t)$ whose components represent known inputs);
- $B^j_{\sigma(t)}$ the matrix obtained by removing the j-th column and the submatrix $B_{C\sigma(t)}$ from the input matrix $B_{\sigma(t)}$;

and let us assume that $b^j_i \neq 0$ for all $i \in I$, that is let us assume that the fault input u_j affects all the modes of Σ_σ. Note that this assumption does not cause any loss of generality, since the j-RSGP, as it has been stated, would not make sense if we had some mode that is not affected by the fault. With the above notations, we can write the state equation of Σ_σ as:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{C\sigma(t)}u_C(t) + b^j_{\sigma(t)}u_j(t) + B^j_{\sigma(t)}u^j(t).$$

Then, we construct the minimal conditioned invariant subspace for Σ_σ that contains $\operatorname{Im} B^j_i$ for all $i \in I$, that is $\mathcal{S}^*_R(\sum_{i \in I} \operatorname{Im} B^j_i)$, by means of the recursive algorithm (7). Letting $P^\top$ be an $n \times q$ matrix whose columns form a basis of $(\mathcal{S}^*_R(\sum_{i \in I} \operatorname{Im} B^j_i))^\perp$, we consider the map $P : \mathcal{X} \to \mathcal{Z}$ with $\mathcal{Z} = \mathbb{R}^q$. Note that $\operatorname{Ker} P = \mathcal{S}^*_R(\sum_{i \in I} \operatorname{Im} B^j_i)$. Then, letting $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ be a friend of $\mathcal{S}^*_R(\sum_{i \in I} \operatorname{Im} B^j_i)$ and letting $\{(L_i, M_i)\}_{i \in I}$ be an indexed family of pairs of matrices that satisfy (5), we can define the switching dynamics

$$\dot{z}(t) = L_{\sigma(t)}z(t) + PB_{C\sigma(t)}u_C(t) - PG_{\sigma(t)}y(t) \qquad (8)$$

with $z \in \mathcal{Z}$. On the basis of the discussion following Proposition 2, we can view the free dynamics in (8) as the dynamics induced on $\mathcal{X}/\mathcal{S}^*_R(\sum_{i \in I} \operatorname{Im} B^j_i)$ by that of $\Sigma_\sigma^{\mathscr{G}}$.

Now, for each $i \in I$, let us consider the subspace image of $\mathcal{S}^*_R(\sum_{i \in I} \operatorname{Im} B^j_i)$ under the map C_i, i.e. $C_i\mathcal{S}^*_R(\sum_{i \in I} \operatorname{Im} B^j_i) \subseteq \mathcal{Y}$, and the quotient space $\mathcal{Y}/C_i\mathcal{S}^*_R(\sum_{i \in I} \operatorname{Im} B^j_i) = \mathcal{Y}_i$. We denote by $\overline{P}_i : \mathcal{Y} \to \mathcal{Y}_i$ the canonical projection and by $\overline{P}_iC_i : \mathcal{X} \to \mathcal{Y}_i$ the map obtained by composing C_i and $\overline{P}_i$. Since $\operatorname{Im} B^j_i \subseteq \mathcal{S}^*_R(\sum_{i \in I} \operatorname{Im} B^j_i) = \operatorname{Ker} P \subseteq \operatorname{Ker} \overline{P}_iC_i$, for each $i \in I$ there exists a unique map $C_{Oi} : \mathcal{Z} \to \mathcal{Y}_i$ such that $C_{Oi}P = \overline{P}_iC_i$, as shown in the commutative diagram below

$$\begin{tikzcd}
\mathcal{S}^*_R(\sum_{i \in I} \operatorname{Im} B^j_i) = \operatorname{Ker} P \arrow[r, "C_i"] \arrow[d, hook] & C_i\mathcal{S}^*_R(\sum_{i \in I} \operatorname{Im} B^j_i) \arrow[d, hook] \\
\mathcal{X} \arrow[r, "C_i"] \arrow[d, "P"'] & \mathcal{Y} \arrow[d, "\overline{P}_i"] \\
\mathcal{Z} \arrow[r, "C_{Oi}"] & \mathcal{Y}_i
\end{tikzcd}$$

First of all, note that the dimension of $\mathcal{Y}_i$ may depend on the index i and vary with it. However, we can define a switching time-signal $r_j(t)$, that depends, in particular, on σ, as

$$r_j(t) = C_{O\sigma(t)}\,z(t) - \overline{P}_{\sigma(t)}\,y(t). \qquad (9)$$

A candidate residual signal generator to solve the j-RSGP is the switching linear system $\Sigma_\sigma^{res_j}$ defined by (8)–(9), i.e.

$$\Sigma_\sigma^{res_j} \equiv \left\{ \begin{aligned} \dot{z}(t) &= L_{\sigma(t)}\, z(t) + P B_{C\sigma(t)}\, u_C(t) - P G_{\sigma(t)}\, y(t), \\ r_j(t) &= C_{O\sigma(t)}\, z(t) - \overline{P}_{\sigma(t)}\, y(t). \end{aligned} \right. \tag{10}$$

As remarked above, the dimension of the output $r_j(t)$ of $\Sigma_\sigma^{res_j}$ may take different values on the different intervals $[t_k, t_{k+1})$. As anticipated in Section II, this fact has no relevant consequences on the definition of the system at issue and, except for signalling it, we do not pay attention to it.

Remark 3: The input/output relation described by (10) does not depend on the choice of the basis of $\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j)$. In fact, any other basis is of the form $P^\top H^\top$ for some invertible $(q \times q)$-matrix H and its use in the above construction gives rise to a different representation of $\Sigma_\sigma^{res_j}$ which is related to (10) by the change of basis in $\mathcal{Z}$ defined by H.

To state the condition under which the system $\Sigma_\sigma^{res_j}$ defined above solves the j-RSGP, we consider, for each $i \in I$, the maximal $(A_i + G_i C_i)$-invariant subspace of $\mathcal{X}$ contained in $\Big(\operatorname{Ker} C_i + \mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j)\Big)$ and we denote such subspace by $\mathcal{W}_i$. Then, the following technical result is in order.

Proposition 3: With the above notations, $\mathcal{W}_i$ does not depend on the choice of the friend $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i\in I}$ of $\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j)$.

Proof: Note that, by construction, $\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j) \subseteq (\operatorname{Ker} C_i + \mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j))$ is $(A_i + G_i C_i)$-invariant. Hence, by maximality of $\mathcal{W}_i$, we have $\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j) \subseteq \mathcal{W}_i$. Also note that any element $w \in \mathcal{W}_i$ can be decomposed as $w = w' + s$, with $w' \in \operatorname{Ker} C_i$ and $s \in \mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j)$ and we have $(A_i + G_i C_i)\, w = (A_i + G_i C_i)\,(w' + s) = A_i\, w' + (A_i + G_i C_i)\, s = A_i\, w' + s'$ with $s' \in \mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j) \subseteq \mathcal{W}_i$ by $(A_i + G_i C_i)$-invariance of $\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j)$. On the other hand, by $(A_i + G_i C_i)$-invariance of $\mathcal{W}_i$, $(A_i\, w' + s')$ belongs to $\mathcal{W}_i$ and we can therefore conclude that $A_i\, w'$ belongs to $\mathcal{W}_i$. Now, if $\mathscr{G}' = \{G_i',\, G_i' : \mathcal{Y} \to \mathcal{X}\}_{i\in I}$ is another friend of $\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j)$, we have $(A_i + G_i' C_i)\, w = (A_i + G_i' C_i)\,(w' + s) = A_i\, w' + (A_i + G_i' C_i)\, s = A_i\, w' + s''$ with $s'' \in \mathcal{S}_R^*(\sum_{i\in I, i\neq j} \operatorname{Im} B_i^j)$, and, consequently, $(A_i + G_i' C_i)\, w$ belongs to $\mathcal{W}_i$. Thus, the subspace $\mathcal{W}_i \subseteq (\operatorname{Ker} C_i + \mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j))$ is also $(A_i + G_i' C_i)$-invariant and its maximality is obvious. ■

The main result that states the condition under which $\Sigma_\sigma^{res_j}$ solves the j-RSGP, making it possible to detect the occurrence of the fault modeled by u_j, is the following.

Theorem 1: Given a switching linear system Σ_σ of the form (1), with $m \geq 2$, let u_j be a fault input that affects all the modes Σ_i of Σ_σ and assume that σ is measurable. Then, letting $\mathcal{W}_i$ be the maximal $(A_i + G_i\, C_i)$-invariant subspace of $\mathcal{X}$ contained in $(\operatorname{Ker} C_i + \mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j))$, where $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i\in I}$ is a friend of $\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j)$, the j-RSGP is solvable if and only if

$$\mathcal{W}_i \cap \operatorname{Im} b_i^j = \{0\} \quad \text{for all } i \in I. \tag{11}$$

Moreover, in that case, the switching linear system $\Sigma_\sigma^{res_j}$ constructed above and described by (10), along with the initialization map $P : \mathcal{X} \to \mathcal{Z}$, where the columns of $P^\top$ are a basis of $(\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j))^\perp$, is a solution to the j-RSGP.

Proof: If. We show the sufficiency of (11) by proving the last statement of the theorem. To this aim, referring to the candidate residual signal generator $\Sigma_\sigma^{res_j}$ defined by (10), let us consider the auxiliary variable

$$e_{aux}(t) = z(t) - P\, x(t). \tag{12}$$

Using the expression of $\dot{x}(t)$ and of $\dot{z}(t)$ given in (1) and (10), respectively, and exploiting the second equality of (6) (which implies $P(A_{\sigma(t)} + G_{\sigma(t)} C_{\sigma(t)}) = L_{\sigma(t)} P$ for all $\sigma \in \mathscr{S}_0$ and for all $t \in \mathbb{R}^+$) and the relation $\operatorname{Im} B_i^j \subseteq \mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j) = \operatorname{Ker} P$ (which implies $P\, B_{\sigma(t)}^j = 0$ for all $\sigma \in \mathscr{S}_0$ and for all $t \in \mathbb{R}^+$), we have that the time evolution of $e_{aux}(t)$ is described by the following equation

$$\begin{aligned} \dot{e}_{aux}(t) &= L_{\sigma(t)} z(t) + P B_{C\sigma(t)} u_C(t) - P G_{\sigma(t)} y(t) - \\ &\quad P A_{\sigma(t)} x(t) - P b_{\sigma(t)}^j u_j(t) - P B_{C\sigma(t)} u_C(t) - P B_{\sigma(t)}^j u^j(t) \\ &= L_{\sigma(t)} z(t) - P b_{\sigma(t)}^j u_j(t) - P(A_{\sigma(t)} + G_{\sigma(t)} C_{\sigma(t)}) x(t) \\ &= L_{\sigma(t)} e_{aux}(t) - P b_{\sigma(t)}^j u_j(t). \end{aligned} \tag{13}$$

Moreover, by adding and subtracting $C_{O\sigma(t)} P x(t)$ to the second member of (9), since $C_{Oi} P = \overline{P}_i C_i$, we have

$$r_j(t) = C_{O\sigma(t)} e_{aux}(t) \tag{14}$$

for all $\sigma \in \mathscr{S}_0$ and for all $t \in \mathbb{R}^+$. Considering (13) and (14) together, we can conclude that $r_j(t)$ depends only on the value of the auxiliary variable $e_{aux}(t)$ at time 0, that is $e_{aux}(0)$, and on $u_j(t)$. In particular, it is independent of $u^j(t)$ for all $\sigma \in \mathscr{S}_0$. Hence, the system $\Sigma_\sigma^{res_j}$ satisfies Requirement $\mathcal{R}1$ of Problem 1.

Assume now for the rest of the proof, that $\Sigma_\sigma^{res_j}$ is correctly initialized at $z(0) = P x(0)$, so that $e_{aux}(0) = 0$. If $u_j(t) = 0$ for all $t \in [0, \bar{t})$, we have by (13) also $e_{aux}(t) = 0$ for all $t \in [0, \bar{t})$ and, therefore, by (14) we get $r_j(t) = 0$ for all $t \in [0, \bar{t})$ and for all $\sigma \in \mathscr{S}_0$. To prove the converse, consider for $i \in I$ the image of $\mathcal{W}_i$ under the map P, namely the subspace $P\mathcal{W}_i \subseteq \mathcal{Z}$. The subspace $P\mathcal{W}_i$ is contained in $\operatorname{Ker} C_{Oi}$ and, since $(A_i + G_i C_i)\mathcal{W}_i \subseteq \mathcal{W}_i$, we have $L_i P\mathcal{W}_i = P(A_i + G_i C_i)\mathcal{W}_i \subseteq P\mathcal{W}_i$. Hence $P\mathcal{W}_i$ is L_i-invariant for all $i \in I$ and, by maximality of $\mathcal{W}_i$, it turns out to be the maximal L_i-invariant subspace of $\mathcal{Z}$ contained in $\operatorname{Ker} C_{Oi}$. From this, we can conclude that, for all $i \in I$, the subspace $P\mathcal{W}_i$ is the unobservability subspace of the i-th mode of the switching linear system described by (13) and (14), namely

$$\left\{ \begin{aligned} \dot{e}_{aux}(t) &= L_i\, e_{aux}(t) - P\, b_i^j\, u_j(t), \\ r_j(t) &= C_i\, e_{aux}(t). \end{aligned} \right. \tag{15}$$

Now, we can state that $P\, b_i^j$ is not an element of $P\mathcal{W}_i$. Otherwise, from $P\, b_i^j \in P\mathcal{W}_i$ we would have $b_i^j = w_i + s$ with $w_i \in \mathcal{W}_i$ and $s \in \operatorname{Ker} P = \mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i) \subseteq \mathcal{W}_i$, where the last inclusion has been shown in the proof of Proposition 3. Hence, b_i^j would belong to $\mathcal{W}_i$ and, since $b_i^j \neq 0$ for all $i \in I$ because u_j has been assumed to affect all the modes, this contradicts (11). The fact that at least one vector, namely

$P\,b_i^j$, is not contained in the unobservability subspace $\mathcal{W}_i$ of system (15), implies that at least one component of the $(\dim \mathcal{Y}_i) \times 1$ transfer function matrix of this latter is different from 0. Otherwise the transfer function matrix would be null and, hence, the unobservability subspace $\mathcal{W}_i$ would include the whole state space. As a consequence, $r_j(t) = 0$ for $t \in [0, \bar{t})$ and for all $\sigma \in \mathscr{S}_0$ implies $u_j(t) = 0$ for $t \in [0, \bar{t})$. Therefore, $\Sigma_\sigma^{res_j}$ satisfies Requirement $\mathcal{R}2$ of Problem 1.

Only if. Let $\Sigma_\sigma^{res_j}$ be a residual signal generator of the form (2), with state space $\mathcal{Z} = \mathbb{R}^q$, that, together with the initialization map $P : \mathcal{X} \to \mathcal{Z}$, solves the j-RSGP. Without loss of generality, we can assume that $\Sigma_\sigma^{res_j}$ has no unobservable states, i.e. no states $z \neq 0$ whose free evolution $z(t) = \phi_\sigma(t, z, 0)$ is such that, for all $\sigma \in \mathscr{S}_0$, $C_{O\sigma(t)} z(t) = 0$ for all $t \in \mathbb{R}^+$. In fact, the unobservable states form a subsystem whose evolution does not influence the residual $r_j(t)$ and that, therefore, can be factored out by taking a suitable quotient system. Since $\Sigma_\sigma^{res_j}$ satisfies Requirement $\mathcal{R}2$ of Problem 1, if, in particular, $u_j(t) = 0$ for $t \geq 0$, for any choice of the initial state $x(0) = x_0 \in \mathcal{X}$ we have $r_j(0) = C_{O\sigma(0)} P x_0 + D_{O\sigma(0)} C_\sigma\, x_0 = 0$ for all $\sigma \in \mathscr{S}_0$. This implies $C_{Oi} P + D_{Oi} C_i = 0$. Hence, by considering the auxiliary error $e_{aux}(t)$ defined by (12) and adding and subtracting $A_{O\sigma(t)} P x(t)$ to its state equation and $C_{O\sigma(t)} P x(t)$ to the output equation of $\Sigma_\sigma^{res_j}$ respectively, we get

$$\left\{ \begin{aligned} \dot{e}_{aux}(t) \;=\;& A_{O\sigma(t)}\, e_{aux}(t) - P b_{\sigma(t)}^j\, u_j(t) - P B_{\sigma(t)}^j\, u^j(t) \\ & + (B_{OC\sigma(t)} - P B_{C\sigma(t)})\, u_C(t) \\ & + (B_{O\sigma(t)} C_{\sigma(t)} - P A_{\sigma(t)} + A_{O\sigma(t)} P)\, x(t), \\ r_j(t) \;=\;& C_{O\sigma(t)}\, e_{aux}(t). \end{aligned} \right. \tag{16}$$

In order to satisfy Requirement $\mathcal{R}1$ of Problem 1 with $z(0) = P x(0)$, in particular, for the identically null input $u(t) = 0$ for $t \geq 0$, the forced component of $e_{aux}(t)$ excited by $x(t)$ must be null for any $x(t) = \phi_\sigma(t, x(0), 0)$ for all $x(0) = x \in \mathcal{X}$, and, hence, we have

$$(B_{Oi} C_i - P A_i + A_{Oi} P) = 0 \quad \text{for all } i \in I. \tag{17}$$

Then, for the same reason, in particular for $u_j(t) = 0$ for $t \geq 0$, since $u^j(t)$ and $u_C(t)$ are mutually independent, we have

$$(B_{OCi} - P B_{Ci}) \;=\; 0 \quad \text{for all } i \in I, \tag{18}$$

$$\operatorname{Im} B_i^j \;\subseteq\; \operatorname{Ker} P \quad \text{for all } i \in I, \tag{19}$$

and (16) reduces to

$$\left\{ \begin{aligned} \dot{e}_{aux}(t) \;=\;& A_{O\sigma(t)}\, e_{aux}(t) - P b_{\sigma(t)}^j\, u_j(t), \\ r_j(t) \;=\;& C_{O\sigma(t)}\, e_{aux}(t). \end{aligned} \right. \tag{20}$$

Since $\operatorname{Im} P = \mathcal{Z}$, there exist suitable maps $G_i : \mathcal{Y} \to \mathcal{X}$ such that $B_{Oi} = -P G_i$ for all $i \in I$ and, by (17), we get $P(A_i + G_i C_i) = A_{Oi} P$ for all $i \in I$. This implies that $\operatorname{Ker} P$ is a conditioned invariant subspace for Σ_σ, of which $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ is a friend, and that the dynamics of $\Sigma_\sigma^{res_j}$ coincides with the dynamics induced on $\mathcal{X}/\operatorname{Ker} P$ by that of $\Sigma_\sigma^{\mathscr{G}}$. Moreover, by (19) and minimality of $\mathcal{S}_R^*(\sum_{i \in I} \operatorname{Im} B_i^j)$, we also have $\mathcal{S}_R^*(\sum_{i \in I} \operatorname{Im} B_i^j) \subseteq \operatorname{Ker} P$. Denoting by $\overline{\mathcal{W}_i}$, for $i \in I$, the maximal $(A_i + G_i C_i)$-invariant subspace of $\mathcal{X}$ contained in $(\operatorname{Ker} C_i + \operatorname{Ker} P)$ and using the same arguments of the proof of Proposition 3 and of the *If*-part of this proof, we can show that $P\overline{\mathcal{W}_i}$ is the unobservability subspace of the i-th mode of the system described by (20). Since Requirement $\mathcal{R}2$ of Problem 1 is satisfied, we have that $P b_i^j$ does not belong to $P\overline{\mathcal{W}_i}$ and, hence, $\overline{\mathcal{W}_i} \cap \operatorname{Im} b_i^j = \{0\}$ for all $i \in I$. Since $(\operatorname{Ker} C_i + \mathcal{S}_R^*(\sum_{i \in I} \operatorname{Im} B_i^j)) \subseteq (\operatorname{Ker} C_i + \operatorname{Ker} P)$, by maximality of $\overline{\mathcal{W}_i}$, we have $\mathcal{W}_i \subseteq \overline{\mathcal{W}_i}$ for all $i \in I$ and we conclude that (11) is satisfied. ■

Remark 4: Assuming that Condition (11) of Theorem 1 holds for all $i \in I$ and that the fault described by u_j occurs at time $\bar{t}$, the theorem says that $r_j(t)$ is completely decoupled from the unknown input $u^j(t)$ and it is not identically equal to 0 on the interval $[\bar{t}, \bar{t} + \epsilon)$ for all $\epsilon > 0$. The value $r_j(\bar{t})$ depends on how the occurrence of the fault is modeled. If $u_j(t) = a\delta(t - \bar{t})$, where $a \in \mathbb{R}$ and $\delta(t)$ denotes the Dirac delta function, we have $r_j(\bar{t}) \neq 0$. Instead, if $u_j(t) = aH(t - \bar{t})$, where $a \in \mathbb{R}$ and $H(t)$ denotes the Heaviside step function, or $u_j(t)$ is a continuous function such that $u_j(t) = 0$ for $t \leq \bar{t}$ and $u_j(t) \neq 0$ for $t > \bar{t}$, we have $r_j(\bar{t}) = 0$ because of the continuity property of $e_{aux}(t)$. In all cases, the time instant $\bar{t}$ at which the fault occurs is the infimum of the set of points on which $r_j(t)$ is different from 0, that is $\bar{t} = \inf\{t,\, t \in \mathbb{R}^+ \text{ such that } r_j(t) \neq 0\}$, the infimum being actually a minimum if $u_j(t)$ is an impulsive signal. By observing $r_j(t)$, it is therefore possible to detect the occurrence of the fault and the time at which it occurs.

Remark 5: Note that the presence in Σ_σ of inputs that are known has no relevance in the j-RSGP (similarly to what happens in the unknown input observation problem studied in [11]), since they do not contribute to the dynamics of $e_{aux}(t)$. The solvability condition, dependending on the matrices B_i^j with $i \neq j$, takes into account only the components of u that are not known, either because they also model faults or because they are unknown inputs of other type (e.g., disturbances).

Remark 6: Note that if the representation of the residual signal generator $\Sigma_\sigma^{res_j}$ that, together with the exact initialization map $P : \mathcal{X} \to \mathcal{Z}$, solves the j-RSGP is modified by a change of basis $z = H^{-1}\xi$, where H is a $q \times q$ invertible matrix, also the exact initialization map is modified into HP. Since $\operatorname{Ker} HP = \operatorname{Ker} P$, recalling Remark 3, we can say that, if Condition (11) is satisfied, Theorem 1 establishes a bijective correspondence between the set of the pairs consisting of a residual signal generator $\Sigma_\sigma^{res_j}$ of the form (2) and an exact initialization map P that solve the j-RSGP and the set of the pairs consisting of a conditioned invariant subspace $\mathcal{S} \subseteq \mathcal{X}$ and a friend $\mathscr{G}$ such that $\mathcal{S}_R^*(\sum_{i \in I} \operatorname{Im} B_i^j) \subseteq \mathcal{S}$ and $W_i \cap b_i^j = \{0\}$ for all $i \in I$. Such correspondence is described by $\mathcal{S} = \operatorname{Ker} P$ and $B_{Oi} = -P G_i$ for all $i \in I$.

A. Analysis in case the solvability condition is not satisfied

Condition (11) is quite tight and when it is not satisfied it is not possible to generate a residual satisfying both $\mathcal{R}1$ and $\mathcal{R}2$. However, a more detailed analysis provides information useful for handling the FDI problem by means of different strategies.

Since $\mathcal{W}_i$ contains the unobservability subspace $\mathcal{O}_i$ of the i-th mode Σ_i of Σ_σ, we have that Condition (11) is not satisfied, in particular, if b_j belongs to $\mathcal{O}_{\bar{i}}$ for some $\bar{i} \in I$, i.e. if b_j is unobservable for the $\bar{i}$-th mode. In such case, if the fault described by $u_j(t)$ occurs when the $\bar{i}$-th mode is active, the residual $r_j(t)$ is not forced to deviate from 0, at least, until the system switches and, so, the fault is not detected when it occurs. Note that unobservability with respect to the $\bar{i}$-th mode does not prevent the fault described by $u_j(t)$ from affecting the output of Σ_σ, except in the case in which the only active mode on $[\bar{t}, +\infty)$, where $\bar{t}$ denotes the time when the fault occurs, is just the $\bar{i}$-th mode. Therefore, this situation represents a non trivial case that is not found in the framework of classical linear systems. The information on the fault contained in the output of Σ_σ is intrinsically poor and the only way to overcome this drawback is to intervene on the physical plant that is modeled by Σ_σ, if it is possible, to modify the location of the sensors or to add some. This action must aim at altering the output map $C_{\bar{i}}$, so that the unobservability subspace of the $\bar{i}$-th mode no longer contains b_j and this obstruction to solve the FDI problem is removed.

Excluding the above situation, we have that Condition (11) is not satisfied only if, for some $\bar{i} \in I$, $A_{\bar{i}}^k b_j$ belongs to $\operatorname{Ker} C_{\bar{i}} + \mathcal{S}_R^*(\sum_{i \in I} B_i^j)$ for all $k \leq \dim A_{\bar{i}} - 1$. The information on the fault contained in the output of Σ_σ is hidden by the effect of the other unknown inputs. In this case, to handle the FDI problem without intervening on the structure of the plant modeled by Σ_σ, one can search for a decomposition of the input matrix $B_{\sigma(t)}^j$ of the form $B_{\sigma(t)}^j = [B_{\sigma(t)1}^j \; B_{\sigma(t)2}^j]$ (possibly after reordering its columns), such that, for all $i \in I$, there exists $k_i \leq \dim A_{\bar{i}} - 1$ which satisfies $A_i^{k_i} b_j \notin \operatorname{Ker} C_i + \mathcal{S}_R^*(\sum_{i \in I} B_{i1}^j)$. If such decomposition exists, Condition (11) is satisfied by replacing $\mathcal{W}_i$ with the maximal $(A_i + G_i C_i)$-conditioned invariant subspace contained in $\operatorname{Ker} C_i + \mathcal{S}_R^*(\sum_{i \in I} B_{i1}^j)$. Then, decomposing $u^j(t)$ accordingly as $u^j(t) = \begin{pmatrix} u^{j1}(t) \\ u^{j2}(t) \end{pmatrix}$ and disregarding the term $B_{\sigma(t)2}^j u^{j2}$ in Σ_σ, it is possible to obtain a residual $r_j(t)$ that satisfies $\mathcal{R}2$ and is independent of (i.e., completely decoupled from) $u_C(t)$ and $u^{j1}(t)$. One can then exploit the degrees of freedom in the choice of a friend $\mathscr{G} = \{G_i, G_i : \mathcal{Y} \to \mathcal{X}\}_{i \in I}$ of $\mathcal{S}_R^*(\sum_{i \in I} B_{i1}^j)$ to attenuate the effects of $u^{j2}(t)$ on $r_j(t)$, while enhancing those of $u_j(t)$, by means of qualitative techniques like, e.g., those employed in [12] and [13]. Note that there may be several alternative ways to decompose $B_{\sigma(t)}^j$ as described above. In choosing one of them, beside minimizing the dimension of $u^{j2}(t)$, one should exploit any available information about norm and/or frequency bounds on the components of $u^j(t)$ to increase the performances of the residual generator by a suitable choice of the friend $\mathscr{G}$.

B. Comparison with the classical linear case

In order to compare the content of Theorem 1 given above with the characterization of solvability of the residual signal generator problem given by [5] in the classical case of linear systems, we assume that Σ_σ consists of a single mode and we write, with an obvious use of the symbols,

$$\Sigma_\sigma = \Sigma \equiv \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t), \\ y(t) = C\,x(t). \end{cases} \tag{21}$$

Letting $u_j(t)$ be a fault input and assuming that Σ is observable, Theorem 3 of [5] states that the j-RSGP is solvable if and only if $C\mathcal{S}_R^*(\operatorname{Im} b^j) \bigcap C\mathcal{S}_R^*(\operatorname{Im} B^j) = \{0\}$, where, consistently with the notations introduced at the beginning of this section, b^j denotes the j-th column of B and B^j denotes the matrix obtained from B by removing its j-th column. The following proposition shows that Theorem 1 is consistent with this statement.

Proposition 4: Given a linear system Σ of the form (21), with $m \geq 2$, for any $j \in \{1, \ldots, m\}$, the following conditions are equivalent:

i) $b^j \in \mathcal{X}$ is an observable state and

$$C\mathcal{S}_R^*(\operatorname{Im} b^j) \cap C\mathcal{S}_R^*(\operatorname{Im} B^j) = \{0\}; \tag{22}$$

ii)

$$\mathcal{W} \cap \operatorname{Im} b^j = \{0\}, \tag{23}$$

where $\mathcal{W}$ is the maximal $(A + GC)$-invariant subspace of $\mathcal{X}$ contained in $(\operatorname{Ker} C + \mathcal{S}_R^*(\operatorname{Im} B^j))$.

Proof: Assume that Condition i) holds and suppose, by contradiction, that b^j belongs to $\mathcal{W}$. Then, since $\mathcal{S}_R^*(\operatorname{Im} b^j)$ is the minimum $(A + GC)$-invariant subspace of $\mathcal{X}$ that contains b^j, we have $\mathcal{S}_R^*(\operatorname{Im} b^j) \subseteq \mathcal{W} \subseteq (\operatorname{Ker} C + \mathcal{S}_R^*(\operatorname{Im} B^j))$. It follows that any $s \in \mathcal{S}_R^*(\operatorname{Im} b^j)$ can be written as $s = k + s'$, with $k \in \operatorname{Ker} C$ and $s' \in \mathcal{S}_R^*(\operatorname{Im} B^j)$, and that $C\,s = C(k + s') = Cs'$. By (22), the last equality implies $Cs = 0$ and, since this holds for any $s \in \mathcal{S}_R^*(\operatorname{Im} b^j)$, it contradicts the observability of b^j.

Assume that Condition ii) holds and let us construct $\mathcal{S}_R^*(\operatorname{Im} b^j)$. If $A^k b^j$ belongs to $\operatorname{Ker} C$ for all $k \geq 0$, by applying the recursive algorithm (7) we have $\mathcal{S}_R^*(\operatorname{Im} b^j) = \operatorname{span}\{A^k b^j\}_{k \geq 0}$. In this case, we would have $\mathcal{S}_R^*(\operatorname{Im} b^j) \subseteq \operatorname{Ker} C \subseteq (\operatorname{Ker} C + \mathcal{S}_R^*(\operatorname{Im} B^j))$ and, hence, by minimality of $\mathcal{S}_R^*(\operatorname{Im} b^j)$, also $\mathcal{S}_R^*(\operatorname{Im} b^j) \subseteq \mathcal{W}$, but this contradicts (23). Therefore, there exists k such that $A^k b^j \notin \operatorname{Ker} C$ and we have $\mathcal{S}_R^*(\operatorname{Im} b^j) = \operatorname{span}\{A^k b^j\}_{k \leq \bar{k}+1}$ where $\bar{k}$ is the minimum integer such that $A^k b^j \in \operatorname{Ker} C$ for $k \leq \bar{k}$. This implies, first of all, $CA^k b^j \neq 0$ and hence $b^j \in \mathcal{X}$ is an observable state. Moreover, assuming, by contradiction, that (22) does not hold, we have that there exists $s \in \mathcal{S}_R^*(\operatorname{Im} B^J)$ such that $CA^{(\bar{k}+1)} b^j = Cs \neq 0$. This implies $A^{(\bar{k}+1)} b^j = x_c + s$ with $x_c \in \operatorname{Ker} C$ and, as a consequence, $\mathcal{S}_R^*(\operatorname{Im} b^j)$ turns out to be contained in $(\operatorname{Ker} C + \mathcal{S}_R^*(\operatorname{Im} B^j))$. As seen above, by minimality of $\mathcal{S}_R^*(\operatorname{Im} b^j)$, this implies also $\mathcal{S}_R^*(\operatorname{Im} b^j) \subseteq \mathcal{W}$, which contradicts (23). ■

Remark 7: Condition ii) of Proposition 4 is just the solvability condition of Theorem 1 specialized to the situation at issue, while Condition i) corresponds to the condition given in [5, Theorem 3] for the case in which one is interested only in detecting and isolating the specific fault input u_j with respect to all the others, with the only difference of requiring explicitely the observability of b^j. We have already pointed out in Remark 2 that this explicit request is satisfied without loss of generality, in the case of classical linear systems, by assuming that Σ is observable. In case one is interested in detecting and isolating each fault input, not only the j-th one, with respect to all the other unknown inputs, namely in solving the Beard-Jones detection filter problem as it is formulated in [5], Theorem 1 says that a solution consisting of a bank of $m-1$ residual generators exists if and only if Condition ii) is satisfied for all $j=1,\ldots,m_1$, and, hence, by Proposition 4, if and only if Condition i) holds for all $j=1,\ldots,m_1$. Focusing the attention on (22), it is possible to prove that it holds for all $j=1,\ldots,m_1$ if and only if $C\mathcal{S}_R^*(\operatorname{Im} b^j)\cap C(\sum_{i\neq j}\mathcal{S}_R^*(\operatorname{Im} b^i))=\{0\}$ for all $j=1,\ldots,m_1$. Then, except again for the additional request of observability, we have that the characterization given by Theorem 1 coincides with that given by [5, Theorem 3] and that they are equivalent if Σ is observable.

V. Example 1

In this section, the devised synthesis procedure is illustrated by a numerical example. Computations and simulations are made with MATLAB and MATLAB-based software for the geometric approach available with [30].

Let us consider system Σ_σ, of the form (1), with $I=\{1,2\}$, where

$$A_1=\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\-1&0&0&1\end{bmatrix},\quad A_2=\begin{bmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\0&1&0&0\end{bmatrix},$$

$$B_1=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}^\top,\quad B_2=\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix}^\top,$$

$$C_1=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix},\quad C_2=\begin{bmatrix}0&0&0&1\\0&0&1&0\end{bmatrix}.$$

Note that Σ_1 is not observable, but this does not prevent solvability of j-RSGP (Remark 2). Let us assume that input u_1 model a fault of interest and input u_2 be unknown (it may be another fault or an input of other nature like, e.g., a disturbance). Then,

$$b_1^1=\begin{bmatrix}1&0&0&0\end{bmatrix}^\top,\quad b_2^1=\begin{bmatrix}0&0&1&0\end{bmatrix}^\top,$$

$$B_1^1=\begin{bmatrix}0&1&0&0\end{bmatrix}^\top,\quad B_2^1=\begin{bmatrix}0&0&0&1\end{bmatrix}^\top.$$

Henceforth, we denote by e_i, with $i=1,\ldots,4$, the i-th vector of the natural basis of $\mathbb{R}^4$. Following the procedure described in Section IV and applying standard methods of linear algebra, by the recursive algorithm (7), where $\mathcal{W}=\operatorname{Im} B_1^1+\operatorname{Im} B_2^1$, we compute $\mathcal{S}=\mathcal{S}_R^*(\mathcal{W})=\operatorname{span}\{e_2,e_4\}$ and we also obtain

$$P=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}.$$

Moreover, for any value of their parameters $g_{\hat{i}\hat{j}}$ and $h_{\hat{i}\hat{j}}$, the matrices

$$G_1=\begin{bmatrix}g_{11}&0\\g_{21}&g_{22}\\g_{31}&-1\\g_{41}&g_{42}\end{bmatrix},\quad G_2=\begin{bmatrix}-1&h_{12}\\h_{21}&h_{22}\\0&h_{32}\\h_{41}&h_{42}\end{bmatrix}$$

define a friend $\mathscr{G}=\{G_1,\,G_2\}$ of $\mathcal{S}$ (see Proposition 1). Then, by solving (5), we obtain

$$L_1=\begin{bmatrix}g_{11}&0\\g_{31}&0\end{bmatrix},\quad L_2=\begin{bmatrix}0&h_{12}\\1&h_{32}\end{bmatrix}.$$

The maximal $(A_i+G_iC_i)$-invariant subspaces contained in $\operatorname{Ker} C_1+\mathcal{S}$ and $\operatorname{Ker} C_2+\mathcal{S}$ are $\mathcal{W}_1=\operatorname{span}\{e_2,e_3,e_4\}$ and $\mathcal{W}_2=\operatorname{span}\{e_2,e_4\}$, respectively. Thus, Condition (11) of Theorem 1 is satisfied with $j=1$ and $i=1,2$.

To complete the construction of the residual generator $\Sigma_\sigma^{res_1}$, that detects the occurrence of the fault modeled by u_1, we use the relation $C_{Oi}\,P=P_i\,C_i$, with $i=1,2$, where the canonical projections $P_1:\mathcal{Y}\to\mathcal{Y}_1=\mathcal{Y}/C_1\mathcal{S}$ and $P_2:\mathcal{Y}\to\mathcal{Y}_2=\mathcal{Y}/C_2\mathcal{S}$ are $P_1=[1\ 0]$ and $P_2=[0\ 1]$. Thus, we obtain $C_{O1}=[1\ 0]$ and $C_{O2}=[0\ 1]$. Consequently, the modes of $\Sigma_\sigma^{res_1}$ are

$$\Sigma_1^{res_1}\equiv\begin{cases}\dot z(t)=\begin{bmatrix}g_{11}&0\\g_{31}&0\end{bmatrix}z(t)-\begin{bmatrix}g_{11}&0\\g_{31}&-1\end{bmatrix}y(t)\\ r_1(t)=\begin{bmatrix}1&0\end{bmatrix}z(t)-\begin{bmatrix}1&0\end{bmatrix}y(t)\end{cases}$$

$$\Sigma_2^{res_1}\equiv\begin{cases}\dot z(t)=\begin{bmatrix}0&h_{12}\\1&h_{32}\end{bmatrix}z(t)-\begin{bmatrix}-1&h_{12}\\0&h_{32}\end{bmatrix}y(t)\\ r_1(t)=\begin{bmatrix}0&1\end{bmatrix}z(t)-\begin{bmatrix}0&1\end{bmatrix}y(t)\end{cases}$$

To run simulations, we choose the parameter values $g_{11}=0.1$, $h_{12}=-1.2$, $g_{31}=-2.5$, $h_{32}=0.7$ and the switching signal

$$\sigma(t)=\begin{cases}1 & \text{for}\quad 0\le t<1,\\ 2 & \text{for}\quad 1\le t<3,\\ 1 & \text{for}\quad 3\le t.\end{cases}$$

We initialize the plant Σ_σ at $x(0)=[0.5\ \ -1\ \ 0.3\ \ 1.2]^\top$. We model the occurrence of a fault at input u_1 at time $\bar t=0.6$ by the signal

$$u_1(t)=\begin{cases}0 & \text{for}\quad 0\le t<0.6,\\ t-0.6 & \text{for}\quad 0.6\le t<1.6,\\ 1 & \text{for}\quad 1.6\le t,\end{cases}$$

and we assume $u_2(t)=0$ for $t\in\mathbb{R}^+$ (see Fig. 1). Consequently, with $\Sigma_\sigma^{res_1}$ initialized at $z(0)=Px(0)=[0.5\ 0.3]$, the residual $r_1(t)$ has the behavior shown in Fig. 2 and, zooming in around time $t=1\,\text{s}$, in Fig. 3. In particular, $r_1(t)$ is equal to 0 on the time interval $[0\ 0.6]$ and, as expected, the instant $\bar t=0.6$ at which the fault occurs corresponds to

$$\bar t=\inf\{t,\ t\in\mathbb{R}^+\ \text{such that}\ r_1(t)\neq 0\}.$$

Note that, although the state of $\Sigma_\sigma^{res_1}$ has a continuous behavior, $r_1(t)$ is discontinuous at the switching times 1 and 3, due to the abrupt change of the output map from $C_{O1}=[1\ 0]$ to $C_{O2}=[0\ 1]$ and viceversa.

To show that the unknown input u_2 has no effect on the residual, let us consider the same situation as above, except for the input $u_2(t)$, which is assumed to be a sinusoid – e.g., $u_2(t)=-1.8\sin(2t+\pi/2)$ as shown in Fig. 4. The behavior of the residual is shown in Fig. 5 and Fig. 6. As is easy to notice, these plots respectively coincide with those shown in Fig. 2 and Fig. 3. Finally, let us consider the same situation as above, except for a delay of $1\,\text{s}$ in the fault input, so that

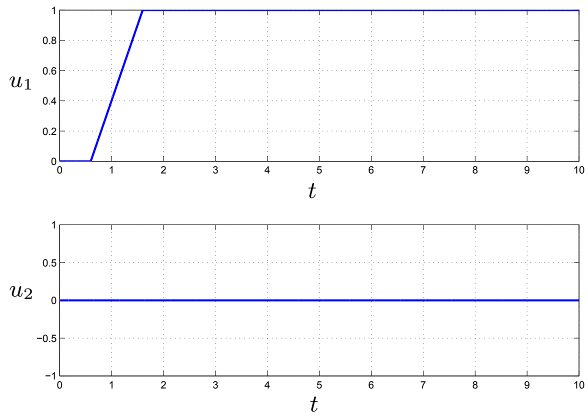

Fig. 1. Saturated ramp at fault input u_1 and zero at input u_2

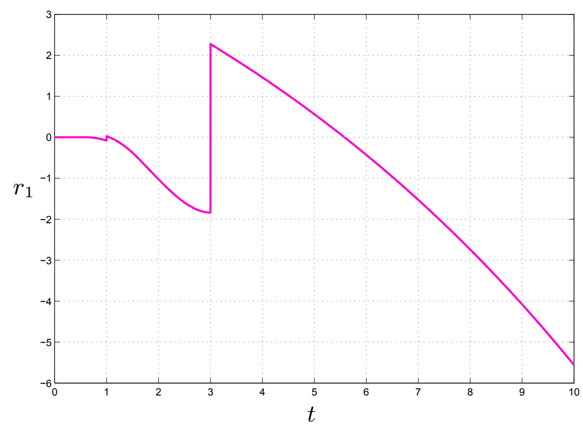

Fig. 2. Residual r_1 with saturated ramp at fault input u_1 and zero at input u_2

the latter affects the system after the first switch has occurred. Namely, let us consider the signal

$$u_1(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t < 1.6, \\ t - 1.6 & \text{for} \quad 1.6 \leq t < 2.6, \\ 1 & \text{for} \quad 2.6 \leq t, \end{cases}$$

(see Fig. 7). As it is shown in Fig. 8 and Fig. 9, the residual remains equal to zero until the occurrence of the fault, and this shows, as expected, that the initial synchronization of the state of the residual generator with that of the system is not compromised by the occurrence of the switch.

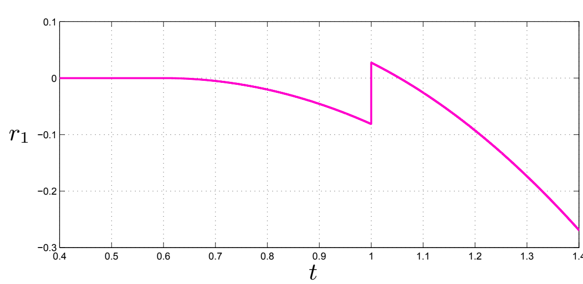

Fig. 3. Zoomed portion of the plot in Fig. 2

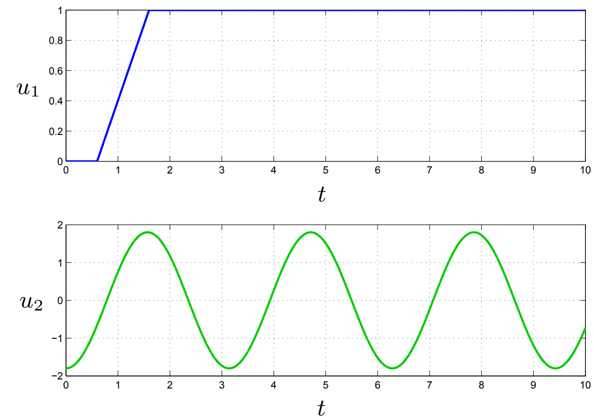

Fig. 4. Saturated ramp at fault input u_1 and sinusoid at input u_2

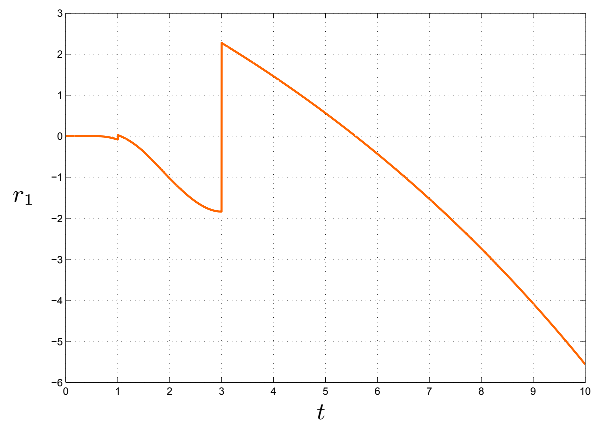

Fig. 5. Residual r_1 with saturated ramp at fault input u_1 and sinusoid at input u_2

VI. Fault Detection and Isolation without Knowledge of the Initial Condition

As pointed out in Section IV and in Remark 4, detection and isolation of the fault is possible provided Condition (11) of Theorem 1 is satisfied and the initial condition $x(0)$ of the system is known. Note that, in principle, it is not necessary to require any stability property for $\Sigma_\sigma^{res_j}$. However, global asymptotic stability over $\mathscr{S}_\alpha$ for some $\alpha \geq 0$, beside being desirable in general, is necessary to deal with the FDI problem in case $x(0)$ is not known and, thus, the residual signal generator cannot be initialized exactly. To analyze this situation, we need the following definition from [11, Definition 2].

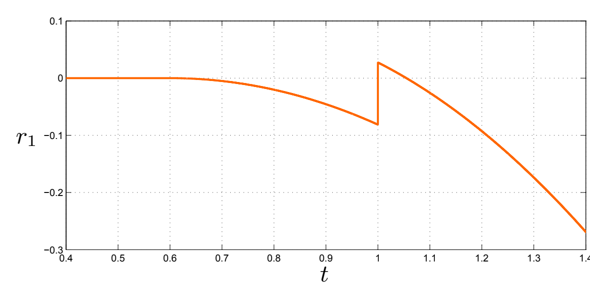

Fig. 6. Zoomed portion of the plot in Fig. 5

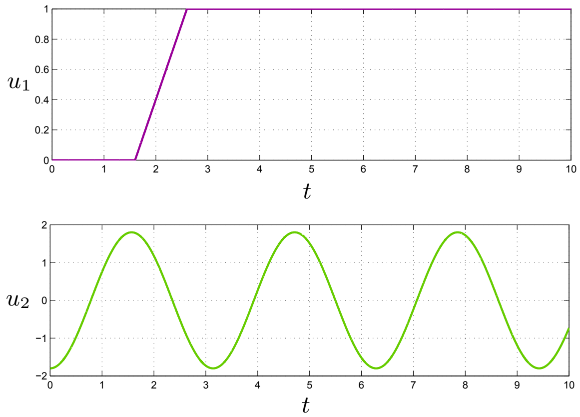

Fig. 7. Delayed saturated ramp at fault input u_1 and sinusoid at input u_2

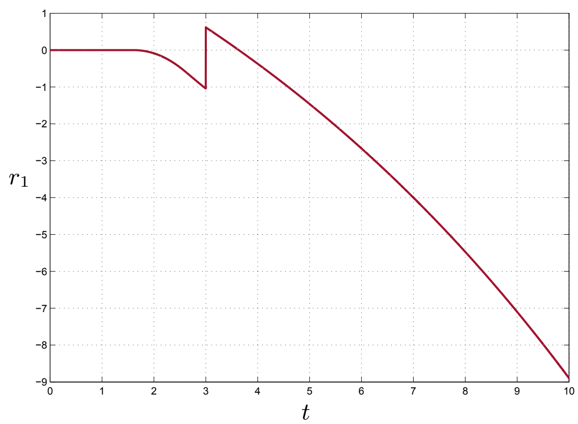

Fig. 8. Residual r_1 with delayed saturated ramp at fault input u_1 and sinusoid at input u_2

Definition 2: Given the switching linear system Σ_σ of the form (1), a conditioned invariant subspace $\mathcal{S}_R \subseteq \mathcal{X}$ is said to be externally stabilizable if there exists a friend $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i\in I}$ such that the switching linear dynamics induced on $\mathcal{X}/\mathcal{S}$ by that of $\Sigma_\sigma^{\mathscr{G}}$ is globally asymptotically stable over $\mathscr{S}_\alpha$ for some $\alpha \geq 0$.

The following proposition completely characterizes the external stabilizability of a conditioned invariant subspace $\mathcal{S}_R$.

Proposition 5: Let $\mathcal{S}_R \subseteq \mathcal{X}$ be a conditioned invariant subspace for Σ_σ and let $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i\in I}$ be a friend of $\mathcal{S}_R$. Let $P^\top$ be a matrix whose columns form a basis of $\mathcal{S}_R^\perp$ and let $\{L_i, M_i\}_{i\in I}$ be an indexed family of matrices that, together with P, satisfy (6). Let $\begin{bmatrix} N_{1i}^\top & N_{2i}^\top \end{bmatrix}^\top$ be, for all $i \in I$, a matrix of suitable dimensions whose columns form a basis of $\operatorname{Ker}\begin{bmatrix} P^\top & C_i^\top \end{bmatrix}$. Then, $\mathcal{S}_R$ is externally stabilizable if and only if the pairs $(L_i, N_{1i}^\top)$ are detectable for all $i \in I$.

Proof: See [11, Proposition 17]. ■

The set of all externally stabilizable conditioned invariant subspaces containing $\sum_{i\in I} \operatorname{Im} B_i^j$ has a minimum element, called the *good* conditioned invariant subspace containing $\sum_{i\in I} \operatorname{Im} B_i^j$ and denoted by $\mathcal{S}_{Rg}(\sum_{i\in I} \operatorname{Im} B_i^j)$. An algorithmic procedure to construct $\mathcal{S}_{Rg}(\sum_{i\in I} \operatorname{Im} B_i^j)$ is given in [11, Section 6.2]. In general, we have $\mathcal{S}_R(\sum_{i\in I} \operatorname{Im} B_i^j) \subseteq \mathcal{S}_{Rg}(\sum_{i\in I} \operatorname{Im} B_i^j)$. Now, we have the following result.

Theorem 2: Given a switching linear system Σ_σ of the form (1), with $m \geq 2$, let u_j be a fault input that affects all the modes Σ_i of Σ_σ and assume that σ is measurable. Then, letting $\mathcal{W}_i'$ be, for all $i \in I$, the maximal $(A_i + G_iC_i)$-invariant subspace of $\mathcal{X}$ contained in $(\operatorname{Ker} C_i + \mathcal{S}_{Rg}^*(\sum_{i\in I} \operatorname{Im} B_i^j))$, where $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i\in I}$ is a friend of $\mathcal{S}_{Rg}^*(\sum_{i\in I} \operatorname{Im} B_i^j)$, the j-RSGP is solvable by means of a residual signal generator that is globally asymptotically stable over $\mathscr{S}_\alpha$ for some $\alpha \geq 0$ if and only if the following condition holds:

$$\mathcal{W}_i' \cap \operatorname{Im} b_i^j = \{0\} \quad \text{for all } i \in I. \tag{24}$$

Proof: If. Since $(\operatorname{Ker} C_i + \mathcal{S}_R(\sum_{i\in I} \operatorname{Im} B_i^j)) \subseteq (\operatorname{Ker} C_i + \mathcal{S}_{Rg}(\sum_{i\in I} \operatorname{Im} B_i^j))$ for all $i \in I$, Theorem 1 applies. Then, the residual signal generator $\Sigma_\sigma^{res_j}$ constructed by using a matrix P such that the columns of $P^\top$ are a basis of $(\mathcal{S}_{Rg}(\sum_{i\in I} \operatorname{Im} B_i^j))^\perp$, a friend $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i\in I}$ of $\mathcal{S}_{Rg}(\sum_{i\in I} \operatorname{Im} B_i^j)$ such that the switching linear dynamics induced on $\mathcal{X}/\mathcal{S}$ by that of $\Sigma_\sigma^{\mathscr{G}}$ is globally asymptotically stable over $\mathscr{S}_\alpha$ for some $\alpha \geq 0$ and an indexed family of pair of matrices $\{L_i, M_i\}_{i\in I}$ that verify (5) solves the j-RSGP and has the required stability property.

Only if. Let $\Sigma_\sigma^{res_j}$ be a residual signal generator of the form (2), with state space $\mathcal{Z} = \mathbb{R}^q$, that, together with the initialization map $P : \mathcal{X} \to \mathcal{Z}$, solves the j-RSGP and that is globally asymptotically stable over $\mathscr{S}_\alpha$ for some $\alpha \geq 0$. Reasoning as in the proof of Theorem 1, we get that $\operatorname{Ker} P$ is a conditioned invariant subspace for Σ_σ that contains $\operatorname{Im} B_i^j$ for all $i \in I$. Moreover, the dynamics induced on $\mathcal{X}/\operatorname{Ker} P$ by that of $\Sigma_\sigma^{\mathscr{G}}$, where the elements of $\mathscr{G} = \{G_i,\, G_i : \mathcal{Y} \to \mathcal{X}\}_{i\in I}$ are defined by $B_{Oi} = -PG_i$ for all $i \in I$, coincides with that of $\Sigma_\sigma^{res_j}$. Since this latter is globally asymptotically stable over $\mathscr{S}_\alpha$ for some $\alpha \geq 0$, it follows that $\operatorname{Ker} P$ is externally stabilizable according to Definition 2. By minimality of $\mathcal{S}_{Rg}(\sum_{i\in I} \operatorname{Im} B_i^j)$, it follows, in particular, $(\operatorname{Ker} C_i + \mathcal{S}_{Rg}(\sum_{i\in I} \operatorname{Im} B_i^j)) \subseteq (\operatorname{Ker} C_i + \operatorname{Ker} P)$ and, denoting by $\overline{\mathcal{W}}_i$ the maximal $(A_i + G_iC_i)$-invariant subspace of $(\operatorname{Ker} C_i + \operatorname{Ker} P)$, we also have $\mathcal{W}_i' \subseteq \overline{\mathcal{W}}_i$ for all $i \in I$. Reasoning again as in the proof of Theorem 1, we obtain that b^j does not belong to $\overline{\mathcal{W}}_i$ for all $i \in I$ and we conclude that (24) is satisfied. ■

Remark 8: Assuming that Condition (24) is satisfied, the correspondence described in Remark 6 specializes to a bijective correspondence between the set of the pairs consisting of a residual signal generator $\Sigma_\sigma^{res_j}$ of the form (2) and an

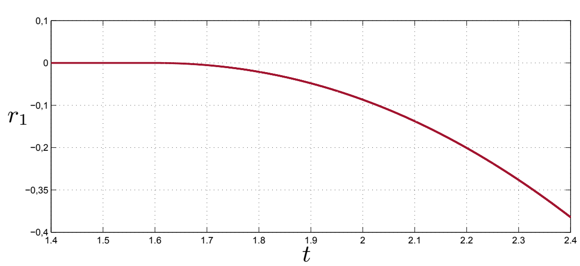

Fig. 9. Zoomed portion of the plot in Fig. 8

exact initialization map P that solve the j-RSGP, with $\Sigma_\sigma^{res_j}$ globally asymptotically stable over $\mathscr{S}_\alpha$ for some $\alpha \geq 0$, and the set of the pairs consisting of an externally stabilizable conditioned invariant subspace $\mathcal{S} \subseteq \mathcal{X}$ and a friend $\mathscr{G}$ such that $\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^j) \subseteq \mathcal{S}$ and $b_i^j \notin \mathcal{S}$ for all $i \in I$.

Assuming that $\Sigma_\sigma^{res_j}$ is a residual signal generator that, along with the exact initialization map P, solves the j-RSGP and that is globally asymptotically stable over $\mathscr{S}_\alpha$ for some $\alpha \geq 0$, let us analyze the behavior of the residual signal $r_j(t)$. As seen in Section IV, $r_j(t)$ can be described in terms of the auxiliary error $e_{aux}(t)$ by (20). Global asymptotic stability implies that $\|e_{aux}(t)\|$ goes to 0 exponentially if no fault occurs and, in particular, that there exists $\beta \in \mathbb{R}$ and $0 < \lambda \in \mathbb{R}$ such that $\|e_{aux}(t)\| \leq e^{(\beta-\lambda)t}\|e_{aux}(0)\|$ for all $t \in \mathbb{R}^+$. Consequently, also $r(t)$ goes to 0 asymptotically if no fault occurs, with a behavior that may be discontinuous at the switching points due to the abrupt change of the output map. Otherwise, letting $T = \{t_0 = 0, t_1, t_2, \ldots\}$ be the finite or countably infinite, ordered set of discontinuity points of the switching signal $\sigma \in \mathscr{S}_\alpha$ and letting $i_k = \sigma(t)$ for $t \in (t_{k-1}, t_k]$, assuming that the fault described by $u_j(t)$ occurs at time $\bar{t} \in (t_{\bar{k}-1}, t_{\bar{k}}]$, we have that for $t \leq t_{\bar{k}}$ the residual signal $r_j(t)$ is given by

$$\begin{aligned} r_j(t) \;=\; & C_{Oi_{\bar{k}}} e^{A_{Oi_{\bar{k}}}(t-t_{\bar{k}-1})} \prod_{h=1}^{\bar{k}-1} e^{L_{i_h}(t_h - t_{h-1})} e_{aux}(0) \\ & + \int_{t_{\bar{k}-1}}^{t} C_{Oi_{\bar{k}}} e^{L_{i_{\bar{k}}}(t-\tau)} P b_{\bar{k}}^j u_j(\tau) d\tau. \end{aligned} \tag{25}$$

If $\Sigma_\sigma^{res_j}$ cannot be exactly initialized since the initial condition $x(0)$ is not known, (25) shows that the only indication of the occurrence of the fault at time $\bar{t}$ that we can gather from the inspection of $r_j(t)$ is some sort of alteration of its asymptotic behavior toward 0, due to the effect of a non zero forced component, namely the second addend on the right-hand side of (25), from time $\bar{t}$ on – note that the first addend on the right-hand side of (25) goes to 0 asymptotically and the second addend is equal to 0 for all $t < \bar{t}$ and for any input $u_j(t)$. In general, since $e_{aux}(0)$ is not known, it may be difficult to detect promptly this phenomenon unless $u_j(t) = a\delta(t\bar{t})$, where $a \in \mathbb{R}$ and $\delta(t)$ is the Dirac delta function. In fact, in that case, the occurrence of the fault causes a left discontinuity at $\bar{t}$ of $r_j(t)$, which otherwise is left-continuous on $(t_{k-1}, t_k]$. If $u_j(t) = aH(t - \bar{t})$, where $a \in \mathbb{R}$ and $H(t)$ is the Heaviside step function, the auxiliary error $e_{aux}(t)$ presents a forced behavior on each interval $(t_{k-1}, t_k]$ with $t_{k-1} \geq \bar{t}$ with initial condition $e_{aux}(t_{k-1})$, whose characteristics depend on the active mode. In particular, if the interval is sufficiently large, after a transient behavior, $e_{aux}(t)$ exhibits a nonzero steady state behavior that, filtered by C_{Oi}, defines the behavior of $r_j(t)$.

Remark 9: Theorem 2 applies obviously also to the classical case of linear systems, for which global asymptotic stability over $\mathscr{S}_\alpha$ for some $\alpha \geq 0$ means asymptotic stability tout-court. It is worth noting that asymptotically stable residual signal generators are considered in [5, Section V], where their construction is discussed, but their existence is not completely characterized in structural terms.

VII. EXAMPLE 2

The aim of this section is to illustrate the devised synthesis procedure when initialization errors are present, through a worked out numerical example.

Let us consider the system Σ_σ of the form (1), with $I = \{1, 2\}$, where

$$A_1 = \begin{bmatrix} 2.60 & 0.40 & -1.90 & -4.60 \\ -0.80 & -0.10 & 0.20 & 0.60 \\ 0 & 0.50 & -0.20 & 3.80 \\ 0 & 0.30 & 0 & -0.60 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.80 & -2.20 & 4.20 & -10.60 \\ 1.20 & 3.30 & -3.9 & 1.60 \\ 0 & 1.52 & -3.36 & -0.48 \\ 0 & 5.0 & 9.0 & 2.0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 & 2 \\ -0.5 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0.1 & 0 & 0.2 \\ 0 & 0.5 & 0 & 3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Let us assume that input u_3 models a fault of interest, while u_1 and u_2 are unknown inputs (i.e., they may be other fault inputs or unknown disturbances), so that

$$b_1^3 = \begin{bmatrix} 1 & 0 & -1 & 2 \end{bmatrix}^\top, \quad b_2^3 = \begin{bmatrix} 2 & 1 & 1 & -1 \end{bmatrix}^\top,$$
$$B_1^3 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^\top, \; B_2^3 = \begin{bmatrix} 0 & -0.5 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^\top.$$

Henceforth, we denote by e_i, with $i = 1, \ldots, 4$, the i-th vector of the natural basis of $\mathbb{R}^4$. Following the procedure described in Section IV, we obtain $\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^3) = \operatorname{span}\{e_1, e_2\}$ and we also get

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using Proposition 5, one can see that $\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^3)$ is externally stabilizable and, therefore, it coincides with $\mathcal{S}_{Rg}^*(\sum_{i\in I} \operatorname{Im} B_i^3)$. In fact, for any value of the parameters,

$$G_1 = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \\ g_{31} & -\dfrac{g_{31}}{5} - 1 \\ g_{41} & -\dfrac{g_{41}}{5} - \dfrac{3}{5} \end{bmatrix}, \quad G_2 = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{31} & -\dfrac{38}{25} - h_{31} \\ h_{41} & -5 - h_{41} \end{bmatrix}$$

define a friend $\mathscr{G} = \{G_1, G_2\}$ of $\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^3)$. By solving (5), we obtain

$$L_1 = \begin{bmatrix} -\dfrac{1}{5} & -\dfrac{2}{5}g_{31} + \dfrac{4}{5} \\ 0 & -\dfrac{2}{5}g_{41} - \dfrac{12}{5} \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 2h_{31} - \dfrac{84}{25} & h_{31} - \dfrac{12}{25} \\ 2h_{41} + 9 & h_{41} + 2 \end{bmatrix}.$$

The matrices L_1 and L_2, along with $N_{11}^\top = [0\ 0.3651]$ and $N_{12}^\top = [0.7559\ 0.3780]$, show that the pairs $(L_i, N_{1i}^\top)$, with $i \in I$, are detectable. In particular, the matrices L_i, with $i \in I$, which represent the dynamics induced by $(A_i + G_i C_i)$, with $i \in I$, on $\mathcal{X}/\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^3)$, turn out to be Hurwitz if, in particular, the following values are assigned to the arbitrary parameters: $g_{31} = 0$, $g_{41} = 1$, $h_{31} = 0$, $h_{41} = -4.5$. In fact, with these values of the parameters,

$$L_1 = \begin{bmatrix} -0.5 & 0.8 \\ 0 & -2.8 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -3.36 & -0.48 \\ 0 & -2.5 \end{bmatrix}$$

and the corresponding switching dynamics is globally asymptotically stable over $\mathscr{S}_\alpha$, in particular with $\alpha = 0.5$, by Lemma 1. The remaining arbitrary parameters of G_1 and G_2 can be taken, for instance, equal to zero. Concerning the computation of the matrices of the output equations of the residual generator, since

$$\begin{aligned} \operatorname{Ker} C_1 + \mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^3) &= \operatorname{span}\{e_1, e_2, e_3\} \\ \operatorname{Ker} C_2 + \mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^3) &= \operatorname{span}\{e_1, e_2, e_3 - 2e_4\} \end{aligned}$$

it is easy to see, also without computing $\mathcal{W}_1'$ and $\mathcal{W}_2'$, that Condition (24) of Theorem 2 is satisfied with $j = 3$ and $i = 1, 2$. Then, it follows that

$$\begin{aligned} C_1\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^3) &= \operatorname{span}\{e_1 + 5e_2\} \subseteq \mathbb{R}^2 \\ C_2\mathcal{S}_R^*(\sum_{i\in I} \operatorname{Im} B_i^3) &= \operatorname{span}\{e_1 + e_2\} \subseteq \mathbb{R}^2. \end{aligned}$$

and also that $\overline{P}_1 = \begin{bmatrix} -5 & 1 \end{bmatrix}$, $\overline{P}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}$. Consequently, from $C_{Oi}\, P = \overline{P}_i\, C_i$, with $i \in I$, one gets

$$C_{O1} = \begin{bmatrix} 0 & 2 \end{bmatrix}, \quad C_{O2} = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$

In conclusion, a residual signal generator that, together with the map P found above, solves the 3-RSGP and is globally asymptotically stable over $\mathscr{S}_{0.5}$ is the system $\Sigma_\sigma^{res_3}$ of the form (2) with $\mathcal{Z} = \mathbb{R}^2$ and modes

$$\Sigma_1^{res_3} \equiv \left\{ \begin{aligned} \dot z(t) &= \begin{bmatrix} -0.2 & 0.8 \\ 0 & -2.8 \end{bmatrix} z(t) + \begin{bmatrix} 0 & 1 \\ -1 & 0.8 \end{bmatrix} y(t) \\ r_3(t) &= \begin{bmatrix} 0 & -2 \end{bmatrix} z(t) - \begin{bmatrix} -5 & 1 \end{bmatrix} y(t) \end{aligned} \right.$$

$$\Sigma_2^{res_3} \equiv \left\{ \begin{aligned} \dot z(t) &= \begin{bmatrix} -3.36 & -0.48 \\ 0 & -2.5 \end{bmatrix} z(t) + \begin{bmatrix} 0 & 1.52 \\ 4.5 & 0.5 \end{bmatrix} y(t) \\ r_3(t) &= \begin{bmatrix} 2 & 1 \end{bmatrix} z(t) - \begin{bmatrix} 1 & -1 \end{bmatrix} y(t) \end{aligned} \right.$$

To run simulations, we choose, e.g., the switching signal

$$\sigma(t) = \begin{cases} 1 & \text{for } 0 \le t < 1, \\ 2 & \text{for } 1 \le t < 4, \\ 1 & \text{for } 4 \le t. \end{cases}$$

We initialize Σ_σ, e.g., at $x(0) = [1\ -1\ 0\ 1]^\top$ and $\Sigma_\sigma^{res_3}$ at $z(0) = [0\ 0]^\top$. Further, we assume that the fault described

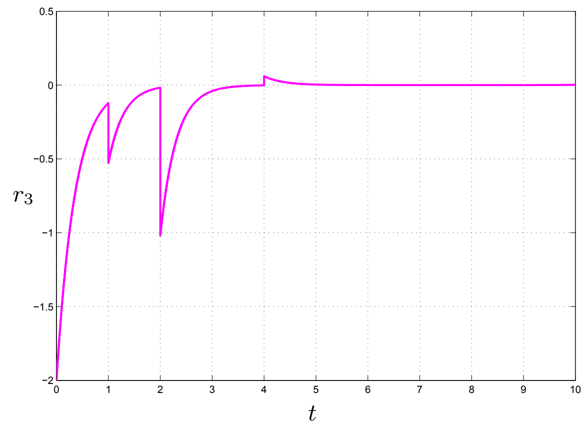

Fig. 10. Residual r_3 with Dirac delta function at fault input u_3, zero at u_1 and u_2

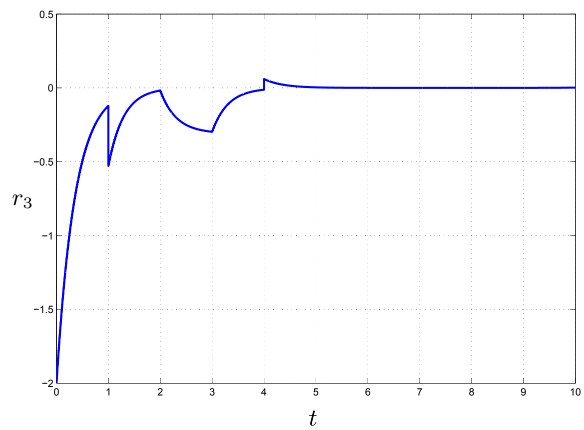

Fig. 11. Residual r_3 with rectangular impulse at fault input u_3, zero at u_1 and u_2

by $u_3(t)$ occurs at time $\bar t = 2$ and, without loss of generality, we take $u_1(t) = u_2(t) = 0$ for all $t \in \mathbb{R}^+$. Then, we consider the behavior of the residual, $r_3(t)$, corresponding to the fault modelled by four different signals.

Case 1

Let the fault be modelled as $u_3(t) = \delta(t-2)$, where $\delta(t)$ is the Dirac delta function. The behavior of $r_3(t)$, shown in Fig. 10, exhibits a discontinuity at $\bar t = 2$ caused by the occurrence of the fault – apart from the discontinuities at the switching times $t = 1$ and $t = 4$ due to the abrupt changes of the output matrix $C_{O\sigma(t)}$. The discontinuity due to the fault occurrence interrupts the asymptotic trend of $r_3(t)$ toward 0. However, the effect of the fault vanishes with time, so that, globally, $r_3(t)$ goes to 0 asymptotically.

Case 2

Let the fault be modelled as $u_3(t) = H(t-2) - H(t-3)$, where $H(t)$ is the Heaviside step function. The behavior of $r_3(t)$, shown in Fig. 11, exhibits an asymptotic trend toward 0 interrupted at $\bar t = 2$ due to the fault occurrence. As the fault signal is discontinuous at $\bar t = 2$ and $t = 3$, correspondingly, the residual $r_3(t)$ is non-differentiable. The fault effect vanishes with time and, globally, $r_3(t)$ goes to 0 asymptotically.

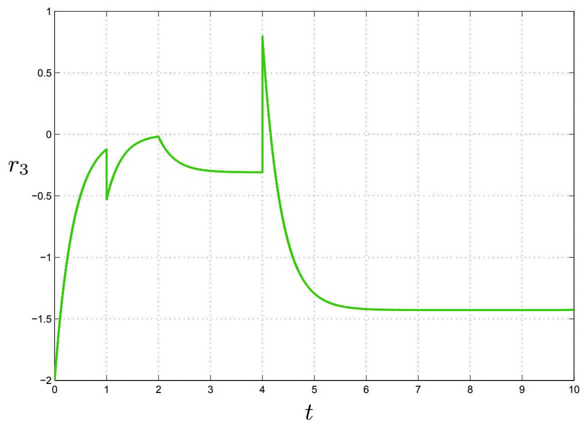

Fig. 12. Residual r_3 with Heaviside step function at fault input u_3, zero at u_1 and u_2

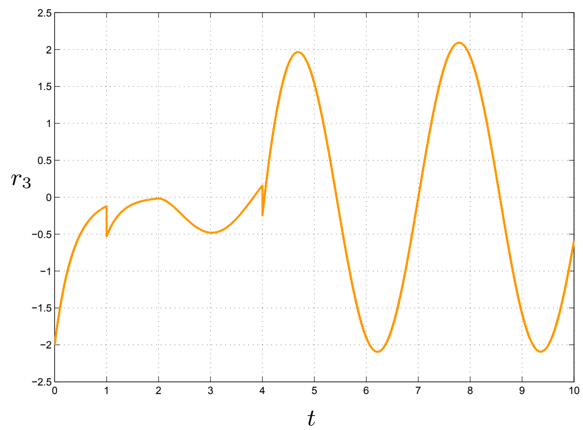

Fig. 13. Residual r_3 with sine wave function at fault input u_3, zero at u_1 and u_2

Case 3
Let the fault be modelled as $u_3(t) = H(t-2)$. The behavior of $r_3(t)$, shown in Fig. 12, has an asymptotic trend toward 0, interrupted at time $\bar{t} = 2$ due to the fault occurrence. Since the fault signal is discontinuous at $\bar{t} = 2$, correspondingly, the residual $r_3(t)$ is non differentiable. Since the fault is equal to 1 for all $t \geq \bar{t}$, the behavior of $r_3(t)$ goes asymptotically toward a steady state value different from 0.

Case 4
Let the fault be modelled as $u_3(t) = H(t-2)\sin(t-2)$. The behavior of $r_3(t)$, shown in Fig. 13, presents an asymptotic trend toward 0, which is interrupted at time $\bar{t} = 2$ due to the fault occurrence. Since the fault is modelled by a continuous function, the behavior of $r_3(t)$ turns out to be differentiable at any point except (possibly, as in this case) at the switching points. This makes it difficult to detect by direct inspection the time at which the fault occurs. However, since the fault input persists in being different from 0, $r_3(t)$ does not go asymptotically to 0 and, in this case, it shows a periodic steady state behavior.

VIII. Conclusions and Future Work

The development of model-based schemes for FDI has been dealt with for plants modeled by switching linear systems. Model uncertainties and disturbances have been assumed to be represented by the effects of a set of unknown inputs. The existence of residual signal generators that achieve a complete decoupling of the residual from the unknown inputs has been characterized in structural terms. Both the case in which the initial condition of the possibly faulty system is known and that in which the initial condition is not known have been considered. This study shows the efficacy of the structural approach in determining the necessary and sufficient conditions under which the problem is solvable and in constructing, when possible, model-based schemes for its solution. Comparison with the literature shows that the results found here are consistent with those found in the classical and structurally simpler case of linear systems.

Since the existence conditions are tight, complete decoupling may not be achievable and it would be of interest, in future works, to develop an algorithmic procedure to find maximal subsets of the set of unknown inputs from which the residual can be completely decoupled. Then, to handle the FDI problem, qualitative techniques can be employed to analyze the residual with respect to the effects of the remaining unknown inputs.

An alternative way to take into account model uncertainties is that of employing polytopic switching systems. Structural methods have already been proved to be suitable for solving control problems for polytopic switching systems (see [31], [32]), while the construction of observers that are robust with respect to polynomial uncertainties has been considered in [33] for the classical linear case. Future work will address the construction of unknown input observers for switching systems with polytopic uncertinties along the lines of [11] and their use in constructing residual generators that are robust with respect to the uncertainties will be explored.

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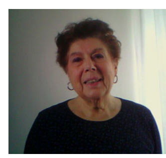

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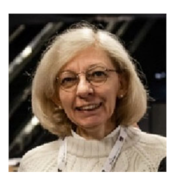

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