Annales Fennici Mathematici
Volumen 48, 2023, 411–428

Lipschitz regularity for solutions of the parabolic p-Laplacian in the Heisenberg group

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Abstract. We prove local Lipschitz regularity for weak solutions to a class of degenerate parabolic PDEs modeled on the parabolic p-Laplacian

$$\partial_t u = \sum_{i=1}^{2n} X_i(|\nabla_0 u|^{p-2} X_i u),$$

in a cylinder $\Omega \times \mathbb{R}^+$, where Ω is domain in the Heisenberg group $\mathbb{H}^n$, and $2 \leq p \leq 4$. The result continues to hold in the more general setting of contact subRiemannian manifolds.

Heisenbergin ryhmän parabolisen p-Laplacen yhtälön ratkaisuiden Lipschitzin-säännöllisyys

Tiivistelmä. Todistamme, että eräiden parabolista p-Laplacen yhtälöä

$$\partial_t u = \sum_{i=1}^{2n} X_i(|\nabla_0 u|^{p-2} X_i u)$$

yleistävien degeneroituneiden parabolisten osittaisdifferentiaaliyhtälöiden heikot ratkaisut lieriössä $\Omega\times\mathbb{R}^+$, missä Ω on Heisenbergin ryhmän $\mathbb{H}^n$ alue, ovat paikallisesti Lipschitzin-säännöllisiä, kun $2 \leq p \leq 4$. Tulos pätee myös ali-Riemannin kontaktimonistoilla muotoillussa yleisemmässä asetelmassa.

1. Introduction

In recent years there has been significant progress in the study of the regularity of the gradient for weak solutions of quasilinear degenerate elliptic PDE modeled on the p-Laplacian in the Heisenberg group $\mathbb{H}^n$. We mention here the contributions of Domokos [10], Manfredi, Mingione [12], Mingione, Zatorska-Goldstein and Zhong [13], Ricciotti [4, 17, 18] and eventually those in [4, 16, 19], where the horizontal $C^{1,\alpha}$ regularity in the full range $1 < p < \infty$, in every contact subRiemannian manifold, is proved. The new insight behind these development are certain mixed type Caccioppoli inequalities which were introduced by one of us in [19]. By contrast, in the degenerate parabolic setting, in view of the differences in homogeneity between the time and spatial derivatives, such inequalities are not available, and the study of non-stationary PDE is at a more primitive stage.

In this paper we present a new way of dealing with such lack of homogeneity and we establish the local Lipschitz regularity of weak solutions of a certain class of quasilinear, degenerate parabolic equations in the Heisenberg group $\mathbb{H}^n$, or more in general in contact subRiemannian manifolds, albeit in the restricted range $2 \leq p \leq 4$. In particular we extend to the non-stationary setting the early work [12, 13].

https://doi.org/10.54330/afm.131227
2020 Mathematics Subject Classification: Primary 35D30, 35H20, 35K65, 35K92.
Key words: Subelliptic p-Laplacian, parabolic gradient estimates, Heisenberg group.
LC was partially supported by NSF award DMS1955992. GC was partially funded by Horizon 2020 Project ref. 777822: GHAIA. XZ was supported by the Academy of Finland, CoE in Randomness and Structures.

To be more specific: In a cylinder $Q = \Omega \times (0,T)$, where $\Omega \subset \mathbb{H}^n$ is an open set and $T > 0$, we consider the equation

$$\partial_t u = \sum_{i=1}^{2n} X_i A_i(x, \nabla_0 u) \quad \text{in} \quad Q = \Omega \times (0,T), \tag{1.1}$$

modeled on the parabolic p-Laplacian

$$\partial_t u = \sum_{i=1}^{2n} X_i \Big(|\nabla_0 u|^{p-2} X_i u \Big), \tag{1.2}$$

where $2 \le p \le 4$ and $X_1, \dots, X_{2n}$ denote the horizontal left invariant frame in $\mathbb{H}^n$ and $\nabla_0 = (X_1, \cdots, X_{2n})$. In a previous study [3], Garofalo and the first two listed authors have extended techniques originally introduced by the third listed author [19] to establish C^∞ smoothness for weak solutions to (2.5) in the range $2 \le p < \infty$ under some additional non-degeneracy hypothesis, where the term $|\nabla_0 u|^{p-2}$ is substituted by $(1+|\nabla_0 u|^2)^{\frac{p-2}{2}}$. We explicitly note that the techniques used in the non-degenerate setting of [3] cannot be adapted to study the degenerate case, where the vanishing of the gradient causes new phenomena.

Similarly, one could naively conjecture that the techniques used in [19] to prove the sophisticated mixed Caccioppoli inequalities that are the key to Zhong's regularity proof, would continue to prove effective in the degenerate parabolic setting. However this is not the case as the lack of homogeneity in the parabolic PDE breaks down the argument completely, and genuinely new ideas are needed.

In the present paper we introduce a new Poincaré type interpolation inequality (4.1), and use it to show that in the restricted range of the nonlinearity $2 \le p \le 4$, one can obtain Lipschitz regularity of weak solutions of (1.1). In our work we make substantial use of subRiemannian analogues of the non-anisotropic space-time cylinders used in the Euclidean setting by DiBenedetto in [8]. To the best of our knowledge, the present paper is the first instance in the literature of the study of higher regularity for weak solutions of the degenerate parabolic p-Laplacian PDE in the subRiemannian setting.

Structural assumptions and main results. We indicate with $x = (x_1, \dots, x_{2n}, x_{2n+1})$ the variable point in $\mathbb{H}^n$. Consequently, we will indicate with ∂_i partial differentiation with respect to the variable x_i, $i = 1, \dots, 2n$, and use the notation Z for the partial derivative $\partial_{x_{2n+1}}$. The notation $\nabla_0 u = \sum_{i=1}^{2n} X_i u X_i \cong (X_1 u, \dots, X_{2n} u)$ denotes the so-called *horizontal gradient* of the function u, where

$$X_i = \partial_i - \frac{x_{n+i}}{2}\partial_{2n+1}, \quad X_{n+i} = \partial_{n+i} + \frac{x_i}{2}\partial_{2n+1}, \quad i = 1, \dots, n.$$

As it is well-known, the $2n+1$ vector fields $X_1, \dots, X_{2n}, Z$ are connected by the following commutation relation: for every couple of index i, j, if $j = i + n$, then $[X_i, X_j] = Z$; all other commutators being trivial.

The relevant assumptions on the vector-valued function

$$(x, \xi) \to A(x, \xi) = (A_1(x,\xi), \dots, A_{2n}(x,\xi))$$

are that there exist $2 \le p \le 4$, and $0 < \lambda' \le \Lambda' < \infty$ such that for a.e. $x \in \Omega, \xi \in \mathbb{R}^{2n}$ and for all $\eta \in \mathbb{R}^{2n}$, one has

$$\begin{cases} \lambda' |\xi|^{p-2} |\eta|^2 \le \partial_{\xi_j} A_i(x,\xi) \eta_i \eta_j \le \Lambda' |\xi|^{p-2} |\eta|^2, \\ |A_i(x,\xi)| + |\partial_{x_j} A_i(x,\xi)| \le \Lambda' |\xi|^{p-1}. \end{cases} \tag{1.3}$$

Given an open set $\Omega \subset \mathbb{H}^n$, we indicate with $W^{1,p}(\Omega)$ the Sobolev space associated with the p-energy $\mathscr{E}_{\Omega,p}(u) = \frac{1}{p}\int_\Omega |\nabla_0 u|^p$, i.e., the space of all functions $u \in L^p(\Omega)$ such that their distributional derivatives $X_i u$, $i = 1, \ldots, 2n$, are also in $L^p(\Omega)$. The corresponding norm is $\|u\|^p_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla_0 u\|_{L^p(\Omega)}$. We will add the subscript *loc* for the local versions of such spaces, and denote by $W^{1,p}_0(\Omega)$ the completion of $C^\infty_0(\Omega)$ with respect to such norm. A function $u \in L^p((0,T), W^{1,p}_{\mathrm{loc}}(\Omega))$ is a weak solution of equation (1.1) in the cylinder $\Omega \times (0,T)$ if

$$\int_0^T \int_\Omega u\phi_t - \sum_{i=1}^{2n} A_i(x, \nabla_0 u) X_i \phi = 0, \tag{1.4}$$

for every $\phi \in C^\infty_0(Q)$. Our main result is a Lipschitz regularity estimate for weak solutions, on parabolic cylinders $Q_{\mu,r}$ (see Definition 5.1). We denote by $N = 2n + 2$ the Hausdorff dimension of $\mathbb{H}^n$.

Theorem 1.1. *Let A_i satisfy the structure conditions* (1.3) *and let $u \in L^p((0,T), W^{1,p}_{\mathrm{loc}}(\Omega))$ be a weak solution of* (1.1) *in $Q = \Omega \times (0,T)$. If $2 \le p \le 4$, then $|\nabla_0 u| \in L^\infty_{\mathrm{loc}}(Q)$ and $\partial_t u, Zu \in L^q_{\mathrm{loc}}(Q)$ for every $1 \le q < \infty$. Moreover, one has that for any $Q_{\mu,2r} \subset Q$,*

$$\sup_{Q_{\mu,r}} |\nabla_0 u| \le C \max\left(\left(\frac{1}{\mu r^{N+2}} \int\int_{Q_{\mu,2r}} (\delta + |\nabla_0 u|^2)^{\frac{p}{2}} \right)^{\frac{1}{2}}, \mu^{\frac{p}{2(2-p)}} \right), \tag{1.5}$$

where $C = C(n, p, \lambda, \Lambda, r, \mu) > 0$. In the special case where there is no direct dependence on the space variable, i.e. $A_i(x,\xi) = A_i(\xi)$, the parameters dependence is more explicit, with $C = C(n, p, \lambda, \Lambda)\mu^{\frac{1}{2}} > 0$.

The boundedness of the gradient of solutions to equation (1.1) in the setting of Euclidean spaces is well-known [9, 5, 8]. The above theorem shows that this is also the case in the setting of Heisenberg group for the range $2 \le p \le 4$. We believe that it is true for all range $1 < p < \infty$ if the solution is bounded as in the Euclidean setting.

Original contributions in the proof. The proof of the main theorem is based on the Caccioppoli type estimate in Proposition 4.4, which is new in the subelliptic context. The Lipschitz regularity follows then through a Moser type iteration, which we present in detail in Section 5. The Caccioppoli estimate in Proposition 4.4 is derived using two main ingredients: The first of these is an approximation scheme, that allows us to invoke the regularity results from [3], thus dealing with smooth approximants that can be differentiated directly, without recurring to fractional difference quotients or to the Hilbert–Haar approach in [19]. One of the original contributions of the present paper is that we can avoid the extra assumption of Riemannian approximation which is needed in [3] (hypotheses (1.6) and (1.7) in [3]), and in fact we prove that our structure hypotheses (1.3) imply that such approximation always exists. The second ingredient is a Poincaré-type inequality for smooth functions, which goes back to an idea originally used to study the Levi mean curvature fully nonlinear PDE in [6]. This latter estimate, in Lemma 4.1, is the only point in the paper where we are forced to impose the limited range $2 \le p \le 4$. While we believe that such constraint may ultimately not be needed for the Caccioppoli inequality in Proposition 4.4, neither our current method, nor the ideas in [19, 3] can overcome the issues due to the combination of the degeneracy and the lack of homogeneity.

Structure of the paper. In Section 2 we review some preliminary definitions and results from [3] and lay out the approximation scheme, thus reducing the problem to finding estimates for smooth solutions u_δ of approximating regularized equations, which are stable as $\delta \to 0$. From that point on, we will simplify the notation by dropping the script δ and by focusing on the case $A_i(x,\xi) = A_i(x)$, thus highlighting how in this case we can obtain more explicit constants in the right hand side of our estimates. In Section 3 we recall some energy type estimates from [3]. In Section 4 we show that derivatives of weak solutions along the center are in every L^q_{loc} space, $q \geq 2$, uniformly in $\delta > 0$ and establish the key Caccioppoli type inequality in Proposition 4.4. In this section we need to use the limitation $2 \leq p \leq 4$ in the proof of a Sobolev type estimate. We conjecture that with the exception of Lemma 4.1, all other estimates continue to hold in the range $2 \leq p < \infty$. Using Proposition 4.4, in Section 5 we prove that the solutions are locally Lipschitz continuous in the sub-Riemannian metric (i.e. the horizontal gradient is in L^∞_{loc}) uniformly in δ. We note explicitly that the Moser iteration in Section 5 involves a Sobolev type estimate, which is also stable as $\delta \to 0$, in view of the results in [2]. Section 6 addresses the higher integrability of the time derivatives $\partial_t u$ of weak solutions.

Acknowledgements. The authors are grateful to Nicola Garofalo for conversations around the topics of this paper. Indeed, the results presented here are a development of some initial work that the authors did jointly with him. The authors thank the referees for valuable comments.

2. Approximating weak solutions via viscosity regularizations

As mentioned in the introduction, our strategy for the proof of the Lipschitz regularity is to locally approximate the weak solutions of (1.1) with smooth solutions u_δ of less degenerate PDE such as (2.5) and prove estimates on such approximate solutions that are uniform as $\delta \to 0$. This approximation is built using both the regularity results in [3], recalled below in Theorem 2.1, and a Riemannian approximation scheme (see [2, 14] and references therein). We start by recalling the main points of the latter. First, we will use interchangeably the notation Z and X_{2n+1} for the generator of the center of the Lie algebra. The left invariant subRiemannian metric $(\mathbb{H}^n, g_0)$ defined by $\langle X_i, X_j\rangle_0 = \delta_{ij}$, for $i,j = 1,\ldots,2n$, can be approximated in the Gromov–Hausdorff sense through a sequence of Riemannian metrics g_ε, for $\varepsilon \to 0^+$, defined by imposing that $X_1,\ldots,X_{2n},\varepsilon Z$ is an orthonormal g_ε-frame for all $\varepsilon > 0$. In the terminology of [14], the metrics g_ε *tame* the metric g_0. We relabel the vectors in this frame as $X_1^\varepsilon,\ldots,X_{2n+1}^\varepsilon$. The corresponding gradient

$$\nabla_\varepsilon u = \sum_{i=1}^{2n} X_i u X_i + \varepsilon^2 Z u Z = \sum_{i=1}^{2n+1} X_i^\varepsilon u X_i^\varepsilon$$

has the obvious property that $\nabla_\varepsilon u \to (\nabla_0 u, 0)$ as $\varepsilon \to 0$. We note explicitly that

$$|\nabla_\varepsilon u|_\varepsilon^2 := |\nabla_\varepsilon u|_{g_\varepsilon}^2 = \sum_{i=1}^{2n} (X_i u)^2 + \varepsilon^2 (Zu)^2 \to |\nabla_0 u|_0,$$

as $\varepsilon \to 0$. For $\delta > 0$, the δ-regularized Riemannian p-Laplacian, i.e. the operator related to the Euler–Lagrange equations for the p-energy $\int |\nabla_\varepsilon u|_\varepsilon^p dx$, is

$$L_p^\varepsilon u := \sum_{i=1}^{2n+1} X_i^\varepsilon([\delta + |\nabla_\varepsilon u|_\varepsilon^2]^{\frac{p-2}{2}} X_i^\varepsilon u), \tag{2.1}$$

and provides a natural (quasilinear) elliptic regularization of the subelliptic p-Laplacian.

Next, we recall the regularity theorem proved in [3].

Theorem 2.1. [3] *For $\Omega \subset \mathbb{H}^n$, $2 \le p < \infty$, and $\delta > 0$, assume that the functions $A_{i,\delta} \colon \Omega \times \mathbb{R}^{2n} \to \mathbb{R}$, $i = 1, \dots, 2n$, satisfy the following structure conditions:*

(i) *For some $\lambda, \Lambda > 0$ depending only on λ', Λ', one has*

$$\begin{cases} \lambda(\delta + |\xi|^2)^{\frac{p-2}{2}} |\eta|^2 \le \partial_{\xi_j} A_{i,\delta}(x,\xi)\eta_i\eta_j \le \Lambda(\delta + |\xi|^2)^{\frac{p-2}{2}} |\eta|^2, \\ |A_i(x,\xi)| + |\partial_{x_j} A_{i,\delta}(x,\xi)| \le \Lambda(\delta + |\xi|^2)^{\frac{p-1}{2}}. \end{cases} \tag{2.2}$$

(ii) *We assume that one can approximate $A_{i,\delta}$ by a 1-parameter family of regularized approximants $A^\varepsilon_\delta(x,\xi) = (A^\varepsilon_{1,\delta}(x,\xi), \dots, A^\varepsilon_{2n+1,\delta}(x,\xi))$ defined for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{2n+1}$, and such that for a.e. $x \in \Omega$, for all $\xi = \sum_{i=1}^{2n} \xi_i X^\varepsilon_i + \xi_{2n+1} X^\varepsilon_{2n+1}$, and $\xi^\varepsilon = \sum_{i=1}^{2n} \xi_i X^\varepsilon_i + \varepsilon \xi_{2n+1} X^\varepsilon_{2n+1}$ one has uniformly on compact subsets of Ω,*

$$\begin{aligned} &(A^\varepsilon_{1,\delta}(x,\xi^\varepsilon), \dots, A^\varepsilon_{2n+1,,\delta}(x,\xi^\varepsilon)) \\ &\underset{\varepsilon\to 0^+}{\longrightarrow} (A_{1,,\delta}(x,\xi_1,\dots,\xi_{2n}), \dots, A_{2n,\delta}(x,\xi_1,\dots,\xi_{2n}), 0), \end{aligned} \tag{2.3}$$

and furthermore

$$\begin{cases} \lambda(\delta + |\xi|^2)^{\frac{p-2}{2}} |\eta|^2 \le \partial_{\xi_j} A^\varepsilon_{i,\delta}(x,\xi)\eta_i\eta_j \le \Lambda(\delta + |\xi|^2)^{\frac{p-2}{2}} |\eta|^2, \\ |A^\varepsilon_{i,\delta}(x,\xi)| + |\partial_{x_j} A^\varepsilon_{i,\delta}(x,\xi)| \le \Lambda(\delta + |\xi|^2)^{\frac{p-1}{2}}, \end{cases} \tag{2.4}$$

for all $\eta \in \mathbb{R}^{2n+1}$, and for some $0 < \lambda \le \Lambda < \infty$ independent of ε. Let $u_\delta \in L^p((0,T), W^{1,p}_{\mathrm{loc}}(\Omega))$ be a weak solution of

$$\partial_t u_\delta = \sum_{i=1}^{2n} X_i A_{i,\delta}(x, \nabla_0 u_\delta). \tag{2.5}$$

in $Q = \Omega \times (0,T)$. If $\delta > 0$ then u_δ is C^∞ smooth in Q.

One of our main contributions in the present paper is that one can avoid the assumptions (2.3), and (2.4), and build the Riemannian approximation using solely the structure condition (2.2). Our result is stated in the following proposition

Proposition 2.2. *Let A_i be as in* (1.3)*. For every $\delta > 0$ there exists $A_{i,\delta}$ such that*

$$A_\delta(x,\xi) \underset{\delta\to 0^+}{\longrightarrow} A(x,\xi), \tag{2.6}$$

satisfying the hypothesis (2.2), (2.3), *and* (2.4) *with constants depending only on the original λ', Λ'. Moreover, if a function $A_{i,\delta}$ satisfies* (2.2), *then it also satisfies* (2.3), *and* (2.4) *with constants depending only on the original λ', Λ'.*

In view of Theorem 2.1, the latter yields immediately the following

Corollary 2.3. *Let u be a weak solution of* (1.1) *in $Q = \Omega \times (0,T)$, with the structure conditions* (1.3)*. For any sub-cylinder $Q_1 = \Omega_1 \times (t_1,t_2) \subset\subset \Omega \times (0,T)$, there exists a sequence $\{u_\delta\}$ of smooth solutions of the regularized problem*

$$\partial_t u_\delta = \sum_{i=1}^{2n} X_i A_{i,\delta}(x, \nabla_0 u_\delta) \quad \text{in } Q_1, \quad \text{and} \quad u_\delta = u \quad \text{on } \partial_p Q_1 \tag{2.7}$$

converging to u, as $\delta \to 0^+$, uniformly on compacts subsets of Q_1 and weakly in the $W^{1,p}$-norm. Here we have denoted by $\partial_p Q_1 = \Omega_1 \times \{t = t_1\} \cup \partial\Omega_1 \times (t_1,t_2)$ the *parabolic boundary of Q_1. The functions A_δ satisfy* (2.6), (2.2), (2.3), *and* (2.4) *with constants depending only on the original λ', Λ'.*

Proof. Let u be a weak solution of (1.1) in Q. In view of the results in [1] we know that the solution is Hölder continuous in compact subsets. For $\delta > 0$, let $A_{i,\delta}$ be as in the statement of Proposition 2.2, and consider the unique weak solution u_δ of (2.7). In view of the comparison principle, the uniform continuity and Caccioppoli inequalities for $\{u_\delta\}$ proved in [1] and of (2.6), (2.2) one can easily see that $u_\delta \to u$ uniformly on compact subsets of Q_1 and weakly in the $W^{1,p}$-norm. In order to conclude the proof, we only need to observe that u_δ are smooth (with the regularity possibly depending on $\delta > 0$ of course) thanks to Theorem 2.2 and Theorem 2.1. In fact, one can apply the results from [3], to derive regularity estimates that are uniform in the parameter ε, thus yielding that the family $u_{\delta,\varepsilon}$ has a subsequence converging to a solution of the problem (2.7) as $\varepsilon \to 0$, which coincides with u_δ in view the comparison principle. $\square$

We are left with the task of proving Proposition 2.2. To better illustrate the argument of the proof we present the special, simpler, case of the p-Laplacian (1.2), i.e. for $\xi = \sum_{i=1}^{2n} \xi_i X_i \in \mathbb{R}^{2n}$, and $x \in \mathbb{H}^n$,

$$A_i(x,\xi) = |\xi|^{p-2}\xi_i, \quad i = 1, \dots, 2n.$$

In this case, we consider for each $\delta > 0$, the following functions

$$A_{i,\delta}(x,\xi) = (\delta + |\xi|^2)^{\frac{p-2}{2}}\xi_i, \quad i = 1, \dots, 2n, \tag{2.8}$$

and for each $\varepsilon > 0$, and $\xi = \sum_{i=1}^{2n+1} \xi_i X_i^\varepsilon \in \mathbb{R}^{2n+1}$,

$$A_{i,\delta}^\varepsilon(x,\xi) = (\delta + \|\xi\|_{g_\varepsilon(x)}^2)^{\frac{p-2}{2}}\xi_i, \quad i = 1, \dots, 2n+1. \tag{2.9}$$

While the quantity $\|\cdot\|_{g_\varepsilon(x)}$ a priori depends on $x \in \mathbb{H}^n$, we remark that when ξ is a left invariant vector field, since g_ε is left invariant as well, the dependence of $\|\xi\|_{g_\varepsilon(x)}$ on the point x vanishes.

Proof of Proposition 2.2. Following the intuition from the example above, we construct the approximates through a two steps process. For $0 < \delta < 1$, let us define

$$A_\delta(x,\xi) = A(x,\xi) + \lambda\delta^{\frac{p-2}{2}}\xi. \tag{2.10}$$

It is clear that

$$A_\delta(x,\xi) \underset{\delta\to 0^+}{\longrightarrow} A(x,\xi)$$

and furthermore, for some $\lambda, \Lambda > 0$ depending only on λ', Λ', one has the estimate (2.2).

For each $\xi = \sum_{i=1}^{2n+1} \xi_i X_i^\varepsilon \in \mathbb{R}^{2n+1}$, and $\varepsilon, \delta > 0$ we set

$$A_{i,\delta,\varepsilon}(x,\xi) = \tilde{A}_i(x,\xi_H) + \lambda(\delta + |\xi|_\varepsilon^2)^{\frac{p-2}{2}}\xi_i, \tag{2.11}$$

for $i = 1, \dots, 2N+1$. Here we have denoted $\xi_H = (\xi_1, \dots, \xi_{2n})$, $\tilde{A} = (A, 0) \in \mathbb{R}^{2n+1}$, and $|\xi|_\varepsilon^2 = \sum_{i=1}^{2n+1} \xi_i^2$.

Clearly for a.e. $x \in \Omega$, and for all $\xi^\varepsilon = \sum_{i=1}^{2n} \xi_i X_i^\varepsilon + \varepsilon\xi_{2n+1}X_{2n+1}^\varepsilon$ one has uniformly on compact subsets of $\Omega \times (0,T)$,

$$(A_{1,\delta,\varepsilon}(x,\xi^\varepsilon), \dots, A_{2n+1,\delta,\varepsilon}(x,\xi^\varepsilon)) \underset{\varepsilon\to 0^+}{\longrightarrow} (A_{1,\delta}(x,\xi_1,\dots,\xi_{2n}), \dots, A_{2n,\delta}(x,\xi_1,\dots,\xi_{2n}), 0),$$

where A_δ is defined as in (2.10). In addition one can see that there exist constants $\lambda, \Lambda > 0$ depending on λ', Λ' such that

$$\begin{cases} \lambda(\delta + |\xi|_\varepsilon^2)^{\frac{p-2}{2}} |\eta|_\varepsilon^2 \le \partial_{\xi_j} A_{i,\delta,\varepsilon}(x,\xi)\eta_i\eta_j \le \Lambda(\delta + |\xi|_\varepsilon^2)^{\frac{p-2}{2}} |\eta|_\varepsilon^2, \\ |A_i(x,\xi)| + |\partial_{x_j} A_{i,\delta,\varepsilon}(x,\xi)| \le \Lambda(\delta + |\xi|_\varepsilon^2)^{\frac{p-1}{2}}, \end{cases}$$

for all $\eta = \sum_{i=1}^{2n+1} \eta_i X_i^\varepsilon \in \mathbb{R}^{2n+1}$. □

Remark 2.4. For the rest of the paper we will always consider solutions u_δ of the Dirichlet problem (2.5) with $\delta > 0$, in a cylinder $D \times (\tau_1, \tau_2) \subset\subset Q_1$, with $D \subset\subset \Omega_1$ and $[\tau_1, \tau_2] \subset (t_1, t_2)$. For the sake of notation we will drop the subscript δ from u_δ and $A_{i,\delta}$, and with a slight abuse of notation write $Q = \Omega \times (0, T)$ instead of $D \times (\tau_1, \tau_2)$. To further simplify the formulation of the estimates, we will assume that $A_i(x,\xi) = A_i(\xi)$, as in this case we can obtain sharper constants, and so we highlight these more involved aspects of the proofs. The more general case is handled in a similar fashion, and does not lead to explicit constants on the right hand side of the estimates. We note explicitly that all constants are independent of the parameter $\delta > 0$.

3. Preliminary energy estimates

We recall the basic Caccioppoli inequalities proved in [3]. These inequalities apply to a smooth solution u of the approximating equation (2.5) with $\delta > 0$, in a cylinder $Q \subset\subset Q_1$. In what follows we will implicitly assume that all constants on the right hand side of the inequalities depend on n, p, on the structure constants, λ, Λ *but not on* δ.

Lemma 3.1. *Let u be a solution of* (2.5) *in Q, with $\delta > 0$. If we set $v_l = X_l u$, with $l = 1, 2, ..., 2n$, and $s_l = (-1)^{[l/n]}$ then the function v_l is a solution of*

$$\partial_t v_l = \sum_{i,j=1}^{2n} X_i\Big(A_{i,\xi_j}(\nabla_0 u) X_l X_j u\Big) + s_l Z(A_{l+s_l n}(\nabla_0 u)). \tag{3.1}$$

Lemma 3.2. *Let u be a solution of* (2.5) *in Q, with $\delta > 0$. The function Zu is then a solution of the equation*

$$\partial_t Zu = \sum_{i,j=1}^{2n} X_i(A_{i,\xi_j}(\nabla_0 u) X_j Zu).$$

First we recall a Caccioppoli estimate for derivatives of the solution along the center of the group.

Lemma 3.3. [3, Lemma 3.4] *Let u be a solution of (2.5) in Q with $\delta > 0$. For every $\beta \geq 0$ and non-negative $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q, one has*

$$\int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_0 u|^2)^{\frac{p-2}{2}} |Zu|^\beta |\nabla_0 Zu|^2 \eta^{4+\beta}$$
$$\leq C\int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_0 u|^2)^{\frac{p-2}{2}} |Zu|^{\beta+2} |\nabla_0 \eta|^2 \eta^{2+\beta} + C\int_{t_1}^{t_2}\int_\Omega |Zu|^{\beta+2}|\partial_t \eta|\eta^{3+\beta},$$

where $C = C(\lambda, \Lambda) > 0$.

Second, we recall a Caccioppoli estimate for the horizontal derivatives.

Lemma 3.4. [3, Lemma 3.5] *Let u be a weak solution of (2.5) in Q, with $\delta > 0$. For every $\beta \geq 0$ and non-negative $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q, we have*

$$\frac{1}{\beta+2}\sup_{t_1<t<t_2}\int_\Omega (\delta + |\nabla_0 u|^2)^{\frac{\beta+2}{2}}\eta^2 + \int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_0 u|^2)^{(p-2+\beta)/2} |\nabla_0^2 u|^2 \eta^2$$
$$\leq C\int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{2}} (|\nabla_0 \eta|^2 + |Z\eta|\eta) + \frac{C}{\beta+2}\int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_0 u|^2)^{\frac{\beta+2}{2}} |\partial_t \eta|\eta$$
$$+ C(\beta+1)^4 \int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_0 u|^2)^{\frac{p-2+\beta}{2}} |Zu|^2\eta^2,$$

where $C = C(n, p, \lambda, \Lambda) > 0$, independent of δ.

4. Main Caccioppoli inequality

The main result of this section is a Caccioppoli inequality, Proposition 4.4, for the horizontal derivatives of the weak solutions of (2.5), with $\delta > 0$. To do this, we first need to prove an estimate for the derivative along the center Zu in Lemma 4.1 and Lemma 4.2. All estimates are uniform in $\delta > 0$, and the constants are stable as $\delta \to 0$.

We begin by recalling a Poincaré-like interpolation inequality from [6]. In the proof, we will need the restriction $2 \leq p \leq 4$ and this is the only use we make of this hypothesis in the paper.

Lemma 4.1. *Assume that $2 \leq p \leq 4$ and let $u \in C^2(Q)$. There exists a constant $C > 0$ depending only on n, p such that for every $\beta \geq 0$ and non-negative $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q, we have*

(4.1)
$$\int_{t_1}^{t_2}\int_\Omega |Zu|^{p+\beta}\eta^{p+\beta} \leq C(p+\beta)\|\nabla_0 \eta\|_{L^\infty} \int\int_{spt(\eta)} (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{2}}$$
$$+ C(p+\beta)\int_{t_1}^{t_2}\int_\Omega (\delta + |\nabla_0 u|^2)^{(p-2)/2} |Zu|^\beta |\nabla_0 Zu|^2 \eta^{4+\beta}.$$

Proof. We denote

$$L = \int_{t_1}^{t_2}\int_\Omega |Zu|^{p+\beta}\eta^{p+\beta}, \quad R = \int\int_{spt(\eta)} (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{2}}. \tag{4.2}$$

We estimate L from above as follows. Fix $l = 1, 2, \ldots, n$. Note that

$$Zu = X_l X_{n+l} u - X_{n+l} X_l u.$$

We can write

$$|Zu|^{p+\beta} = |Zu|^{p-2+\beta} Zu(X_l X_{n+l} u - X_{n+l} X_l u).$$

Then integration by parts gives us

$$\begin{aligned} L &= \int_{t_1}^{t_2} \int_\Omega |Zu|^{p-2+\beta} Zu(X_l X_{n+l} u - X_{n+l} X_l u) \eta^{p+\beta} \\ &= -(p-1+\beta) \int_{t_1}^{t_2} \int_\Omega |Zu|^{p-2+\beta} (X_l Zu X_{n+l} u - X_{n+l} Zu X_l u) \eta^{p+\beta} \\ &\quad - (p+\beta) \int_{t_1}^{t_2} \int_\Omega |Zu|^{p-2+\beta} Zu(X_{n+l} u X_l \eta - X_l u X_{n+l} \eta) \eta^{p-1+\beta} \\ &\le 2(p+\beta) \int_{t_1}^{t_2} \int_\Omega |\nabla_0 u| |Zu|^{p-2+\beta} |\nabla_0 Zu| \eta^{p+\beta} \\ &\quad + 2(p+\beta) \int_{t_1}^{t_2} \int_\Omega |\nabla_0 u| |Zu|^{p-1+\beta} |\nabla_0 \eta| \eta^{p-1+\beta} = I_1 + I_2. \end{aligned} \tag{4.3}$$

We will estimate the integrals I_1, I_2 on the right hand side of (4.3) by Hölder's inequality. First for I_2, we have

$$I_2 \le 2(p+\beta) \|\nabla_0 \eta\|_{L^\infty} R^{\frac{1}{p+\beta}} L^{\frac{p-1+\beta}{p+\beta}}, \tag{4.4}$$

where L and R are as in (4.2).

Second, for I_1, we have

$$I_1 \le 2(p+\beta) M^{\frac{1}{2}} R^{\frac{4-p}{2(p+\beta)}} L^{\frac{2p-4+\beta}{2(p+\beta)}}, \tag{4.5}$$

where

$$M = \int_{t_1}^{t_2} \int_\Omega |\nabla_0 u|^{p-2} |Zu|^\beta |\nabla_0 Zu|^2 \eta^{4+\beta}.$$

This yields

$$L \le C(p+\beta) \|\nabla_0 \eta\|_{L^\infty} R^{\frac{1}{p+\beta}} L^{\frac{p-1+\beta}{p+\beta}} + C(p+\beta) M^{\frac{1}{2}} R^{\frac{4-p}{2(p+\beta)}} L^{\frac{2p-4+\beta}{2(p+\beta)}}, \tag{4.6}$$

from which the conclusion follows immediately through Young's inequality. □

The previous Poincaré-like inequality can be applied to solutions of (2.5) and through invoking Lemma 3.3 leads us to the following key estimate.

Lemma 4.2. *Let u be a solution of* (2.5) *in Q, with $\delta > 0$ and $2 \le p \le 4$. Then for every $\beta \ge 0$ and non-negative $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q, we have*

$$\begin{aligned} &\left(\int_{t_1}^{t_2} \int_\Omega |Zu|^{p+\beta} \eta^{p+\beta} \right)^{\frac{1}{p+\beta}} \\ &\le C(p+\beta) \|\nabla_0 \eta\|_{L^\infty} \left(\int \int_{\mathrm{spt}(\eta)} (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{2}} \right)^{\frac{1}{p+\beta}} \\ &\quad + C(p+\beta) \|\eta \partial_t \eta\|_{L^\infty}^{\frac{1}{2}} |\operatorname{spt}(\eta)|^{\frac{p-2}{2(p+\beta)}} \left(\int \int_{\mathrm{spt}(\eta)} (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{2}} \right)^{\frac{4-p}{2(p+\beta)}}. \end{aligned} \tag{4.7}$$

Remark 4.3. Suspending temporarily the notation established in Remark 2.4, we denote by u_δ the solutions of the approximating equation (2.5). In particular, Lemma 4.2 establishes the local L^q integrability of Zu_δ, the derivative along the center of the approximating solutions, with uniform L^q bounds as $\delta \to 0$, for all

$1 \le q < \infty$. This implies that one can find a subsequence, Zu_{δ_k} converging to a L^q_{loc} function, which in view of the definition of weak derivative, is also a derivative along the center of the uniform limit of the u_δ. Since such limit is the original solution of (1.1), this proves the local integrability of Zu in every L^q class as stated in Theorem 1.1.

Proof. We apply the inequality (4.6) in the previous lemma to the solution u and invoke Lemma 3.3 to estimate the integral

$$M = \int_{t_1}^{t_2} \int_\Omega |\nabla_0 u|^{p-2} |Zu|^\beta |\nabla_0 Zu|^2 \eta^{4+\beta},$$

obtaining

$$\begin{aligned} M &\le C \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_0 u|^2)^{\frac{p-2}{2}} |Zu|^{\beta+2} |\nabla_0 \eta|^2 \eta^{2+\beta} + C \int_{t_1}^{t_2} \int_\Omega |Zu|^{\beta+2} |\partial_t \eta| \eta^{3+\beta} \\ &\le C \|\nabla_0 \eta\|^2_{L^\infty} R^{\frac{p-2}{p+\beta}} L^{\frac{\beta+2}{p+\beta}} + C \|\eta \partial_t \eta\|_{L^\infty} |\operatorname{spt}(\eta)|^{\frac{p-2}{p+\beta}} L^{\frac{\beta+2}{p+\beta}}, \end{aligned} \tag{4.8}$$

where $C = C(\lambda, \Lambda) > 0$, and L is as in (4.3). In the second inequality of (4.8), we used Hölder's inequality. Combining (4.5), (4.6) and (4.8), we obtain the estimate for I_1,

$$\begin{aligned} I_1 &\le C(p+\beta) \|\nabla_0 \eta\|_{L^\infty} R^{\frac{1}{p+\beta}} L^{\frac{p-1+\beta}{p+\beta}} \\ &\quad + C(p+\beta) \|\eta \partial_t \eta\|^{\frac12}_{L^\infty} |\operatorname{spt}(\eta)|^{\frac{p-2}{2(p+\beta)}} R^{\frac{4-p}{2(p+\beta)}} L^{\frac{p-1+\beta}{2(p+\beta)}}. \end{aligned} \tag{4.9}$$

Next, we substitute the latter in the estimate (4.5) for I_1 and the estimate (4.4) for I_2 to (4.3), and conclude

$$L \le C(p+\beta) \|\nabla_0 \eta\|_{L^\infty} R^{\frac{1}{p+\beta}} L^{\frac{p-1+\beta}{p+\beta}} + C(p+\beta) \|\eta \partial_t \eta\|^{\frac12}_{L^\infty} |\operatorname{spt}(\eta)|^{\frac{p-2}{2(p+\beta)}} R^{\frac{4-p}{2(p+\beta)}} L^{\frac{p-1+\beta}{2(p+\beta)}},$$

which yields immediately (4.7). ☐

The following result follows from Lemma 4.2, and the energy estimate in Lemma 3.4. It yields a Caccioppoli inequality for the horizontal derivatives of weak solutions, which extends to the subRiemannian setting the analogue Euclidean estimate proved in [8, Proposition 3.2 (3.7), page 225].

Proposition 4.4. *Let u be a weak solution of (2.5) in Q, with $\delta > 0$ and $2 \le p \le 4$. Then for every $\beta \ge 0$ and non-negative $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q, we have*

$$\begin{aligned} &\sup_{t_1 < t < t_2} \int_\Omega (\delta + |\nabla_0 u|^2)^{\frac{\beta+2}{2}} \eta^2 + \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_0 u|^2)^{(p-2+\beta)/2} |\nabla_0^2 u|^2 \eta^2 \\ &\le C(p+\beta)^7 \big(\|\nabla_0 \eta\|^2_{L^\infty} + \|\eta Z \eta\|_{L^\infty} \big) \int \int_{\operatorname{spt}(\eta)} (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{2}} \\ &\quad + C(p+\beta)^7 \|\eta \partial_t \eta\|_{L^\infty} |\operatorname{spt}(\eta)|^{\frac{p-2}{p+\beta}} \left(\int \int_{\operatorname{spt}(\eta)} (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{2}} \right)^{\frac{\beta+2}{p+\beta}}, \end{aligned} \tag{4.10}$$

where $C = C(n, p, \lambda, \Lambda) > 0$.

Remark 4.5. Although the statement addresses the approximate solution u_δ, in view of arguments analogue to those in Remark 4.3, the same estimate holds for weak solutions of (1.1).

Proof. Lemma 3.4 gives us the following estimate for the left hand side of (4.10)

$$
\begin{aligned}
&\sup_{t_1<t<t_2}\int_\Omega(\delta+|\nabla_0u|^2)^{\frac{\beta+2}{2}}\eta^2+\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_0u|^2)^{(p-2+\beta)/2}|\nabla_0^2u|^2\eta^2\\
&\le C(p+\beta)\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_0u|^2)^{\frac{p+\beta}{2}}(|\nabla_0\eta|^2+|Z\eta|\eta)\\
&\quad+C\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_0u|^2)^{\frac{\beta+2}{2}}|\partial_t\eta|\eta\\
&\quad+C(p+\beta)^5\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_0u|^2)^{\frac{p-2+\beta}{2}}|Zu|^2\eta^2.
\end{aligned}
\tag{4.11}
$$

To obtain the desired estimate (4.11), we will show that each integral on the right hand side of (4.11) can be bounded from above by the right hand side of (4.10). For the first integral on the right hand of (4.11), it is obviously bounded from above by the first item on the right hand side of (4.10). For the second integral, Hölder's inequality gives us

$$
\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_0u|^2)^{\frac{\beta+2}{2}}|\partial_t\eta|\eta\le\|\eta\partial_t\eta\|_{L^\infty}|\operatorname{spt}(\eta)|^{\frac{p-2}{p+\beta}}\left(\int\int_{\operatorname{spt}(\eta)}(\delta+|\nabla_0u|^2)^{\frac{p+\beta}{2}}\right)^{\frac{\beta+2}{p+\beta}},
$$

which shows that it is bounded from above by the second item on the right hand side of (4.10).

For the third integral on the right hand side of (4.11), we use Hölder's inequality and our main lemma, Lemma 4.2, and we have

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_0u|^2)^{\frac{p-2+\beta}{2}}|Zu|^2\eta^2\\
&\le\left(\int\int_{\operatorname{spt}(\eta)}(\delta+|\nabla_0u|^2)^{\frac{p+\beta}{2}}\right)^{\frac{p-2+\beta}{p+\beta}}\left(\int_{t_1}^{t_2}\int_\Omega|Zu|^{p+\beta}\eta^{p+\beta}\right)^{\frac{2}{p+\beta}}\\
&\le C(p+\beta)^2\|\nabla_0\eta\|_{L^\infty}^2\int\int_{\operatorname{spt}(\eta)}(\delta+|\nabla_0u|^2)^{\frac{p+\beta}{2}}\\
&\quad+C(p+\beta)^2\|\eta\partial_t\eta\|_{L^\infty}|\operatorname{spt}(\eta)|^{\frac{p-2}{p+\beta}}\left(\int\int_{\operatorname{spt}(\eta)}(\delta+|\nabla_0u|^2)^{\frac{p+\beta}{2}}\right)^{\frac{\beta+2}{p+\beta}},
\end{aligned}
$$

which concludes the proof of the lemma. □

5. Boundedness of the horizontal gradient

In this section we conclude the proof of Theorem 1.1, i.e. we establish that weak solutions of the approximating equation (2.5) with $\delta>0$ are Lipschitz continuous with respect to the subRiemannian distance, uniformly in the parameter δ. The proof follows immediately from Proposition 4.4 and from the Moser type iteration in Theorem 5.2 below. The proof of Theorem 5.2 should be known, but we can not find the precise reference in the literature. It is similar to the proof of Theorem 4 in [5] for the case $1<p<2$. The proof is included for the reader's convenience.

First, we recall a few definitions needed in the proof. We will denote by $d_0(x,y) = \|y^{-1}x\|$ the subRiemannian distance, where

$$\|x\|^4 = \left(\sum_{i=1}^{2n} x_i^2\right)^2 + 16x_{2n+1}^2,$$

is the Koranyi gauge. The corresponding parabolic metric is $d_0((x,t),(y,s)) = d_0(x,y) + |t-s|^2$.

Definition 5.1. A parabolic cylinder $Q_r(x_0,t_0) \subset Q$ is a set of the form $Q_r(x_0,t_0) = B(x_0,r) \times (t_0 - r^2, t_0)$, where $r > 0$, $B(x_0,r) = \{y \mid \|yx_0^{-1}\| < r\} \subset \Omega$ denotes the gauge ball of center x_0. The parabolic boundary of the cylinder $Q_r(x_0,t_0) \subset Q$ is the set $B(x_0,r) \times \{t_0 - r^2\} \cup \partial B(x_0,r) \times [t_0 - r^2, t_0)$. For r, $\mu > 0$ we also define the cylinders

$$Q_{\mu,r} := B(x,r) \times [t_0 - \mu r, t_0].$$

Theorem 5.2. *Let $u \in C^\infty(Q)$, with $Q = \Omega \times (0,T)$. If u satisfies the Caccioppoli type inequality* (4.10)*, then for every $p \geq 2$, and for any $Q_{\mu,2r} \subset Q$, we have*

$$\sup_{Q_{\mu,r}} |\nabla_0 u| \leq C\mu^{\frac{1}{2}} \max\left(\left(\frac{1}{\mu r^{N+2}} \int\int_{Q_{\mu,2r}} (\delta + |\nabla_0 u|^2)^{\frac{p}{2}}\right)^{\frac{1}{2}}, \mu^{\frac{p}{2(2-p)}}\right), \tag{5.1}$$

where $C = C(n,p,\lambda,\Lambda) > 0$.

Remark 5.3. Suspending temporarily the notation established in Remark 2.4, we denote by u_δ the solutions of the approximating equation (2.5). As mentioned earlier, there is a subsequence $u_\delta \to u$ converging uniformly in compact subsets of Q to the weak solution u of (1.1). In view of the uniform bound on the Lipschitz constant of u_δ in (5.1), then the Lipschitz regularity of u follows immediately.

Proof. Let $\eta \in C^1([0,T], C_0^\infty(\Omega))$ be a non-negative cut-off function vanishing on the parabolic boundary of Q such that $|\eta| \leq 1$ in Q. For $\beta \geq 0$, we set

$$v = (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{4}} \eta^2.$$

Then the Caccioppoli inequality (4.10) gives us

$$\begin{aligned}
&\sup_{t_1<t<t_2} \int_\Omega v^m + \int_{t_1}^{t_2}\int_\Omega |\nabla_0 v|^2 \\
&\leq C(p+\beta)^7 \big(\|\nabla_0\eta\|_{L^\infty}^2 + \|\eta Z\eta\|_{L^\infty}\big) \int\int_{\mathrm{spt}(\eta)} v^2 \\
&\quad + C(p+\beta)^7 \|\eta\partial_t\eta\|_{L^\infty} |\operatorname{spt}(\eta)|^{\frac{p-2}{p+\beta}} \left(\int\int_{\mathrm{spt}(\eta)} v^2\right)^{\frac{\beta+2}{p+\beta}},
\end{aligned} \tag{5.2}$$

where $C = C(n,p,\lambda,\Lambda) > 0$. Here $m = 2(\beta+2)/(p+\beta)$. Note that $4/p < m \leq 2$. Now let $q = 2(m+N)/N$, where $N = 2n+2$. We have

$$\int_{t_1}^{t_2}\int_\Omega v^q \leq \int_{t_1}^{t_2} \left(\int_\Omega v^m\right)^{\frac{2}{N}} \left(\int_\Omega v^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} \leq C\left(\sup_{t_1<t<t_2}\int_\Omega v^m\right)^{\frac{2}{N}} \left(\int_{t_1}^{t_2}\int_\Omega |\nabla_0 v|^2\right),$$

where $C = C(n) > 0$. Here in the second inequality, we used the Sobolev inequality in the space variables where the sharp exponent $2N/(N-2)$ involves the Hausdorff dimension of $\mathbb{H}^n$. Now we plug the estimate (5.2) into the above inequality and we obtain that

$$\left(\int_{t_1}^{t_2}\int_\Omega v^q\right)^{\frac{N}{N+2}} \le C(p+\beta)^7\big(\|\nabla_0\eta\|_{L^\infty}^2+\|\eta Z\eta\|_{L^\infty}\big)\int\int_{spt(\eta)} v^2 + C(p+\beta)^7\|\eta\partial_t\eta\|_{L^\infty}|\operatorname{spt}(\eta)|^{\frac{p-2}{p+\beta}}\left(\int\int_{spt(\eta)} v^2\right)^{\frac{\beta+2}{p+\beta}}, \tag{5.3}$$

where $C = C(n,p,\lambda,\Lambda) > 0$. Here $q = 2+4(\beta+2)/(N(p+\beta))$. This is the inequality on which our iteration is based.

Let $Q_{\mu,2r}\subset Q$. We define, for $i = 0, 1, 2, \ldots$, a sequence of radius $r_i = (1+2^{-i})r$ and a sequence of exponents β_i such that $\beta_0 = 0$ and

$$p+\beta_{i+1} = (p+\beta_i)\left(1+\frac{2(\beta_i+2)}{N(p+\beta_i)}\right),$$

that is,

$$\beta_i = 2(\kappa^i-1),\quad \kappa = \frac{N+2}{N}.$$

We denote $Q_i = Q_{\mu,r_i}$. Note that $Q_0 = Q_{\mu,2r}$ and $Q_\infty = Q_{\mu,r}$. Then we choose a standard parabolic cut-off function $\eta_i \in C^\infty(Q_i)$ such that $\eta_i = 1$ in Q_{i+1} with

$$|\nabla_0\eta_i| \le \frac{2^{i+8}}{r},\quad |Z\eta_i| \le \frac{2^{2i+8}}{r^2},\quad |\partial_t\eta_i| \le \frac{2^{2i+8}}{\mu r^2}\quad \text{in } Q_i.$$

Now we let $\eta = \eta_i$ and $\beta = \beta_i$ in (5.3) and we obtain that for $i = 0, 1, \ldots$

$$\begin{aligned}&\left(\int\int_{Q_{i+1}}(\delta+|\nabla_0u|^2)^{\frac{\alpha_{i+1}}{2}}\right)^{\frac{N}{N+2}}\\ &\le C2^{2i}\alpha_i^7r^{-2}\left[\left(\int\int_{Q_i}(\delta+|\nabla_0u|^2)^{\frac{\alpha_i}{2}}\right)^{\frac{p-2}{\alpha_i}}+\mu^{-1}\left(\mu r^{N+2}\right)^{\frac{p-2}{\alpha_i}}\right]\\ &\quad\times\left(\int\int_{Q_i}(\delta+|\nabla_0u|^2)^{\frac{\alpha_i}{2}}\right)^{\frac{\alpha_i-p+2}{\alpha_i}},\end{aligned} \tag{5.4}$$

where $C = C(n,p,\lambda,\Lambda) > 0$ and $\alpha_i = p+\beta_i = p-2+2\kappa^i$. We denote

$$M_i = \left(\frac{1}{\mu r^{N+2}}\int\int_{Q_i}(\delta+|\nabla_0u|^2)^{\frac{\alpha_i}{2}}\right)^{\frac{1}{\alpha_i}}.$$

Then we can write (5.4) as

$$M_{i+1}^{\frac{\alpha_{i+1}}{\kappa}} \le C\mu^{\frac{2}{N+2}}2^{2i}\alpha_i^7\big(M_i^{p-2}+\mu^{-1}\big)M_i^{\alpha_i-p+2}.$$

We set

$$\overline{M}_i = \max\big(M_i, \mu^{\frac{1}{2-p}}\big).$$

Then it follows from the above inequality that

$$\overline{M}_{i+1}^{\frac{\alpha_{i+1}}{\kappa}} \le C\mu^{\frac{2}{N+2}}2^{2i}\alpha_i^7\overline{M}_i^{\alpha_i}, \tag{5.5}$$

since we may assume that $C = C(n,p,\lambda,\Lambda) \ge 1$. Iterating (5.5), we obtain that

$$\overline{M}_{i+1} \le \left(\prod_{j=0}^{i}K_j^{\frac{\kappa^{i+1-j}}{\alpha_{i+1}}}\right)\overline{M}_0^{\frac{\alpha_0\kappa^{i+1}}{\alpha_{i+1}}},$$

where

$$K_i = C\mu^{\frac{2}{N+2}} 2^{2i} \alpha_i^7.$$

Recall that $\alpha_i = p-2+2\kappa^i$ and $\kappa = (N+2)/N$. Let i go to infinity. We obtain that

$$\overline{M}_\infty = \limsup_{i\to\infty} \overline{M}_i \le C\mu^{\frac{1}{2}} \overline{M}_0^{\frac{p}{2}}, \tag{5.6}$$

where $C = (n, p, \lambda, \Lambda) > 0$. Note that

$$\overline{M}_\infty \ge \sup_{Q_{\mu,r}} |\nabla_0 u|, \quad \overline{M}_0 = \max\left(\left(\frac{1}{\mu r^{N+2}} \int\int_{Q_{\mu,2r}} (\delta + |\nabla_0 u|^2)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \mu^{\frac{1}{2-p}}\right).$$

Thus (5.6) gives us the desired inequality (5.1), completing the proof. □

6. Higher integrability of $\partial_t u$

In this section, we prove that the time derivative $\partial_t u$ of weak solutions of (2.5) in the range $2 \le p \le 4$ belongs to $L^q_{\mathrm{loc}}(\Omega \times (0,T))$ for every $q \ge 1$. As observed before, once we establish uniform estimates for $\partial_t u_\delta$, then arguing as in Remark 4.3, one can readily conclude the integrability of $\partial_t u$.

Lemma 6.1. *Let u be a solution of equation (2.5) in $Q = \Omega \times (0,T)$. Then we have*

$$\partial_t u \in L^q_{\mathrm{loc}}(\Omega \times (0,T))$$

for every $q \ge 1$. Moreover, for every $\beta \ge 0$, and non-negative $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary, we have

$$\begin{aligned} &\sup_{t_1<t<t_2} \int_{\Omega_2} |\partial_t u|^{\frac{\beta+2}{2}} \eta^2 + \int_{t_1}^{t_2} \int_\Omega |\partial_t u|^{\beta+2} \eta^{\beta+2} \\ &\quad \le C\big(M^{2p-2} \|\nabla_0 \eta\|^2_{L^\infty} + M^p \|\eta \partial_t \eta\|_{L^\infty}\big)^{\frac{\beta+2}{2}} |\operatorname{spt}(\eta)|, \end{aligned} \tag{6.1}$$

where $C = C(p, \lambda, \Lambda, \beta) > 0$ and $M = \sup_{\operatorname{spt}(\eta)} (\delta + |\nabla_0 u|^2)^{\frac{1}{2}}$.

Proof. Let $\beta \ge 0$. Since u is a solution of (2.5), we can write

$$|\partial_t u|^{\beta+2} = |\partial_t u|^\beta \partial_t u X_i(A_{i,\delta}(x, \nabla_0 u)).$$

We denote by L the integral on the left hand side of (6.1), which is the object we will estimate. Let $\eta \in C^1([0,T], C_0^\infty(\Omega))$ be a non-negative cut-off function, vanishing on the parabolic boundary. Since $\eta(\cdot, t) \in C_0^\infty(\Omega)$ for every $t \in [0,T]$, integration by parts gives us that

$$\begin{aligned} L &= \int_{t_1}^{t_2} \int_\Omega |\partial_t u|^{\beta+2} \eta^{\beta+2} = \int_{t_1}^{t_2} \int_\Omega |\partial_t u|^\beta \partial_t u X_i(A_{i,\delta}(x, \nabla_0 u)) \eta^{\beta+2} \\ &= -(\beta+2) \int_{t_1}^{t_2} \int_\Omega |\partial_t u|^\beta \partial_t u A_{i,\delta}(x, \nabla_0 u) \eta^{\beta+1} X_i \eta \\ &\quad - (\beta+1) \int_{t_1}^{t_2} \int_\Omega |\partial_t u|^\beta X_i(\partial_t u) A_{i,\delta}(x, \nabla_0 u) \eta^{\beta+2} = I_1 + I_2. \end{aligned} \tag{6.2}$$

We will estimate the integrals I_1, I_2 in the right hand side of the above equality as follows. First, we use the structure condition and Hölder's inequality to estimate I_1.

We have

$$
\begin{aligned}
|I_1| &\le (\beta+2)\Lambda \int_{t_1}^{t_2}\int_\Omega (\delta+|\nabla_0 u|^2)^{\frac{p-1}{2}}|\partial_t u|^{\beta+1}\eta^{\beta+1}|\nabla_0\eta| \\
&\le (\beta+2)\Lambda\left(\int_{t_1}^{t_2}\int_\Omega |\partial_t u|^{\beta+2}\eta^{\beta+2}\right)^{\frac{\beta+1}{\beta+2}} \\
&\quad\times\left(\int_{t_1}^{t_2}\int_\Omega (\delta+|\nabla_0 u|^2)^{\frac{p-1}{2}(\beta+2)}|\nabla_0\eta|^{\beta+2}\right)^{\frac{1}{\beta+2}} \\
&= (\beta+2)\Lambda\|\nabla_0\eta\|_{L^\infty}|\operatorname{spt}(\eta)|^{\frac{1}{\beta+2}}M^{p-1}L^{\frac{\beta+1}{\beta+2}},
\end{aligned}
\tag{6.3}
$$

where $M=\sup_{\operatorname{spt}(\eta)}(\delta+|\nabla_0 u|^2)^{\frac12}$.

Second, we also use the structure condition and Hölder's inequality to estimate I_2. We have

$$
\begin{aligned}
|I_2| &\le (\beta+1)\Lambda\int_{t_1}^{t_2}\int_\Omega (\delta+|\nabla_0 u|^2)^{\frac{p-1}{2}}|\partial_t u|^{\beta}|\nabla_0\partial_t u|\eta^{\beta+2} \\
&\le (\beta+1)\Lambda\left(\int_{t_1}^{t_2}\int_\Omega |\partial_t u|^{\beta+2}\eta^{\beta+2}\right)^{\frac{\beta}{2(\beta+2)}}\left(\iint_{\operatorname{spt}(\eta)}(\delta+|\nabla_0 u|^2)^{\frac{p}{4}(\beta+2)}\right)^{\frac{1}{\beta+2}}J^{\frac12} \\
&\le (\beta+1)\Lambda|\operatorname{spt}(\eta)|^{\frac{1}{\beta+2}}M^{\frac{p}{2}}L^{\frac{\beta}{2(\beta+2)}}J^{\frac12},
\end{aligned}
\tag{6.4}
$$

where

$$
J=\int_{t_1}^{t_2}\int_\Omega (\delta+|\nabla_0 u|^2)^{\frac{p-2}{2}}|\partial_t u|^{\beta}|\nabla_0\partial_t u|^2\eta^{\beta+4}.
$$

To estimate the integral J, we differentiate equation (2.5) with respect to t and we obtain that

$$
\partial_t(\partial_t u)=X_i(\partial_{\xi_j}A_{i,\delta}(x,\nabla_0 u)X_j(\partial_t u)).
\tag{6.5}
$$

Then we use $\varphi=|\partial_t u|^\beta\partial_t u\eta^{\beta+4}$ as a test function to the above equation and we obtain the following Caccioppoli inequality by the structure condition and the Cauchy–Schwarz inequality

$$
\begin{aligned}
&\sup_{t_1<t<t_2}\int_{\Omega_2}|\partial_t u|^{\frac{\beta+2}{2}}\eta^2+J\le \\
&C\int_{t_1}^{t_2}\int_\Omega(\delta+|\nabla_0 u|^2)^{\frac{p-2}{2}}|\partial_t u|^{\beta+2}\eta^{\beta+2}|\nabla_0\eta|^2+C\int_{t_1}^{t_2}\int_\Omega|\partial_t u|^{\beta+2}\eta^{\beta+3}|\partial_t\eta| \\
\le& C(M^{p-2}\|\nabla_0\eta\|_{L^\infty}^2+\|\eta\partial_t\eta\|_{L^\infty})L,
\end{aligned}
\tag{6.6}
$$

where $C=C(p,\lambda,\Lambda,\beta)>0$. Here we used the fact that η vanishes on the parabolic boundary. Combining (6.4) and (6.6), we obtain the following estimate for I_2.

$$
|I_2|\le C|\operatorname{spt}(\eta)|^{\frac{1}{\beta+2}}M^{\frac{p}{2}}L^{\frac{\beta+1}{\beta+2}}(M^{p-2}\|\nabla_0\eta\|_{L^\infty}^2+\|\eta\partial_t\eta\|_{L^\infty})^{\frac12}.
\tag{6.7}
$$

Now we combine (6.2) with the estimates (6.3) and (6.7) and we end up with

$$
\begin{aligned}
L &\le CM^{p-1}\|\nabla_0\eta\|_{L^\infty}|\operatorname{spt}(\eta)|^{\frac{1}{\beta+2}}L^{\frac{\beta+1}{\beta+2}} \\
&\quad + C|\operatorname{spt}(\eta)|^{\frac{1}{\beta+2}}M^{\frac{p}{2}}L^{\frac{\beta+1}{\beta+2}}(M^{p-2}\|\nabla_0\eta\|_{L^\infty}^2+\|\eta\partial_t\eta\|_{L^\infty})^{\frac12},
\end{aligned}
$$

from which (6.1) follows. This completes the proof. □

7. Concluding remarks and some open problems

There are a number of immediate extensions which we want to highlight, as well as some more involved, plausible extensions which we listen as open problems.

First of all, the prototype for the class of operators in (1.1) is the regularized p-Laplacian operator

$$L_p u = \mathrm{div}_{g_0,\mu_0}((\delta + |\nabla_0 u|^2_{g_0})^{\frac{p-2}{2}} \nabla_0 u)$$

in a subRiemannian contact manifold (M, ω, g_0), where M is the underlying differentiable manifold, ω is the contact form and g_0 is a Riemannian metric on the contact distribution. The measure μ_0 is the corresponding Popp measure. Since the structure conditions (1.3) and equation (1.1) are invariant by contact diffeomorphisms, then invoking Darboux coordinates one can pull-back the PDE from the setting of contact subRiemannian manifolds to that of the Heisenberg group. Consequently all our results extends to the more general contact subRiemannian setting. For a more detailed description, see [4, Section 6.1]. As an immediate corollary of Theorem 1.1 one has the following.

Theorem 7.1. *Let (M, ω, g_0) be a contact, subRiemannian manifold and let $\Omega \subset M$ be an open set. For $2 \le p \le 4$, $\delta \ge 0$, consider $u \in L^p((0,T), W^{1,p}_0(\Omega))$ be a weak solution of*

$$\partial_t u = \mathrm{div}_{g_0,\mu_0}((\delta + |\nabla_0 u|^2_{g_0})^{\frac{p-2}{2}} \nabla_0 u),$$

in $Q = \Omega \times (0,T)$. For any open ball $B \subset\subset \Omega$ and $T > t_2 \ge t_1 \ge 0$, and $q \ge 1$, there exist constants $C = C(n, p, d(B, \partial\Omega), T - t_2, \delta) > 0$ and $C_q = C(n, p, q, d(B, \partial\Omega), T - t_2, \delta) > 0$ such that

$$||\nabla_0 u||_{L^\infty(B\times(t_1,t_2))} \le C \text{ and } ||\partial_t u||_{L^q(B\times(t_1,t_2))} + ||Zu||_{L^q(B\times(t_1,t_2))} \le C_q.$$

Of course, if $\delta > 0$ then in view of the results in [3], the solutions are smooth in Q.

Some of the following extensions seem challenging, and we list them as open problems in increasing order of their perceived difficulty.

(1) Standard, but technically involved, modifications should allow to extend our work to the case of equations of the type

$$\partial_t u - X_i A_i(x, t, u, \nabla_0 u) = B(x, t, u, \nabla_0 u)$$

with structure conditions similar to those in [8, Section 1, Chapter VIII].

(2) We feel it should be possible to weaken the bounds in the structure conditions for $\partial_{x_k} A_i$ and request instead only horizontal derivatives bounds, bounds on $X_k A_i$, although this would require some additional work in the proof of Lemma 4.1.

(3) This paper only deals with scalar equations, however in the Euclidean case the results continue to hold also for systems of equations with additional structure (see [8]). The extension in the subelliptic setting would involve first extending the results of [3], and all the regularity theory literature that is used there.

(4) Because our argument rests in a crucial way on Lemma 4.1, the Lipschitz regularity for the range $4 < p < \infty$ is currently beyond our reach. We conjecture that our main Caccioppoli inequality (4.10) still holds with exactly the same statement in this extended range.

(5) Proof of the Hölder regularity of horizontal derivatives, in any range of $p \neq 2$. Even in the range $2 \leq p \leq 4$ the methods of Zhong [19] and the Euclidean proofs in [8] break down and new ideas are needed.
(6) Just as in the Euclidean case, the regularity problem in the range $1 < p < 2$ is more challenging, and would require completely different arguments. In the stationary case this has been solved by Mukherjee and Zhong in [16].
(7) Our work extends easily to any step two Carnot group. Beyond this setting, in the stationary case, there is promising work by Domokos and Manfredi [11] dealing with regularity in higher step groups, while the papers [4] and [7] show extensions beyond the group setting, but within the step two hypothesis. The problem is completely open in the non-stationary case.

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Received 2 September 2022 • Revision received 21 April 2023 • Accepted 1 May 2023
Published online 22 June 2023

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