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EXISTENCE AND DENSITY OF GENERAL COMPONENTS OF THE NOETHER-LEFSCHETZ LOCUS ON NORMAL THREEFOLDS

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Abstract. We consider the Noether-Lefschetz problem for surfaces in $\mathbb{Q}$-factorial normal 3-folds with rational singularities. We show the existence of components of the Noether-Lefschetz locus of maximal codimension, and that there are indeed infinitely many of them. Moreover, we show that their union is dense in the natural topology.

1. Introduction

Let Y be a smooth complex variety and let D be a smooth ample divisor. Among several classical results in this setting, stand for importance the Noether-Lefschetz type results, namely that the natural restriction map $i_D : \mathrm{Pic}(Y) \to \mathrm{Pic}(D)$ is an isomorphism if $\dim Y \geq 4$, and, in many cases, if $\dim Y = 3$ and D is very general in its linear system.

In the latter case, the locus of smooth surfaces D such that i_D is not surjective, is called the Noether-Lefschetz locus of $|D|$. This gives rise to countably many subvarieties of $|D|$, called components of the Noether-Lefschetz locus. The study of the geometry of such components is nowadays itself a classical subject (see, to mention a few, [9, 20, 21, 42, 43, 10, 26, 11, 35, 34, 27]) and is basically divided in two parts: the study of low or high, in fact maximal, codimension components.

In the present paper we consider components of maximal codimension, the main goal being to study their existence, the fact that there are infinitely many such components and that they are dense in the natural topology. Moreover, we work on an ambient threefold with mild singularities. To our knowledge this is a novelty, if we exclude [13] and the toric case [6, 7], from which this work drew inspiration.

Let X be a complex normal irreducible threefold with rational singularities (we shall always consider varieties over the complex numbers), and let L be a very ample line bundle on X. Given a normal surface $S \in |L|$ it follows, by Mumford's vanishing [31, Thm. 2], that

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$H^1(S, -mL_{|S}) = 0$ for every $m \geq 1$, whence, the restriction map

$$i_S : \mathrm{Pic}(X) \to \mathrm{Pic}(S)$$

is injective by [22, Exposé XII, Cor. 3.6].

Recall that for a normal variety Y we define $\rho(Y)$ to be the rank of $\mathrm{Pic}(Y) \otimes \mathbb{Q}$. We can therefore define (in analogy with the smooth case):

Definition 1.1. *Let X be a normal irreducible threefold with rational singularities, and let L be a very ample line bundle on X. Let $U(L)$ be the open subset of $|L|$ parametrizing irreducible normal surfaces with rational singularities.*

The **Noether-Lefschetz locus** *of (X, L) is*

$$\mathrm{NL}(L) = \{S \in U(L) : \rho(S) > \rho(X)\}.$$

If, for a very general $S \in |L|$, we have that $\rho(S) = \rho(X)$, then $\mathrm{NL}(L)$ is a countable union of proper subvarieties of $U(L)$, which we call **components of the Noether-Lefschetz locus***.*

As in the case of $\mathbb{P}^3$, assuming that $\omega_X(L)$ is globally generated and $h^2(\mathcal{O}_X) = h^3(\mathcal{O}_X)$, it is not difficult to see (Proposition 3.2) that the components of the Noether-Lefschetz locus $\mathrm{NL}(L)$ exist and have a maximal possible codimension $h^0(\omega_X(L))$ in $U(L)$.

Our first result is that, in many cases, we can get the same results as for $\mathbb{P}^3$, namely that components of maximal codimension exist:

Theorem 1.
Let X be a normal, $\mathbb{Q}$-factorial, irreducible threefold with rational singularities, and let H be a very ample line bundle on X. Suppose that

(i) $H^i(\mathcal{O}_X) = 0$ *for* $i > 0$*;*
(ii) $H^1(H) = 0$*;*
(iii) $H^0(\omega_X(H)) = 0$.

Let $d \geq 2$ be an integer such that

(iv) $\omega_X(dH)$ *is globally generated.*

Then there is a component $W(dH)$ of the Noether-Lefschetz locus $\mathrm{NL}(dH)$ such that

$$\mathrm{codim}_{U(dH)} W(dH) = h^0(\omega_X(dH)).$$

Moreover, this gives density in the natural topology:

Corollary 1.
Let X be a normal, $\mathbb{Q}$-factorial, irreducible threefold with rational singularities, let H be a very ample line bundle on X and let $d \geq 2$ be an integer such that (i)-(iv) of Theorem 1

are satisfied. Then the Noether-Lefschetz locus $\mathrm{NL}(dH)$ *is dense, in the natural topology, in* $U(dH)$*.*

In the special case of toric threefolds, we obtain:

Theorem 2.
Let $\mathbb{P}_\Sigma$ *be a projective simplicial Gorenstein toric threefold and let* H *be a very ample line bundle on* X *such that* $-K_{\mathbb{P}_\Sigma} - 2H$ *is nef. Then, for every* $d \geq 0$*, there is a component* $W(d)$ *of the Noether-Lefschetz locus* $\mathrm{NL}(-K_{\mathbb{P}_\Sigma} + dH)$ *such that*

$$\operatorname{codim} W(d) = h^0(dH).$$

Note that the hypotheses in the above theorem imply that $\mathbb{P}_\Sigma$ is a Fano threefold. Moreover, combining with [7]:

Corollary 2.
Let $\mathbb{P}_\Sigma$ *be a projective simplicial Gorenstein toric threefold and let* H *be a very ample line bundle on* X *such that* $-K_{\mathbb{P}_\Sigma} - 2H$ *is nef. Then, for every integer* $d \geq 0$*, the Noether-Lefschetz locus* $\mathrm{NL}(-K_{\mathbb{P}_\Sigma} + dH)$ *is dense, in the natural topology, in* $U(-K_{\mathbb{P}_\Sigma} + dH)$*.*

If $-K_{\mathbb{P}_\Sigma} \neq 2H$ *and* $d \geq 3$ *then* $d \leq \operatorname{codim} \mathrm{NL}(-K_{\mathbb{P}_\Sigma} + dH) \leq h^0(dH)$*.*

It can be easily verified that several families of varieties satisfy the hypotheses of the above Theorems and Corollaries. We present some examples in Section 2; we also discuss the relation with Castelnuovo-Mumford regularity.

As this paper was being completed, we received a preprint from O. Benoist [4] that contains an application of density results for Noether-Lefschetz loci in the context of studying properties of real polynomials which are a sum of squares, related to “Hilbert’s 17th problem”. Even though both papers obtain density results by using determinantal curves, there are substantial differences in both the results and the methods. Benoist’s paper, as well as [28] and [5] use the density results for Noether-Lefschetz loci in smooth ambient varieties. The current paper opens the way to study such problems in a more general context.

We would like to thank the referee for insightful comments.

2. Examples

Let X be a projective variety and let H be a very ample line bundle. Recall the definition of Castelnuovo-Mumford regularity:

Definition 2.1. H *is* m*-regular if* $H^q(X, (m + 1 - q)H) = 0$ *for all* $q > 0$*.*

Proposition 2.2. *Let* X *be a threefold with klt singularities and let* H *be a very ample line bundle. Then* H *is* 0*-regular if and only if* $h^1(\mathcal{O}_X) = 0$ *and* $H^0(\omega_X(2H)) = 0$*.*

Proof. Since klt singularities are Cohen-Macaulay, by Serre's duality we have $H^3(-2H) = H^0(\omega_X(2H))$ and $H^2(-H) = H^1(\omega_X(H))$; however $H^1(\omega_X(H)) = 0$ by Kawamata-Viehweg's vanishing theorem [17]. □

Proposition 2.3. *Let X be a normal irreducible threefold with rational singularities and let H be a very ample line bundle. The hypotheses (i)-(iii) of Theorem 1 are satisfied if and only if H is 1-regular and $h^1(\mathcal{O}_X) = 0$.*

Proof. The condition $q = 1$ for 1-regularity is (ii) of Theorem 1, $q = 2$ is the first part of (i) and $q = 3$ becomes (iii) with Serre's duality. □

Proposition 2.4. *Let X be a normal irreducible threefold and let H be a very ample line bundle. If H is 0-regular then the hypotheses (i)-(iii) of Theorem 1 are satisfied.*

Proof. $H^1(\mathcal{O}_X) = 0$ is the 0-regularity condition for $q = 1$, and we conclude by Proposition 2.3, since H is also 1-regular as it is 0-regular. □

Note that many varieties with mild singularities are Cohen-Macaulay, such as the ones with klt singularities or normal toric varieties [25].

Example 2.5. The weighted projective spaces

(2.1.1) The infinite series $\mathbb{P}[1,1,1,q]$, $\mathbb{P}[1,2,2q-1,2q-1]$ and $\mathbb{P}[2,2,2q-1,2q-1]$, $q \in \mathbb{N}$
(2.1.2) $\mathbb{P}[1,1,2,3]$, $\mathbb{P}[3,3,4,4]$, $\mathbb{P}[3,3,5,5]$, $\mathbb{P}[1,2,2,3]$

satisfy the hypotheses of Theorem 1. In fact, let $\mathbb{P}_\Sigma = \mathbb{P}[q_0, q_1, q_2, q_3]$ be a weighted projective 3-space with reduced weights $\{q_i\}$ [15] and let η_0 be the effective generator of the class group of $\mathbb{P}_\Sigma$. Then $\eta = \delta\eta_0$ is the very ample generator of the Picard group $\mathrm{Pic}(\mathbb{P}_\Sigma)$, and $\sigma\eta_0 = -K_{\mathbb{P}_\Sigma}$ is the anti-canonical class, where $\delta = \mathrm{l.c.m.}(q_i)$, and $\sigma = \sum_i q_i$. The 3-fold $\mathbb{P}_\Sigma$ is normal, $\mathbb{Q}$-factorial, irreducible, it has rational singularities, and satisfies conditions (i), (ii) in Theorem 1 and also condition (iv) for d big enough. If we take $H = \eta$, condition (iii) is equivalent to $\delta < \sigma$ and this is satisfied precisely in the cases (2.1.1) and (2.1.2).

Note that $\mathbb{P}[1,1,1,2]$ and $\mathbb{P}[1,1,2,2]$ also satisfy the hypotheses of Corollary 2. △

Remark 2.6. Cox in [13] studies the locus $\mathrm{NL}(L)$ where $L = 6q\eta_0$ in $X = \mathbb{P}[1,1,2q,3q]$, with $q \geq 3$. The results of [13] and this paper are somewhat complementary. In fact, the starting point of our analysis, as well as [12], is that L should be of high enough degree in X to assure that $\rho(S) = \rho(X)$ for a very general $S \in |L|$ (condition (iv) of Theorem 1). When $X = \mathbb{P}[1,1,2q,3q]$ and $L = 6q\eta_0$ this condition can only be satisfied when $q = 2$ since $\omega_X(L) = (q-2)\eta_0$ is not globally generated except when $q = 2$. Cox considers instead the case when $q \geq 3$ and proves directly in Proposition 3.2 that the general surface S has $\rho(S) = 1$. The minimal resolution of a surface $S \in |L|$ is a regular elliptic surface $\tilde{S}$ with $h^0(\tilde{S}, K_{\tilde{S}}) \geq 2$ and a section; the section is the exceptional divisor of the minimal resolution $\tilde{S} \to S$. Cox proves then that all but one component of $\mathrm{NL}(L)$ are of maximal codimension and that they are dense in the natural topology.

The case $q = 2$ corresponds to elliptic K3 surfaces with section, for which all the components of the Noether-Lefschetz locus have codimension 1 and are known to be dense. Our methods do not apply to this case: our construction depends in particular upon finding a suitable curve $C \subset X$ and a very ample line bundle H such that condition (v) in Lemma 4.1 is satisfied; but there is no such line bundle H when $q = 2$.

Note also that for $q = 1$, the system $|L| = |6\eta_0|$ corresponds to rational elliptic surfaces and $\rho(S) = 9 > \rho(X)$ for every S. In the Example 2.5 above we consider instead $X = \mathbb{P}[1,1,2,3]$ and $L = k\eta_0$ with $k > 6$.

△

Example 2.7. The quasi-Fano variety $\mathbb{P}_\Sigma$, which is the resolution of the cone over a quadric surface in $\mathbb{P}^3$, also satisfies the hypotheses of Theorem 1. In addition for any $d \geq 0$ the bounds $d \leq \operatorname{codim} \mathrm{NL}(-K_{\mathbb{P}_\Sigma} + dH) \leq h^0(dH)$ are also satisfied [7]. △

Example 2.8. Other examples are provided by Fano varieties. Indeed, using [40, Thm. 7.80 (c)], it is easily seen that the hypotheses of Theorem 1 and Corollary 1 are satisfied by a normal, $\mathbb{Q}$-factorial, irreducible Fano threefold with rational singularities X having a very ample line bundle H such that $H^0(K_X + H) = 0$ and $\omega_X(dH)$ is globally generated. In particular this happens when $-K_X = rH$ with $r \geq 2$ and $d \geq r$.

Smooth Fano threefolds of index 2 were classified by [45]. Fano threefolds with high index and singularities are studied in [18] and [37]. △

Example 2.9. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is such a Fano manifold of index 2. In addition, with the methods of Section 5 in [7] (Proposition 5.2 and Lemma 5.3) we find that the codimension of the smooth surfaces in $|-K_{\mathbb{P}_\Sigma} + dH|$ for $d \geq 0$ which contain any of the rulings of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is $d + 1$.

△

Example 2.10. $\mathbb{P}^1 \times \mathbb{P}^2$ satisfies the hypotheses of Corollary 2 with $H = p^*_{\mathbb{P}^1}(O_{\mathbb{P}^1}(1)) \oplus p^*_{\mathbb{P}^2}(O_{\mathbb{P}^2}(1))$. Moreover, the bounds $d \leq \operatorname{codim} \mathrm{NL}(-K_{\mathbb{P}_\Sigma} + dH) \leq h^0(dH)$ are satisfied, for $d \geq 0$ [7]. △

Example 2.11. A rational threefold with $\mathbb{Q}$-factorial klt singularities and $-K_X$ nef satisfies the hypotheses (i)-(iii) of Theorem 1 if $H^0(\omega_X(H)) = 0$, as Kawamata-Viehweg's vanishing theorem applies. △

Example 2.12. The projective 3-space blown-up along a line $\widehat{\mathbb{P}}^3$ is one such example. The nef cone is generated by η_1, the pullback of a plane in $\mathbb{P}^3$ and $\eta_2 = \eta_1 - E$, where E is the exceptional divisor. Any $H = s_1\eta_1 + s_2\eta_2$ with $s_1 = 1, 2,\ s_2 \geq 1$ is very ample, while $h^0(K_{\widehat{\mathbb{P}}^3} + H) = 0$. Also $-K_{\widehat{\mathbb{P}}^3} + H$ is very ample, H is 0-regular and the hypotheses of

Theorem 1 are satisfied for $s_1 = 1, d \geq 3$ and $s_1 = 2, d \geq 2$. Moreover, we have the bounds [7]

$$d \leq \operatorname{codim} \mathrm{NL}(-K_{\mathbb{P}_\Sigma} + dH) \leq h^0(dH).$$

Note, however, that $-K_{\mathbb{P}_\Sigma} - 2H$ is not nef and thus the hypotheses of Corollary 2 are not satisfied; in fact the cone of effective divisors includes the nef cone. △

3. Existence and maximal codimension of components

Unless otherwise specified, throughout this paper X will be a normal complex $\mathbb{Q}$-factorial irreducible threefold with rational singularities. We shall denote by ω_X its dualizing sheaf.

When X is smooth, there are well-known conditions that assure the existence of components of the Noether-Lefschetz locus, namely that $h^{2,0}_{ev}(S, \mathbb{C}) > 0$ for $S \in |L|$ general [30], [44, Thm. 15.33]. If X is a toric threefold, the same is assured by a suitable combinatorial condition [6].

Remark 3.1.

(i) Since X has rational singularities, it is Cohen-Macaulay, and $p_*\omega_{\overline{X}} \simeq \omega_X$, where $p : \overline{X} \to X$ is any desingularization [25, Thm. 5.10].

(ii) For every projective normal variety X with rational singularities, the group $H^2(X, \mathbb{Z})$ has a pure Hodge structure induced by that of a desingularization [2, Lemma 2.1], [41].

(iii) The general hyperplane section of a variety with rational singularities has rational singularities [16, Rmk. 3.4.11(3)], and a general hyperplane section of a normal variety is normal [39, Thm. 7'].

Proposition 3.2. *Let X be as above, and let L be a very ample line bundle on X. Assume that $\omega_X(L)$ is globally generated. Then:*

(i) $\mathrm{Cl}(X) \simeq \mathrm{Cl}(S)$*, for a very general* $S \in |L|$.

(ii) $\rho(S) = \rho(X)$ *for a very general* $S \in |L|$ *(thus one can define the Noether-Lefschetz locus* $\mathrm{NL}(L)$*).*

(iii) *For every component V of* $\mathrm{NL}(L)$*, and for every $S \in V$, we have*

$$\operatorname{codim}_{U(L)} V \leq h^{2,0}(S) = h^0(\omega_X(L)) + h^2(\mathcal{O}_X) - h^3(\mathcal{O}_X).$$

Proof. (i) Let $f : X \to \mathbb{P}^N$ be the embedding given by the line bundle L. It was shown in [38, Thm. 1] that $\mathrm{Cl}(X) \simeq \mathrm{Cl}(S)$ for a very general surface S in $|L|$ whenever the line bundle $(f_*\omega_X)(1)$ is globally generated. To show that this condition holds, we write the exact sequence

$$H^0(X, \omega_X(L)) \otimes \mathcal{O}_X \to \omega_X(L) \to 0.$$

We apply the functor f_* obtaining a surjective morphism $H^0(X, \omega_X(L)) \otimes f_*\mathcal{O}_X \to (f_*\omega_X)(1)$, and, by composing with the evaluation morphism $\mathcal{O}_{\mathbb{P}^N} \to f_*\mathcal{O}_X$, we obtain a surjective morphism $H^0(X, \omega_X(L)) \otimes \mathcal{O}_{\mathbb{P}^N} \to (f_*\omega_X)(1)$. Hence $(f_*\omega_X)(1)$ is globally generated.

(ii) Since S is normal, we have two injections $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(S)$ (as in the Introduction), and $\mathrm{Pic}(S) \hookrightarrow \mathrm{Cl}(S)$, whence, using the $\mathbb{Q}$-factoriality of X, we get

$$\begin{aligned} \rho(X) &\leq \rho(S) = \mathrm{rk}(\mathrm{Pic}(S) \otimes \mathbb{Q}) \leq \mathrm{rk}(\mathrm{Cl}(S) \otimes \mathbb{Q}) \\ &= \mathrm{rk}(\mathrm{Cl}(X) \otimes \mathbb{Q}) = \mathrm{rk}(\mathrm{Pic}(X) \otimes \mathbb{Q}) = \rho(X). \end{aligned}$$

(iii) Now let V be a component of $\mathrm{NL}(L)$ and let $S \in V$, so that $\rho(S) > \rho(X)$. In the smooth case, as is well known [9, pages 71-72], this gives $h^{2,0}(S)$ conditions. By Remark 3.1 (ii), one can reason as in [7, Prop. 4.6] and obtain, using [40, Thm. 7.80 (c)],

$$\mathrm{codim}_{U(L)} V \leq h^{2,0}(S) = h^0(\omega_X(L)) + h^2(\mathcal{O}_X) - h^3(\mathcal{O}_X).$$

□

4. Components of maximal codimension from curves

In the case of $\mathbb{P}^3$, components of maximal codimension have been constructed in two ways: by a degeneration argument in [10], and by choosing suitable components of the Hilbert scheme in [11]. We consider here the second approach.

We first show that we can construct components of maximal codimension as soon as we have some curve in X with good properties.

Lemma 4.1. *Let X be as above, and let L be a very ample line bundle on X. Let W be a component of the Hilbert scheme of curves on X such that there is a smooth irreducible curve C representing a point of W, and with $C \cap \mathrm{Sing}(X) = \emptyset$. Moreover, suppose that:*

(i) $H^i(\mathcal{O}_X) = 0$ *for* $i > 0$*;*
(ii) $H^1(N_{C/X}) = 0$*;*
(iii) $H^1(\mathcal{J}_{C/X}(L)) = 0$*;*
(iv) $H^0(\mathcal{J}_{C/X} \otimes \omega_X(L)) = H^1(\mathcal{J}_{C/X} \otimes \omega_X(L)) = 0$*;*
(v) *there is a very ample line bundle H on X such that $\mathcal{J}_{C/X}(L-H)$ is globally generated;*
(vi) $\omega_X(L)$ *is globally generated.*

Then W defines a component $W(L)$ of maximum codimension of $\mathrm{NL}(L)$*, that is,*

$$\mathrm{codim}_{U(L)} W(L) = h^0(\omega_X(L)).$$

Proof. By (vi) we can apply Proposition 3.2, that is, the components of the Noether-Lefschetz locus $\mathrm{NL}(L)$ exist. Note that $\mathcal{J}_{C/X}(L)$ is globally generated by (v). Let $S \in |\mathcal{J}_{C/X}(L)|$ be very general. We claim that:

a) the conditions

(1) $$S \in U(L) \text{ and } \rho(S) = \rho(X) + 1$$

hold;

b) the same conditions of the Lemma and (a) hold for a curve C_η representing a generic point in W, and a very general surface S_η in the linear system $|\mathcal{J}_{C_\eta/X}(L)|$.

To prove this let $\pi : \widetilde{X} \to X$ be the blow-up of X along C with exceptional divisor E, and let $\widetilde{S}$ be the strict transform of S, so that $\widetilde{S} \simeq S$. Note that $\widetilde{L} := \pi^*L - E = \pi^*(L-H) - E + \pi^*H$ is very ample by (v) (and, for example, [36, 4.1] or [3, Proof of Thm.2.1]). Since $\widetilde{S}$ is general in $\widetilde{L}$ and $\widetilde{X}$ is also normal with rational singularities, it follows that $\widetilde{S}$ is irreducible, normal with rational singularities, whence so is S, and therefore $S \in U(L)$. Now $\omega_{\widetilde{X}}(\widetilde{L})$ is globally generated by (vi), and moreover, $\mathrm{Cl}(\widetilde{X}) \simeq \mathbb{Z}E \oplus \mathrm{Cl}(X)$; thus, as in Proposition 3.2, we get

$$\rho(\widetilde{X}) = \mathrm{rk}(\mathrm{Cl}(\widetilde{X}) \otimes \mathbb{Q}) = \mathrm{rk}(\mathrm{Cl}(X) \otimes \mathbb{Q}) + 1 = \rho(X) + 1.$$

Moreover, as $\widetilde{S}$ is normal, we have $\mathrm{Pic}(\widetilde{X}) \hookrightarrow \mathrm{Pic}(\widetilde{S})$ (as in the Introduction), and $\mathrm{Pic}(\widetilde{S}) \hookrightarrow \mathrm{Cl}(\widetilde{S})$, whence

$$\rho(X) + 1 = \rho(\widetilde{X}) \le \rho(\widetilde{S}) \le \mathrm{rk}(\mathrm{Cl}(\widetilde{S}) \otimes \mathbb{Q}) = \mathrm{rk}(\mathrm{Cl}(\widetilde{X}) \otimes \mathbb{Q}) = \rho(X) + 1$$

and (1)(a) is proved.

Let g be the genus of C. From the exact sequence

$$0 \to \mathcal{J}_{C/X}(L) \to L \to L_{|C} \to 0$$

using (iii) we get

$$h^0(L) - h^0(\mathcal{J}_{C/X}(L)) = h^0(L_{|C}) \ge L \cdot C - g + 1. \tag{2}$$

Now consider C_η. This curve is smooth and irreducible, $C_\eta \cap \mathrm{Sing}(X) = \emptyset$, and by semicontinuity the conditions (ii)-(iv) hold for C_η. The exact sequence

$$0 \to \mathcal{J}_{C_\eta/X}(L) \to L \to L_{|C_\eta} \to 0$$

gives, by semicontinuity

$$h^0(\mathcal{J}_{C/X}(L)) \ge h^0(\mathcal{J}_{C_\eta/X}(L)) = h^0(L) - h^0(L_{|C_\eta}) \ge h^0(L) - h^0(L_{|C}) = h^0(\mathcal{J}_{C/X}(L)).$$

whence we get equality.

Now let $S_\eta \in |\mathcal{J}_{C_\eta/X}(L)|$ be very general; then (1)(a) holds for S_η. For ease of notation, in the sequel of the proof we will replace C_η with C and S_η with S. From (ii) we get

$$\dim W = h^0(N_{C/X}) = \chi(N_{C/X}) = \deg N_{C/X} + 2 - 2g = \deg T_{X|C} = -\deg \omega_{X|C}. \tag{3}$$

Consider the incidence correspondence

$$\mathfrak{J} = \{(S', C') : C' \subset S'\} \subset U(L) \times W$$

together with its projections

$$\begin{array}{ccc} & \mathfrak{J} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ U(L) & & W \end{array}$$

and let $W(L) = \operatorname{Im} \pi_1$. Now (1) implies that π_2 is dominant, hence, using (3) we find

$$\begin{aligned} \dim W(L) &= \dim \mathfrak{J} - (h^0(\mathcal{O}_S(C)) - 1) = \dim W + (h^0(\mathcal{J}_{C/X}(L)) - 1) - (h^0(\mathcal{O}_S(C)) - 1) \\ &= -\deg \omega_{X|C} + h^0(\mathcal{J}_{C/X}(L)) - h^0(\mathcal{O}_S(C)) \end{aligned}$$

whence

$$\operatorname{codim}_{U(L)} W(L) = h^0(L) - 1 - h^0(\mathcal{J}_{C/X}(L)) + \deg \omega_{X|C} + h^0(\mathcal{O}_S(C)). \tag{4}$$

Since $H^2(\mathcal{O}_X(-L)) = 0$ by [40, Thm. 7.80 (c)], the exact sequence

$$0 \to \mathcal{O}_X(-L) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

and (i) give that $H^1(\mathcal{O}_S) = 0$ and then the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$$

gives

$$h^0(\mathcal{O}_S(C)) - 1 = h^0(\mathcal{O}_C(C)) = h^1(\omega_{S|C}) = h^1(\omega_X(L)_{|C}) \tag{5}$$

(here we use the adjunction formula for S in X, see e.g. [24, Eq. 4.2.9]).

Moreover, note that, by the hypothesis $C \cap \operatorname{Sing}(X) = \emptyset$, the following sequence

$$0 \to \mathcal{J}_{C/X} \otimes \omega_X(L) \to \omega_X(L) \to \omega_X(L)_{|C} \to 0$$

is exact, so that, using (iv), we get

$$h^0(\omega_X(L)) = h^0(\omega_X(L)_{|C}). \tag{6}$$

Putting together (4), (2), (5) and (6) we have

$$\begin{aligned} \operatorname{codim}_{U(L)} W(L) &\geq L \cdot C - g + 1 + \deg \omega_{X|C} + h^1(\omega_X(L)_{|C}) = \\ &= \deg \omega_X(L)_{|C} - g + 1 + h^1(\omega_X(L)_{|C}) = h^0(\omega_X(L)_{|C}) = h^0(\omega_X(L)). \end{aligned}$$

It remains to prove that $W(L)$ is a component of $\operatorname{NL}(L)$. This, together with Proposition 3.2, will give that $\operatorname{codim}_{U(L)} W(L) = h^0(\omega_X(L))$.

Let V be a component of $\operatorname{NL}(L)$ containing $W(L)$ and let S' be a surface representing its general point, so that (1) gives $\rho(S') = \rho(X) + 1$. Then we can assume that there is a line bundle L' on S' that specializes to $\mathcal{O}_S(C)$ when S' specializes, in V, to S. It will therefore suffice to prove that $h^0(L') = h^0(\mathcal{O}_S(C))$ (so that L' is effective and therefore corresponds to a deformation of C). By semicontinuity we have $h^0(L') \leq h^0(\mathcal{O}_S(C))$ and $h^2(L') \leq h^2(\mathcal{O}_S(C))$, and then

$$h^1(L') \leq h^1(\mathcal{O}_S(C)) = h^1(\omega_S(-C)) = h^1(\mathcal{J}_{C/S} \otimes \omega_X(L)) \tag{7}$$

where the last equality follows by the adjunction formula. Now we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{J}_{S/X} \otimes \omega_X(L) \to \mathcal{J}_{C/X} \otimes \omega_X(L) \to \mathcal{J}_{C/S} \otimes \omega_X(L) \to 0$$

where $\mathcal{F}$ is a sheaf with support of dimension at most 1. Since $\mathcal{J}_{S/X} \otimes \omega_X(L) \simeq \omega_X$, we get $H^2(\mathcal{J}_{S/X} \otimes \omega_X(L)) = H^2(\omega_X) = H^1(\mathcal{O}_X) = 0$ by (i). Then (iv) gives $h^1(\mathcal{J}_{C/S} \otimes \omega_X(L)) = 0$, so that $h^1(L') = 0$ by (7). Therefore

$$\begin{aligned} h^0(L') = \chi(L') + h^1(L') - h^2(L') &= \chi(\mathcal{O}_S(C)) - h^2(L') \\ &\geq \chi(\mathcal{O}_S(C)) - h^2(\mathcal{O}_S(C)) = h^0(\mathcal{O}_S(C)) \end{aligned}$$

and we are done. □

Now we shall see how the conditions in Lemma 4.1 can be met. To get condition (ii) of Lemma 4.1 we will adapt a result of Kleppe [23].

Lemma 4.2. *Let X be a Cohen-Macaulay projective threefold such that $H^i(\mathcal{O}_X) = 0$ for $0 < i < 3$. Let Γ be a Cohen-Macaulay equidimensional subscheme of X of dimension 1 such that X is smooth along Γ. Then*

$$H^1(N_{\Gamma/X}) \simeq \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{J}_{\Gamma/X}, \mathcal{J}_{\Gamma/X}).$$

Proof. We apply [23, Remark 2.2.6]. Setting, in Kleppe's notation, $\mathbb{P} = X$ and $X = \Gamma$, we need to satisfy the conditions in [23, Thm. 2.2.1], with the exception of the requirement that Γ is generically complete intersection. Hence it suffices to verify that there is an embedding $X \subset \mathbb{P}^N$ such that the cone is Cohen-Macaulay. Since $H^i(\mathcal{O}_X) = 0$ for $0 < i < 3$, this can be obtained via a sufficiently ample embedding, in the following, probably well-known, way. Let H be very ample on X. Then there exists $m_1 \in \mathbb{N}$ such that $H^i(mH) = 0$ for $i > 0$ and $m \geq m_1$. By Serre duality there exists $m_2 \in \mathbb{N}$ such that $H^i(-mH) = 0$ for $i < 3$ and $m \geq m_2$. Moreover, let $m_3 \in \mathbb{N}$ be such that $S^k H^0(mH) \twoheadrightarrow H^0(kmH)$ for every $k \in \mathbb{N}$ and for every $m \geq m_3$. Then, setting $m_0 = \max\{m_1, m_2, m_3\}$, and embedding $X \subset \mathbb{P}^N = \mathbb{P}H^0(m_0 H)$ we have that $H^i(\mathcal{O}_X(j)) = 0$ for every $j \in \mathbb{Z}$ and for all i such that $0 < i < 3$. Now we can apply Corollary 3.11 in [24]. □

Next, to construct curves having the properties of Lemma 4.1, we use degeneracy loci of morphisms of vector bundles.

Proposition 4.3. *Let X be a normal projective irreducible threefold, let H be a very ample line bundle on X and let $\mathcal{E} = \mathcal{O}_X(-dH)^{\oplus(d-1)}, \mathcal{F} = \mathcal{O}_X((1-d)H)^{\oplus d}$ for $d \geq 2$. Let $\phi : \mathcal{E} \to \mathcal{F}$*

be a general morphism and let $C = D_{\mathrm{rk}(\mathcal{E})-1}(\phi)$ *be its degeneracy locus. Then* C *is a smooth irreducible curve such that* $C \cap \mathrm{Sing}(X) = \emptyset$.

Proof. By [33, Thm. 2.8] or [8, Thm. 1] and [19, Thm. II] we see that C is a smooth irreducible curve. We need to prove that C does not pass though Γ, the singular locus of X. Note that $\dim(\Gamma) \leq 1$. Recall that a general morphism $\phi : \mathcal{E} \to \mathcal{F}$ is represented by a $(d, d-1)$ matrix M_d with general entries $\Phi_{i,j} \in H^0(X, H)$.

For $i = 1, \dots, d$ let F_i^d be hypersurface on X defined by the minor D_i^d of M_d obtained by removing the i-th row. We will prove, by induction on d, that for a general M_d

$$F_{d-1}^d \cap F_d^d \cap \Gamma = \emptyset. \tag{8}$$

Equation (8) proves that $C \cap \mathrm{Sing}(X) = \emptyset$ since $C \subseteq F_{d-1}^d \cap F_d^d$.

If $d = 2$, $D_1^2 = \Phi_{2,1}, D_2^2 = \Phi_{1,1}$ whence (8) holds since H is very ample and $\Phi_{1,1}, \Phi_{2,1}$ are general.

Next suppose $d \geq 3$ and that (8) holds for M_{d-1}. Then it clearly also holds for the $(d-2, d-1)$ transpose matrix M_{d-1}^T, that is

$$F_{d-2}^{d-1} \cap F_{d-1}^{d-1} \cap \Gamma = \emptyset \tag{9}$$

where F_i^{d-1} is the hypersurface defined by the minor D_i^{d-1} of M_{d-1}^T obtained by removing the i-th column.

Let M_d be the $(d, d-1)$ matrix obtained by adding to M_{d-1}^T two bottom rows with general entries $\Phi_{d-1,j}$ and $\Phi_{d,j}$ in $H^0(X, H)$.

The $(d-1, d-1)$ minors D_{d-1}^d and D_d^d of M_d can be computed as:

$$D_{d-1}^d = \sum_{i=1}^{d-1} (-1)^{i+d-1} D_i^{d-1} \Phi_{d,i} \quad \text{and} \quad D_d^d = \sum_{i=1}^{d-1} (-1)^{i+d-1} D_i^{d-1} \Phi_{d-1,i}. \tag{10}$$

Note that for every $x \in \Gamma$ it follows by (9) that

$$((-1)^{1+d-1} D_1^{d-1}(x), \dots, (-1)^{d-1+d-1} D_{d-1}^{d-1}(x)) \neq (0, \dots, 0),$$

whence setting $a_i = (-1)^{i+d-1} D_i^{d-1}(x)$, the linear system

$$V(x) := \{s \in H^0(X, H) : \exists\ s_1, \dots, s_{d-1} \in H^0(X, H) \text{ with } s = a_1 s_1 + \dots + a_{d-1} s_{d-1}\}$$

is the whole $H^0(X, H)$, whence base-point free. Therefore, choosing general $\Phi_{d,i}$'s and using (10), we see that the hypersurface F_{d-1}^d does not contain Γ and will therefore intersect Γ at finitely many points $\{x_1, \dots, x_s\}$.

Again by (9) the linear systems $V(x_k)$ are base-point free for every $1 \leq k \leq s$, whence choosing general $\Phi_{d-1,i}$'s and using (10), we see that $D_d^d(x_k) \neq 0$ for all k, that is $x_k \notin F_d^d$.

This proves (8). Note that a linear algebra argument shows also that

$$F_i^d \cap F_j^d \cap \Gamma = \emptyset, \quad \forall i, j.$$

□

Corollary 4.4. *Let X be a normal Cohen-Macaulay projective irreducible threefold, and let L be a very ample line bundle on X. Let $\mathcal{E}, \mathcal{F}$ be two locally free sheaves on X such that* $\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{E}) + 1, \det \mathcal{E} \simeq \det \mathcal{F}$ *and $\mathcal{E}^* \otimes \mathcal{F}$ is ample and globally generated. Let $\phi : \mathcal{E} \to \mathcal{F}$ be a general morphism and let $C = D_{\mathrm{rk}(\mathcal{E})-1}(\phi)$ be its degeneracy locus. Suppose that*

(a) $H^i(\mathcal{O}_X) = 0$ *for* $i > 0$
(b) $H^1(\mathcal{F}(L)) = 0$
(c) $H^2(\mathcal{E}(L)) = 0$
(d) $H^0(\mathcal{F} \otimes \omega_X(L)) = H^1(\mathcal{F} \otimes \omega_X(L)) = 0$
(e) $H^1(\mathcal{E} \otimes \omega_X(L)) = H^2(\mathcal{E} \otimes \omega_X(L)) = 0$
(f) $H^2(\mathcal{F} \otimes \mathcal{F}^*) = H^3(\mathcal{E} \otimes \mathcal{F}^*) = 0$
(g) $H^1(\mathcal{F} \otimes \mathcal{E}^*) = H^2(\mathcal{E} \otimes \mathcal{E}^*) = 0$
(h) *there is a very ample line bundle H on X such that $\mathcal{F}(L - H)$ is globally generated.*

Then conditions (i)-(v) of Lemma 4.1 are satisfied.

Proof. First note that (i) of Lemma 4.1 is (a).

Moreover, [1, Ch. VI, §4, page 257] implies that the ideal sheaf of C has a resolution

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{J}_{C/X} \to 0 \tag{11}$$

so that (v) of Lemma 4.1 follows by (h) of this Corollary. Then we get the exact sequences

$$0 \to \mathcal{E} \otimes \mathcal{F}^* \to \mathcal{F} \otimes \mathcal{F}^* \to \mathcal{J}_{C/X} \otimes \mathcal{F}^* \to 0$$

and

$$0 \to \mathcal{E} \otimes \mathcal{E}^* \to \mathcal{F} \otimes \mathcal{E}^* \to \mathcal{J}_{C/X} \otimes \mathcal{E}^* \to 0.$$

Using (f) and (g) we deduce that $H^2(\mathcal{J}_{C/X} \otimes \mathcal{F}^*) = H^1(\mathcal{J}_{C/X} \otimes \mathcal{E}^*) = 0$. Applying $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{J}_{C/X})$ to (11) we get the exact sequence

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{J}_{C/X}) \to \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{J}_{C/X}, \mathcal{J}_{C/X}) \to \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}_{C/X}). \tag{12}$$

Now $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{J}_{C/X}) \simeq H^1(\mathcal{J}_{C/X} \otimes \mathcal{E}^*) = 0$, and $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}_{C/X}) \simeq H^2(\mathcal{J}_{C/X} \otimes \mathcal{F}^*) = 0$. By (12) and Lemma 4.2 it follows that $H^1(N_{C/X}) = 0$, that is (ii) of Lemma 4.1.

From (11) we also have the exact sequence

$$0 \to \mathcal{E}(L) \to \mathcal{F}(L) \to \mathcal{J}_{C/X}(L) \to 0$$

and, using (b) and (c), we get (iii) of Lemma 4.1.

Finally (11) gives an exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \otimes \omega_X(L) \to \mathcal{F} \otimes \omega_X(L) \to \mathcal{J}_{C/X} \otimes \omega_X(L) \to 0$$

where $\mathcal{G}$ is a sheaf with support of dimension at most 1. Using (d) and (e), we get (iv) of Lemma 4.1. □

5. Proof of main results

Putting together our tools, Lemma 4.1, Proposition 4.3 and Corollary 4.4, we now proceed to the proofs.

5.1. **Proof of Theorem 1.**

Proof. Let $\mathcal{E} = \mathcal{O}_X(-dH)^{\oplus(d-1)}, \mathcal{F} = \mathcal{O}_X((1-d)H)^{\oplus d}$ and let $\phi : \mathcal{E} \to \mathcal{F}$ be a generic morphism. Note that $H^1(\omega_X(H)) = H^2(\mathcal{O}_X(-H)) = 0$ by [40, Thm. 7.80 (c)]. Setting $C = D_{d-2}(\phi)$, it follows by the hypotheses that all conditions (a)-(h) of Corollary 4.4 are satisfied. Moreover, by Proposition 4.3, C is smooth irreducible, $C \cap \operatorname{Sing}(X) = \emptyset$ and all conditions (i)-(vi) of Lemma 4.1 are satisfied. We then conclude by Lemma 4.1. □

5.2. **Proof of Corollary 1.**

Proof. We just note that, since we are working with irreducible normal surfaces with rational singularities, the proof of [10, §5] works verbatim on the open subset $U(L)$ of $|L|$. □

5.3. **Proof of Theorem 2.**

Proof. Note that $\mathbb{P}_\Sigma$ is normal and $\mathbb{Q}$-factorial, because it is toric and simplicial. Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}_\Sigma}(-(d+2)H)^{\oplus(d+1)}, \mathcal{F} = \mathcal{O}_{\mathbb{P}_\Sigma}(-(d+1))H)^{\oplus(d+2)}$ and let $\phi : \mathcal{E} \to \mathcal{F}$ be a generic morphism. We set $L = -K_{\mathbb{P}_\Sigma} + dH$ and check the conditions of Corollary 4.4.

Note that $-K_{\mathbb{P}_\Sigma} - 2H$ is globally generated by [29, Thm. 1.6], whence $L = -K_{\mathbb{P}_\Sigma} - 2H + (d+2)H$ is very ample. Now also $\mathcal{F}(L-H) \simeq \mathcal{O}_{\mathbb{P}_\Sigma}(-K_{\mathbb{P}_\Sigma} - 2H)^{\oplus(d+2)}$ is globally generated, and this gives (h). Using the nefness of $-K_{\mathbb{P}_\Sigma} - 2H$ we see that conditions (a)-(c), (g) and the first vanishing in (f), follow by Demazure's vanishing theorem [14, Thm. 9.2.3]. Also conditions (d) and (e) follow by toric Serre duality [14, Thm. 9.2.10] and by Bott-Danilov-Steeenbrink's vanishing theorem [32, Chapt. 3]. Let us see that also the second vanishing in (f) holds, namely that $H^3(\mathcal{O}_{\mathbb{P}_\Sigma}(-H)) = 0$. In fact if $H^3(\mathcal{O}_{\mathbb{P}_\Sigma}(-H)) \neq 0$, then, by toric Serre duality, $H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(K_{\mathbb{P}_\Sigma} + H)) \neq 0$ and therefore also $H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(2K_{\mathbb{P}_\Sigma} + 2H)) \neq 0$. But the latter is dual to $H^3(\mathcal{O}_{\mathbb{P}_\Sigma}(-K_{\mathbb{P}_\Sigma} - 2H)) = 0$ by Demazure's vanishing theorem, a contradiction.

Therefore all the conditions of Proposition 4.3 and Corollary 4.4 are satisfied and we deduce that conditions (i)-(v) of Lemma 4.1 are also satisfied. Since $K_{\mathbb{P}_\Sigma} + L = dH$ is globally generated we also have (vi) of Lemma 4.1. We then conclude by Lemma 4.1. □

5.4. **Proof of Corollary 2.**

Proof. The first part of the statement is proved as in Corollary 1. Note that H is 0-regular, see Section 2. Corollary 4.13 and Proposition 3.6 in [7] then imply the lower bound estimate on the codimension. The upper bound follows by Proposition 3.2. □

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