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This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

Published Version:
Cherubini, U., Mulinacci, S. (2021). Hierarchical Archimedean Dependence in Common Shock Models. METHODOLOGY AND COMPUTING IN APPLIED PROBABILITY, 23(1 (March)), 143-163 [10.1007/s11009-020-09816-8].

Availability:
This version is available at: https://hdl.handle.net/11585/774092 since: 2021-03-03

Published:

DOI: http://doi.org/10.1007/s11009-020-09816-8

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(Article begins on next page)

10 November 2026

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https://doi.org/10.1007/s11009-020-09816-8

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Hierarchical Archimedean Dependence in Common Shock Models

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Abstract

In this paper we show how to extend a simple common shock model with Archimedean dependence of the hidden variables to the non-exchangeable case. The assumption is that the hidden risk factors are linked by a hierarchical Archimedean dependence structure, possibly fully nested . We give directions about how to implement the model and to address the issue that the hidden variables must be put in descending dependence order. We show how the model can be simplified in the Gumbel-Marshall-Olkin distribution in Cherubini and Mulinacci (2017), the only case in which exponential distribution of the observed variables is preserved.

Keywords: Common shock models, Marshall-Olkin distribution, Hierarchical Archimedean copulas, Systemic risk

1 Introduction

Common shock models are extremely useful tools for many applications in reliability theory, insurance and risk management, but are tools that are quite difficult to exploit in full generality. The reason for this complexity is that the common feature of these models is that most of the assunptions must be made about the hidden risk factors that trigger the events observed in reality. Since these risk factors are in the background, the task of identifying their dependence structure from the dependence of the events observed may be very involved.

The seminal paper of this literature, due to Marshall and Olkin (1967), was based on the simplest possible assumptions, that the occurrance times of the hidden factors be independent and exponentially distributed. Even in this simple setting, the model may very quickly grow in complexity if one considers the extension to higher dimensions. It is immediate to see that addressing in full generality the common shocks linking all the possible subsets of a large number of hidden factors and observed events runs into a classical curse of dimensionality problem. But this is not the complexity that we are going to address in this paper. So, throughout the paper we will keep the model to the lowest possible dimensional complexity. Our model will consist of a system of d components, in which the lifetime of each of them can come to an end either for idiosyncratic or a common systemic shock.

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Different streams of literature have addressed other extensions of the classical Marshall-Olkin model. Some of these contributions have focussed on the shape of the marginal distributions, providing extensions beyond the exponential model: this is the case of the bivariate model of Li and Pellerey (2011) and the multivatiate extension in Lin and Li (2014). A large part of the extensions presented in the literature was instead devoted to the need to drop the independence assumption of the hidden shocks. The first contribution in this direction was due to Li (2009), in which the dependence induced by a mixing procedure is considered. More general frameworks in which both the assumption of independence as well as that of marginal distributions of exponential type are relaxed, are considered, among the others, in Durante et al. (2010), Bernhart et al. (2013), Mai et al. (2013) and Mulinacci (2015, 2018).

In many applications, allowing for dependence in the hidden factor is crucial because it enables to take into account contagion effects among idiosyncratic shocks, that is shocks that only trigger the end of one component in the system, and the others hidden shocks, included the common shock that would trigger the failure the system. So, in systemic risk analysis it is reasonable to assume that a trouble affecting a large bank could trigger the default of many other companies, financial and non financial, in the system. In reliability theory, it may happen that the failure of one element in a system may impact on the lifetime of the others, and of the system as a whole, for example because of a surcharge of labor.

The Archimedean dependence represents the natural extension of the Marshall Olkin model. The reason is that Archimedean dependence is obtained by conditional independence of exponential variables. The exponential distribution assumed by the original Marshall-Olkin model is then only extended by introducing dependence of these exponential variables on the same common factor. The distribution of this common factor linking the hidden shock times and its Laplace transform generate a specific model of the Archimedean family. One question is whether the result of the original Marshall-Olkin model, that is the exponential distribution of the observed times, may be preserved in the analysis with Archimedean dependence case. Cherubini and Mulinacci (2017) prove that the answer to this question is generally negative. The only exception is the case of a specific kind of Archimedean dependence, that is the Gumbel copula.

There remains an open issue that is the object of the current paper. The issue is that the Archimedean dependence assumed among the hidden shock times is exchangeable. All the hidden components are assumed to have the same dependence with each one of the others, both idiosyncratic and common. So, an increase of fragility of one component has the same effect on the whole system, no matter what the relevance and the role of the component. In systemic risk applications, this would imply for example that the default of any bank would have the same impact on the rest of the system, no matter what the dimension and degree of interconnectedness of the bank. This is far from consistent with reality, where banks of different dimensions clearly represent different risks for the system as a whole.

A possible solution to this problem could be to assume a *hierarchical Archimedean* structure among the hidden variables. Possibly, the best solution would be to assume a *fully-nested hierarchical* structure, so that each element of the system could be associated to a different level of dependence with the others. The task is involved, because a well known fact is that the hierarchical Archimedean structures require that the variables should be placed in descending order of dependence. To put it in other terms, the dependence structure among elements in the same set (or closer elements) must be higher than that among elements of different sets (or distant elements). The fact that in the common shock models these variables are not observed make the issue particularly complex. The aim of this paper is to give a set of directions to address this problem, on a theoretical level.

The plan of the paper is as follows. In Section 2 we review the case of a set underlying shocks whose dependence structure is given by an Archimedean copula function. As a particular specification of the model we consider the case in which the distortions are of linear type and the generator belongs to the Gumbel family: in this case we recover lifetimes marginally exponentially distributed as in the original Marshall-Olkin model. In Section 3 we discuss the theoretical features of the extension to non-exchangeable dependence of the underlying observable shocks arrival times: in order to compare the results with the exchangeable case we consider, as an example, the case of linear distortions and of Gumbel generator.

2 The exchangeable dependence case model

In this section we briefly review the case of unobservable shocks linked by an Archimedean copula. The model presented is a particular case of that studied in Mulinacci (2015) and Mulinacci (2018) where the possibility of more than one shock affecting different subsets of the considered lifetimes is allowed. We summarize here the model, restricted to one common shock, for the sake of completeness and easyness of the reader. Moreover, the simple structure of the exchangeable model will allow to better understand the degree of complexity that may be induced by an Archimedean structure.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a $d+1$-vector, $(X_0, X_1, \dots, X_d)$ whose components have $[0, +\infty)$ as support. X_0 denotes the arrival time of the systemic shock and $(X_1, \dots, X_d)$ are those of the idiosyncratic ones. We assume that the joint survival dependence structure is represented by a strict Archimedean copula, that is

$$\bar{F}(x_0, x_1, \dots, x_d) = \psi\left(\psi^{-1}(\bar{F}_0(x_0)) + \cdots + \psi^{-1}(\bar{F}_d)\right)$$

for $(x_0, \dots, x_d) \in [0, +\infty)^{d+1}$, where $\bar{F}_i$ (that is assumed to be continuous and strictly decreasing) is the marginal survival function of X_i, $i = 1, \dots, d$, and ψ is the generator of a strict $d+1$-dimensional Archimedean copula.

We recall that ψ is the generator of a $d+1$-Archimedean copula if and only if $\psi : [0, +\infty) \to [0, 1]$ is $d+1$-monotone on $[0, +\infty)$ that is

- it is differentiable on $(0, +\infty)$ up to order $d-1$ and the derivatives satisfy $(-1)^k \psi^{(k)}(x) \geq 0$ for $k = 0, 1, \ldots, d-1$ and $x \in (0, +\infty)$
- $(-1)^{d-1}\psi^{(d-1)}$ is non-increasing and convex in $(0, +\infty)$.

(see McNeil and Nešlehová, 2009, for more details on multidimensional Archimedean copulas).

Since we restrict ourselves to the strict case, we assume $\psi(x) > 0$ for all $x \in [0, +\infty)$.

Let us define

$$\tau_k = \min\{X_0, X_k\},\ k = 1, \ldots, d.$$

The observed default times τ_k represent the first arrival time between a common (systemic) shock affecting all the system and the idiosyncratic shocks. We then add an Archimedean type of dependence among the arrival times of the shocks, in order to represent contagion.

The joint survival function of the random vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_d)$ can be easily recovered

$$\bar{F}_{\boldsymbol{\tau}}(t_1, \ldots, t_d) = \psi\left(\psi^{-1}(\bar{F}_0(\max_{1\leq k\leq d}\{t_k\})) + \sum_{k=1}^{d}\psi^{-1}(\bar{F}_k(t_k))\right) \tag{1}$$

for $t_1, \ldots, t_d \in [0, +\infty)^d$, while the marginal survival distribution functions are

$$\bar{F}_{\tau_k}(t) = \psi\left(\psi^{-1}(\bar{F}_0(t)) + \psi^{-1}(\bar{F}_k(t))\right) = \psi\left(H_{0,k}(t)\right),\ t \in [0, +\infty) \tag{2}$$

where $H_{0,k}(x) = \psi^{-1}(\bar{F}_0(x)) + \psi^{-1}(\bar{F}_k(x))$.

Applying Sklar's theorem it is also easy to extract the copula function of the observed default times.

More precisely, the survival copula $\hat{C}$ of the vector of default times $\boldsymbol{\tau}$ is, for $\mathbf{u} \in [0, 1]^d$,

$$\hat{C}(\mathbf{u}) = \sum_{j=1}^{d}\psi\left(\psi^{-1}(u_j) + \sum_{k=1, k\neq j}^{d} D_k \circ \psi^{-1}(u_k)\right)\mathbf{1}_{A_j}(\mathbf{u}) \tag{3}$$

where $D_k(x) = \psi^{-1} \circ \bar{F}_k \circ H_{0,k}^{-1}(x)$ and

$$A_j = \left\{\mathbf{u} \in [0, 1]^d : \max_{1\leq i\leq d}\{H_{0,i}^{-1} \circ \psi^{-1}(u_i)\} = H_{0,j}^{-1} \circ \psi^{-1}(u_j)\right\}$$

with the convention that if $\mathbf{u}$ satisfies the required condition for more than one index j, it is assumed to belong to the A_j with the smallest index j.

The main feature of our model, right from the most general setting, is to increase the degree of dependence among the default times, both with respect to the standard Archimedean copula without any systemic risk factor and the Marshall-Olkin copula in which the systemic risk factor is independent of the others. In fact, the copula in (3) can be seen as a distorted Archimedean copula: in fact, in the case in which the distortions D_k satisfy $D_k(x) = x$ for all k (that corresponds to the case of absence of the common shock), we recover the Archimedean copula with generator ψ. On the other hand, if we consider the generator $\psi(x) = e^{-x}$ (that corresponds to independence), we recover the Lin and Li (2014) case in the particular situation in which only one common shock is assumed.

The dependence structure of the model encompasses both the sensitivity of the default times to the systemic shock, and the dependence among the shocks, represented by Archimedean copulas. Both these elements interact to determine the dependence among default times.

In the general setting, the Kendall's tau $\tau_{j,k}$ measuring the dependence of the pair of default times (τ_j, τ_k) can be written as

$$\tau_{j,k} = \tau^{\psi} + 4 \int_0^{\infty} (\psi'(x))^2 \cdot T(x) dx$$

where τ^{ψ} denotes the Archimedean Kendall's tau corresponding to the generator ψ and

$$T(x) = \psi^{-1} \circ \bar{F}_0 \circ \left(\psi^{-1} \circ \bar{F}_0 + \psi^{-1} \circ \bar{F}_j + \psi^{-1} \circ \bar{F}_k\right)^{-1}(x)$$

where we refer the reader to Mulinacci (2018) for the derivation.
Notice that if we are interested in representing the dependence structure between the systemic shock and default times, we have that the Kendall's tau $\tau_{j,0}$ of the pair (τ_j, X_0) is

$$\tau_{j,0} = \tau^{\psi} + 4 \int_0^{\infty} (\psi'(x))^2 \cdot \left(\psi^{-1} \circ \bar{F}_0 \circ \left(\psi^{-1} \circ \bar{F}_0 + \psi^{-1} \circ \bar{F}_j\right)^{-1}(x)\right) dx$$

The first term is simply the Kendall's tau of the Archimedean copula used in the analysis, while the other term, that is more complex, involves both the generator of the Archimedean copula and the relative relevance of systemic and idiosyncratic shocks.

Example 2.1. Linear distortions

A possible assumption about the functions $\psi^{-1}(\bar{F}_i(x))$, in the spirit of the paper by Muliere and Scarsini (1987), is that they are all proportional to the same function $K(x)$: that is, $\psi^{-1}(\bar{F}_i(x)) = \lambda_i K(x)$ for $\lambda_i > 0$, for $i = 0, 1, \ldots, d$. This is equivalent to

$$\bar{F}_i(x) = \psi\left(\lambda_i K(x)\right)$$

and distortions of type

$$D_i(x) = (1 - \alpha_i)x \text{ where } \alpha_i = \frac{\lambda_0}{\lambda_i + \lambda_0} \in [0, 1)$$

that imply that the resulting copula (see (3)) is independent of K.

In the more specific case in which ψ is completely monotone (that is ψ is the Laplace transform of some positive random variable), we recover the Scale-Mixture of Marshall-Olkin distributions and copulas models (SMMO) studied in Li(2009). The exchangeable case of SMMO model is studied in Mai and Scherer (2013) and in Bernhart et al. (2013) where it is applied to the pricing of CDOs.

The Gumbel generator case: As an example, we consider the case in which ψ is the Gumbel generator, that is $\psi(x) = e^{-x^{\frac{1}{\theta}}}$, $\theta \geq 1$. Now, equations (1), (2) and (3) take the form

$$\bar{F}_\tau(t_1, \ldots, t_d) = \exp\left\{ -\left(\lambda_0 K\left(\max_{1\leq i\leq d}\{t_i\} \right) + \sum_{k=1}^{d} \lambda_k K(t_k) \right)^{\frac{1}{\theta}} \right\}$$

$$\bar{F}_{\tau_k}(t) = \exp\left(-(\lambda_0 + \lambda_k)^{\frac{1}{\theta}} K^{\frac{1}{\theta}}(t) \right)$$

$$\hat{C}(\mathbf{u}) = \sum_{j=1}^{d} \exp\left\{ -\left[(-\ln u_j)^\theta + \sum_{k=1, k\neq j}^{d} (1-\alpha_k)(-\ln u_k)^\theta \right]^{\frac{1}{\theta}} \right\} \mathbf{1}_{A_j}(\mathbf{u}).$$

This case represents a particular specification of the model considered in Cherubini and Mulinacci (2017) where more than one systemic shock is allowed affecting different subsets of the components. As proved in that paper, if $K(x) = x^\theta$, this setting implies that the observed lifetimes τ_k are exponentially distributed with intensities

$$\mu_k = (\lambda_0 + \lambda_k)^{\frac{1}{\theta}}.$$

Moreover in this case (since this family of copulas at a bivariate level represents a particular specification of the Archimax copulas, see also Capéraà et al. (2000)) the pairwise Kendall's taus are known to be

$$\tau_{j,k} = \frac{\theta - 1}{\theta} + \frac{\tau_{j,k}^{MO}}{\theta} \tag{4}$$

where

$$\tau_{MO}^{j,k} = \frac{\alpha_j \alpha_k}{\alpha_j + \alpha_k - \alpha_j \alpha_k}$$

is the Kendall's tau of the Marshall-Olkin copula.

Now, the dependence between each default time and the time of a systemic shock is linear

$$\tau_{0,j} = \frac{\theta - 1}{\theta} + \frac{\alpha_j}{\theta}. \tag{5}$$

3 The hierarchical Archimedean risk factors model

In this Section we will consider a possible extension of the model with exchangeable dependence structure presented above. Clearly, any $d+1$-dimensional copula can be considered in place of the Archimedean one and the same construction implemented. Among the possible reasonable choices, vine- Archimedean copulas and hierarchical Archimedean copulas (HAC) could be considered as natural non-exchangeable extensions.
In this paper we will consider $d+1$-dimensional HAC copulas. These are obtained through the composition of simple Archimedean copulas: such composition is recursively applied using different segmentations of the random variables involved. Starting from the initial variables $u_1, \ldots, u_{d+1}$, these are grouped in l_1 copulas $C_{1,1}, \ldots, C_{1,l_1}$. Then, these copulas are grouped in l_2 copulas $C_{2,1}, \ldots, C_{2,l_2}$, and up to the last level where we have just one copula. In order to ensure that the so obtained HAC copula is indeed a copula, the generators $\psi_{i,j}$ of the copulas involved have to be completely monotone and the same must hold for their compositions $\psi_{i+1,j}^{-1} \circ \psi_{i,k}$ whenever $C_{i,k}$ is an argument of $C_{i+1,j}$. When the generators $\psi_{i,j}$ are in the same parametrized family, the described procedure yields a copula if inner copulas have a parameter higher than the outer ones: in this paper we will consider generators belonging to the same family (see Savu and Trede 2008 and McNeil 2008 amog the others as references on this topic).

The *fully nested* HAC is given by the particurar configuration

$$C(\mathbf{u}) = C_d\left(C_{d-1}\left(\ldots C_3\left(C_2\left(C_1(u_1,u_2),u_3\right),u_4\right),\ldots,u_d\right),u_{d+1}\right).$$

If the probability distribution of the systemic shock X_0 corresponds to u_1, then, the idiosyncratic risks X_i, $i \geq 1$, can be decreasingly ordered with respect to the dependence to X_0 being

$$C_{X_0,X_i}(u,v) = C_{i-1}(u,v).$$

If, instead, the probability of X_0 corresponds to u_{d+1}, then

$$C_{X_0,X_i}(u,v) = C_d(u,v)$$

and the dependence structure between each idiosyncratic risk and the systemic one is the same for all the idiosyncratic triggers.
In the intermediate case in which the probability of X_0 corresponds to u_j for some $j = 2, \ldots, d$, we have that

$$C_{X_i,X_0}(u,v) = C_{j-1}(u,v)$$

for those X_i that correspond to those u_i with $i < j$, and

$$C_{X_0,X_i}(u,v) = C_{i-1}(u,v)$$

for those X_i that correspond to u_i with $i > j$.

Of course, under other hierarchical configurations, completely different relationships among the systemic and the idiosyncratic risks can be modeled. For example if

$$C(\mathbf{u}) = C\left(C_{h,1}\left(u_1, \ldots, u_{j-1}\right), C_{h,2}\left(u_j, u_{j+1}, \ldots, u_{d+1}\right)\right)$$

where $C_{h,1}$ e $C_{h,2}$ are again HAC copulas, and X_0 corresponds to u_j, we have that

$$C_{X_i,X_0}(u,v) = C(u,v)$$

for those X_i so that $i < j$ and

$$C_{X_0,X_i}(u,v) = C_{h,2}(u,v)$$

for those X_i so that $i > j$. Hence, in the first case, the dependence structure between X_i and X_0 is constant and weaker than that in the second case where however it varies according to the structure of $C_{h,2}$.

Notice that, however, whatever is the case, the dependence structure between X_0 and X_i is always Archimedean, exactly as in the exchangeable case investigated in Section 2. As a consequence, the formulas there presented for the Kendall's tau between the systemic shock and every default time continue to hold.

Notice, then, between the fully exchangeable system, and the fully non-exchangeable one, we can identify an intermediate case in which the exchangeability concept is only applied to the bivariate relationships between the systemic shock arrival times and the idiosyncratic shocks, whatever the dependence among the idiosyncratic shocks could be.

In next subsection we will study the dependence structure induced on the pairs of the observed lifetimes by the HAC model: this will be done through the analysis of the pairwise Kendall's function and Kendall's tau.

We just recall that the Kendall's function of a copula C, is the cumulative distribution function $\mathcal{K}$ of the random variable $C(U,V)$ with respect to the measure induced by C itself and that the Kendall's tau τ of a copula C can be computed as (see, Nelsen 2007)

$$\tau = 3 - 4\int_0^1 \mathcal{K}(t)dt.$$

3.1 Dependence structure of observed default times

Clearly, the shocks involved are the systemic one and the two idiosyncratic ones that correspond to the default times we are considering. Formally, let X_i, X_j, X_k be the three shocks arrival times we are considering. Whatever the hierarchical structure is, their joint survival distribution is of type

$$\bar{F}(x_i, x_j, x_k) = C_{\psi_\phi}\left(C_{\psi_\theta}\left(\bar{F}_i(x_i), \bar{F}_j(x_j)\right), \bar{F}_k(x_k)\right)$$

where C_{ψ_ϕ} and C_{ψ_θ} are bivariate Archimedean copula functions with generators ψ_ϕ and ψ_θ.

Here below we will analyze the two relevant cases in which the systemic shock is represented by X_i and the case in which it is represented by X_k. In both cases, we will compute the survival distribution of the resulting pair of observable lifetimes and in order to identify their dependence structure, we show the associated copula function and we compute their Kendall's function and Kendall's tau. As an example, the particular case of linear distortions is considered and explicit formulas are provided in the specific case of the Gumbel generators family, that, as in the classical Marshall-Olkin case, generates exponentially distributed lifetimes.

For the sake of simplicity, we will assume that all marginal survival distributions are differentiable when needed.

3.1.1 X_i is the arrival time of the systemic shock

Assume X_i be the systemic shock's arrival time and

$$\tau_j = \min(X_i, X_j),\ \tau_k = \min(X_i, X_k)$$

be the considered default times. Then the joint survival distribution function of (τ_j, τ_k) is

$$\bar{F}_{\tau_j,\tau_k}(t_j, t_k) = \psi_\phi\left(\psi_\phi^{-1} \circ \psi_\theta\left(\psi_\theta^{-1} \circ \bar{F}_i(\max(t_j, t_k)) + \psi_\theta^{-1} \circ \bar{F}_j(t_j)\right) + \psi_\phi^{-1} \circ \bar{F}_k(t_k)\right),$$

while the marginal survival distribution functions are

$$\bar{F}_{\tau_j}(t) = \psi_\theta\left(\psi_\theta^{-1} \circ \bar{F}_i(t) + \psi_\theta^{-1} \circ \bar{F}_j(t)\right) = \psi_\theta \circ H_{0,j}(t)$$

and

$$\bar{F}_{\tau_k}(t) = \psi_\phi\left(\psi_\phi^{-1} \circ \bar{F}_i(t) + \psi_\phi^{-1} \circ \bar{F}_k(t)\right) = \psi_\phi \circ H_{0,k}(t)$$

where $H_{0,j}(t) = \psi_\theta^{-1} \circ \bar{F}_i(t) + \psi_\theta^{-1} \circ \bar{F}_j(t)$ and $H_{0,k}(t) = \psi_\phi^{-1} \circ \bar{F}_i(t) + \psi_\phi^{-1} \circ \bar{F}_k(t)$.

Hence, thanks to Sklar's Theorem, from

$$t_j = H_{0,j}^{-1} \circ \psi_\theta^{-1}(u_j) \text{ and } t_k = H_{0,k}^{-1} \circ \psi_\phi^{-1}(u_k)$$

we get that the associated survival copula is

$$\begin{aligned}
&\hat{C}_{\tau_j,\tau_k}(u_j, u_k) = \\
&= \psi_\phi\left(\psi_\phi^{-1} \circ \psi_\theta\left(\psi_\theta^{-1} \circ \bar{F}_i(\max(H_{0,j}^{-1} \circ \psi_\theta^{-1}(u_j), H_{0,k}^{-1} \circ \psi_\phi^{-1}(u_k))) + \right.\right. \\
&\left.\left. + \psi_\theta^{-1} \circ \bar{F}_j \circ H_{0,j}^{-1} \circ \psi_\theta^{-1}(u_j)\right) + \psi_\phi^{-1} \circ \bar{F}_k \circ H_{0,k}^{-1} \circ \psi_\phi^{-1}(u_k)\right).
\end{aligned}$$

If we set

$$D_{ij} = \psi_\theta^{-1} \circ \bar{F}_i \circ H_{0,j}^{-1}, D_{ik} = \psi_\theta^{-1} \circ \bar{F}_i \circ H_{0,k}^{-1}, D_{ji} = \psi_\theta^{-1} \circ \bar{F}_j \circ H_{0,j}^{-1}, D_{ki} = \psi_\phi^{-1} \circ \bar{F}_k \circ H_{0,k}^{-1},$$

then the associated copula can be rewritten as

$$\begin{aligned}
&\hat{C}_{\tau_j,\tau_k}(u_j,u_k) = \\
&= \psi_\phi\left(\psi_\phi^{-1}\circ\psi_\theta\left(\max(D_{ij}\circ\psi_\theta^{-1}(u_j), D_{ik}\circ\psi_\phi^{-1}(u_k)) + D_{ji}\circ\psi_\theta^{-1}(u_j)\right) + D_{ki}\circ\psi_\phi^{-1}(u_k)\right) = \\
&= \begin{cases} \psi_\phi\left(\psi_\phi^{-1}(u_j) + D_{ki}\circ\psi_\phi^{-1}(u_k)\right), & u_k \geq h(u_j) \\ \psi_\phi\left(\psi_\phi^{-1}\circ\psi_\theta\left(D_{ik}\circ\psi_\phi^{-1}(u_k) + D_{ji}\circ\psi_\theta^{-1}(u_j)\right) + D_{ki}\circ\psi_\phi^{-1}(u_k)\right), & u_k < h(u_j) \end{cases}
\end{aligned} \tag{6}$$

where

$$h(x) = \psi_\phi \circ D_{ik}^{-1} \circ D_{ij} \circ \psi_\theta^{-1}(x).$$

As for the Kendall's function and the Kendall's tau, we have the following results:

Theorem 3.1. *If $\rho = \psi_\phi^{-1}\circ\psi_\theta$, let (see (6))*

$$C(u,v) = \begin{cases} \psi_\phi\left(\psi_\phi^{-1}(u) + D_{ki}\circ\psi_\phi^{-1}(v)\right), v \geq h(u) \\ \psi_\phi\left(\rho\left(D_{ik}\circ\psi_\phi^{-1}(v) + D_{ji}\circ\psi_\theta^{-1}(u)\right) + D_{ki}\circ\psi_\phi^{-1}(v)\right), v < h(u) \end{cases}$$

where

$$h(x) = \psi_\phi \circ D_{ik}^{-1} \circ D_{ij} \circ \psi_\theta^{-1}(x).$$

We have that the corresponding Kendall's function $\mathcal{K}(t) = \mathbb{P}(C(U,V) \leq t)$ and Kendall's tau are, respectively,

$$\mathcal{K}(t) = t - \psi_\phi'\circ\psi_\phi^{-1}(t)\cdot\left[D_{ki}\circ\psi_\phi^{-1}(t) - \int_{D_{ik}\circ\psi_\phi^{-1}(t)}^{D_{ik}\circ G^{-1}\circ\psi_\phi^{-1}(t)} \rho'\circ\rho^{-1}(\psi_\phi^{-1}(t) - D_{ki}\circ D_{ik}^{-1}(z))dz\right] \tag{7}$$

and

$$\begin{aligned}
\tau = 1 &+ 4\int_0^1 \psi_\phi'\circ\psi_\phi^{-1}(t)\cdot D_{ki}\circ\psi_\phi^{-1}(t)dt - \\
&- 4\int_0^1 \psi_\phi'\circ\psi_\phi^{-1}(t)\int_{D_{ik}\circ\psi_\phi^{-1}(t)}^{D_{ik}\circ G^{-1}\circ\psi_\phi^{-1}(t)} \rho'\circ\rho^{-1}(\psi_\phi^{-1}(t) - D_{ki}\circ D_{ik}^{-1}(z))dzdt
\end{aligned}$$

where

$$G(x) = \psi_\phi^{-1}\circ\psi_\theta\circ D_{ij}^{-1}\circ D_{ik}(x) + D_{ki}(x).$$

Proof. See Appendix 5. □

Example 3.1. The linear distortion case

Assume that there exist two functions K and $\hat{K}$ such that

$$\bar{F}_i(t) = \psi_\theta\left(\hat{\lambda}_i\hat{K}(t)\right), \qquad \bar{F}_j(t) = \psi_\theta\left(\lambda_j\hat{K}(t)\right)$$

and

$$\bar{F}_i(t) = \psi_\phi\left(\lambda_i K(t)\right), \qquad \bar{F}_k(t) = \psi_\phi\left(\lambda_k K(t)\right)$$

which implies that

$$\hat{K}(t) = \frac{1}{\hat{\lambda}_i}\psi_\theta^{-1} \circ \psi_\phi \left(\lambda_i K(t)\right). \tag{8}$$

Now, setting $\mu_{ij} = \hat{\lambda}_i + \lambda_j$ *and* $\mu_{ik} = \lambda_i + \lambda_k$,

$$H_{0,j}(t) = \hat{\mu}_{ij}\hat{K}(t) \text{ and } H_{0,k}(t) = \mu_{ik}K(t)$$

and

$$D_{ij}(x) = \frac{\hat{\lambda}_i}{\mu_{ij}}x,\ D_{ik}(x) = \frac{\hat{\lambda}_i}{\mu_{ik}}x,\ D_{ji}(x) = \frac{\lambda_j}{\mu_{ij}}x,\ D_{ki}(x) = \frac{\lambda_k}{\mu_{ik}}x,$$

from which

$$\bar{F}_{\tau_j}(t) = \psi_\theta\left(\mu_{ij}\hat{K}(t)\right) \text{ and } \bar{F}_{\tau_k}(t) = \psi_\phi\left(\mu_{ik}K(t)\right)$$

and

$$\hat{C}_{\tau_j,\tau_k}(u_j,u_k) =$$
$$= \psi_\phi\left(\psi_\phi^{-1} \circ \psi_\theta\left(\max\left(\frac{\hat{\lambda}_i}{\mu_{ij}}\psi_\theta^{-1}(u_j), \frac{\hat{\lambda}_i}{\mu_{ik}}\psi_\phi^{-1}(u_k)\right) + \frac{\lambda_j}{\mu_{ij}}\psi_\theta^{-1}(u_j)\right) + \frac{\lambda_k}{\mu_{ik}}\psi_\phi^{-1}(u_k)\right). \tag{9}$$

The Gumbel generators case: Assume $\psi_\theta(x) = e^{-x^{\frac{1}{\theta}}}$ *and* $\psi_\phi(x) = e^{-x^{\frac{1}{\phi}}}$, *with* $\theta \geq \phi \geq 1$. *Then* $\psi_\phi^{-1} \circ \psi_\theta(x) = x^{\frac{\phi}{\theta}}$ *and (9) writes*

$$\hat{C}_{\tau_j,\tau_k}(u_j,u_k) =$$
$$= \exp\left\{-\left(\left(\max\left(\frac{\hat{\lambda}_i}{\mu_{ij}}(-\log(u_j))^\theta, \frac{\hat{\lambda}_i}{\mu_{ik}}(-\log(u_k))^\phi\right) + \frac{\lambda_j}{\mu_{ij}}(-\log(u_j))^\theta\right)^{\frac{\phi}{\theta}} + \right.$$
$$\left. + \frac{\lambda_k}{\mu_{ik}}(-\log(u_k))^\phi\right)^{\frac{1}{\phi}}\right\}.$$

Necessarily, by (8), $\hat{\lambda}_i\hat{K} = \lambda_i^{\frac{\theta}{\phi}}K^{\frac{\theta}{\phi}}$ *and an admissible choice is*

$$\hat{\lambda}_i = \lambda_i^{\frac{\theta}{\phi}} \text{ and } \hat{K} = K^{\frac{\theta}{\phi}}.$$

In particular, if $K(t) = t^\phi$ *and* $\hat{K}(t) = t^\theta$ *we recover exponential marginal distributions, that is*

$$\bar{F}_{\tau_j}(t) = e^{-\mu_{ij}^{\frac{1}{\theta}}t} \text{ and } \bar{F}_{\tau_k}(t) = e^{-\mu_{ik}^{\frac{1}{\phi}}t}.$$

In this specific framework, the Kendall's function and the Kendall's tau are given by:

$$\mathcal{K}(t) = t - \frac{t}{\phi}(-\log t)^{1-\phi}\cdot\left[\frac{\lambda_k}{\mu_{ik}}(-\log t)^\phi - \frac{\phi}{\theta}\int_{\frac{\hat{\lambda}_i}{\mu_{ik}}(-\log t)^\phi}^{\frac{\hat{\lambda}_i}{\mu_{ik}}G^{-1}\left((-\log t)^\phi\right)}\left((-\log t)^\phi - \frac{\lambda_k}{\hat{\lambda}_i}z\right)^{1-\frac{\theta}{\phi}}dz\right]$$

and

$$\tau = 1 + \frac{\lambda_k}{\mu_{ik}} \frac{1}{\phi} - \frac{4}{\theta} \int_0^1 t(\log t)^{1-\phi} \left(\int_{\frac{\hat{\lambda}_i}{\mu_{ik}}(-\log t)^{\phi}}^{\frac{\hat{\lambda}_i}{\mu_{ik}} G^{-1}\left((-\log t)^{\phi}\right)} \left((-\log t)^{\phi} - \frac{\lambda_k}{\hat{\lambda}_i} z \right)^{1-\frac{\theta}{\phi}} dz \right) dt$$

where $G(x) = \left(\frac{\mu_{ij}}{\mu_{ik}}\right)^{\frac{\phi}{\theta}} x^{\frac{\phi}{\theta}} + \frac{\lambda_k}{\mu_{ik}} x$.

3.1.2 X_k is the arrival time of the systemic shock

Here we assume that X_k is the arrival time of the systemic shock and

$$\tau_i = \min(X_i, X_k) \text{ and } \tau_j = \min(X_j, X_k)$$

the observable lifetimes. Then their joint survival distribution is

$$\bar{F}_{\tau_i,\tau_j}(t_i, t_j) = \psi_\phi \left[\psi_\phi^{-1} \circ \psi_\theta \left(\psi_\theta^{-1}(\bar{F}_i(t_i)) + \psi_\theta^{-1}(\bar{F}_j(t_j)) \right) + \psi_\phi^{-1}(\bar{F}_k(\max(t_i, t_j)) \right]$$

while the marginal survival distributions are

$$\bar{F}_{\tau_i}(t_i) = \psi_\phi \circ H_{0,i}(t_i) \text{ and } \bar{F}_{\tau_j}(t_j) = \psi_\phi \circ H_{0,j}(t_j)$$

where

$$H_{0,i} = \psi_\phi^{-1} \circ \bar{F}_i + \psi_\phi^{-1} \circ \bar{F}_k \text{ and } H_{0,j} = \psi_\phi^{-1} \circ \bar{F}_j + \psi_\phi^{-1} \circ \bar{F}_k.$$

If $\rho = \psi_\phi^{-1} \circ \psi_\theta$, the joint survival distribution can be rewritten as

$$\bar{F}_{\tau_i,\tau_j}(t_i, t_j) = \psi_\phi \left[\rho \left(\rho^{-1} \circ \psi_\phi^{-1} \circ \bar{F}_i(t_i) + \rho^{-1} \circ \psi_\phi^{-1} \circ \bar{F}_j(t_j) \right) + \psi_\phi^{-1} \circ \bar{F}_k(\max(t_i, t_j)) \right]$$

and, applying Sklar's Theorem, we recover the associated survival copula

$$\begin{aligned} \hat{C}_{\tau_i,\tau_j}(u_i, u_j) = \psi_\phi \Big[& \rho \left(\rho^{-1} \circ \psi_\phi^{-1} \circ \bar{F}_i \circ H_{0,i}^{-1} \circ \psi_\phi^{-1}(u_i) + \rho^{-1} \circ \psi_\phi^{-1} \circ \bar{F}_j \circ H_{0,j}^{-1} \circ \psi_\phi^{-1}(u_j) \right) + \\ & + \psi_\phi^{-1} \circ \bar{F}_k(\max(H_{0,i}^{-1} \circ \psi_\phi^{-1}(u_i), H_{0,j}^{-1} \circ \psi_\phi^{-1}(u_j))) \Big] . \end{aligned}$$

If we set

$D_{ik} = \psi_\phi^{-1} \circ \bar{F}_i \circ H_{0,i}^{-1}$, $D_{jk} = \psi_\phi^{-1} \circ \bar{F}_j \circ H_{0,j}^{-1}$, $D_{ki} = \psi_\phi^{-1} \circ \bar{F}_k \circ H_{0,i}^{-1}$ and $D_{kj} = \psi_\phi^{-1} \circ \bar{F}_k \circ H_{0,j}^{-1}$,

the copula can be rewritten as

$$\begin{aligned} & \hat{C}_{\tau_i,\tau_j}(u_i, u_j) = \\ & \psi_\phi \left[\rho \left(\rho^{-1} \circ D_{ik} \circ \psi_\phi^{-1}(u_i) + \rho^{-1} \circ D_{jk} \circ \psi_\phi^{-1}(u_j) \right) + \max(D_{ki} \circ \psi_\phi^{-1}(u_i), D_{kj} \circ \psi_\phi^{-1}(u_j)) \right] = \\ & = \begin{cases} \psi_\phi \left[\rho \left(\rho^{-1} \circ D_{ik} \circ \psi_\phi^{-1}(u_i) + \rho^{-1} \circ D_{jk} \circ \psi_\phi^{-1}(u_j) \right) + D_{ki} \circ \psi_\phi^{-1}(u_i) \right], & u_j \geq h(u_i) \\ \psi_\phi \left[\rho \left(\rho^{-1} \circ D_{ik} \circ \psi_\phi^{-1}(u_i) + \rho^{-1} \circ D_{jk} \circ \psi_\phi^{-1}(u_j) \right) + D_{kj} \circ \psi_\phi^{-1}(u_j) \right], & u_j < h(u_i) \end{cases} \end{aligned} \tag{10}$$

where

$$h(x) = \psi_\phi \circ D_{kj}^{-1} \circ D_{ki} \circ \psi_\phi^{-1}(x).$$

In our present framework the Kendall's function and the Kendall's tau are given by the following result:

Theorem 3.2. *Let (see (10))*

$$C(u,v) = \\ = \begin{cases} \psi_\phi \left[\rho\left(\rho^{-1}\circ D_{ik}\circ\psi_\phi^{-1}(u)+\rho^{-1}\circ D_{jk}\circ\psi_\phi^{-1}(v)\right)+D_{ki}\circ\psi_\phi^{-1}(u)\right], & v \geq h(u) \\ \psi_\phi \left[\rho\left(\rho^{-1}\circ D_{ik}\circ\psi_\phi^{-1}(u)+\rho^{-1}\circ D_{jk}\circ\psi_\phi^{-1}(v)\right)+D_{kj}\circ\psi_\phi^{-1}(v)\right], & v < h(u) \end{cases}$$

where

$$h(x) = \psi_\phi \circ D_{kj}^{-1} \circ D_{ki} \circ \psi_\phi^{-1}(x).$$

We have that the Kendall's function $\mathcal{K}(t) = \mathbb{P}(C(U,V) \leq t)$ *and the Kendall's tau respectively are*

$$\begin{aligned} \mathcal{K}(t) = t + \psi_\phi' \circ \psi_\phi^{-1}(t) \cdot \\ \cdot \left[\int_{\rho^{-1}\circ D_{jk}\circ\psi_\phi^{-1}(t)}^{\rho^{-1}\circ\psi_\phi^{-1}\circ\bar{F}_j\circ G^{-1}\circ\psi_\phi^{-1}(t)} \rho' \circ \rho^{-1}\left\{\psi_\phi^{-1}(t) - \psi_\phi^{-1}\circ\bar{F}_k\circ\bar{F}_j^{-1}\circ\psi_\phi\circ\rho(z)\right\} dz + \right. \\ + \int_{\rho^{-1}\circ D_{ik}\circ\psi_\phi^{-1}(t)}^{\rho^{-1}\circ\psi_\phi^{-1}\circ\bar{F}_i\circ G^{-1}\circ\psi_\phi^{-1}(t)} \rho' \circ \rho^{-1}\left\{\psi_\phi^{-1}(t) - \psi_\phi^{-1}\circ\bar{F}_k\circ\bar{F}_i^{-1}\circ\psi_\phi\circ\rho(z)\right\} dz + \\ \left. -(D_{kj}+D_{ki})\circ\psi_\phi^{-1}(t) + 2\psi_\phi^{-1}\circ\bar{F}_k\circ G^{-1}\circ\psi_\phi^{-1}(t)\right] \end{aligned} \tag{11}$$

and

$$\begin{aligned} \tau = 1 - 4\int_0^1 \psi_\phi' \circ \psi_\phi^{-1}(t) \cdot \left[\int_{\rho^{-1}\circ D_{jk}\circ\psi_\phi^{-1}(t)}^{\rho^{-1}\circ\psi_\phi^{-1}\circ\bar{F}_j\circ G^{-1}\circ\psi_\phi^{-1}(t)} \rho' \circ \rho^{-1}\left\{\psi_\phi^{-1}(t) - \psi_\phi^{-1}\circ\bar{F}_k\circ\bar{F}_j^{-1}\circ\psi_\phi\circ\rho(z)\right\} dz \right. \\ \left. + \int_{\rho^{-1}\circ D_{ik}\circ\psi_\phi^{-1}(t)}^{\rho^{-1}\circ\psi_\phi^{-1}\circ\bar{F}_i\circ G^{-1}\circ\psi_\phi^{-1}(t)} \rho' \circ \rho^{-1}\left\{\psi_\phi^{-1}(t) - \psi_\phi^{-1}\circ\bar{F}_k\circ\bar{F}_i^{-1}\circ\psi_\phi\circ\rho(z)\right\} dz \right] dt + \\ - 4\int_0^1 \psi_\phi' \circ \psi_\phi^{-1}(t) \cdot (2\psi_\phi^{-1}\circ\bar{F}_k\circ G^{-1} - (D_{kj}+D_{ki}))\circ\psi_\phi^{-1}(t) dt \end{aligned}$$

where

$$G(z) = \rho\left\{\rho^{-1}\circ\psi_\phi^{-1}\circ\bar{F}_i(z) + \rho^{-1}\circ\psi_\phi^{-1}\circ\bar{F}_j(z)\right\} + \psi_\phi^{-1}\circ\bar{F}_k(z). \tag{12}$$

Proof. See Appendix 5. □

Example 3.2. The linear distortion case

Assume there exists a function K *such that*

$$\bar{F}_v(x) = \psi_\phi\left(\lambda_v K(x)\right) \text{ for } v = i,j,k,$$

and set $\mu_{ik} = \lambda_i + \lambda_k$ *and* $\mu_{jk} = \lambda_j + \lambda_k$.

It follows that

$$D_{ik}(x) = \frac{\lambda_i}{\mu_{ik}}x,\ D_{jk}(x) = \frac{\lambda_j}{\mu_{jk}}x,\ D_{ki}(x) = \frac{\lambda_k}{\mu_{ik}}x \text{ and } D_{kj}(x) = \frac{\lambda_k}{\mu_{jk}}x$$

and the marginal survival distributions can be written as

$$\bar{F}_{\tau_s}(t) = \psi_\phi\left(\mu_{sk}K(t)\right),\ s = i, j$$

while the associated survival copula takes the form

$$\hat{C}_{\tau_i,\tau_j}(u_i, u_j) =$$
$$\psi_\phi\left[\rho\left(\rho^{-1}\left(\frac{\lambda_i}{\mu_{ik}}\psi_\phi^{-1}(u_i)\right) + \rho^{-1}\left(\frac{\lambda_j}{\mu_{jk}}\psi_\phi^{-1}(u_j)\right)\right) + \max(\frac{\lambda_k}{\mu_{ik}}\psi_\phi^{-1}(u_i), \frac{\lambda_k}{\mu_{jk}}\psi_\phi^{-1}(u_j))\right].$$

The Gumbel generators case: Assume $\psi_\theta(x) = e^{-x^{\frac{1}{\theta}}}$ *and* $\psi_\phi(x) = e^{-x^{\frac{1}{\phi}}}$*, with* $\theta \geq \phi \geq 1$*. Then* $\rho(x) = x^{\frac{\phi}{\theta}}$ *and*

$$\hat{C}_{\tau_i,\tau_j}(u_i, u_j) = \exp\left\{-\left[\left(\left(\frac{\lambda_i}{\mu_{ik}}\right)^{\frac{\theta}{\phi}}(-\log(u_i))^\theta + \left(\frac{\lambda_j}{\mu_{jk}}\right)^{\frac{\theta}{\phi}}(-\log(u_j))^\theta\right)^{\frac{\phi}{\theta}} + \right.\right.$$
$$\left.\left. + \max\left(\frac{\lambda_k}{\mu_{ik}}(-\log(u_i))^\phi, \frac{\lambda_k}{\mu_{jk}}(-\log(u_j))^\phi\right)\right]^{\frac{1}{\phi}}\right\}$$

while, if $K(t) = t^\phi$*, we get exponential marginal distributions*

$$\bar{F}_{\tau_i}(t) = e^{-\mu_{ik}^{\frac{1}{\phi}}t} \text{ and } \bar{F}_{\tau_j}(t) = e^{-\mu_{jk}^{\frac{1}{\phi}}t}.$$

Moreover, in this framework, the Kendall's function is given by

$$\mathcal{K}(t) = t - \frac{t}{\phi}(-\log t)^{1-\phi}\cdot\left[\frac{\phi}{\theta}\int_{\left(\frac{\lambda_j}{\mu_{jk}}\right)^{\frac{\theta}{\phi}}(-\log t)^\theta}^{\lambda_j^{\frac{\theta}{\phi}}\left(G^{-1}((-\log t)^\phi)\right)^{\frac{\theta}{\phi}}}\left((-\log t)^\phi - \frac{\lambda_k}{\lambda_j}z^{\frac{\phi}{\theta}}\right)^{1-\frac{\theta}{\phi}}dz+\right.$$
$$+\frac{\phi}{\theta}\int_{\left(\frac{\lambda_i}{\mu_{ik}}\right)^{\frac{\theta}{\phi}}(-\log t)^\theta}^{\lambda_i^{\frac{\theta}{\phi}}\left(G^{-1}((-\log t)^\phi)\right)^{\frac{\theta}{\phi}}}\left((-\log t)^\phi - \frac{\lambda_k}{\lambda_i}z^{\frac{\phi}{\theta}}\right)^{1-\frac{\theta}{\phi}}dz+$$
$$\left.-\left(\frac{\lambda_k}{\mu_{ik}} + \frac{\lambda_k}{\mu_{jk}}\right)(-\log t)^\phi + 2\lambda_k G^{-1}((-\log t)^\phi)\right]$$

while the Kendall's tau is

$$\tau = 1 - \frac{4}{\theta}\int_0^1 t(-\log t)^{1-\phi}\cdot\left[\int_{\left(\frac{\lambda_j}{\mu_{jk}}\right)^{\frac{\theta}{\phi}}(-\log t)^\theta}^{\lambda_j^{\frac{\theta}{\phi}}\left(G^{-1}((-\log t)^\phi)\right)^{\frac{\theta}{\phi}}}\left((-\log t)^\phi - \frac{\lambda_k}{\lambda_j}z^{\frac{\phi}{\theta}}\right)^{1-\frac{\theta}{\phi}}dz+\right.$$
$$\left.+\int_{\left(\frac{\lambda_i}{\mu_{ik}}\right)^{\frac{\theta}{\phi}}(-\log t)^\theta}^{\lambda_i^{\frac{\theta}{\phi}}\left(G^{-1}((-\log t)^\phi)\right)^{\frac{\theta}{\phi}}}\left((-\log t)^\phi - \frac{\lambda_k}{\lambda_i}z^{\frac{\phi}{\theta}}\right)^{1-\frac{\theta}{\phi}}dz\right]dt+$$
$$-\frac{4}{\phi}\int_0^1 t(-\log t)^{1-\phi}\left(2\lambda_k G^{-1}((-\log t)^\phi) - \left(\frac{\lambda_k}{\mu_{ik}} + \frac{\lambda_k}{\mu_{jk}}\right)(-\log t)^\phi\right)dt$$

where

$$G(z) = \left(\lambda_i^{\frac{\theta}{\phi}} + \lambda_j^{\frac{\theta}{\phi}} \right)^{\frac{\phi}{\theta}} z^{\phi} + \lambda_k z.$$

4 Conclusions

This paper presents in full generality the extension of the Marshall Olkin model to the case in which the hidden factors are linked by an Archimedean dependence structure. The ultimate contribution that we provide to this problem is that of a fully nested Archimedean structure in which the dependence structure of the hidden factors is fully flexible, within the set of Archimedean copulas. The problem is made complex by the fact that not every nested Archimedean structure preserves the properties of a multivariate distribution. For this reason, in practical applications different models should be adopted depending on the degree of dependence between the common shock and any of the idiosyncratic ones. The contribution of this paper is to derive the dependence structure between the observed shocks consistent with the chosen Archimedean hierarchical structure. In many cases the theoretical dependence structure of the observed components would require numerical integration or simulation, but this is not much more involved than in the exchangeable case, when the analytical method of moment approach can be used only in a specific case, the Gumbel-Marshall-Olkin copula. For the rest, both in the Archimedean exchangeable and hierarchical cases, whether to stick to the methods of moments, implemented with numerical methods, or to resort to simulation is an empirical problem to be left for future research. Here we help that choice giving the formulas of the link between a hierarchical Archimedean structure and the Kendall tau matrix defining the dependence of the observed shocks.

5 Appendix

Proof of Theorem 3.1

Proof. In the sequel we set $\partial_1 C(u,v) = \frac{\partial}{\partial u} C(u,v)$ and $\partial_2 C(u,v) = \frac{\partial}{\partial v} C(u,v)$.

We want to compute the C-measure of the set

$$S_t = \{(u,v) \in [0,1]^2 : C(u,v) \leq t\}.$$

Notice that the level curve $C(u,v) = t$ intersects the graph of the function $v = h(u)$ in a unique point that we denote with (u_t, v_t). Hence S_t can be decomposed as $S_t = R_t + R_{1,t} + R_{2,t}$ where $R_t = [0,u_t] \times [0,v_t]$, $R_{1,t} = \{(u,v) : v \in (v_t,1], C(u,v) \leq t\}$ and $R_{2,t} = \{(u,v) : u \in (u_t,1], C(u,v) \leq t\}$.
Clearly, the C-measure of R_t is t. In order to compute the C-measure of $R_{1,t}$ and $R_{2,t}$, we compute u_t and v_t. Since (u_t, v_t) satisfies $\psi_\phi \left(\psi_\phi^{-1}(u_t) + D_{ki} \circ \psi_\phi^{-1}(v_t) \right) = t$ and $v_t = h(u_t)$, we get

$$\psi_\phi^{-1} \circ \psi_\theta \circ D_{ij}^{-1} \circ D_{ik} \circ \psi_\phi^{-1}(v_t) + D_{ki} \circ \psi_\phi^{-1}(v_t) = \psi_\phi^{-1}(t)$$

from which

$$v_t = \psi_\phi \circ G^{-1} \circ \psi_\phi^{-1}(t)$$

and

$$u_t = \psi_\theta \circ D_{ij}^{-1} \circ D_{ik} \circ G^{-1} \circ \psi_\phi^{-1}(t).$$

Let us start with $R_{1,t}$. Notice that here, $C(u,v) \leq t$ is equivalent to $u \leq F_1(t,v)$ where $F_1(t,v) = \psi_\phi\left(\psi_\phi^{-1}(t) - D_{ki} \circ \psi_\phi^{-1}(v)\right)$. Hence

$$\begin{aligned}
\mathbb{P}(R_{1,t}) &= \int_{v_t}^1 \mathbb{P}(U \leq F_1(t,v)|V=v)dv = \\
&= \int_{v_t}^1 \partial_2 C(F_1(t,v),v)dv = \\
&= \int_{v_t}^1 \psi_\phi' \circ \psi_\phi^{-1}(t) \cdot \frac{d}{dv} D_{ki} \circ \psi_\phi^{-1}(v)dv = \\
&= -\psi_\phi' \circ \psi_\phi^{-1}(t) \cdot D_{ki} \circ \psi_\phi^{-1}(v_t).
\end{aligned}$$

Let us now consider $R_{2,t}$. Notice that here, the inequality $C(u,v) \leq t$, is equivalent to $u \leq F_2(t,v)$ where

$$F_2(t,v) = \psi_\theta \circ D_{ji}^{-1}\left(\rho^{-1}\left(\psi_\phi^{-1}(t) - D_{ki} \circ \psi_\phi^{-1}(v)\right) - D_{ik} \circ \psi_\phi^{-1}(v)\right).$$

But

$$R_{2,t} = \{(u,v) : u_t < u \leq 1, t < v, C(u,v) \leq t\} \cup \{(u,v) : u_t < u \leq 1, v \leq t\}$$

and

$$\begin{aligned}
&\mathbb{P}\left(u_t < U \leq 1, V \leq t\right) = t - C(u_t,t) = \\
&= \mathbb{P}\left(U \leq u_t, t < V \leq v_t\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{P}(R_{2,t}) = \int_t^{v_t} \mathbb{P}(U \leq F_2(t,v)|V=v)dv = \\
&= \int_t^{v_t} \partial_2 C(F_2(t,v),v)dv = \\
&= \int_t^{v_t} \psi_\phi' \circ \psi_\phi^{-1}(t)\{\rho' \circ \rho^{-1}\left(\psi_\phi^{-1}(t) - D_{ki} \circ \psi_\phi^{-1}(v)\right) \frac{d}{dv} D_{ik} \circ \psi_\phi^{-1}(v) + \frac{d}{dv} D_{ki} \circ \psi_\phi^{-1}(v)\}dv = \\
&= \psi_\phi' \circ \psi_\phi^{-1}(t) \left\{ \int_{D_{ik} \circ \psi_\phi^{-1}(t)}^{D_{ik} \circ \psi_\phi^{-1}(v_t)} \rho' \circ \rho^{-1}(\psi_\phi^{-1}(t) - D_{ki} \circ D_{ik}^{-1}(z))dz + \right. \\
&\left. + D_{ki} \circ \psi_\phi^{-1}(v_t) - D_{ki} \circ \psi_\phi^{-1}(t) \right\}.
\end{aligned}$$

From $\mathbb{P}(S_t) = t + \mathbb{P}(R_{1,t}) + \mathbb{P}(R_{2,t})$ we get (7).

As a consequence, the Kendall's tau is

$$\begin{aligned}
\tau &= 3 - 4\int_0^1 \mathcal{K}(t)dt = \\
&= 3 - 4\int_0^1 \Big\{ t - \psi'_\phi \circ \psi_\phi^{-1}(t) \cdot \\
&\cdot \left[D_{ki} \circ \psi_\phi^{-1}(t) - \int_{D_{ik}\circ\psi_\phi^{-1}(t)}^{D_{ik}\circ G^{-1}\circ\psi_\phi^{-1}(t)} \rho' \circ \rho^{-1}(\psi_\phi^{-1}(t) - D_{ki} \circ D_{ik}^{-1}(z))dz \right] \Big\} dt = \\
&= 1 + 4\int_0^1 \psi'_\phi \circ \psi_\phi^{-1}(t) \cdot D_{ki} \circ \psi_\phi^{-1}(t)dt - \\
&- 4\int_0^1 \psi'_\phi \circ \psi_\phi^{-1}(t) \int_{D_{ik}\circ\psi_\phi^{-1}(t)}^{D_{ik}\circ G^{-1}\circ\psi_\phi^{-1}(t)} \rho' \circ \rho^{-1}(\psi_\phi^{-1}(t) - D_{ki} \circ D_{ik}^{-1}(z))dzdt.
\end{aligned}$$

□

Proof of Theorem 3.2

Proof. In the sequel we set $\partial_1 C(u,v) = \frac{\partial}{\partial u}C(u,v)$ and $\partial_2 C(u,v) = \frac{\partial}{\partial v}C(u,v)$.

The proof is similar to the one of Theorem 3.1.

Again we decompose the set $S_t = \{(u,v) \in [0,1]^2 : C(u,v) \leq t\}$ as $S_t = R_t + R_{1,t} + R_{2,t}$ where, if (u_t, v_t) is the intersection point of the curves $C(u,v) = t$ and $v = h(u)$, $R_t = [0,u_t] \times [0,v_t]$, $R_{1,t} = \{(u,v) : u \in (u_t,1], C(u,v) \leq t\}$ and $R_{2,t} = \{(u,v) : v \in (v_t,1], C(u,v) \leq t\}$.
Clearly, the C-measure of R_t is t. In order to compute the C-measure of $R_{1,t}$ and $R_{2,t}$, we compute u_t and v_t. Since

$$\psi_\phi \left[\rho \left(\rho^{-1} \circ D_{ik} \circ \psi_\phi^{-1}(u_t) + \rho^{-1} \circ D_{jk} \circ \psi_\phi^{-1}(v_t) \right) + D_{kj} \circ \psi_\phi^{-1}(v_t) \right] = t$$

and $v_t = h(u_t)$, we get

$$v_t = \psi_\phi \circ H_{0,j} \circ G^{-1} \circ \psi_\phi^{-1}(t)$$

and

$$u_t = \psi_\phi \circ H_{0,i} \circ G^{-1} \circ \psi_\phi^{-1}(t)$$

where G is given by (12).
Let us start with $R_{1,t}$. Notice that here, $C(u,v) \leq t$ is equivalent to $u \leq F_1(t,v)$ where

$$F_1(t,v) = \psi_\phi \circ D_{ik}^{-1} \circ \rho \left(\rho^{-1} \left(\psi_\phi^{-1}(t) - D_{kj} \circ \psi_\phi^{-1}(v) \right) - \rho^{-1} \circ D_{jk} \circ \psi_\phi^{-1}(v) \right).$$

By similar arguments as those used in the proof of Theorem 3.1, we have

$$\mathbb{P}(R_{1,t}) = \int_t^{v_t} \mathbb{P}(U \leq F_1(t,v)|V=v)dv =$$
$$= \int_t^{v_t} \partial_2 C(F_1(t,v),v)dv =$$
$$= \int_t^{v_t} \psi'_\phi \circ \psi_\phi^{-1}(t) \left\{ \rho' \circ \rho^{-1}\left(\psi_\phi^{-1}(t) - D_{kj} \circ \psi_\phi^{-1}(v)\right) \frac{d}{dv}\rho^{-1} \circ D_{jk} \circ \psi_\phi^{-1}(v) + \right.$$
$$\left. + \frac{d}{dv} D_{kj} \circ \psi_\phi^{-1}(v) \right\} dv =$$
$$= \psi'_\phi \circ \psi_\phi^{-1}(t) \left\{ \int_{\rho^{-1}\circ D_{jk}\circ\psi_\phi^{-1}(t)}^{\rho^{-1}\circ D_{jk}\circ\psi_\phi^{-1}(v_t)} \rho' \circ \rho^{-1}(\psi_\phi^{-1}(t) - \psi_\phi^{-1} \circ \bar{F}_k \circ \bar{F}_j^{-1} \circ \psi_\phi \circ \rho(z))dz + \right.$$
$$\left. + D_{kj} \circ \psi_\phi^{-1}(v_t) - D_{kj} \circ \psi_\phi^{-1}(t) \right\}.$$

Substituting v_t we get

$$\mathbb{P}(R_{1,t}) =$$
$$= \psi'_\phi \circ \psi_\phi^{-1}(t) \left\{ \int_{\rho^{-1}\circ D_{jk}\circ\psi_\phi^{-1}(t)}^{\rho^{-1}\circ\psi_\phi^{-1}\circ\bar{F}_j\circ G^{-1}\psi_\phi^{-1}(t)} \rho' \circ \rho^{-1}(\psi_\phi^{-1}(t) - \psi_\phi^{-1} \circ \bar{F}_k \circ \bar{F}_j^{-1} \circ \psi_\phi \circ \rho(z))dz + \right.$$
$$\left. + \psi_\phi^{-1} \circ \bar{F}_k \circ G^{-1} \circ \psi_\phi^{-1}(t) - D_{kj} \circ \psi_\phi^{-1}(t) \right\}.$$

With similar computations we get

$$\mathbb{P}(R_{2,t}) =$$
$$= \psi'_\phi \circ \psi_\phi^{-1}(t) \left\{ \int_{\rho^{-1}\circ D_{ik}\circ\psi_\phi^{-1}(t)}^{\rho^{-1}\circ\psi_\phi^{-1}\circ\bar{F}_i\circ G^{-1}\psi_\phi^{-1}(t)} \rho' \circ \rho^{-1}(\psi_\phi^{-1}(t) - \psi_\phi^{-1} \circ \bar{F}_k \circ \bar{F}_i^{-1} \circ \psi_\phi \circ \rho(z))dz + \right.$$
$$\left. + \psi_\phi^{-1} \circ \bar{F}_k \circ G^{-1} \circ \psi_\phi^{-1}(t) - D_{ki} \circ \psi_\phi^{-1}(t) \right\}.$$

From $\mathbb{P}(S_t) = t + \mathbb{P}(R_{1,t}) + \mathbb{P}(R_{2,t})$ we get (11).

As a consequence, the Kendall's tau is

$$\tau = 3 - 4\int_0^1 \mathcal{K}(t)dt =$$
$$= 1 - 4\int_0^1 \psi'_\phi \circ \psi_\phi^{-1}(t) \cdot$$
$$\cdot \left[\int_{\rho^{-1}\circ D_{jk}\circ\psi_\phi^{-1}(t)}^{\rho^{-1}\circ\psi_\phi^{-1}\circ\bar{F}_j\circ G^{-1}\circ\psi_\phi^{-1}(t)} \rho' \circ \rho^{-1}\left\{\psi_\phi^{-1}(t) - \psi_\phi^{-1} \circ \bar{F}_k \circ \bar{F}_j^{-1} \circ \psi_\phi \circ \rho(z)\right\} dz + \right.$$
$$\left. + \int_{\rho^{-1}\circ D_{ik}\circ\psi_\phi^{-1}(t)}^{\rho^{-1}\circ\psi_\phi^{-1}\circ\bar{F}_i\circ G^{-1}\circ\psi_\phi^{-1}(t)} \rho' \circ \rho^{-1}\left\{\psi_\phi^{-1}(t) - \psi_\phi^{-1} \circ \bar{F}_k \circ \bar{F}_i^{-1} \circ \psi_\phi \circ \rho(z)\right\} dz \right] dt +$$
$$- 4\int_0^1 \psi'_\phi \circ \psi_\phi^{-1}(t) \cdot (2\psi_\phi^{-1} \circ \bar{F}_k \circ G^{-1} - (D_{kj} + D_{ki})) \circ \psi_\phi^{-1}(t)dt.$$

□

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