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(Article begins on next page)

Bounding the order of a verbal subgroup in a residually finite group

Eloisa Detomi, Marta Morigi, and Pavel Shumyatsky

Abstract. Let w be a group-word. Given a group G, we denote by $w(G)$ the verbal subgroup corresponding to the word w, that is, the subgroup generated by the set G_w of all w-values in G. The word w is called concise in a class of groups $\mathcal{X}$ if $w(G)$ is finite whenever G_w is finite for a group $G \in \mathcal{X}$. It is a long-standing problem whether every word is concise in the class of residually finite groups. In this paper we examine several families of group-words and show that all words in those families are concise in residually finite groups.

1. Introduction

Let $w = w(x_1, \ldots, x_k)$ be a group-word. Given a group G, we denote by $w(G)$ the verbal subgroup corresponding to the word w, that is, the subgroup generated by the set G_w of all values $w(g_1, \ldots, g_k)$, where $g_1, \ldots, g_k$ are elements of G. The word w is called concise if $w(G)$ is finite whenever the set of w-values in G is finite. More generally, the word w is called concise in a class of groups $\mathcal{X}$ if $w(G)$ is finite whenever the set of w-values in G is finite for a group $G \in \mathcal{X}$. In the sixties Hall raised the problem whether all words are concise, but in 1989 S. Ivanov [**16**] (see also [**21**, p. 439]) solved the problem in the negative. On the other hand, the problem for residually finite groups remains open (cf. Segal [**24**, p. 15] or Jaikin-Zapirain [**17**]). In recent years several new positive results with respect to this problem were obtained (see [**1, 13, 10, 6, 7, 8**]).

A word w is boundedly concise in a class of groups $\mathcal{X}$ if for every integer m there exists a number $\nu = \nu(\mathcal{X}, w, m)$ such that whenever $|G_w| \leq m$ for a group $G \in \mathcal{X}$ it always follows that $|w(G)| \leq \nu$. In [**9**] it is shown that every word which is concise in the class of all groups is actually boundedly concise. It was conjectured in [**10**] that every word which is concise in the class of residually finite groups is boundedly concise.

We showed in [**6**] that words implying virtual nilpotency are boundedly concise in the class of residually finite groups. Recall that a word w is said to imply virtual nilpotency if every finitely generated metabelian group G where w is a law, that is $w(G) = 1$, has a nilpotent subgroup of finite index. Such words admit several important characterizations (see [**2, 3, 11**]). Moreover, if w is a word implying virtual nilpotency, then for a large class of groups G, including all finitely generated residually finite groups, $w(G) = 1$ implies that G is nilpotent-by-finite [**3**].

Our first result in this paper is the following theorem.

Theorem 1.1. *Let $u = u(x_1, \dots, x_k)$ and $v = v(y_1, \dots, y_r)$ be words implying virtual nilpotency. Then the word $[u, v]$ is concise in the class of residually finite groups.*

Unfortunately, the proof of Theorem 1.1 sheds no light on the question whether the word $[u, v]$ is boundedly concise in the class of residually finite groups. With regard to this question, the situation remains absolutely unclear. One purpose of this paper is to show that for some particular words v, the word $[u, v]$ is indeed boundedly concise in residually finite groups. It is easy to see that the word $v = [y_1^{d_1}, \dots, y_r^{d_r}]$, where $d_1, \dots, d_r$ are positive integers, implies virtual nilpotency (see Section 5).

Theorem 1.2. *Let $u = u(x_1, \dots, x_k)$ be a word implying virtual nilpotency and let $v = [y_1^{d_1}, \dots, y_r^{d_r}]$ for some positive integers $d_1, \dots, d_r$. Then the word $[u, v]$ is boundedly concise in the class of residually finite groups.*

The proofs of both above results rely on Zelmanov's positive solution of the restricted Burnside problem [**26, 27**].

The techniques employed in the proofs of Theorems 1.1 and 1.2 might be helpful for establishing conciseness of some other words. In particular, we show how the techniques can be applied in the treatment of certain words related to 'weakly rational' ones. Following [**13**] we say that a group word w is weakly rational if for every finite group G and for every positive integer e relatively prime with the order of G, the set G_w is closed under taking eth powers of its elements. It is not difficult to see that weakly rational words are boundedly concise in residually finite groups (see [**13**]). Here we will prove:

Theorem 1.3. *Let $v = v(x_1, \dots, x_k)$ be a weakly rational word and set $v_i = v(x_{i,1}, \dots, x_{i.k})$ for $i = 1, \dots, n$. Then the word $w = [v_1, \dots, v_n]$ is boundedly concise in the class of residually finite groups.*

Thus, Theorem 1.3 provides a sufficient condition for $\gamma_n(v(G))$ to have bounded order. Throughout, we write $\gamma_n = \gamma_n(x_1, \dots, x_n)$ for the lower central word $[x_1, \dots, x_n]$ and $\gamma_n(G)$ for the corresponding verbal subgroup of a group G. Of course $\gamma_n(G)$ is the nth term of lower central series of G.

It is easy to see that the problem on conciseness of words in residually finite groups is equivalent to the same problem in profinite groups. On the other hand, in the case of profinite groups there are interesting modifications of the very concept of conciseness of words, allowing the set G_w to be infinite (see [**5**] and [**4**]).

2. Preliminaries

Throughout the paper we denote by G' the commutator subgroup of a group G and by $\langle M \rangle$ the subgroup generated by a subset $M \subseteq G$. We use the expression "$(a, b, \dots)$-bounded" to mean that a quantity is bounded by a certain number depending only on the parameters $a, b, \dots$. In this section we collect some preliminary results which will be needed for the proofs of the main theorems.

A proof of the following lemma can for example be found in [**6**].

Lemma 2.1. [**6**, Lemma 4] *Let v be a word and G a group such that the set of v-values in G is finite with at most m elements. Then the order of the commutator subgroup $v(G)'$ is m-bounded.*

We will now quote a lemma from [**6**] that provides a valuable technical tool for bounding the order of verbal subgroups. It played an important role in proofs of several results on conciseness of words in residually finite groups.

Lemma 2.2. [**6**, Lemma 10] *Let $w = w(x_1, x_2, \dots, x_k)$ be a word and let p be a prime. Let G be a nilpotent group of class c generated by k elements $a_1, a_2, \dots, a_k$. Denote by X the set of all conjugates in G of elements of the form $w(a_1^i, a_2^i, \dots, a_k^i)$, where i ranges over the set of all integers not divisible by p, and assume that $|X| \leq m$ for some integer m. Then $|\langle X \rangle|$ is (c, m)-bounded.*

As mentioned in the introduction, a group word w is weakly rational if for every finite group G and $g \in G_w$, the power g^e belongs to G_w whenever e is relatively prime to $|G|$. By [**13**, Lemma 1], the word w is weakly rational if and only if for every finite group G and $g \in G_w$, the power g^e belongs to G_w whenever e is relatively prime to $|g|$. As usual, $F(G)$ denotes the Fitting subgroup of a group G.

Corollary 2.3. *Let G be a finite group with at most m values of the word $[u, v]$, where u and v are words on disjoint sets of variables. Assume that v is weakly rational. Let $a \in G_u$ and $b \in G_v \cap F(G)$ and suppose that the subgroup $\langle b^a, b\rangle$ is nilpotent of class at most c. Then the order of the element $[a, b]$ is (c, m)-bounded.*

Proof. First, consider the case where $F(G)$ is a p-group for some prime p. Let $\nu(x_1, x_2)$ be the word $x_1^{-1}x_2$ and set $H = \langle b^a, b\rangle$. Since v is a weakly rational word, for every integer i relatively prime to p the element b^i is again a v-value. Therefore, all elements of the form $b^{-ia}b^i = [a, b^i]$ are $[u, v]$-values whenever i is relatively prime to p. The same holds for all conjugates of $[a, b^i]$. So, the set X of all conjugates in H of elements of the form $\nu(b^{ia}, b^i)$, with $(i, p) = 1$, contains at most m elements. Thus, we deduce from Lemma 2.2 that there is a (c, m)-bounded number B such that the order of $[a, b]$ is at most B.

We will now deal with the case where $F(G)$ is not necessarily a p-group. Let $p_1, \ldots, p_s$ be the prime divisors of the order of $F(G)$. Since the case where $s = 1$ was already dealt with, we assume that $s \geq 2$. For each $j = 1, \ldots, s$ let N_j denote the Hall p_j'-subgroup of $F(G)$. The result obtained in the case where $F(G)$ is a p-group implies that for any j the image of $[a, b]$ in G/N_j has order at most B. Since the intersection of the subgroups N_j is trivial, we deduce that the order of $[a, b]$ is at most $B!$, which is (c, m)-bounded. The proof is complete. □

3. Proof of Theorem 1.1

The main tools employed in the proof of Theorem 1.1 are P. Hall's theorem stating that finitely generated abelian-by-nilpotent groups are residually finite [**14**], and a classical result of Turner-Smith that every word is concise in the class of groups all of whose quotients are residually finite [**25**]. It follows that every word is concise in the class of virtually abelian-by-nilpotent groups (see also [**24**, Theorem 2.3.1]). Recall that if $\mathcal{X}$ is a class of groups, then a virtually-$\mathcal{X}$ group is a group having an $\mathcal{X}$-subgroup of finite index.

We will also use a theorem of Burns and Medvedev stating that if a word w implies virtual nilpotency and G is a finite group in which w is a law, then G has a normal nilpotent subgroup N such that the nilpotency class of N and the exponent of G/N are bounded in terms of w only [**3**, Theorem A].

Lemma 3.1. *Let $u = u(x_1, \ldots, x_k)$ and $v = v(y_1, \ldots, y_r)$ be words implying virtual nilpotency. Assume G is a d-generated finite group in which the word $w = [u, v]$ is a law. Then G has two normal subgroups*

N and M, with $M \leq N$, such that N has (w,d)-bounded index, N/M is nilpotent of w-bounded class, and M is abelian.

Proof. Set $M = u(G) \cap v(G)$ and note that M is abelian. As u is a word implying virtual nilpotency and it is a law in the quotient $G/u(G)$, the theorem of Burns and Medvedev [**3**, Theorem A] says that $G/u(G)$ has a normal nilpotent subgroup $N_1/u(G)$ such that the nilpotency class of $N_1/u(G)$ and the exponent of G/N_1 are bounded in terms of u only. The solution of the restricted Burnside problem [**26, 27**] now tells us that G/N_1 has (u,d)-bounded order. Similarly, there exists a normal nilpotent subgroup $N_2/v(G)$ such that the nilpotency class of $N_2/v(G)$ and the order of G/N_2 are bounded in terms of v and d only. It follows that G/M, being a subdirect product of $G/u(G)$ and $G/v(G)$, has a normal nilpotent subgroup N of (w,d)-bounded index and which is nilpotent of w-bounded class. □

Proof of Theorem 1.1. Assume that G is a residually finite group in which the word $w = [u,v]$ takes only finitely many, say m, values. We want to show that $w(G)$ is finite.

Without loss of generality we may assume that G is finitely generated, namely by at most $m(k+r)$ elements, where $k+r$ is the number of variables appearing in w. As G_w is finite and G is residually finite, there exists a normal subgroup K of finite index in G such that $K \cap G_w = 1$. In particular, K is finitely generated, say by d elements, and $w(K) = 1$.

Let $\bar{K}$ be any finite image of K. By Lemma 3.1, $\bar{K}$ has two normal subgroups N and M, with $M \leq N$, such that N has (w,d)-bounded index, N/M is nilpotent of w-bounded class, and M is abelian. Since the bounds here do not depend on the choice of $\bar{K}$, it follows that K is virtually abelian-by-nilpotent.

This proves that G is virtually abelian-by-nilpotent. As mentioned above, combining the results of P. Hall [**14**] and Turner-Smith [**25**], we conclude that $w(G)$ is finite. Hence $[u,v]$ is concise in the class of residually finite groups. □

4. Virtually (abelian-by-nilpotent) groups

Following the lines of [**9**, Appendix] and [**6**, Section 6] we will establish the following proposition.

Proposition 4.1. *Let c,t be positive integers and let $\mathcal{X}$ be the class of groups having a normal subgroup N of finite index at most t which is abelian-by-(nilpotent of class at most c), i.e. $\gamma_{c+1}(N)$ is abelian. Then every word is boundedly concise in the class $\mathcal{X}$.*

The proof of the above proposition uses the concept of ultraproducts. The details concerning this construction can be found, for example, in [**9**].

If w is a word and S is a subset of $w(G)$, we say that w has width k over S, where k is a natural number, if every element of S can be expressed as the product of at most k elements in $G_w \cup G_w^{-1}$.

Proposition 4.2. *Let $\mathcal{X}$ be a class of groups with the property that if $\mathcal{G} = \{G_i\}_{i\in\mathbb{N}}$ is a family of groups in $\mathcal{X}$ and $\mathcal{U}$ is an ultrafilter over $\mathbb{N}$, then the ultraproduct $\mathcal{G}_{\mathcal{U}}$ is again in $\mathcal{X}$. Assume also that the word w is concise in $\mathcal{X}$. Then the word w is boundedly concise in $\mathcal{X}$.*

Proof. We need to prove that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if G is a group in $\mathcal{X}$ with $|G_w| \leq m$, then $|w(G)| \leq f(m)$. By way of contradiction, assume that there is a family $\{G_i\}_{i\in\mathbb{N}}$ of groups in $\mathcal{X}$ such that $|(G_i)_w| \leq m$ for all i but nevertheless $\lim_{i\to\infty} |w(G_i)| = \infty$. Let us fix an arbitrary positive integer k. According to Lemma A.7 in [**9**], if i is big enough, there is a subset S_i of $w(G_i)$ such that $|S_i| \geq k$ and w has width less than k over S_i. We complete the sequence $\{S_i\}_{i\in\mathbb{N}}$ by choosing the first terms equal to 1. So, in particular, the width of w can be uniformly bounded over all the subsets S_i. Now, if $G = \prod_{i\in\mathbb{N}} G_i$ and $S = \prod_{i\in\mathbb{N}} S_i$, we have

$$G_w = \prod_{i\in\mathbb{N}} (G_i)_w$$

and Lemma A.6 in [**9**] yields that

$$S \subseteq w(G).$$

Consider now a non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$, and let $Q = \mathcal{G}_{\mathcal{U}}$ be the corresponding ultraproduct. By assumption Q is in $\mathcal{X}$. Let $\overline{S}$, $\overline{G_w}$, $\overline{w(G)}$ be the images of S, G_w and $w(G)$ in Q, respectively. Then $\overline{G_w} = Q_w$ and $\overline{S} \subseteq \overline{w(G)} = w(Q)$. Moreover

$$|\overline{G}_w| = \sup_{J\in\mathcal{U}} \left(\min_{i\in J} |(G_i)_w| \right) \leq m, \tag{1}$$

and

$$|\overline{S}| = \sup_{J\in\mathcal{U}} \left(\min_{i\in J} |S_i| \right) \geq k, \tag{2}$$

(see Lemma A.5 in [**9**]), thus $|Q_w| \leq m$ and $|w(Q)| \geq k$. Since k is arbitrary, we conclude that the verbal subgroup $w(Q)$ is infinite. This is a contradiction, since $Q \in \mathcal{X}$ and w is concise in $\mathcal{X}$. □

The next lemma shows that under the hypotheses of Proposition 4.1 the class $\mathcal{X}$ is closed under taking ultraproducts of its members.

Lemma 4.3. *Let $\mathcal{X}$ be as in Proposition 4.1. If $\mathcal{G} = \{G_i\}_{i\in\mathbb{N}}$ is a family of groups in $\mathcal{X}$ and $\mathcal{U}$ is an ultrafilter over $\mathbb{N}$, then the ultraproduct $\mathcal{G}_{\mathcal{U}}$ is again in $\mathcal{X}$.*

Proof. For each $i \in \mathbb{N}$ let N_i be a normal subgroup of G_i of finite index at most t such that $\gamma_{c+1}(N_i)$ is abelian. Then $N = \prod_{i\in\mathbb{N}} N_i$ is a normal subgroup of the Cartesian product $\prod_{i\in\mathbb{N}} G_i$. Since $\gamma_{c+1}(N)$, is abelian, so is its image $\gamma_{c+1}(\bar{N})$ in the ultraproduct $\mathcal{G}_{\mathcal{U}}$.

It remains to prove that $\bar{N}$ has index at most t in $\mathcal{G}_{\mathcal{U}}$. This amounts to proving that the order of the quotient group $\mathcal{G}_{\mathcal{U}}/\bar{N}$ is at most t. But $\mathcal{G}_{\mathcal{U}}/\bar{N}$ is isomorphic to the ultraproduct modulo $\mathcal{U}$ of the family of groups $\{G_i/N_i\}_{i\in I}$, thus

$$|\mathcal{G}_{\mathcal{U}}/\bar{N}| = \sup_{J\in\mathcal{U}}\left(\min_{i\in J}|G_i/N_i|\right) \le t, \tag{3}$$

(see Lemma A.5 (2) in [**9**]) and the result follows. □

The proof of Proposition 4.1 will now be short.

Proof of Proposition 4.1. Let w be a word. We have already noted that every word w is concise in the class of virtually (abelian-by-nilpotent) groups. In particular, w is concise in the class $\mathcal{X}$. In view of Lemma 4.3 and Proposition 4.2, we conclude that w is boundedly concise in $\mathcal{X}$. □

5. On bounded conciseness of some words

In the first part of this section we will prove a special case of Theorem 1.2, namely that the word $w = [u, \gamma_r]$ is boundedly concise in the class of residually finite groups. Recall that here $u = u(x_1, \ldots, x_k)$ is a word implying virtual nilpotency and $\gamma_r = [y_1, \ldots, y_r]$.

We start with the following lemma.

Lemma 5.1. *Let e, r be positive integers. Let G be a finite group and set $H = \gamma_r(G)$. Suppose that all γ_r-values of G have order dividing e and H^e contains no nontrivial γ_r-values. Then H has e-bounded exponent.*

Proof. Note that H^e centralizes $H = \gamma_r(G)$, because if $a \in G_{\gamma_r}$ and $b \in H^e$ then $[a, b] \in G_{\gamma_{r+1}} \cap H^e \le G_{\gamma_r} \cap H^e = 1$. This means that H^e is central in H and so $H/Z(H)$ has exponent dividing e. A theorem of Mann [**18**] states that if B is a finite group such that $B/Z(B)$ has exponent e, then the exponent of B' is bounded by a function depending on e only. Hence H' has e-bounded exponent. Since H is generated by elements of order dividing e, it follows that also H has e-bounded exponent, as claimed. □

In the sequel we will require the fact that the words $[\dots[x_1^{n_1}, x_2]^{n_2}, \dots, x_k]^{n_k}$ are weakly rational for any integers $n_1, n_2, \dots, n_k$ (see [**13**]). In particular, the lower central words $\gamma_k = [x_1, x_2, \dots, x_k]$ and their powers γ_k^q are weakly rational.

Proposition 5.2. *The word $w = [u, \gamma_r]$ is boundedly concise in residually finite groups.*

Proof. It suffices to prove that if G is a finite group with only m values of w, then $w(G)$ has bounded order. Without loss of generality we may assume that G is generated by a (w, m)-bounded number of elements. Using Lemma 2.1, we may pass to the quotient $G/w(G)'$ and assume that $w(G)$ is abelian.

Set $H = \gamma_r(G)$. By Theorem 1.2 in [**19**], there exists an integer $k = k(w, m)$ depending only on w and m, such that each element of H is a product of at most k values of the word γ_r. Let $a \in H_u$ and $b \in H$. Writing b as a product of at most k values of the word γ_r and using the usual commutator identities we see that each commutator $[a, b]$ is a product of at most k values of the word w. Therefore the word $[u, x]$ takes only (w, m)-boundedly many values in H. In the other notation, $|H_{[u,x]}|$ is (w, m)-bounded.

As the word u implies virtual nilpotency, it is easy to see that the same holds for the word $[u, x]$. In view of [**6**, Theorem 2] we deduce that $[u, x]$ is boundedly concise in the class of residually finite groups. So the corresponding verbal subgroup in H has (w, m)-bounded order. Passing to the quotient over this subgroup we assume, without loss of generality, that $[u, x]$ is a law in H. Hence there exists two (w, m)-bounded integers e and c such that H^e is nilpotent of class c.

Choose a γ_r-value y and a u-value z in G. Since y^e belongs to H^e, the subgroup $\langle y^{ez}, y^e \rangle$ is nilpotent of class at most c. Taking into account that $[z, y]$ and all its conjugates are w-values and using the usual commutator identities we derive that $[z, y^e]$ is a product of at most e elements from G_w. Therefore the word $[u, \gamma_r^e]$ takes only (w, m)-boundedly many values in G. Recall that the word γ_r^e is weakly rational. Thus, we apply Corollary 2.3 with the word $[u, \gamma_r^e]$ and deduce that $[z, y^e]$ has order bounded by an integer, say t, which depends only on m and c.

We now consider the special case where $y \in G_{\gamma_r} \cap H^e$. Then the subgroup $\langle y^z, y \rangle$ is contained in H^e and therefore is nilpotent of class at most c. Thus, as above, apply Corollary 2.3 and we deduce that the order of $[z, y]$ is bounded by a number (we can assume that this is the same number t as above) depending only on m and c.

Recall that $w(G)$ is an abelian m-generated subgroup. The subgroup generated by the elements of order at most t of $w(G)$ has order at most t^m. Passing to the quotient over this subgroup, we are reduced to the case where every γ_r-value lying in H^e, and every eth power of a γ_r-value, commutes with $u(G)$.

Summarizing, now $C = C_G(u(G))$ contains every γ_r-value lying in H^e and every e-th power of a γ_r-value. We deduce from Lemma 5.1 that the image of $H = \gamma_r(G)$ in G/C has e-bounded exponent.

Let $N = u(G) \cap H$. As $Z(N) \geq N \cap C$, it follows that $N/Z(N)$ has e-bounded exponent. By the already mentioned result of Mann [**18**], N' has e-bounded exponent. Since $N' \cap w(G)$ has (e, m)-bounded order, we can pass to the quotient group over N' and assume that N is abelian. Arguing as in Lemma 3.1, we deduce that G/N has a nilpotent subgroup of (w, m)-bounded index and w-bounded class, since G/N is a subdirect product of $G/u(G)$ and $G/\gamma_r(G)$. Therefore, for some (w, m)-bounded integers j and c_0, the group G belongs to the class $\mathcal{X}$ of groups having a normal subgroup of finite index at most j which is abelian-by-(nilpotent of class at most c_0). Now we deduce from Proposition 4.1 that the order of $w(G)$ is bounded by an integer that depends only on w and m. □

Now we will prove Theorem 1.2, which states that if $u = u(x_1, \ldots, x_k)$ is a word implying virtual nilpotency and $v = [y_1^{d_1}, \ldots, y_r^{d_r}]$, for some positive integers $d_1, \ldots, d_r$, then the word $[u, v]$ is boundedly concise in residually finite groups. Note that the word $[y_1^{d_1}, \ldots, y_r^{d_r}]$ implies virtual nilpotency. Indeed, if G is a finitely generated metabelian group and $[g_1^{d_1}, \ldots, g_r^{d_r}] = 1$ for every $g_i \in G$, then the subgroup $H = G^e$ generated by all eth powers in G, where $e = \mathrm{l.c.m}(d_1, \ldots, d_r)$, has finite index and it is nilpotent of class at most $r - 1$.

Proof of Theorem 1.2. Set $w = [u, v]$. As usual, it suffices to prove that if G is finite group with only m values of the word w, then $w(G)$ has bounded order. Here and in the rest of the proof bounded will always mean (w, m)-bounded. Without loss of generality we may assume that G is generated by boundedly many elements. Using Lemma 2.1, we may further assume that $w(G)$ is abelian.

Let $e = \mathrm{l.c.m}(d_1, \ldots, d_r)$ and let $H = G^e$ be the subgroup generated by all eth powers in G. By the solution of the Restricted Burnside Problem, the index of H in G is bounded and hence H is generated by a bounded number of elements. The main result in [**20**] shows that there exists an integer $t = t(w, m)$, depending only on w and m, such that each element of H can be written as a product of at most t eth powers. Consider the word $w_0 = [u, \gamma_r]$ in H and let $h_1, \ldots, h_{k+r} \in H$. Writing each $h_{k+1}, \ldots, h_{k+r}$ as a product of at most t eth powers and using the usual commutator identities we can write $w_0(h_1, \ldots, h_{k+r})$ as a product of (w, m)-boundedly many elements of the form

$$[u(b_1, \dots, b_k), \gamma_r(b_{k+1}^e, \dots, b_{k+r}^e)]$$

where $b_j \in G$. Since $e = \mathrm{l.c.m}(d_1, \dots, d_r)$, each of these elements is a w-value. Hence $w_0(h_1, \dots, h_{k+r})$ is a product of boundedly many w-values. So w_0 has only boundedly many values in H. Since, by Proposition 5.2, w_0 is boundedly concise, $w_0(H)$ has bounded order. Passing to the quotient over $w_0(H)$, without loss of generality we can assume that w_0 is a law in H. Hence, $u(H) \cap \gamma_r(H)$ is abelian. As H is generated by boundedly many elements and u is a word implying virtual nilpotency, $H/u(H)$ has a normal nilpotent subgroup of bounded index and bounded nilpotency class. Clearly $H/\gamma_r(H)$ has nilpotency class at most $r-1$. Hence $H/(u(H) \cap \gamma_r(H))$ has a normal nilpotent subgroup of bounded index and bounded nilpotency class. Taking into account that the index of H in G is bounded, we conclude that, for some bounded integers l and c, the group G belongs to the class $\mathcal{X}$ of groups having a normal subgroup N of finite index at most l which is abelian-by-(nilpotent of class at most c). The theorem follows from Proposition 4.1. □

Recall that multilinear commutator words, also known as outer-commutator words, are the ones obtained by nesting commutators and using each variable only once. For example, $[[x_1, x_2], [x_3, x_4, x_5], x_6]$ is a multilinear commutator word while the 3-Engel word $[x, y, y, y]$ is not.

Remark 5.3. *Some arguments employed in this section can be used to obtain a result of independent interest: the word $w(x_1^e, \dots, x_k^e)$ is boundedly concise in the class of residually finite groups whenever w is a multilinear commutator word.*

Indeed, let G be a finite group with only m values of $\tilde{w} = w(x_1^e, \dots, x_k^e)$. We need to show that $\tilde{w}(G)$ has bounded order. Without loss of generality we may assume that G can be generated by (w, m)-boundedly many elements. Using Lemma 2.1, we may further assume that $\tilde{w}(G)$ is abelian.

Let $H = G^e$ be the subgroup generated by all eth powers in G. By the solution of the Restricted Burnside Problem, the index of H in G is $(\tilde{w}, m)$-bounded and hence H is generated by a $(\tilde{w}, m)$-bounded number of elements. The main result in [**20**] shows that each element of H is a product of $(\tilde{w}, m)$-boundedly many eth powers. Writing each of the elements $h_1, \dots, h_k \in H$ as a product of $(\tilde{w}, m)$-boundedly many eth powers and using the usual commutator identities we can write $w(h_1, \dots, h_k)$ as a product of $(\tilde{w}, m)$-boundedly many w-values. So w has only $(\tilde{w}, m)$-boundedly many values in H. Since multilinear commutator words are boundedly concise (see [**9**]), $w(H)$ has $(\tilde{w}, m)$-bounded order. Hence also the order of $\tilde{w}(G)$ is $(\tilde{w}, m)$-bounded, as claimed.

It would be interesting to see if this can be extended to words of the form $w(x_1^{e_1}, \dots, x_k^{e_k})$, where w is a multilinear commutator word and the exponents $e_1, \dots, e_k$ are not necessarily equal.

6. Proof of Theorem 1.3

Set $[x, {}_0 y] = x$, $[x, {}_1 y] = [x, y] = x^{-1}y^{-1}xy$ and $[x, {}_{i+1} y] = [[x, {}_i y], y]$ for $i \geq 1$. The word $[x, {}_n y]$ is called the nth Engel word. An element g of a group G is called a (left) Engel element if for any $x \in G$ there exists $n = n(g, x) \geq 1$ such that $[x, {}_n g] = 1$. If n here does not depend on x, then g is a (left) n-Engel element.

The next result is a well-known theorem due to Baer (see [**15**, Satz III.6.15]).

Lemma 6.1. *Let G be a finite group generated by Engel elements. Then G is nilpotent.*

A well-known theorem of Gruenberg says that a soluble group generated by finitely many Engel elements is nilpotent [**12**]. The following lemma shows that the nilpotency class of the group is bounded in terms of the relevant parameters.

Lemma 6.2. [**23**, Lemma 4.1] *Let G be a group generated by d elements which are n-Engel. Suppose that G is soluble with derived length s. Then G is nilpotent with (d, n, s)-bounded class.*

Let $v = v(x_1, \dots, x_k)$ be a weakly rational word and consider the word $w = [v_1, \dots, v_n]$ where $v_i = v(x_{i,1}, \dots, x_{i.k})$, for $i = 1, \dots, n$. We will now prove that w is boundedly in concise the class of residually finite groups. Obviously, for any group G we have $w(G) = \gamma_n(v(G))$.

Proof of Theorem 1.3. As usual, it is sufficient to prove that, given a finite group G such that G_w contains only m elements, $w(G)$ has (w, m)-bounded order. Without loss of generality, we can also assume that $w(G)$ is abelian (see Lemma 2.1).

Let $x \in G$ and $y \in G_v$. Note that, since $[a^{-1}, b] = [b, a]^{a^{-1}} = [b^{a^{-1}}, a]$, the element

$$[x, y, y] = [y^{-x}y, y] = [y^{-xy}, y] = [y^{y^{-xy}}, y^{xy}]$$

is a $[v_1, v_2]$-value. Thus

$$[x, {}_n y] = [[x, y, y], {}_{n-2} y] \in G_w.$$

Moreover, as $w(G)$ is abelian, whenever $x \in w(G)$ we have

$$[x^i, {}_n y] = [x, {}_n y]^i.$$

So $[x,_n y]^i \in G_w$ for every integer i. As $|G_w| \leq m$, it follows that the elements of the form $[x, {}_n y]$, with $x \in w(G)$ and $y \in G_v$, have order at most m.

Since $w(G)$ is abelian and m-generated, the order of the subgroup of $w(G)$ generated by the elements of order at most m is at most m^m. Passing to the quotient over this subgroup we may assume that $[x, {}_n y] = 1$ for all $x \in w(G)$ and $y \in G_v$. Taking into account that $w(G) = \gamma_n(v(G))$ we deduce that every $y \in G_v$ is a $2n$-Engel element in $v(G)$. It follows from Lemma 6.1 that $v(G)$ is nilpotent.

Write $w = [u, v]$, where $u = [v_1, \ldots, v_{n-1}]$, and choose a nontrivial w-value $[x, y]$, where x is a u-value and y is a v-value. Our next claim is that $H = \langle y^x, y \rangle$ is nilpotent of n-bounded class. Note that the quotient group $v(G)/w(G)$ is nilpotent of class at most n. Since $w(G)$ is abelian, we deduce that $v(G)$ is soluble with derived length at most $n+1$. As H is generated by two $2n$-Engel elements, by applying Lemma 6.2, we obtain that H is nilpotent with n-bounded class, as claimed.

Since v is a weakly rational word and $v(G)$ is nilpotent, we are in a position to apply Corollary 2.3 and deduce that the order of $[x, y]$ is bounded by some number f depending only on m and n. Thus, $w(G)$ is an abelian subgroup generated by at most m elements of (w, m)-bounded order. We conclude that the order of $w(G)$ is (w, m)-bounded. This establishes the theorem. □

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