



ALMA MATER STUDIORUM  
UNIVERSITÀ DI BOLOGNA

ARCHIVIO ISTITUZIONALE  
DELLA RICERCA

## Alma Mater Studiorum Università di Bologna Archivio istituzionale della ricerca

Minimal Gevrey Regularity for Hörmander Operators

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

*Published Version:*

Minimal Gevrey Regularity for Hörmander Operators / Bove, Antonio; Mughetti, Marco. - In: INTERNATIONAL MATHEMATICS RESEARCH NOTICES. - ISSN 1073-7928. - STAMPA. - 2024:4(2024), pp. 2790-2832. [10.1093/imrn/rnad055]

*Availability:*

This version is available at: <https://hdl.handle.net/11585/954337> since: 2024-03-06

*Published:*

DOI: <http://doi.org/10.1093/imrn/rnad055>

*Terms of use:*

Some rights reserved. The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (<https://cris.unibo.it/>).  
When citing, please refer to the published version.

(Article begins on next page)

This is the final peer-reviewed accepted manuscript of:

**Antonio Bove, Marco Mughetti, Minimal Gevrey Regularity for Hörmander Operators, *International Mathematics Research Notices*, 2023;, rnad055**

The final published version is available online at: <https://doi.org/10.1093/imrn/rnad055>

Terms of use:

Some rights reserved. The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (<https://cris.unibo.it/>)

**When citing, please refer to the published version.**

# MINIMAL GEVREY REGULARITY FOR HÖRMANDER OPERATORS

ANTONIO BOVE AND MARCO MUGHETTI

ABSTRACT. We prove a minimal Gevrey regularity theorem for Hörmander's sum of squares type operators (1.1), improving the result of Derridj and Zuily [10]. The Gevrey index given here is optimal, in the sense that there are operators of this type that just attain that regularity and not any better.

## 1. INTRODUCTION

Consider the following operator

$$(1.1) \quad P(x, D) = \sum_{j=1}^N (X_j(x, D))^2 + iX_0(x, D) + g(x),$$

where  $g \in G^s(\Omega)$ , the operators  $X_j(x, D)$ ,  $j = 0, \dots, N$ , are vector fields in  $\Omega$  with real  $G^s$  coefficients having the form

$$(1.2) \quad X_j(x, D) = \sum_{k=1}^n a_{jk}(x)D_k, \quad D_k = \frac{1}{i} \frac{\partial}{\partial x_k} \quad j = 0, 1, \dots, N,$$

We remark that  $iX_0(x, D)$  is a real vector field in the usual sense. We point out that when the vector field  $X_0$  is complex the problem is much more involved. We refer to [16], [20], [3] for papers devoted to that case.

If  $X, Y$ , are two vector fields we write  $[X, Y]$ , the commutator, or Lie bracket, of  $X$  and  $Y$  as

$$[X, Y] = XY - YX = \sum_{j=1}^n ((Xb_j) - (Ya_j)) D_j,$$

where  $X(x, D) = \sum_{j=1}^n a_j(x)D_j$ ,  $Y(x, D) = \sum_{j=1}^n b_j(x)D_j$ .

---

*Date:* January 25, 2023.

*2020 Mathematics Subject Classification.* 35H10, 35H20 (primary), 35B65, 35A20, 35A27 (secondary).

*Key words and phrases.* Hörmander operators; Gevrey hypoellipticity; subelliptic operators.

Let  $r$  be a positive integer and let  $i_1, \dots, i_r \in \{0, \dots, N\}$ . Then we denote by  $I$  the multiindex  $I = (i_1, \dots, i_r)$  and by  $X_I$  the  $r$  times iterated commutator

$$X_I = [X_{i_1}, [X_{i_2}, \dots [X_{i_{r-1}}, X_{i_r}] \dots]].$$

We denote by  $|I|$ , according to Rothschild and Stein, [24], the weighted length of the commutator  $X_I$  defined as

$$(1.3) \quad |I| = \#\{i_\ell \mid i_\ell \geq 1, \ell = 1, \dots, r\} + 2 \#\{i_\ell \mid i_\ell = 0, \ell = 1, \dots, r\}.$$

We assume that the vector fields  $X_0, \dots, X_N$  verify Hörmander hypothesis, i.e. that

- (H) For every  $x_0 \in \Omega$ , there is a neighborhood  $U = U_{x_0} \Subset \Omega$  and a positive integer  $m$ , depending on  $U$ , such that the vector fields  $X_0, \dots, X_N$ , as well as their commutators of length  $\leq m$ , generate the  $n$ -dimensional Lie algebra on  $U$ .

Hörmander, [12], and Rothschild and Stein, [24], proved that the operator (1.1) satisfying hypothesis (H) is  $C^\infty$  hypoelliptic and that the following a priori estimate holds:

$$(1.4) \quad \|u\|_{\frac{2}{m}} + \sum_{i,j=1}^N \|X_i X_j u\|_0 + \|X_0 u\|_0 \leq C (\|Pu\|_0 + \|u\|_0),$$

where  $u \in C_0^\infty(\Omega_1)$ ,  $\Omega_1 \Subset \Omega$ ,  $m$  denotes the minimum length of the Poisson bracket needed to generate the Lie algebra in  $\Omega_1$  and  $C = C(\Omega_1)$  is a suitable positive constant.

Let us first recall the definition of Gevrey class and of Gevrey hypoellipticity on an open subset  $U$  of  $\Omega$ .

**Definition 1.1.** Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ . We say that the function  $u \in C^\infty(\Omega)$  is in the Gevrey class  $G^s(\Omega)$ , with  $s \geq 1$ , real number, if for every compact set  $K \subset \Omega$  there is a positive constant  $C_K$  such that

$$|\partial^\alpha u(x)| \leq C_K^{|\alpha|+1} \alpha!^s, \quad \text{for every } x \in K,$$

and for every multiindex  $\alpha$ .

We also define  $G_0^s(\Omega)$  as the intersection  $G^s(\Omega) \cap C_0^\infty(\Omega)$ .

**Remark 1.2.** We observe that if  $u \in G_0^s(\Omega)$ , then there exist positive constants,  $M, C$ , such that, for every  $\xi \in \mathbb{R}^n$ , we have

$$(1.5) \quad |\hat{u}(\xi)| \leq M e^{-C|\xi|^{\frac{1}{s}}}.$$

**Definition 1.3.** We say that the operator  $P$  is  $C^\infty$  (Gevrey  $s$ ,  $s \geq 1$ ) hypoelliptic in the open subset  $U \subset \Omega$  if for every  $u \in \mathcal{D}'(U)$  and for every open subset  $U_1 \subset U$ ,  $Pu \in C^\infty(U_1)$  ( $Pu \in G^s(U_1)$ ) implies that  $u \in C^\infty(U_1)$  ( $u \in G^s(U_1)$ ).

The purpose of this paper is to give the Gevrey regularity of the solutions to  $Pu = f$ ,  $f$  real analytic, implied by (1.4). We point out that this is the minimal Gevrey regularity for this type of equations in the sense that it is obtained for every operator in the class. However it not difficult to make examples of operators having a higher regularity; for instance the anharmonic oscillator

$$\Delta_{x'} + |x'|^{2(q-1)}\Delta_{x''}, \quad (x', x'') = x \in \mathbb{R}^n,$$

has minimal Gevrey regularity  $q$ , but is analytic hypoelliptic. An important role is played by the symplecticity of the characteristic manifold. However it is known that even in two variables a symplectic characteristic manifold does not imply analytic hypoellipticity (see G. Chinni, [6], for a result in this case.)

On the other hand the Baouendi, Goulaouic operator,

$$P_{BG}(x, D) = D_1^2 + D_2^2 + x_1^2 D_3^2,$$

and  $P_1$  in (1.6) give instances of operators for which the minimal regularity is optimal.

In this paper we prove the theorem

**Theorem 1.4.** *Let  $x_0 \in \Omega$  and assume that  $(\mathbf{H})$  holds. Denote by  $U$  the open set associated to  $x_0$  in  $(\mathbf{H})$  and by  $m$  the maximum length of the commutators generating the Lie algebra on  $U$ , then the operator  $P$  is  $G^s$  hypoelliptic on  $U$ , for  $s \geq m$ .*

We explicitly remark that Theorem 1.4 coincides with Theorem 1.6 of Derridj and Zuily, [10], when the Lie algebra is generated using only the vector fields  $X_1, \dots, X_N$ . On the other hand Theorem 1.4 is sharper than Theorem 1.5 of [10]. More precisely when  $X_0$  is needed to generate the Lie algebra we get a Gevrey regularity being one half of that found in [10].

Recently Derridj, [11], studied the influence of the vector field  $X_0$  on the regularity of Gevrey vectors for  $P$ .

As a consequence once we agree to weigh the vector fields according to (1.3) the Gevrey minimal regularity is given by the length of the longest bracket generating the Lie algebra, no matter if this uses the vector field  $X_0$  or not.

**Remark 1.5.** *The hypoellipticity defined in Definition 1.3 is the analytic (Gevrey) hypoellipticity in distributions. Fix a number  $s > 1$ ; an analogous definition can be given in the ultradistributions space of order  $s$ ,  $\mathcal{D}^{\{s\}'}$ , (the dual space of  $G_0^s$ .) i.e. in Definition 1.3 we take  $u \in \mathcal{D}^{\{s\}'}(U)$ .*

*However in [7] it has been shown that in general there may be an ultradistribution  $u \in \mathcal{D}^{\{s\}'} \setminus \mathcal{D}'$ , for a suitable  $s > 1$ , such that  $Pu \in C^\infty$ ,*

*i.e. Hörmander bracket condition does not imply  $C^\infty$  hypoellipticity in ultradistributions.*

*Actually Cordaro and Hanges in [7] prove Lemma 2.1, which, when applied to an operator  $Q$  of the form given in (1.1), satisfying Hörmander hypothesis, can be stated as*

**Lemma 1.6.** *[[7], Lemma 2.1] Let  $Q$  be as in (1.1) satisfying the assumptions of Theorem 1.4 and  $s > 1$ . Assume that  $Q$  is Gevrey  $s$  hypoelliptic in  $\Omega$  in distributions. Then  ${}^tQ$  is  $C^\infty$  hypoelliptic in  $\mathcal{D}^{\{s\}'}$ .*

*Assume that  $P$  satisfies the assumptions of Theorem 1.4, then  ${}^tP$  satisfies the same assumptions. Assume that  $s \geq m$ . By Theorem 1.4, applying Lemma 1.6 to  ${}^tP$ , we obtain that  $P$  is  $C^\infty$  hypoelliptic in  $\mathcal{D}^{\{s\}'}$ . The same argument of the following sections then can be used to show that  $P$  is  $G^s$  hypoelliptic in  $\mathcal{D}^{\{s\}'}$  for  $s \geq m$ .*

*We are grateful to one of the referees for suggesting to include this remark.*

Next we give some examples to illustrate the role of the vector field  $X_0$ .

**1.** Let  $P_1$  denote the operator

$$(1.6) \quad P_1 = \sum_{j=1}^{n-1} D_j^2 + ix_1^k D_n, \quad n \geq 2,$$

where  $k$  is a positive integer. Theorem 1.4 implies that  $P_1$  is Gevrey  $s$  hypoelliptic for  $s \geq k + 2$ . Ōkaji, [21], proved that the regularity  $G^{k+2}$  is optimal when  $n \geq 3$  and  $k$  is even or  $k = 3$ . On the other hand if  $n = 2$  or  $k = 1$  one has the  $G^2$  regularity ([21].)

When  $n \geq 3$  and  $k = 2$  there is an elementary proof that the value  $k+2 = 4$  gives the optimal regularity. To this end we use an explicit representation of the solution of the equation  $P_1 u = 0$ . Define

$$u(x) = \int_0^{+\infty} e^{ix_n \tau} e^{-\omega \frac{x_1^2}{2} \tau^{\frac{1}{2}} + zx_2 \tau^{\frac{1}{4}} - \tau^{\frac{1}{4}}} d\tau,$$

where  $\omega = e^{i\frac{\pi}{4}}$ ,  $z = e^{i\frac{\pi}{8}}$  and  $x$  is in a neighborhood of the origin. We have

$$|D_n^\ell u(0)| = \int_0^{+\infty} \tau^\ell e^{-\tau^{\frac{1}{4}}} d\tau \sim \ell!^4.$$

This shows that the theorem gives the optimal regularity, however in particular cases a better regularity can be attained.

**2.** In the next example we show that the vector field  $X_0$  has no influence on the hypoellipticity of the operator.

Consider now the operator

$$(1.7) \quad P_2 = D_1^2 + x_1^{2(q-1)} D_2^2 + ix_1^k D_2, \quad q \geq 2.$$

The Lie algebra is generated by brackets of length equal to  $r = \min\{k+2, q\}$ . Theorem 1.4 gives then that  $P_2$  is Gevrey  $r$  hypoelliptic. However in section 4 we prove that  $P_2$  is actually  $C^\omega$  hypoelliptic for any  $q, k$ . This is no surprise, since even for sums of squares the operator may actually attain a higher regularity than the minimal.

The technique of proof consists in using iteratively the a priori estimate for  $P$ . Next we give a sketchy idea of the proof, neglecting errors and the more gory details.

We start with the norm  $\|X_j^2|D|^p\varphi_p u\|_0$ , where  $\varphi_p$  is a cutoff function of the type defined in Definition 3.2 and  $|D|$  means  $(-\Delta)^{\frac{1}{2}}$ . Applying the a priori estimate (1.4) we are led to compute

$$(1.8) \quad \|P|D|^p\varphi_p u\|_0 \leq \| |D|^p\varphi_p Pu\|_0 + \|[P, |D|^p\varphi_p]u\|_0.$$

The first term is good since  $Pu = f \in G^m$  has the good estimates by assumption. Consider the second. We have

$$\begin{aligned} \|[P, |D|^p\varphi_p]u\|_0 &\leq \sum_{j=1}^N \left( 2\|X_j[X_j, |D|^p\varphi_p]u\|_0 \right. \\ &\quad \left. + \|[X_j, [X_j, |D|^p\varphi_p]]u\|_0 \right). \end{aligned}$$

The norms containing the double commutator are treated as the first norm. Since  $\|X_j[X_j, |D|^p\varphi_p]u\|_0 \leq \|X_j[X_j, |D|^p]\varphi_p u\|_0 + \|X_j|D|^p[X_j, \varphi_p]u\|_0$ , we have

$$\begin{aligned} \|X_j[X_j, |D|^p\varphi_p]u\|_0 &\leq C \left( p\|X_j|D|^p\varphi_p u\|_0 + \|X_j|D|^p\varphi_p' u\|_0 \right) \\ &\leq C_1 \left( p\|X_j\langle D \rangle^{\frac{1}{m}}|D|^{p-\frac{1}{m}}\varphi_p u\|_0 + \|X_j\langle D \rangle^{\frac{1}{m}}|D|^{p-\frac{1}{m}}\varphi_p' u\|_0 \right) \\ &\leq C_2 \left( p\|P|D|^{p-\frac{1}{m}}\varphi_p u\|_0 + \|P|D|^{p-\frac{1}{m}}\varphi_p' u\|_0 \right). \end{aligned}$$

The last estimate is a consequence of Lemma 2.1. Here we forgot about the support of the functions  $|D|^{p-\frac{1}{m}}\varphi_p u$ ,  $|D|^{p-\frac{1}{m}}\varphi_p' u$ . This issue is addressed in Proposition 3.5.

The latter terms have the same form as that in (1.8), where  $|D|^p$  has been replaced by  $|D|^{p-\frac{1}{m}}$  paying a factor  $p$ . Iterating the argument we get that  $|D|^p$  becomes  $p^{mp}$  which yields the desired Gevrey order.

## 2. SOME PREPARATIONS

For the proof we need a slight modification of the a priori estimate (1.4). More precisely, if  $\omega \in C_0^\infty(\Omega)$ ,  $\omega \equiv 1$  in  $\Omega_1$ , we consider the vector fields  $\omega X_j$ ,  $j = 0, 1, \dots, N$ . We observe that (1.4) applies to the vector fields  $\omega X_j$ ,

for every  $u \in C_0^\infty(\Omega_1)$ . From what follows it shall be apparent that we may assume that the vector fields  $X_j$  are compactly supported.

**Lemma 2.1.** *We use the same notations of the preceding section and assume that the vector fields  $X_j$  have compact support. Then we have the a priori estimate*

$$(2.1) \quad \|u\|_{\frac{2}{m}} + \sum_{j=1}^N (\|\Lambda X_j u\|_0 + \|X_j \Lambda u\|_0) + \sum_{i,j=1}^N \|X_i X_j u\|_0 + \|X_0 u\|_0 \\ \leq C (\|Pu\|_0 + \|u\|_0),$$

where  $\Lambda = \langle D \rangle^{\frac{1}{m}}$ ,  $\langle D \rangle = (|D|^2 + 1)^{\frac{1}{2}}$  and  $u \in C_0^\infty(\Omega_1)$ , where, as in (1.4),  $\Omega_1 \Subset \Omega$ .

*Proof.* For  $j = 1, \dots, N$  consider the norm

$$\|\Lambda X_j u\|_0^2 = \langle \Lambda X_j u, \Lambda X_j u \rangle.$$

We have

$$\begin{aligned} \langle \Lambda X_j u, \Lambda X_j u \rangle &= \langle X_j \Lambda u, \Lambda X_j u \rangle + \langle [\Lambda, X_j] u, \Lambda X_j u \rangle \\ &= \langle \Lambda X_j \Lambda u, X_j u \rangle + \langle \Lambda [\Lambda, X_j] u, X_j u \rangle \\ &= \langle X_j \Lambda^2 u, X_j u \rangle + \langle [\Lambda, X_j] \Lambda u, X_j u \rangle + \langle \Lambda [\Lambda, X_j] u, X_j u \rangle \\ &= \langle \Lambda^2 u, X_j^2 u \rangle + \langle \Lambda^2 u, g_j X_j u \rangle + \langle [\Lambda, X_j] \Lambda u, X_j u \rangle + \langle \Lambda [\Lambda, X_j] u, X_j u \rangle, \end{aligned}$$

where we wrote  $X_j^* = X_j + g_j$ ,  $g_j$  denoting a smooth function.

Since

$$\begin{aligned} \|X_j u\|_0^2 &= \langle X_j^* X_j u, u \rangle \leq |\langle X_j^2 u, u \rangle| + |\langle X_j u, g_j u \rangle| \\ &\leq \|X_j^2 u\|_0^2 + \|u\|_0^2 + \varepsilon \|X_j u\|_0^2 + \frac{1}{\varepsilon} C_g \|u\|_0^2. \end{aligned}$$

Taking  $\varepsilon < 1$  we obtain that

$$\|X_j u\|_0^2 \leq C_1 \left( \|X_j^2 u\|_0^2 + \|u\|_0^2 \right).$$

Moreover the operators  $[\Lambda, X_j] \Lambda$ ,  $\Lambda [\Lambda, X_j]$  are pseudodifferential operators of order  $\frac{2}{m}$ , so that

$$\begin{aligned} \langle \Lambda X_j u, \Lambda X_j u \rangle &\leq |\langle \Lambda^2 u, X_j^2 u \rangle| + |\langle \Lambda^2 u, g_j X_j u \rangle| + |\langle [\Lambda, X_j] \Lambda u, X_j u \rangle| \\ &\quad + |\langle \Lambda [\Lambda, X_j] u, X_j u \rangle| \leq C_2 \left( \|X_j^2 u\|_0^2 + \|u\|_{\frac{2}{m}}^2 \right). \end{aligned}$$



To treat the term  $\|X_j \Lambda u\|_0^2$ , we observe that

$$(2.2) \quad \|X_j \Lambda u\|_0 \leq \|\Lambda X_j u\|_0 + \|[X_j, \Lambda]u\|_0 \leq \|\Lambda X_j u\|_0 + C\|u\|_{\frac{1}{m}}.$$

Hence, using estimate (1.4) we conclude as desired.  $\square$

Proving Theorem 1.4 for  $s = m$  is enough, since Theorem 3.1 in Métivier [18] implies that from  $G^m$  hypoellipticity we deduce  $G^s$  hypoellipticity for  $s > m$ , provided the coefficients of the operator have the same regularity.

From now on we denote by  $U_1$  the open set  $\Omega_1$  in (1.4). Let  $f \in G^m(U_1)$ . Because of Hörmander's theorem we may consider a function  $u \in C^\infty(U_1)$  such that

$$(2.3) \quad Pu = f, \quad \text{in } U_1.$$

Let us show that we may reduce ourselves to the case where  $u$ ,  $f$  and the coefficients of the vector fields in  $P$  have compact support contained in  $U_1$ .

Let  $\chi \in G_0^m(U_1) = C_0^\infty(U_1) \cap G^m(U_1)$ , be such that  $\chi \equiv 1$  in  $U_2$ , where  $U_2 \Subset U_1$ . Then

$$(2.4) \quad P(\chi u) = \chi f + [P, \chi]u \in G^m(U_2).$$

Moreover we have that  $\chi u \in C_0^\infty(U_1)$ . Hence we have to solve the problem

$$Pv = h, \quad \text{in } U_1,$$

where  $h \in C_0^\infty(U_1) \cap G^m(U_2)$ , for  $v \in C_0^\infty(U_1)$ . Furthermore we may suppose without loss of generality that the coefficients of the vector fields in  $P$  as well as  $g$  (see eq. (1.1)) are functions in  $G_0^m(U_1)$ . In fact if  $a$  denotes one of these coefficients, we may write  $a = \psi a + (1 - \psi)a$ , where  $\psi \in G_0^m(U_1)$  and  $\psi \equiv 1$  on  $K = \text{supp } \chi$ . We immediately see that the contribution of the  $1 - \psi$  part of a vector field is zero when applied to  $v = \chi u$ . For the sake of simplicity we revert to the old notation with  $u \in C_0^\infty(K)$ ,

$$(2.5) \quad Pu = f \in C_0^\infty(U_1) \cap G^m(U_2),$$

and  $g$  (see equation (1.1)) as well as the coefficients of the vector fields belong to  $G_0^m(U_1)$ . Finally we point out that estimate (2.1) holds for any  $u \in C_0^\infty(K)$  when the vector fields have coefficients with compact support, since the cutoff on the coefficients of the vector fields has no effect when  $u \in C_0^\infty(K)$ , by (2.2).

### 3. PROOF OF THE THEOREM

The main tool of the proof is the estimate (2.1). The basic idea of the proof is to replace  $u$  in the left hand side of (2.1) with  $D^p \varphi u$ , where  $\varphi$  denotes a suitable cutoff function and  $D^p$  denotes a derivative of order  $p$ . Our strategy is to shift derivatives from  $u$  to the known Gevrey function  $Pu$ .

This requires some definitions to make things more precise and allow for an inductive proof.

In what follows we are going to denote by  $C_j$ ,  $j = 0, 1, 2, \dots$ , suitable positive constants independent of  $p$ . It is understood that these constants may change from one proof to another, even if the subscript does not.

Throughout the proof we will use a particular type of cutoff functions supported near a point  $x_0 \in U_2$ , defined e.g. in Ehrenpreis [9] (see also Hörmander [14]).

**Lemma 3.1** (Lemma 2.2 in [14]). *Let  $K$  be a compact subset of  $\mathbb{R}^n$ ,  $R, r > 0$  and  $p$  a positive integer. Then one can find a function  $\varphi \in C_0^\infty$  equal to 1 on  $K$ , such that  $\varphi$  vanishes at all points with distance larger than  $r$  from  $K$  and*

$$(3.1) \quad |D^\alpha \varphi(x)| \leq C^{|\alpha|+1} \left(\frac{R}{r}\right)^{|\alpha|} p^{|\alpha|}, \quad \text{for } |\alpha| \leq R(p+1),$$

where  $C$  is a positive constant depending on  $n$ .

**Definition 3.2.** *For any natural number  $p$ , denote by  $\varphi_p = \varphi_p(x)$  a function in  $C_0^\infty(\mathbb{R}^n)$  such that*

- (S)  $\varphi_p$  is equal to 1 in a small neighborhood,  $W$ , of  $x_0 \in U_2$  and supported in  $W_r \Subset U_2$  (see (2.5),) where  $W_r$  denotes the set of all points with distance from  $W$  less or equal to  $r$ ,  $r$  suitably chosen.

We say that  $\varphi_p$  is an Ehrenpreis sequence of cutoff functions if there is a positive constant  $R$  such that for  $|\alpha| \leq R(p+1)$  we have, for every  $p$

$$(3.2) \quad |\partial_x^\alpha \varphi_p(x)| \leq C_\varphi^{|\alpha|+1} p^{|\alpha|},$$

where  $C_\varphi > 0$  and independent of  $p$ .

Next we define the type of derivatives we take on  $u$ .

**Definition 3.3.** *Let  $p$  denote a natural number,  $q$  a rational number with  $q \leq p$ ,  $R_0 > 0$ . We denote by  $\Lambda_q^p$  the class of smooth functions,  $L_q(\xi)$ , in  $S_{1,0}^q(\mathbb{R}^n)$  (see Definition A.7 and [15] eq. (18.1.1)'') such that*

$$(3.3) \quad |\partial_\xi^\alpha L_q(\xi)| \leq C_\Lambda^{1+|\alpha|} p^{|\alpha|} \langle \xi \rangle^{q-|\alpha|},$$

for every  $\alpha$ ,  $|\alpha| \leq R_0(q+1)$ , where  $C_\Lambda$  denotes a positive constant independent of  $p$ ,  $q$ ,  $\alpha$  and  $L_q$ . Since the estimate (3.3) matters when  $p$  is large, we may choose  $R_0$  large enough to allow a fixed shift in the multiindex  $\alpha$ .

We remark that  $\xi^\alpha$ , with  $|\alpha| = q$ , belongs to  $\Lambda_q^p$  and the same holds for  $C^{-q}|\xi|^q$  and  $C^{-q}\langle \xi \rangle^q$ , for a suitable positive constant  $C$  independent of  $q$ .

**Definition 3.4.** We define the norm (see (2.1))

$$(3.4) \quad |||u|||^2 = \|u\|_{\frac{2}{m}}^2 + \sum_{j=1}^N \|X_j u\|_{\frac{1}{m}}^2 + \sum_{i,j=1}^N \|X_i X_j u\|_0^2 + \|X_0 u\|_0^2,$$

for  $u \in \mathcal{S}(\mathbb{R}^n)$ .

In what follows we are going to prove an estimate of the form

$$|||L_q(D)\varphi_p u||| \leq C_{\#}^{q+1} p^{qm},$$

where  $q \leq p$ ,  $L_q \in \Lambda_q^p$ ,  $C_{\#} = C_{\#}(C_{\Lambda})$  is a positive constant independent of  $p$ ,  $q$  and  $\varphi_p$  is a cutoff function of the type defined in Definition 3.2.

The above estimate implies that  $u \in G^m$  in a neighborhood of the point  $x_0$  and ultimately that  $u \in G^m(U_1)$ .

For technical reasons we are going to prove an estimate of the form

$$(3.5) \quad |||L_q(D)\varphi_p^{(k)} u||| \leq C_{\#}^{k+2mq+1} p^{mq+k+\sigma},$$

where  $\varphi_p^{(k)} = \partial_x^{\alpha} \varphi_p$  with  $|\alpha| = k$ ,  $C_{\#}$  does not depend on  $k$ ,  $q$ ,  $p$ ,  $mq+k \leq mp$  and  $\sigma$  is a positive constant independent of  $q$ ,  $p$ .

This is done by arguing by induction with respect to  $q$ : we start with  $q = 0$ .  $L_0$  is a bounded operator in  $L^2(\mathbb{R}^n)$  whose norm depends only on  $C_{\Lambda}$  in Definition 3.3 and on a fixed power of  $p$ .

Then for  $k \leq mp$  we have that  $|||\varphi_p^{(k)} u||| \leq C^{k+1} p^{k+2}$ , where  $C$  is a suitable constant depending on  $C_{\varphi}$  (see definition 3.2) and on the problem data. This allows us to obtain (3.5) when  $q = 0$  and  $\sigma$  is a suitable positive number depending on  $n$ .

Our induction has steps of  $\frac{1}{m}$  and thus it is convenient to write  $q$  as  $\tilde{q}/m$  and  $p$  as  $\tilde{p}/m$ . Then the relation  $mq+k \leq mp$  becomes  $\tilde{q}+k \leq \tilde{p}$ . Assume that (3.5) is satisfied for every  $\tilde{q}' \leq \tilde{q}-1$  and  $k$  such that  $\tilde{q}'+k \leq \tilde{p}$ . Then we have to prove that (3.5) holds for every  $\tilde{q}' \leq \tilde{q}$  and  $k$  such that  $\tilde{q}'+k \leq \tilde{p}$ .

It is convenient to simplify the notation a bit by writing  $L_q(D)$  as  $L_{\tilde{q}}(D)$ , where  $q = \tilde{q}/m$ . Hence the order of  $L_{\tilde{q}}(D)$  is  $\tilde{q}/m$ .

Estimate (3.5) is then rewritten as

$$(3.6) \quad |||L_{\tilde{q}}(D)\varphi_p^{(k)} u||| \leq C_{\#}^{k+2\tilde{q}+1} p^{\tilde{q}+k+\sigma}.$$

In order to apply (2.1) we use the cutoff function  $\chi$  of equation (2.4). We have

$$(3.7) \quad |||L_{\tilde{q}}(D)\varphi_p^{(k)} u||| \leq |||\chi L_{\tilde{q}}(D)\varphi_p^{(k)} u||| + |||(1-\chi)L_{\tilde{q}}(D)\varphi_p^{(k)} u|||.$$

We are going to apply (2.1) with  $u$  replaced by  $\chi L_{\tilde{q}}(D)\varphi_p^{(k)} u$  to the first term on the right hand side above. Before doing this let us discuss the second summand above:

$$\begin{aligned}
(3.8) \quad & \| (1 - \chi) L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_{\frac{2}{m}}^2 \\
&= \| (1 - \chi) L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_{\frac{2}{m}}^2 + \sum_{j=1}^N \| X_j (1 - \chi) L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_{\frac{1}{m}}^2 \\
&+ \sum_{i,j=1}^N \| X_i X_j (1 - \chi) L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0^2 + \| X_0 (1 - \chi) L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0^2.
\end{aligned}$$

Consider the first term on the right hand side and apply Lemma A.3 to  $L_{\tilde{q}}(D) \varphi_p^{(k)}$  remarking that  $\chi \equiv 1$  on  $U_2$  and that, by **(S)**,  $\text{supp } \varphi_p \subset U_2$ . We get

$$\begin{aligned}
\| (1 - \chi) L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_{\frac{2}{m}} &= \| (1 - \chi) R_{\tilde{q}, \varphi, k} u \|_{\frac{2}{m}} \\
&\leq \| (1 - \chi) R_{\tilde{q}, \varphi, k} u \|_1 \leq C_\chi \| R_{\tilde{q}, \varphi, k} u \|_1 \leq C_\chi C_0^{k+1} p^{k+\tilde{q}+\nu} \| u \|_0,
\end{aligned}$$

where the last estimate is due to (A.14).

Consider next  $\| X_i X_j (1 - \chi) L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0$ . Again by Lemma A.3 we have

$$\| X_i X_j (1 - \chi) L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 = \| X_i X_j (1 - \chi) R_{\tilde{q}, \varphi, k} u \|_0.$$

Let us write  $X_i X_j$  as a finite sum of terms of the form  $a \partial_\ell \partial_r$  and  $c \partial_\ell$ . Then we have to estimate a finite number of norms of the form  $\| a \partial_\ell \partial_r (1 - \chi) R_{\tilde{q}, \varphi, k} u \|_0$  and  $\| c \partial_\ell (1 - \chi) R_{\tilde{q}, \varphi, k} u \|_0$ . Let us discuss the first norm, the other being completely analogous. We have

$$\begin{aligned}
\| a \partial_\ell \partial_r (1 - \chi) R_{\tilde{q}, \varphi, k} u \|_0 \\
\leq \| a (1 - \chi) R_{\tilde{q}, \varphi, k} \partial_\ell \partial_r u \|_0 + \| a [\partial_\ell \partial_r, (1 - \chi) R_{\tilde{q}, \varphi, k}] u \|_0.
\end{aligned}$$

The first term is bound as  $\tilde{C}_\chi C_0^{k+1} p^{k+\tilde{q}+\nu} \| u \|_2$  by Lemma A.3. As for the second term we observe that it is a finite sum of terms containing an order zero pseudodifferential operator whose symbol is a  $x$ -derivative of order at most 2 of the symbol of  $R_{\tilde{q}, \varphi, k}$ . By (A.13) of Lemma A.3 and Theorem A.8, we obtain an analogous estimate  $C_a C_0^{k+1} p^{k+\tilde{q}+\nu} \| u \|_0$ .

The other terms in (3.8) are treated analogously and we skip them. Hence the term in (3.8) has the estimate (3.6). We are left with the term involving the function  $\chi$  in (3.7). By (2.1) we have

$$\| \chi L_{\tilde{q}}(D) \varphi_p^{(k)} u \| \leq C \left( \| P \chi L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 + \| \chi L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 \right).$$

The last term is bound by  $C_\chi \| L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0$ .

Consider then  $\| P \chi L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0$ . We have

$$\| P \chi L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 \leq C_\chi \| P L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 + \| [P, \chi] L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0.$$

Consider the second term, containing

$$[P, \chi] = \sum_{j=1}^N \left( 2[X_j, \chi]X_j + [X_j, [X_j, \chi]] \right) + [X_0, \chi].$$

Thus

$$\begin{aligned} & \| [P, \chi] L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 \\ & \leq \sum_{j=1}^N \left( 2 \| [X_j, \chi] X_j L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 + \| [X_j, [X_j, \chi]] L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 \right) \\ & \quad + \| [X_0, \chi] L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 \\ & \leq C_1 \sum_{j=1}^N \left( \| X_j L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 + \| L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 \right). \end{aligned}$$

Hence we are left with the estimate of norms of the form  $\| X_j L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0$ . By Lemma A.6 we have

$$\begin{aligned} \| X_j L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 & = K \| X_j \langle D \rangle^{\frac{1}{m}} \frac{1}{K} \langle D \rangle^{-\frac{1}{m}} L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 \\ & = K \| X_j \langle D \rangle^{\frac{1}{m}} L_{\tilde{q}-1}(D) \varphi_p^{(k)} u \|_0. \end{aligned}$$

To obtain a term present in the expression (3.4), we observe that

$$\begin{aligned} (3.9) \quad & K \| X_j \langle D \rangle^{\frac{1}{m}} L_{\tilde{q}-1}(D) \varphi_p^{(k)} u \|_0 \\ & \leq K \| X_j L_{\tilde{q}-1}(D) \varphi_p^{(k)} u \|_{\frac{1}{m}} + K \| [X_j, \langle D \rangle^{\frac{1}{m}}] L_{\tilde{q}-1}(D) \varphi_p^{(k)} u \|_0 \\ & = K \| X_j L_{\tilde{q}-1}(D) \varphi_p^{(k)} u \|_{\frac{1}{m}} + K \| [X_j, \langle D \rangle^{\frac{1}{m}}] \langle D \rangle^{-\frac{1}{m}} \langle D \rangle^{\frac{1}{m}} L_{\tilde{q}-1}(D) \varphi_p^{(k)} u \|_0 \\ & \leq K_1 \left( \| X_j L_{\tilde{q}-1}(D) \varphi_p^{(k)} u \|_{\frac{1}{m}} + \| L_{\tilde{q}-1}(D) \varphi_p^{(k)} u \|_{\frac{1}{m}} \right), \end{aligned}$$

where we used the fact that  $[X_j, \langle D \rangle^{\frac{1}{m}}] \langle D \rangle^{-\frac{1}{m}}$  has order zero and is continuous on  $L^2$  with norm independent of  $p$ . Applying the inductive hypothesis (see (3.6)), we obtain an estimate of the form

$$\| X_j L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 \leq \varepsilon_1 C_{\#}^{k+2\tilde{q}+1} p^{k+\tilde{q}-1+\sigma},$$

where  $\varepsilon_1, 0 < \varepsilon_1 < 1$ , will be chosen later.

We may summarize what has been done in the

**Proposition 3.5.** *We have*

$$(3.10) \quad \| \| L_{\tilde{q}}(D) \varphi_p^{(k)} u \| \| \leq C \left( \| PL_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 + \| L_{\tilde{q}}(D) \varphi_p^{(k)} u \|_0 \right)$$

$$+ \varepsilon C_{\#}^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma},$$

where  $0 < \varepsilon < 1$ .

The above proposition allows us to perform the following estimate

$$(3.11) \quad \begin{aligned} & \|L_{\tilde{q}}(D)\varphi_p^{(k)}u\| \\ & \leq C \left( \|L_{\tilde{q}}(D)\varphi_p^{(k)}Pu\|_0 + \|[P, L_{\tilde{q}}(D)\varphi_p^{(k)}]u\|_0 + \|L_{\tilde{q}}(D)\varphi_p^{(k)}u\|_0 \right) \\ & \quad + \varepsilon C_{\#}^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma}. \end{aligned}$$

The last term in the second line above is estimated using the induction. We apply Lemma A.6 with  $\theta = \frac{2}{m}$ . We have (for  $\tilde{q} \geq 2$ )

$$(3.12) \quad \begin{aligned} \|L_{\tilde{q}}(D)\varphi_p^{(k)}u\|_0 & \leq K \|L_{\tilde{q}-2}(D)\varphi_p^{(k)}u\|_{\frac{2}{m}} \leq K \|L_{\tilde{q}-2}(D)\varphi_p^{(k)}u\| \\ & \leq K C_{\#}^{-4} C_{\#}^{k+2\tilde{q}+1} p^{\tilde{q}+k+\sigma-2} \leq \varepsilon C_{\#}^{k+2\tilde{q}+1} p^{\tilde{q}+k+\sigma}, \end{aligned}$$

provided  $C_{\#}^4 \geq \varepsilon^{-1}K$ , where  $\varepsilon$  is a small positive constant to be chosen later. We point out that, at the end of this process,  $\varepsilon$  will be chosen small depending on a number of constants given by the problem. As a consequence  $C_{\#}$  is chosen large depending only on the problem's data.

Let us consider the first term in the right hand side of (3.11).

Let  $\psi \in G_0^m(U_2)$  denote a cutoff function with support contained in  $U_2$  and such that  $\psi \equiv 1$  on  $W_r$  (see Definition 3.2.) Then

$$\|L_{\tilde{q}}(D)\varphi_p^{(k)}Pu\|_0 = \|L_{\tilde{q}}(D)\varphi_p^{(k)}\psi Pu\|_0.$$

Then

$$\|L_{\tilde{q}}(D)\varphi_p^{(k)}\psi Pu\|_0 \leq \|\varphi_p^{(k)}L_{\tilde{q}}(D)\psi Pu\|_0 + \|[L_{\tilde{q}}(D), \varphi_p^{(k)}]\psi Pu\|_0 = N_1 + N_2.$$

Consider  $N_1$ . We have from (3.3)

$$\begin{aligned} N_1 & \leq \|\varphi_p^{(k)}\|_{\infty} \|L_{\tilde{q}}(D)\psi Pu\|_0 \\ & = \|\varphi_p^{(k)}\|_{\infty} (2\pi)^{\frac{n}{2}} \|L_{\tilde{q}}(\xi)\widehat{\psi Pu}\|_0 \leq (2\pi)^{\frac{n}{2}} C_{\varphi}^{k+1} C_{\Lambda} p^k \|\langle \xi \rangle^{\frac{\tilde{q}}{m}} \widehat{\psi Pu}\|_0. \end{aligned}$$

Because of the decaying property of the compactly supported Gevrey functions, by (1.5) we have the estimate ( $q = \tilde{q}/m$ )

$$(3.13) \quad N_1 \leq (2\pi)^{\frac{n}{2}} C_{\varphi}^{k+1} C_{\Lambda} p^k \|\langle \xi \rangle^{\frac{\tilde{q}}{m}} C_u e^{-\delta|\xi|^{\frac{1}{m}}}\|_0 \leq C_{N_1}^{k+\tilde{q}+1} p^{k+\tilde{q}}.$$

Consider now  $N_2$ . Applying Lemma A.2 we have

$$\begin{aligned} N_2 & = \|[L_{\tilde{q}}(D), \varphi_p^{(k)}]\psi Pu\|_0 \\ & \leq \sum_{\ell=1}^{[q]} \sum_{|\beta|=\ell} \frac{1}{\beta!} \|\varphi_p^{(k+|\beta|)}(x) (\partial_{\xi}^{\beta} L_{\tilde{q}})(D)\psi Pu\|_0 + \|R_{q,\varphi,k}(x, D)\psi Pu\|_0. \end{aligned}$$

Consider the last term in the right hand side above. Since  $R_{q,\varphi,k}(x,\xi) \in S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$ , its corresponding pseudodifferential operator is bounded in  $L^2(\mathbb{R}^n)$ , with a norm bounded by a seminorm of its symbol (see Theorem 18.1.11 of [15] and Theorem A.8.) By (A.8) we obtain that

$$(3.14) \quad \|R_{q,\varphi,k}(x,D)\psi Pu\|_0 \leq C_0^{k+1} p^{k+\tilde{q}+\nu} \|\psi Pu\|_0 \leq C_1^{k+1} p^{k+\tilde{q}+\nu},$$

where  $C_1$  depends on the problem data only.

Consider then a summand of the form

$$\frac{1}{\beta!} \|\varphi_p^{(k+\ell)}(x)(\partial_\xi^\beta L_{\tilde{q}})(D)\psi Pu\|_0,$$

with  $|\beta| = \ell$ ,  $1 \leq \ell \leq [q]$ . Since  $k + \ell \leq k + q \leq k + mq \leq mp \leq R(p + 1)$  provided  $R$  is large enough, we may apply Definition 3.2, Lemma A.4 and, arguing as above, we get

$$\begin{aligned} & \frac{1}{\beta!} \|\varphi_p^{(k+\ell)}(x)(\partial_\xi^\beta L_{\tilde{q}})(D)\psi Pu\|_0 \\ & \leq \frac{1}{\beta!} C_\varphi^{k+\ell+1} p^{k+\ell} C_\Lambda^\ell p^\ell \|L_{\tilde{q}-m\ell}(D)\psi Pu\|_0 \\ & \leq \frac{1}{\beta!} C_\varphi^{k+\ell+1} C_\Lambda^\ell (2\pi)^{\frac{n}{2}} p^{k+2\ell} \|L_{\tilde{q}-m\ell}(\xi)\widehat{\psi Pu}\|_0 \\ & \leq \frac{1}{\beta!} C_2^{k+q+1} p^{k+\tilde{q}-(m-2)\ell}. \end{aligned}$$

We observe that the following formula holds:

$$(3.15) \quad \#\{\beta \mid |\beta| = \ell\} = \binom{\ell + n - 1}{n - 1}.$$

Thus

$$\sum_{\ell=1}^{[q]} \sum_{|\beta|=\ell} \frac{1}{\beta!} \|\varphi_p^{(k+|\beta|)}(x)(\partial_\xi^\beta L_{\tilde{q}})(D)\psi Pu\|_0 \leq C_3^{k+q+1} p^{k+\tilde{q}},$$

since  $\psi Pu$  is in  $G_0^m$ .

Summing up, from (3.13), (3.14) and the above inequality, we have the following bound for the first term on the right hand side of (3.11)

$$(3.16) \quad \|L_{\tilde{q}}(D)\varphi_p^{(k)} Pu\|_0 \leq \varepsilon C_\#^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma},$$

provided  $C_\#, \sigma$  are large enough.

Let us finally consider the term

$$(3.17) \quad \|[P, L_{\tilde{q}}(D)\varphi_p^{(k)}]u\|_0.$$

Replacing  $P$  with its expression (1.1) we have

$$\begin{aligned}
(3.18) \quad & \| [P, L_{\bar{q}}(D)\varphi_p^{(k)}]u \|_0 \\
& \leq \sum_{j=1}^N \| [X_j^2, L_{\bar{q}}(D)\varphi_p^{(k)}]u \|_0 + \| [X_0, L_{\bar{q}}(D)\varphi_p^{(k)}]u \|_0 + \| [g, L_{\bar{q}}(D)\varphi_p^{(k)}]u \|_0 \\
& = \sum_{j=1}^N A_j + A_0 + A_g.
\end{aligned}$$

Using the fact that

$$\begin{aligned}
(3.19) \quad [X_j^2, L_{\bar{q}}(D)\varphi_p^{(k)}] &= 2X_j[X_j, L_{\bar{q}}(D)\varphi_p^{(k)}] - [X_j, [X_j, L_{\bar{q}}(D)\varphi_p^{(k)}]] \\
&= \mathcal{A}_{j1} + \mathcal{A}_{j2},
\end{aligned}$$

we define  $A_{ji} = \|\mathcal{A}_{ji}u\|_0$ ,  $i = 1, 2$ .

**3.1. The single commutator with  $X_j$ ,  $A_{j1}$ ,  $j \geq 1$ , in (3.19).** We start our analysis with  $A_{j1}$ :

$$A_{j1} \leq 2 \left( \|X_j[X_j, L_{\bar{q}}(D)]\varphi_p^{(k)}u\|_0 + \|X_j L_{\bar{q}}(D)[X_j, \varphi_p^{(k)}]u\|_0 \right).$$

Plugging (1.2) in the above expression we have

$$\begin{aligned}
(3.20) \quad A_{j1} &\leq 2 \sum_{h=1}^n \left( \|X_j[a_{jh}, L_{\bar{q}}(D)]D_h\varphi_p^{(k)}u\|_0 + \|X_j L_{\bar{q}}(D)a_{jh}[D_h, \varphi_p^{(k)}]u\|_0 \right) \\
&\leq 2 \sum_{h=1}^n \left( \|X_j[a_{jh}, L_{\bar{q}}(D)]D_h\varphi_p^{(k)}u\|_0 + \|X_j a_{jh} L_{\bar{q}}(D)\varphi_p^{(k+1)}u\|_0 \right. \\
&\quad \left. + \|X_j[L_{\bar{q}}(D), a_{jh}]\varphi_p^{(k+1)}u\|_0 \right) = 2 \sum_{h=1}^n \left( A_{j11}^{(h)} + A_{j12}^{(h)} + A_{j13}^{(h)} \right),
\end{aligned}$$

where with some abuse of language we denoted  $D_h\varphi_p^{(k)}$  by  $\varphi_p^{(k+1)}$ .

To treat the term  $A_{j11}^{(h)}$  we use Lemma A.1.

$$\begin{aligned}
(3.21) \quad A_{j11}^{(h)} &\leq \sum_{\ell=1}^{[q]} \sum_{|\beta|=\ell} \frac{1}{\beta!} \|X_j(D_x^\beta a_{jh}(x))\partial_\xi^\beta L_{\bar{q}}(D)D_h\varphi_p^{(k)}u\|_0 \\
&\quad + \|X_j R_{\bar{q}, a_{jh}} D_h\varphi_p^{(k)}u\|_0.
\end{aligned}$$

Let us start by bounding the last term above:

$$(3.22) \quad \|X_j R_{\bar{q}, a_{jh}} D_h\varphi_p^{(k)}u\|_0 \leq \sum_{r=1}^n \|a_{jr}\|_\infty \|D_r R_{\bar{q}, a_{jh}} D_h\varphi_p^{(k)}u\|_0$$



$$\begin{aligned} &\leq \sum_{r=1}^n \|a_{jr}\|_\infty \left( \|R_{\tilde{q}, a_{jh}} D_r D_h \varphi_p^{(k)} u\|_0 + \|\text{Op}(\partial_{x_r} R_{\tilde{q}, a_{jh}}(x, \xi)) D_h \varphi_p^{(k)} u\|_0 \right) \\ &\leq C_0^{q+1} p^{\tilde{q}+\nu} \left( \|D_r D_h \varphi_p^{(k)} u\|_0 + \|D_h \varphi_p^{(k)} u\|_0 \right), \end{aligned}$$

where we used Lemma A.1, (A.2), (A.3), and the fact that the coefficients of the vector fields are uniformly bounded. By (3.2) we conclude that

$$(3.23) \quad \|X_j R_{\tilde{q}, a_{jh}} D_h \varphi_p^{(k)} u\|_0 \leq C_1^{q+k+1} p^{\tilde{q}+k+2+\nu}.$$

Consider now a summand in (3.21).

$$\begin{aligned} &\frac{1}{\beta!} \|X_j (D_x^\beta a_{jh}(x)) \partial_\xi^\beta L_{\tilde{q}}(D) D_h \varphi_p^{(k)} u\|_0 \\ &\leq \frac{1}{\beta!} \left( \|(D_x^\beta a_{jh}(x)) X_j \partial_\xi^\beta L_{\tilde{q}}(D) D_h \varphi_p^{(k)} u\|_0 \right. \\ &\quad \left. + \|(X_j (D_x^\beta a_{jh}(x))) \partial_\xi^\beta L_{\tilde{q}}(D) D_h \varphi_p^{(k)} u\|_0 \right) = B_1 + B_2. \end{aligned}$$

Consider  $B_1$ . We have, by Lemma A.5 with  $|\gamma| = 1$  and  $|\beta| = \ell$ , and Lemma A.6,

$$B_1 \leq K^2 C_\Lambda^\ell C_a^{\ell+1} p^\ell \beta!^{m-1} \|X_j \langle D \rangle^{\frac{1}{m}} L_{\tilde{q}-\ell m+m-1}(D) \varphi_p^{(k)} u\|_0.$$

Arguing as in (3.9), by induction we get (see (3.6), (3.4))

$$\begin{aligned} (3.24) \quad B_1 &\leq C_1^{\ell+1} p^\ell \beta!^{m-1} C_\#^{k+2\tilde{q}-2\ell m+2m-1} p^{k+\tilde{q}-\ell m+m-1+\sigma} \\ &\leq C_\#^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma} C_1^{\ell+1} C_\#^{-2m(\ell-1)-2} \beta!^{m-1} p^{-(\ell-1)(m-1)} \\ &\leq C_\#^{k+\tilde{q}+1} p^{k+\tilde{q}+\sigma} C_\#^{-(2m-1)(\ell-1)-1} \left( \frac{C_1^2}{C_\#} \right)^\ell \\ &\leq C_\#^{k+\tilde{q}+1} p^{k+\tilde{q}+\sigma} \left( \frac{C_1^2}{C_\#} \right)^\ell, \end{aligned}$$

where we used the inequality

$$(3.25) \quad \frac{\beta!}{p^{\ell-1}} \leq \prod_{i=1}^n \frac{\beta_i!}{p^{(\beta_i-1)_+}} \leq \prod_{i=1}^n \left( \frac{2}{p} \dots \frac{\beta_i}{p} \right) \leq 1.$$

Analogously,  $B_2$  is estimated as

$$B_2 \leq K^2 C_\Lambda^\ell C_a^{\ell+2} p^\ell \sum_{r=1}^n \frac{(\beta + e_r)!^m}{\beta!} \|L_{\tilde{q}-\ell m+m-2}(D) \varphi_p^{(k)} u\|_{\frac{2}{m}}.$$

By induction we get, as above,

$$\begin{aligned}
(3.26) \quad B_2 &\leq C_1^{\ell+1} p^\ell C_\#^{k+2\tilde{q}-2\ell m+2m-3} p^{k+\tilde{q}-\ell m+m-2+\sigma} \sum_{r=1}^n \frac{(\beta + e_r)!^m}{\beta!} \\
&\leq C_\#^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma} C_1^{\ell+1} C_\#^{-2m(\ell-1)-4} p^{-(\ell-1)(m-1)-1} \sum_{r=1}^n \frac{(\beta + e_r)!^m}{\beta!} \\
&\leq C_\#^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma} C_1^{\ell+1} C_\#^{-2m(\ell-1)-4} \sum_{r=1}^n \frac{\beta_r + 1}{p} \left( \frac{(\beta + e_r)!}{p^{\ell-1}} \right)^{m-1} \\
&\leq C_\#^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma} \left( \frac{C_1^2}{C_\#} \right)^\ell \frac{n2^{n(m-1)+1}}{C_\#} \leq C_\#^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma} \left( \frac{C_1^2}{C_\#} \right)^\ell,
\end{aligned}$$

where we used the inequality

$$\begin{aligned}
\sum_{r=1}^n \frac{\beta_r + 1}{p} \left( \frac{(\beta + e_r)!}{p^{\ell-1}} \right)^{m-1} &\leq \sum_{r=1}^n 2 \left( \frac{(\beta + e_r)!}{p^{\ell-1}} \right)^{m-1} \\
&\leq \sum_{r=1}^n 2 \left( \prod_{i=1}^n \left( 2 \frac{3}{p} \dots \frac{\beta_i + 1}{p} \right) \right)^{m-1} \leq \sum_{r=1}^n 2^{n(m-1)+1} = n2^{n(m-1)+1},
\end{aligned}$$

and we chose  $C_\# > n2^{n(m-1)+1}$ .

Plugging (3.23), (3.24), (3.26) into (3.21), we obtain the estimate

$$\begin{aligned}
(3.27) \quad A_{j11}^{(h)} &\leq C_\#^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma} 2 \sum_{\ell=1}^{[q]} \binom{\ell + n - 1}{n - 1} \left( \frac{C_1^2}{C_\#} \right)^\ell + C_1^{q+k+1} p^{\tilde{q}+k+2+\nu} \\
&\leq C_\#^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma} 2^n \sum_{\ell=1}^{\infty} \left( \frac{2C_1^2}{C_\#} \right)^\ell + C_1^{q+k+1} p^{\tilde{q}+k+2+\nu} \\
&\leq \varepsilon C_\#^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma},
\end{aligned}$$

provided  $C_\#$  is large enough,  $\sigma \geq 2 + \nu$ . Here we used (3.15).

Let us now turn back to (3.20). The term  $A_{j13}^{(h)}$  is treated analogously to  $A_{j11}^{(h)}$  and has the estimate

$$(3.28) \quad A_{j13}^{(h)} \leq \varepsilon C_\#^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma}.$$

Consider now  $A_{j12}^{(h)}$  in (3.20). We have, applying Lemma A.6,

$$\begin{aligned}
\|X_j a_{jh} L_{\tilde{q}}(D) \varphi_p^{(k+1)} u\|_0 &\leq \|a_{jh} X_j L_{\tilde{q}}(D) \varphi_p^{(k+1)} u\|_0 \\
&\quad + \|(X_j a_{jh}) L_{\tilde{q}}(D) \varphi_p^{(k+1)} u\|_0 \\
&\leq C_0 K \left( \|X_j \langle D \rangle^{\frac{1}{m}} L_{\tilde{q}-1}(D) \varphi_p^{(k+1)} u\|_0 + \|L_{\tilde{q}-2}(D) \varphi_p^{(k+1)} u\|_{\frac{2}{m}} \right).
\end{aligned}$$

Using (3.9) and arguing by induction we obtain the estimate (see (3.6))

$$(3.29) \quad A_{j12}^{(h)} = \|X_j a_{jh} L_{\bar{q}}(D) \varphi_p^{(k+1)} u\|_0 \leq 2C_0 K C_{\#}^{k+2\bar{q}} p^{k+\bar{q}+\sigma} \\ \leq \varepsilon C_{\#}^{k+2\bar{q}+1} p^{k+\bar{q}+\sigma},$$

provided  $C_{\#}$  is large enough.

Plugging (3.27), (3.29), (3.28) into (3.20) we obtain

$$(3.30) \quad A_{j1} \leq 6n\varepsilon C_{\#}^{k+2\bar{q}+1} p^{k+\bar{q}+\sigma}.$$

**3.2. The double commutator,  $A_{j2}$ , in (3.19).** Next we have to estimate

$$A_{j2} = \|[X_j, [X_j, L_{\bar{q}}(D) \varphi_p^{(k)}]]u\|_0.$$

We have

$$(3.31) \quad A_{j2} \leq \sum_{\ell,r=1}^n \|[a_{jr} D_r, [a_{j\ell} D_{\ell}, L_{\bar{q}}(D) \varphi_p^{(k)}]]u\|_0.$$

Now

$$[a_{j\ell} D_{\ell}, L_{\bar{q}}(D) \varphi_p^{(k)}] = a_{j\ell} L_{\bar{q}}(D) \varphi_p^{(k+1)} + [a_{j\ell}, L_{\bar{q}}(D)] \varphi_p^{(k)} D_{\ell}.$$

Hence

$$\begin{aligned} & [a_{jr} D_r, [a_{j\ell} D_{\ell}, L_{\bar{q}}(D) \varphi_p^{(k)}]] \\ &= [a_{jr} D_r, a_{j\ell} L_{\bar{q}}(D) \varphi_p^{(k+1)} + [a_{j\ell}, L_{\bar{q}}(D)] \varphi_p^{(k)} D_{\ell}] \\ &= a_{jr} [D_r, a_{j\ell} L_{\bar{q}}(D) \varphi_p^{(k+1)}] + [a_{jr}, a_{j\ell} L_{\bar{q}}(D) \varphi_p^{(k+1)}] D_r \\ &+ a_{jr} [D_r, [a_{j\ell}, L_{\bar{q}}(D)] \varphi_p^{(k)} D_{\ell}] + [a_{jr}, [a_{j\ell}, L_{\bar{q}}(D)] \varphi_p^{(k)} D_{\ell}] D_r \\ &= a_{jr} a_{j\ell}^{(1)} L_{\bar{q}}(D) \varphi_p^{(k+1)} + a_{jr} a_{j\ell} L_{\bar{q}}(D) \varphi_p^{(k+2)} + a_{j\ell} [a_{jr}, L_{\bar{q}}(D)] \varphi_p^{(k+1)} D_r \\ &+ a_{jr} [D_r, [a_{j\ell}, L_{\bar{q}}(D)]] \varphi_p^{(k)} D_{\ell} + a_{jr} [a_{j\ell}, L_{\bar{q}}(D)] \varphi_p^{(k+1)} D_{\ell} \\ &+ [a_{jr}, [a_{j\ell}, L_{\bar{q}}(D)]] \varphi_p^{(k)} D_{\ell} D_r - [a_{j\ell}, L_{\bar{q}}(D)] \varphi_p^{(k)} a_{jr}^{(1)} D_r \\ &= \sum_{i=1}^7 E_i. \end{aligned}$$

Here, to keep the notation simple, we forgot about the dependence of each  $E_i$  on  $r, \ell, j, q, k$ .

By Lemma A.6, we may apply the inductive assumption to  $\|E_i u\|_0$ , for  $i = 1, 2$ , and get

$$(3.32) \quad \|E_1 u\|_0 + \|E_2 u\|_0 \leq C_0 \left( \|L_{\bar{q}-2}(D) \varphi_p^{(k+1)} u\|_{\frac{2}{m}} + \|L_{\bar{q}-2}(D) \varphi_p^{(k+2)} u\|_{\frac{2}{m}} \right)$$

$$\leq C_0 C_{\#}^{k+2\tilde{q}-1} p^{k+\tilde{q}+\sigma} \leq \varepsilon C_{\#}^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma},$$

provided  $C_{\#}$  is large enough.

Consider now  $\|E_3u\|_0, \|E_5u\|_0$ . These term are dealt with as we did for  $A_{j11}^{(h)}$  in (3.21), but without the vector field in front. Then by Lemma A.1,

$$(3.33) \quad \|E_3u\|_0 \leq C_0 \left( \sum_{\ell=1}^{[q]} \sum_{|\beta|=\ell} \frac{1}{\beta!} \|(D_x^\beta a_{jr}(x)) \partial_\xi^\beta L_{\tilde{q}}(D) \varphi_p^{(k+1)} D_r u\|_0 \right. \\ \left. + \|R_{\tilde{q}, a_{jr}} \varphi_p^{(k+1)} D_r u\|_0 \right).$$

Arguing as for the summands in (3.21), we obtain that

$$(3.34) \quad \|E_3u\|_0 \leq \varepsilon C_{\#}^{k+\tilde{q}+1} p^{k+\tilde{q}+\sigma}.$$

The norm containing  $E_5$  is bound in a completely analogous way. Consider further  $E_7$ . We have

$$E_7 = [a_{j\ell}, L_{\tilde{q}}(D)] \varphi_p^{(k)} a_{jr}^{(1)} D_r \\ = a_{jr}^{(1)} [a_{j\ell}, L_{\tilde{q}}(D)] \varphi_p^{(k)} D_r - [a_{jr}^{(1)}, [a_{j\ell}, L_{\tilde{q}}(D)]] \varphi_p^{(k)} D_r.$$

The first term is discussed as  $E_5$ , while the second as  $E_6$ .

We are left with the analysis of  $E_4$  and  $E_6$ . Let us start with  $\|E_4u\|_0$ . We have

$$-[D_r, [a_{j\ell}, L_{\tilde{q}}(D)]] = [D_r, \sum_{s=1}^{[q]} \sum_{|\beta|=s} \frac{1}{\beta!} D_x^\beta a_{j\ell}(x) \partial_\xi^\beta L_{\tilde{q}}(D) + R_{q, a_{j\ell}}(x, D)] \\ = \sum_{s=1}^{[q]} \sum_{|\beta|=s} \frac{1}{\beta!} D_x^{\beta+e_r} a_{j\ell}(x) \partial_\xi^\beta L_{\tilde{q}}(D) + (D_r R)_{q, a_{j\ell}}(x, D).$$

Hence

$$(3.35) \quad \|E_4u\|_0 = \|a_{jr} [D_r, [a_{j\ell}, L_{\tilde{q}}(D)]] \varphi_p^{(k)} D_\ell u\|_0 \\ \leq \|a_{jr} [D_r, [a_{j\ell}, L_{\tilde{q}}(D)]] D_\ell \varphi_p^{(k)} u\|_0 + \|a_{jr} [D_r, [a_{j\ell}, L_{\tilde{q}}(D)]] \varphi_p^{(k+1)} u\|_0 \\ \leq C_0 \sum_{s=1}^{[q]} \sum_{|\beta|=s} \frac{1}{\beta!} \|D_x^{\beta+e_r} a_{j\ell}(x) \partial_\xi^\beta L_{\tilde{q}}(D) D_\ell \varphi_p^{(k)} u\|_0 \\ + C_0 \|(D_r R)_{q, a_{j\ell}}(x, D) D_\ell \varphi_p^{(k)} u\|_0$$

$$\begin{aligned}
& + C_0 \sum_{s=1}^{[q]} \sum_{|\beta|=s} \frac{1}{\beta!} \|D_x^{\beta+e_r} a_{j\ell}(x) \partial_\xi^\beta L_{\bar{q}}(D) \varphi_p^{(k+1)} u\|_0 \\
& \quad + C_0 \|(D_r R)_{q,a_{j\ell}}(x, D) \varphi_p^{(k+1)} u\|_0.
\end{aligned}$$

The norms containing a remainder term are bound as in (3.22), (3.23), while the summands are bound exactly as in the estimate of  $B_2$  in (3.26).

We are left with the estimate of the norm containing  $E_6$ . We have, by Lemma A.1,

$$\begin{aligned}
& [a_{jr}, [a_{j\ell}, L_{\bar{q}}(D)]] = [a_{jr}, \sum_{s=1}^{[q]} \sum_{|\beta|=s} \frac{1}{\beta!} D_x^\beta a_{j\ell}(x) (\partial_\xi^\beta L_{\bar{q}}(D) + R_{q,a_{j\ell}}(x, D))] \\
& = \sum_{s=1}^{[q]} \sum_{|\beta|=s} \frac{1}{\beta!} D_x^\beta a_{j\ell}(x) [a_{jr}, (\partial_\xi^\beta L_{\bar{q}}(D))] + a_{jr} R_{q,a_{j\ell}}(x, D) - R_{q,a_{j\ell}}(x, D) a_{jr}.
\end{aligned}$$

By Lemma A.4,

$$\frac{1}{C_\Lambda^{|\beta|} p^{|\beta|}} (\partial_\xi^\beta L_{\bar{q}}(D)) \in \Lambda_{q-|\beta|}^p.$$

Hence

$$\begin{aligned}
& [a_{jr}, [a_{j\ell}, L_{\bar{q}}(D)]] \\
& = \sum_{s_1=1}^{[q]} \sum_{|\beta|=s_1} \sum_{s_2=1}^{[q]-s_1} \sum_{|\gamma|=s_2} \frac{1}{\beta! \gamma!} D_x^\beta a_{j\ell}(x) D_x^\gamma a_{jr}(x) (\partial_\xi^{\beta+\gamma} L_{\bar{q}}(D)) \\
& \quad + \sum_{s_1=1}^{[q]} \sum_{|\beta|=s_1} C_\Lambda^{s_1} \frac{1}{\beta!} p^{s_1} D_x^\beta a_{j\ell}(x) R_{q-s_1, a_{jr}}(x, D) \\
& \quad \quad + a_{jr} R_{q,a_{j\ell}}(x, D) - R_{q,a_{j\ell}}(x, D) a_{jr} \\
& = \sum_{s_1=1}^{[q]} \sum_{|\beta|=s_1} \sum_{s_2=1}^{[q]-s_1} \sum_{|\gamma|=s_2} \frac{1}{\beta! \gamma!} D_x^\beta a_{j\ell}(x) D_x^\gamma a_{jr}(x) (\partial_\xi^{\beta+\gamma} L_{\bar{q}}(D)) \\
& \quad + \sum_{s_1=1}^{[q]} \sum_{|\beta|=s_1} C_\Lambda^{s_1} \frac{1}{\beta!} p^{s_1} D_x^\beta a_{j\ell}(x) R_{q-s_1, a_{jr}}(x, D) \\
& \quad \quad + a_{jr} R_{q,a_{j\ell}}(x, D) - R_{q,a_{j\ell}}(x, D) a_{jr}.
\end{aligned}$$

Then the term containing  $E_6$  becomes

$$(3.36) \quad \|E_6 u\|_0 \leq \sum_{s_1=1}^{[q]} \sum_{|\beta|=s_1} \sum_{s_2=1}^{[q]-s_1} \sum_{|\gamma|=s_2} \frac{1}{\beta! \gamma!}$$

$$\begin{aligned}
& \cdot \|D_x^\beta a_{j\ell}(x) D_x^\gamma a_{jr}(x) (\partial_\xi^{\beta+\gamma} L_{\bar{q}})(D) \varphi_p^{(k)} D_\ell D_r u\|_0 \\
& + \sum_{s_1=1}^{[q]} \sum_{|\beta|=s_1} C_\Lambda^{s_1} \frac{1}{\beta!} p^{s_1} \|D_x^\beta a_{j\ell}(x) R_{q-s_1, a_{jr}}(x, D) \varphi_p^{(k)} D_\ell D_r u\|_0 \\
& + \|a_{jr} R_{q, a_{j\ell}}(x, D) \varphi_p^{(k)} D_\ell D_r u\|_0 + \|R_{q, a_{j\ell}}(x, D) a_{jr} \varphi_p^{(k)} D_\ell D_r u\|_0 \\
& = \sum_{i=1}^4 E_{6i}.
\end{aligned}$$

Arguing as in (3.22) we have that

$$(3.37) \quad E_{63} + E_{64} \leq C_0^{q+k+1} p^{\bar{q}+k+\nu} \leq \varepsilon C_\#^{2\bar{q}+k+1} p^{\bar{q}+k+\sigma}.$$

Consider  $E_{61}$ .

$$\begin{aligned}
(3.38) \quad E_{61} &= \sum_{s_1=1}^{[q]} \sum_{|\beta|=s_1} \sum_{s_2=1}^{[q]-s_1} \sum_{|\gamma|=s_2} \frac{1}{\beta! \gamma!} \\
& \cdot \|D_x^\beta a_{j\ell}(x) D_x^\gamma a_{jr}(x) (\partial_\xi^{\beta+\gamma} L_{\bar{q}})(D) \varphi_p^{(k)} D_\ell D_r u\|_0 \\
& \leq \sum_{s_1=1}^{[q]} \sum_{|\beta|=s_1} \sum_{s_2=1}^{[q]-s_1} \sum_{|\gamma|=s_2} C_a^{s_1+s_2+2} (\beta! \gamma!)^{m-1} \|(\partial_\xi^{\beta+\gamma} L_{\bar{q}})(D) \varphi_p^{(k)} D_\ell D_r u\|_0 \\
& \leq \sum_{s_1=1}^{[q]} \sum_{|\beta|=s_1} \sum_{s_2=1}^{[q]-s_1} \sum_{|\gamma|=s_2} C_a^{s_1+s_2+2} (\beta! \gamma!)^{m-1} \|\langle D \rangle^{-\frac{2}{m}} (\partial_\xi^{\beta+\gamma} L_{\bar{q}})(D) \varphi_p^{(k)} D_\ell D_r u\|_{\frac{2}{m}}.
\end{aligned}$$

In the last term of (3.38) we bring the derivatives  $D_\ell D_r$  to the left. As a result we have a sum of terms; the principal part of this sum is  $D_\ell D_r \varphi_p^{(k)}$ . We are going to estimate the principal part since the other terms are bounded in an analogous way.

$$\begin{aligned}
& \|\langle D \rangle^{-\frac{2}{m}} (\partial_\xi^{\beta+\gamma} L_{\bar{q}})(D) D_\ell D_r \varphi_p^{(k)} u\|_{\frac{2}{m}} \\
& \leq K C_\Lambda^{|\beta|+|\gamma|} p^{|\beta|+|\gamma|} \|L_{\bar{q}-m(|\beta|+|\gamma|-2)-2} \varphi_p^{(k)} u\|_{\frac{2}{m}} \\
& \leq K C_\Lambda^{|\beta|+|\gamma|} C_\#^{k+2(\bar{q}-m(|\beta|+|\gamma|-2)-2)+1} p^{k+\bar{q}-(m-1)(|\beta|+|\gamma|-2)+\sigma}.
\end{aligned}$$

Here we used Lemma A.5 as well as the inductive hypothesis. Plugging this into (3.38) we obtain

$$\begin{aligned}
E_{61} &\leq \sum_{s_1=1}^{[q]} \sum_{|\beta|=s_1} \sum_{s_2=1}^{[q]-s_1} \sum_{|\gamma|=s_2} C_\Lambda^{s_1+s_2} C_a^{s_1+s_2+2} (\beta! \gamma!)^{m-1} \\
& \cdot K C_\#^{k+2(\bar{q}-m(s_1+s_2-2)-2)+1} p^{k+\bar{q}-(m-1)(s_1+s_2-2)+\sigma}.
\end{aligned}$$

Applying (3.25) we obtain that

$$\frac{(\beta!\gamma!)^{m-1}}{p^{(m-1)(s_1+s_2-2)}} = \left( \frac{\beta!}{p^{s_1-1}} \frac{\gamma!}{p^{s_2-1}} \right)^{m-1} \leq 1.$$

Hence the sum in the estimate for  $E_{61}$  becomes, using (3.15),

$$\begin{aligned} E_{61} &\leq \sum_{s_1=1}^{[q]} \sum_{s_2=1}^{[q]-s_1} \binom{n-1+s_1}{n-1} \binom{n-1+s_2}{n-1} \\ &\quad \cdot K C_{\Lambda}^{s_1+s_2} C_a^{s_1+s_2+2} C_{\#}^{k+2(\bar{q}-m(s_1+s_2-2)-2)+1} p^{k+\bar{q}+\sigma} \\ &\leq C_{\#}^{k+2\bar{q}+1} p^{k+\bar{q}+\sigma} \frac{1}{C_{\#}} \sum_{\lambda=2}^{[q]} (\lambda-1) C_{\#}^{\lambda-2} C_{\#}^{-2m(\lambda-2)} \leq \varepsilon C_{\#}^{k+2\bar{q}+1} p^{k+\bar{q}+\sigma}, \end{aligned}$$

provided  $C_{\#}$  is large enough. Here we used (3.6) and the fact that the binomials are bounded by  $2^{2(n-1)+s_1+s_2}$ . Moreover we changed the variable so that  $\lambda = s_1 + s_2$ .

Finally consider  $E_{62}$ .

$$\begin{aligned} E_{62} &\leq \sum_{s_1=1}^{[q]} \sum_{|\beta|=s_1} C_{\Lambda}^{s_1} \frac{1}{\beta!} C_a^{|\beta|+1} p^{s_1} \beta!^m \|R_{q-s_1, a_{j_r}}(x, D) \varphi_p^{(k)} D_{\ell} D_r u\|_0 \\ &\leq \sum_{s_1=1}^{[q]} C_0^{s_1+1} p^{s_1} s_1!^{m-1} C_0^{k+q-s_1+1} p^{k+\bar{q}-ms_1+\nu} \\ &\leq C_0^{k+\bar{q}+1} p^{k+\bar{q}+\nu} q C_0 \leq \varepsilon C_{\#}^{k+2\bar{q}+1} p^{k+\bar{q}+\sigma}, \end{aligned}$$

provided  $C_{\#}$  is large enough and  $\sigma \geq \nu$ .

Plugging the above estimates in (3.36), we finally obtain

$$(3.39) \quad E_6 \leq 4\varepsilon C_{\#}^{k+2\bar{q}+1} p^{k+\bar{q}+\sigma}.$$

As a consequence of the estimates of the  $\|E_i u\|_0$ , (3.31) is bounded by

$$(3.40) \quad A_{j2} \leq \sum_{\ell, r=1}^n \sum_{i=1}^7 \|E_i u\|_0 \leq C_{j2} \varepsilon C_{\#}^{k+\bar{q}+1} p^{k+\bar{q}+\sigma},$$

for a suitable positive constant  $C_{j2}$ , independent of  $k, q$ .

Thus far we have established the following estimate for  $\sum_{j=1}^N A_j$  in (3.18)

$$(3.41) \quad \sum_{j=1}^N \|[X_j^2, L_{\bar{q}}(D) \varphi_p^{(k)}] u\|_0 \leq C_A \varepsilon C_{\#}^{k+\bar{q}+1} p^{k+\bar{q}+\sigma},$$

for a suitable positive constant  $C_A$ , independent of  $k, q$ .

**3.3. The term containing the  $X_0$  commutator in (3.18).** We are going to discuss the bound for the term

$$\|[X_0, L_{\tilde{q}}(D)\varphi_p^{(k)}]u\|_0$$

in (3.18). We have

$$\begin{aligned} \|[X_0, L_{\tilde{q}}(D)\varphi_p^{(k)}]u\|_0 &\leq \sum_{\ell=1}^n \|[a_{0\ell}(x)D_\ell, L_{\tilde{q}}(D)\varphi_p^{(k)}]u\|_0 \\ &\leq \sum_{\ell=1}^n \left( \|a_{0\ell}(x)L_{\tilde{q}}(D)\varphi_p^{(k+1)}u\|_0 + \|[a_{0\ell}(x), L_{\tilde{q}}(D)]\varphi_p^{(k)}D_\ell u\|_0 \right) \\ &\leq \sum_{\ell=1}^n \left( \|a_{0\ell}(x)L_{\tilde{q}}(D)\varphi_p^{(k+1)}u\|_0 + \|[a_{0\ell}(x), L_{\tilde{q}}(D)]D_\ell\varphi_p^{(k)}u\|_0 \right. \\ &\quad \left. + \|[a_{0\ell}(x), L_{\tilde{q}}(D)]\varphi_p^{(k+1)}u\|_0 \right) = \sum_{\ell=1}^n \sum_{i=1}^3 F_{\ell i}. \end{aligned}$$

Consider first  $F_{\ell 1}$ . We have, by Lemma A.6 and the inductive hypothesis,

$$(3.42) \quad \begin{aligned} F_{\ell 1} = \|a_{0\ell}(x)L_{\tilde{q}}(D)\varphi_p^{(k+1)}u\|_0 &\leq C_a K \|L_{\tilde{q}-2}(D)\varphi_p^{(k+1)}u\|_{\frac{2}{m}} \\ &\leq C_a K C_{\#}^{k+1+2(\tilde{q}-2)+1} p^{k+1+\tilde{q}-2+\sigma} \leq \varepsilon C_{\#}^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma}. \end{aligned}$$

Consider  $F_{\ell 2}$ . Arguing as we did for (3.21) we have, applying Lemmas A.1, A.5,

$$\begin{aligned} F_{\ell 2} &= \|[a_{0\ell}(x), L_{\tilde{q}}(D)]D_\ell\varphi_p^{(k)}u\|_0 \\ &\leq \sum_{r=1}^{[q]} \sum_{|\beta|=r} \frac{1}{\beta!} \|D_x^\beta a_{0\ell}(x)(\partial_\xi^\beta L_{\tilde{q}})(D)D_\ell\varphi_p^{(k)}u\|_0 + \|R_{q,a_{0\ell}}D_\ell\varphi_p^{(k)}u\|_0 \\ &\leq \sum_{r=1}^{[q]} \sum_{|\beta|=r} C_a^{r+1} \beta!^{m-1} K C_\Lambda^r p^r \|L_{\tilde{q}-rm-2+m}(D)\varphi_p^{(k)}u\|_{\frac{2}{m}} + C_0^{\tilde{q}+1} p^{\tilde{q}+\nu} \|D_\ell\varphi_p^{(k)}u\|_0 \\ &\leq \sum_{r=1}^{[q]} \sum_{|\beta|=r} C_a^{r+1} \beta!^{m-1} K C_\Lambda^r C_{\#}^{k+2(\tilde{q}-rm-2+m)+1} p^{k+\tilde{q}-(r-1)(m-1)-1+\sigma} \\ &\quad + C_0^{\tilde{q}+1} C_\varphi^{k+2} p^{k+\tilde{q}+1+\nu}. \end{aligned}$$

The first sum is treated applying (3.25) and arguing as for  $E_{61}$ ; we finally get

$$(3.43) \quad F_{\ell 2} \leq \varepsilon C_{\#}^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma}.$$



The quantity  $F_{\ell 3}$  is treated exactly as  $F_{\ell 2}$ , getting an estimate of the form (3.43).

Using (3.42), (3.43) we finally obtain

$$(3.44) \quad \|[X_0, L_{\tilde{q}}(D)\varphi_p^{(k)}]u\|_0 \leq C_{X_0} \varepsilon C_{\#}^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma},$$

for a suitable positive constant  $C_{X_0}$  independent of  $k, q, p$ .

**3.4. The term containing the commutator with  $g$  in (3.18).** Next we consider  $A_g = \|[g, L_{\tilde{q}}(D)\varphi_p^{(k)}]u\|_0 = \|[g, L_{\tilde{q}}(D)]\varphi_p^{(k)}u\|_0$ . By Lemmas A.1, A.4, A.6, we have

$$\begin{aligned} A_g &\leq \sum_{\ell=1}^{[q]} \sum_{|\beta|=\ell} \frac{1}{\beta!} \|D_x^\beta g(x) (\partial_\xi^\beta L_{\tilde{q}})(D)\varphi_p^{(k)}u\|_0 + \|R_{q,g}(x, D)\varphi_p^{(k)}u\|_0 \\ &\leq \sum_{\ell=1}^{[q]} \sum_{|\beta|=\ell} C_g^{\ell+1} \beta!^{m-1} \|(\partial_\xi^\beta L_{\tilde{q}})(D)\varphi_p^{(k)}u\|_0 + C_R^{q+1} C_\varphi^{k+1} p^{k+\tilde{q}+\nu} \|u\|_0 \\ &\leq \sum_{\ell=1}^{[q]} \sum_{|\beta|=\ell} K C_\Lambda^\ell C_g^{\ell+1} \beta!^{m-1} p^\ell \|L_{\tilde{q}-\ell m-2}(D)\varphi_p^{(k)}u\|_{\frac{2}{m}} + C_R^{q+1} C_\varphi^{k+1} p^{k+\tilde{q}+\nu} \|u\|_0. \end{aligned}$$

We now apply the inductive hypothesis and use the inequality  $\beta! \leq p^{|\beta|}$  to get

$$\begin{aligned} (3.45) \quad A_g &\leq \sum_{\ell=1}^{[q]} K C_\Lambda^\ell C_g^{\ell+1} \binom{\ell+n-1}{n-1} p^{\ell m} C_{\#}^{k+2(\tilde{q}-\ell m-2)+1} p^{k+\tilde{q}-\ell m-2+\sigma} \\ &\quad + C_1^{q+k+1} p^{k+\tilde{q}+\nu} \|u\|_0 \\ &\leq C_{\#}^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma} \frac{2^{n-1} K C_g}{C_{\#}^4} \sum_{\ell=1}^{\infty} \left( \frac{2C_g C_\Lambda}{C_{\#}^{2m}} \right)^\ell + \varepsilon C_{\#}^{k+\tilde{q}+1} p^{k+\tilde{q}+\sigma} \\ &\leq M_g \varepsilon C_{\#}^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma}, \end{aligned}$$

provided  $C_{\#}$  is large enough.

**3.5. End of the proof of Theorem 1.4.** By inequalities (3.41), (3.44), (3.45), we obtain that

$$(3.46) \quad \|[P, L_{\tilde{q}}(D)\varphi_p^{(k)}]u\|_0 \leq M_1 \varepsilon C_{\#}^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma},$$

with  $M_1 > 0$  and suitable. By (3.11), (3.12), (3.16) and (3.46) we finally get that

$$(3.47) \quad \|L_{\tilde{q}}(D)\varphi_p^{(k)}u\| \leq C_{\#}^{k+2\tilde{q}+1} p^{k+\tilde{q}+\sigma},$$

if  $C_{\#}$  is chosen large enough depending only on the problem's data and  $k + \tilde{q} \leq \tilde{p}$ . This concludes the induction argument.

Choosing in (3.47)  $k = 0$  and  $q = p$  we deduce that

$$(3.48) \quad \|L_{\tilde{p}}(D)\varphi_p u\|_{\frac{2}{m}} \leq C_{\#}^{2\tilde{p}+1} p^{\tilde{p}+\sigma} \leq C_1^{p+1} p^{mp+\sigma}.$$

In particular when  $L_{\tilde{p}}(D) = C^{-p}\langle D \rangle^p$ ,  $C$  suitable, we have that  $u \in G^m(W)$ , where  $W$  has been defined in (S).

This ends the proof of the theorem.

#### 4. REGULARITY FOR $P_2$ IN (1.7)

In this section we prove

**Proposition 4.1.** *The operator*

$$(4.1) \quad P_2 = D_1^2 + x_1^{2(q-1)} D_2^2 + ix_1^k D_2,$$

is analytic hypoelliptic for every  $k > 0$ ,  $q \geq 2$ .

*Proof.* We have that  $\text{Char}(P_2) = \{x_1 = \xi_1 = 0, \xi_2 \neq 0\}$ . Fix a point in  $\text{Char}(P_2)$ , e.g.  $\rho_0 = (0, 0; 0, \tilde{\xi}_2)$ . Fix a large integer  $p$  and denote by  $\varphi_p = \varphi_p(x_2)$  an Ehrenpreis cutoff supported near the origin (see Definition 3.2.)

We are going to exploit the a priori estimate ([24])

$$(4.2) \quad \|u\|_{\frac{2}{q}} + \|D_1^2 u\|_0 + \|x_1^{2(q-1)} D_2^2 u\|_0 + \|x_1^k D_2 u\|_0 \leq C (\|P_2 u\|_0 + \|u\|_0).$$

Denote by  $\|u\|$  the left hand side of (4.2). First we prove an estimate of the form

$$(4.3) \quad \|\|D_2^h \varphi_p^{(\ell)} u\|\| \leq C_{\#}^{2h+\ell+1} p^{h+\ell},$$

for  $h + \ell \leq p$ . Here  $\varphi_p^{(\ell)} = D_2^{\ell} \varphi_p$ . We point out that the exponent  $2h$  instead of just  $h$  is a technical trick which is harmless for the conclusion and plays a role in the inductive process.

The second step of the proof consists in deducing the estimate

$$(4.4) \quad \|\|D^{\alpha} \varphi_p^{(\ell)} u\|\| \leq C_{\#}^{2|\alpha|+\ell+1} p^{|\alpha|+\ell},$$

where  $|\alpha| + \ell \leq p$  and  $\alpha_1 \neq 0$ .

Estimate (4.3) holds evidently when  $h = 0$ , because of the properties of  $\varphi_p$ . We argue by induction on  $h$ : assume that (4.3) holds for  $h' \leq h - 1$ ,  $h' + \ell \leq p$ ; we want to prove it for  $h$ ,  $h + \ell \leq p$ . To this end consider  $\|\|D_2^h \varphi_p^{(\ell)} u\|\|$  and apply (4.2)

$$(4.5) \quad \|\|D_2^h \varphi_p^{(\ell)} u\|\| \leq C \left( \|P_2 D_2^h \varphi_p^{(\ell)} u\|_0 + \|D_2^h \varphi_p^{(\ell)} u\|_0 \right)$$

To begin with, consider the last term on the right hand side of the above inequality. We are going to show that this term,  $\|D_2^h \varphi_p^{(\ell)} u\|_0$ , can be actually absorbed on the left hand side of (4.5), modulo a term with an analytic growth estimate. To this end, denote by  $\chi$  a smooth cutoff function such that  $\chi(t) = 1$  if  $|t| \geq 2$  and  $\chi(t) = 0$  if  $|t| \leq 1$ . It turns out that  $\chi(p^{-1}D_2) \in OPS_{0,0}^0$ . For the Definition of these classes we refer to [17] and to [2] for an application in a similar context.

We have then

$$\|D_2^h \varphi_p^{(\ell)} u\|_0 \leq \|(1 - \chi(p^{-1}D_2))D_2^h \varphi_p^{(\ell)} u\|_0 + \|\chi(p^{-1}D_2)D_2^h \varphi_p^{(\ell)} u\|_0.$$

By the Calderón–Vaillancourt theorem (see Appendix) we have that

$$\|(1 - \chi(p^{-1}D_2))D_2^h \varphi_p^{(\ell)} u\|_0 \leq C_\chi^{h+1} p^h \|\varphi_p^{(\ell)} u\|_0 \leq C_1^{h+\ell+1} p^{h+\ell}.$$

Further

$$\begin{aligned} \|\chi(p^{-1}D_2)D_2^h \varphi_p^{(\ell)} u\|_0 &\leq p^{-\frac{2}{q}} \|p^{\frac{2}{q}} \chi(p^{-1}D_2) \langle D \rangle^{-\frac{2}{q}} \langle D \rangle^{\frac{2}{q}} D_2^h \varphi_p^{(\ell)} u\|_0 \\ &\leq C_2 p^{-\frac{2}{q}} \|D_2^h \varphi_p^{(\ell)} u\|_{\frac{2}{q}}. \end{aligned}$$

Hence we get

$$(4.6) \quad \|D_2^h \varphi_p^{(\ell)} u\|_0 \leq C_2 p^{-\frac{2}{q}} \|D_2^h \varphi_p^{(\ell)} u\|_{\frac{2}{q}} + C_1^{h+\ell+1} p^{h+\ell}.$$

The first norm above can be absorbed on the left hand side of (4.5), while the second is the desired bound, modulo an adjustment of the constant.

Consider now the term

$$\|P_2 D_2^h \varphi_p^{(\ell)} u\|_0 \leq \|D_2^h \varphi_p^{(\ell)} P_2 u\|_0 + \|[P_2, D_2^h \varphi_p^{(\ell)}] u\|_0.$$

We need to discuss only the second term since, on  $\text{supp } \varphi_p$ ,  $P_2 u$  has the good analytic bounds. Since  $D_1$  commutes with  $D_2^h \varphi_p^{(\ell)}$ ,  $[P_2, D_2^h \varphi_p^{(\ell)}] = [X_2^2, D_2^h \varphi_p^{(\ell)}] + [X_0, D_2^h \varphi_p^{(\ell)}]$ .

Now

$$\begin{aligned} [X_2^2, D_2^h \varphi_p^{(\ell)}] &= x_1^{2(q-1)} D_2^h [D_2^2, \varphi_p^{(\ell)}] \\ &= x_1^{2(q-1)} D_2^h \left( 2D_2 \varphi_p^{(\ell+1)} - \varphi_p^{(\ell+2)} \right) \\ &= 2X_2^2 D_2^{h-1} \varphi_p^{(\ell+1)} - X_2^2 D_2^{h-2} \varphi_p^{(\ell+2)}. \end{aligned}$$

Hence

$$\|[X_2^2, D_2^h \varphi_p^{(\ell)}] u\|_0 \leq C \left( \|X_2^2 D_2^{h-1} \varphi_p^{(\ell+1)} u\|_0 + \|X_2^2 D_2^{h-2} \varphi_p^{(\ell+2)} u\|_0 \right),$$

to which we may apply the inductive hypothesis.

Consider  $[X_0, D_2^h \varphi_p^{(\ell)}]$ . A computation gives

$$ix_1^k D_2^h [D_2, \varphi_p^{(\ell)}] = ix_1^k D_2^h \varphi_p^{(\ell+1)},$$

so that

$$\| [X_2^2, D_2^h \varphi_p^{(\ell)}] u \|_0 = \| x_1^k D_2^h \varphi_p^{(\ell+1)} u \|_0 = \| X_0 D_2^{h-1} \varphi_p^{(\ell+1)} u \|_0,$$

to which we may apply the inductive hypothesis.

As a consequence (4.5) becomes

$$\begin{aligned} \| \| D_2^h \varphi_p^{(\ell)} u \| \| &\leq C_2 \left( \| D_2^h \varphi_p^{(\ell)} P_2 u \|_0 \right. \\ &\quad + \| X_2^2 D_2^{h-1} \varphi_p^{(\ell+1)} u \|_0 + \| X_2^2 D_2^{h-2} \varphi_p^{(\ell+2)} u \|_0 \\ &\quad \left. + \| X_0 D_2^{h-1} \varphi_p^{(\ell+1)} u \|_0 \right) + C_1^{h+\ell+1} p^{h+\ell} \end{aligned}$$

Applying the inductive hypothesis gives the desired conclusion, provided  $C_\#$  is chosen large enough, independent of  $h$ .

Consider now  $\| \| D^\alpha \varphi_p^{(\ell)} u \| \|$ . We have

$$(4.7) \quad \| \| D^\alpha \varphi_p^{(\ell)} u \| \| \leq C \left( \| P_2 D^\alpha \varphi_p^{(\ell)} u \|_0 + \| D^\alpha \varphi_p^{(\ell)} u \|_0 \right).$$

The  $L^2$ -error term can be absorbed on the left as above. Consider the norm  $\| P_2 D^\alpha \varphi_p^{(\ell)} u \|_0$ . As above we have to study the commutator  $[P_2, D^\alpha \varphi_p^{(\ell)}] = [X_2^2, D^\alpha \varphi_p^{(\ell)}] + [X_0, D^\alpha \varphi_p^{(\ell)}]$ .

Let us examine the first term

$$[X_2^2, D^\alpha \varphi_p^{(\ell)}] = 2X_2 [X_2, D^\alpha \varphi_p^{(\ell)}] - [X_2, [X_2, D^\alpha \varphi_p^{(\ell)}]].$$

We have

$$\begin{aligned} 2X_2 [X_2, D^\alpha \varphi_p^{(\ell)}] &= 2X_2 x_1^{q-1} [D_2, D^\alpha \varphi_p^{(\ell)}] + 2X_2 [x_1^{q-1}, D^\alpha \varphi_p^{(\ell)}] D_2 \\ &= 2X_2 x_1^{q-1} D^\alpha \varphi_p^{(\ell+1)} - 2X_2 D_2^{\alpha_2} [D_1^{\alpha_1}, x_1^{q-1}] \varphi_p^{(\ell)} D_2 \\ &= 2X_2 x_1^{q-1} D^\alpha \varphi_p^{(\ell+1)} - 2X_2 \sum_{j=1}^{\min\{q-1, \alpha_1\}} \binom{\alpha_1}{j} (D_1^j x_1^{q-1}) D^{\alpha - j e_1} \varphi_p^{(\ell)} D_2 \\ &= 2x_1^{q-1} X_2 D_1 D^{\alpha - e_1} \varphi_p^{(\ell+1)} - 2 \sum_{\substack{j=1 \\ j+1 \leq \alpha_1}}^{\min\{q-1, \alpha_1\}} \binom{\alpha_1}{j} (D_1^j x_1^{q-1}) X_2 D_1 D^{\alpha - (j+1)e_1 + e_2} \varphi_p^{(\ell)} \\ &\quad + 2 \sum_{\substack{j=1 \\ j+1 \leq \alpha_1}}^{\min\{q-1, \alpha_1\}} \binom{\alpha_1}{j} (D_1^j x_1^{q-1}) X_2 D_1 D^{\alpha - (j+1)e_1} \varphi_p^{(\ell+1)}, \end{aligned}$$

where we assumed that  $\alpha_1 \geq q$ . If  $\alpha_1 < q$  then for  $j = \alpha_1$  we cannot bring a  $D_1$  near  $X_2$ , so that  $D^{\alpha - (j+1)e_1 + e_2}$  is replaced by  $D_2^{\alpha_2 + e_2}$  and we use (4.3). Then

$$\begin{aligned}
2\|X_2[X_2, D^\alpha \varphi_p^{(\ell)}]u\|_0 &\leq C_0 \left( \|X_2 D_1 D^{\alpha-e_1} \varphi_p^{(\ell+1)} u\|_0 \right. \\
&+ \left. \sum_{\substack{j=1 \\ j+1 \leq \alpha_1}}^{\min\{q-1, \alpha_1\}} \binom{\alpha_1}{j} \left[ \|X_2 D_1 D^{\alpha-(j+1)e_1+e_2} \varphi_p^{(\ell)} u\|_0 + \|X_2 D_1 D^{\alpha-(j+1)e_1} \varphi_p^{(\ell+1)} u\|_0 \right] \right) \\
&\leq C_0 \left( \|D^{\alpha-e_1} \varphi_p^{(\ell+1)} u\| \right. \\
&+ \left. \sum_{\substack{j=1 \\ j+1 \leq \alpha_1}}^{\min\{q-1, \alpha_1\}} \alpha_1^j \left[ \|D^{\alpha-(j+1)e_1+e_2} \varphi_p^{(\ell)} u\| + \|D^{\alpha-(j+1)e_1} \varphi_p^{(\ell+1)} u\| \right] \right).
\end{aligned}$$

We now argue by induction with respect to  $\alpha_1$ : (4.4) holds when  $\alpha_1 = 0$  because of (4.3). Assume that (4.4) holds for  $\alpha'_1 \leq \alpha_1 - 1$ ,  $|\alpha'| + \ell \leq p$  and we want to show that (4.4) holds for  $\alpha_1$ ,  $|\alpha| + \ell \leq p$ .

The above expression can then be bounded as

$$\begin{aligned}
2\|X_2[X_2, D^\alpha \varphi_p^{(\ell)}]u\|_0 &\leq C_0 \left( C_\#^{2|\alpha|+\ell} p^{|\alpha|+\ell} \right. \\
&+ \left. \sum_{\substack{j=1 \\ j+1 \leq \alpha_1}}^{\min\{q-1, \alpha_1\}} \alpha_1^j \left[ C_\#^{2|\alpha|-2j+\ell+1} p^{|\alpha|-j+\ell} + C_\#^{2|\alpha|-2j+\ell} p^{|\alpha|-j+\ell} \right] \right) \\
&\leq \varepsilon C_\#^{2|\alpha|+\ell+1} p^{|\alpha|+\ell},
\end{aligned}$$

where  $\varepsilon$  is a small constant to be determined later, provided  $C_\#$  is large enough.

The double commutator  $[X_2, [X_2, D^\alpha \varphi_p^{(\ell)}]]$  as well as the commutator with  $X_0$  are treated in a similar fashion.

Choosing  $\ell = 0$  and using the Sobolev immersion theorem we conclude the proof.  $\square$

#### A. APPENDIX: SOME TECHNICAL RESULTS

We gather in this appendix the proofs of some lemmas concerning the class  $\Lambda_q^p$ .

Let us start by proving

**Lemma A.1.** *Let  $a \in G_0^m(U_1)$ , i.e.  $|\partial_x^\alpha a(x)| \leq C_a^{|\alpha|+1} \alpha!^m$ , and let  $L_q \in \Lambda_q^p$ . Then*

$$(A.1) \quad [L_q, a](x, \xi) = \sum_{\ell=1}^{[q]} \sum_{|\beta|=\ell} \frac{1}{\beta!} D_x^\beta a(x) \partial_\xi^\beta L_q(\xi) + R_{q,a}(x, \xi),$$

where  $R_{q,a}(x, \xi)$  denotes a symbol in  $S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$  satisfying estimates of the form

$$(A.2) \quad |\partial_x^\gamma \partial_\xi^\beta R_{q,a}(x, \xi)| \leq C_R^{q+1} p^{qm+\nu} \langle \xi \rangle^{-|\beta|},$$

where  $[q]$  is the integer part of  $q$ ,  $|\gamma|+|\beta| \leq C_n$ , where  $C_n$  denotes a constant depending on the ambient space dimension  $n$ ,  $C_R > 0$  depending only on  $n$ ,  $m$ ,  $C_a$ ,  $C_\Lambda$  (see Definition 3.3) and  $\nu$  is a positive constant independent of  $q$ ,  $p$ .

In particular the operator  $R_{q,a}(x, D)$  is bounded in  $L^2(\mathbb{R}^n)$  by Theorem A.8 with the estimate

$$(A.3) \quad \|R_{q,a}(x, D)u\|_0 \leq C_0^{q+1} p^{qm+\nu} \|u\|_0,$$

for any  $u \in \mathcal{S}(\mathbb{R}^n)$ .

*Proof.* From the composition formula for two pseudodifferential operators  $B, C$  with symbols  $b(x, \xi), c(x, \xi)$  respectively,

$$\sigma(B \circ C)(x, \xi) = \int e^{iz\zeta} b(x, \xi + \zeta) c(x - z, \xi) \bar{d}\zeta dz,$$

where  $\bar{d}\zeta = (2\pi)^{-n} d\zeta$ , we obtain

$$(A.4) \quad \begin{aligned} \sigma([L_q, a])(x, \xi) &= \sum_{k=1}^{N-1} \sum_{|\alpha|=k} \frac{1}{\alpha!} D_x^\alpha a(x) \partial_\xi^\alpha L_q(\xi) \\ &+ N \sum_{|\alpha|=N} \frac{1}{\alpha!} \int \int_0^1 e^{iz\zeta} \zeta^\alpha (1-s)^{N-1} \partial_\xi^\alpha L_q(\xi + s\zeta) a(x-z) \bar{d}\zeta dz ds. \end{aligned}$$

The above integral has to be understood as an oscillating integral with respect to  $\zeta$ .

We choose  $N = [q] + 1$ . Let us denote by  $R_{q,a}$  the symbol in the last line above, then we have

$$\begin{aligned} &\partial_x^\gamma \partial_\xi^\beta R_{q,a}(x, \xi) \\ &= ([q]+1) \sum_{|\alpha|=[q]+1} \frac{1}{\alpha!} \int \int_0^1 e^{iz\zeta} \zeta^\alpha (1-s)^{[q]} \partial_\xi^{\alpha+\beta} L_q(\xi + s\zeta) \partial_x^\gamma a(x-z) \bar{d}\zeta dz ds \\ &= ([q]+1) \sum_{|\alpha|=[q]+1} \frac{1}{\alpha!} i^{-|\alpha|} \int \int_0^1 e^{iz\zeta} (1-s)^{[q]} \partial_\xi^{\alpha+\beta} L_q(\xi + s\zeta) \partial_x^{\gamma+\alpha} a(x-z) \bar{d}\zeta dz ds. \end{aligned}$$

Since  $(1 - \Delta_z)^M e^{iz\zeta} = e^{iz\zeta} (1 + |\zeta|^2)^M$  we may write the above integral as

$$\frac{1}{\alpha!} \int \int_0^1 e^{iz\zeta} (1+|\zeta|^2)^{-M} (1-s)^{[q]} \partial_\xi^{\alpha+\beta} L_q(\xi + s\zeta) (1 - \Delta_x)^M \partial_x^{\gamma+\alpha} a(x-z) \bar{d}\zeta dz ds,$$

which makes the integral a convergent one provided  $M \geq \frac{n+1}{2}$ , since the function  $a$  has compact support.

In order to show that the above integral defines a symbol in  $S_{1,0}^0$ , we split the domain of integration into two regions:  $|\zeta| \leq \varepsilon|\xi|$  and  $|\zeta| \geq \varepsilon|\xi|$ , where  $\varepsilon$  denotes a positive number less than 1.

Consider the integral over the first region:

$$(A.5) \quad R_1(x, \xi) = \frac{1}{\alpha!} \int_{|\zeta| \leq \varepsilon|\xi|} \int_0^1 e^{iz\zeta} (1+|\zeta|^2)^{-M} (1-s)^{[q]} \partial_\xi^{\alpha+\beta} L_q(\xi+s\zeta) (1-\Delta_x)^M \partial_x^{\gamma+\alpha} a(x-z) d\zeta dz ds.$$

We observe that  $|\xi + s\zeta| \leq (1 + \varepsilon)|\xi|$  and that  $|\xi| \leq |\xi + s\zeta| + |\zeta| \leq |\xi + s\zeta| + \varepsilon|\xi|$ , so that  $|\xi + s\zeta| \sim |\xi|$ . We choose  $M \sim \frac{n+1}{2}$ .

The integral in (A.5) satisfies the estimates for a symbol in  $S_{1,0}^0$  (see [15], Definition 18.1.1):

$$|R_1(x, \xi)| \leq C_{\beta, \gamma} (1 + |\xi|)^{q-|\alpha|-|\beta|} \leq C_{\beta, \gamma} (1 + |\xi|)^{-|\beta|}.$$

Let us now prove estimate (A.2) for  $R_1$ . We point out that  $|\alpha| + |\beta| = [q] + 1 + |\beta| \leq q + 1 + C_n \leq R_0(q + 1)$  if  $R_0$  is chosen large enough. Hence the derivatives on  $L_q$  are admissible according to Definition 3.3.

$$\begin{aligned} |R_1(x, \xi)| &\leq \frac{1}{\alpha!} C_\Lambda^{1+|\alpha|+|\beta|} C_a^{|\alpha|+|\gamma|+2[\frac{n}{2}]+2+1} p^{|\alpha|+|\beta|} \langle \xi \rangle^{q-|\alpha|-|\beta|} (|\alpha|+|\gamma|+2[\frac{n}{2}]+2)!^m \\ &\quad \cdot \int (1 + |\zeta|^2)^{-M} d\zeta \int_{x-\text{supp } a} dz \\ &\leq \frac{1}{|\alpha!} C_\Lambda^{1+|\alpha|+|\beta|} (2nC_a)^{|\alpha|+|\gamma|+2[\frac{n}{2}]+2+1} p^{|\alpha|+|\beta|} \langle \xi \rangle^{q-|\alpha|-|\beta|} |\alpha|!^m (|\gamma|+2[\frac{n}{2}]+2)!^m \\ &\quad \cdot \int (1 + |\zeta|^2)^{-M} d\zeta \int_{x-\text{supp } a} dz \\ &\leq C_\Lambda^{1+|\alpha|+|\beta|} (2nC_a)^{|\alpha|+|\gamma|+2[\frac{n}{2}]+2+1} p^{|\alpha|m} p^{|\beta|} \langle \xi \rangle^{q-|\alpha|-|\beta|} (C_n + 2[\frac{n}{2}] + 2)!^m \\ &\quad \cdot \int (1 + |\zeta|^2)^{-M} d\zeta \int_{x-\text{supp } a} dz \\ &\leq C_{R_1}^{q+1} p^{qm+c} \langle \xi \rangle^{-|\beta|}, \end{aligned}$$

where  $C_{R_1}$  verifies the same conditions of  $C_R$  in the statement of the lemma, and  $c$  is a positive constant independent of  $p$  and  $q$ .

Consider the integral over the second region:

$$(A.6) \quad R_2(x, \xi) = \frac{1}{\alpha!}$$

$$\int_{|\zeta| \geq \varepsilon |\xi|} \int_0^1 e^{iz\zeta} (1+|\zeta|^2)^{-M} (1-s)^{[q]} \partial_\xi^{\alpha+\beta} L_q(\xi+s\zeta) (1-\Delta_x)^M \partial_x^{\gamma+\alpha} a(x-z) d\zeta dz ds.$$

We increase the value of  $M$  by  $\frac{|\beta|}{2} + 1$  integrations by parts with respect to  $z$  in order to get a better decay of the integrand. Set  $M \sim \left[\frac{n}{2}\right] + 1 + \left[\frac{|\beta|}{2}\right] + 1$ .

It is easy to see that  $R_2$  verifies the estimates for the class  $S_{1,0}^0$ . Let us show that  $R_2$  satisfies (A.2). Arguing as we did above we obtain

$$\begin{aligned} |R_2(x, \xi)| &\leq \frac{1}{\alpha!} C_\Lambda^{1+|\alpha|+|\beta|} C_a^{|\alpha|+|\gamma|+2\left[\frac{n}{2}\right]+4+2\left[\frac{|\beta|}{2}\right]} p^{|\alpha|+|\beta|} (|\alpha|+|\gamma|+2\left[\frac{n}{2}\right]+4+2\left[\frac{|\beta|}{2}\right])!^m \\ &\cdot \int_0^1 \int_{|\zeta| \geq \varepsilon |\xi|} (1+|\zeta|^2)^{-\left[\frac{n}{2}\right]-\left[\frac{|\beta|}{2}\right]-2} \langle \xi + s\zeta \rangle^{q-|\alpha|-|\beta|} d\zeta ds \int_{x-\text{supp } a} dz \\ &\leq C_{R_2}^{q+1} p^{qm+c_1} \langle \xi \rangle^{-|\beta|}, \end{aligned}$$

where  $c_1$  is a positive constant independent of  $p$  and  $q$ .

The estimate (A.3) is a consequence of Theorem 18.1.11 of [15].

This completes the proof of the lemma.  $\square$

**Lemma A.2.** *Let  $\varphi_p$  be an Ehrenpreis cutoff as that in (3.5), and let  $L_q \in \Lambda_q^p$ . Let  $k$  denote an integer such that  $mq + k \leq mp$ . Then*

$$(A.7) \quad [L_q, \varphi_p^{(k)}](x, \xi) = \sum_{\ell=1}^{[q]} \sum_{|\beta|=\ell} \frac{1}{\beta!} \varphi_p^{(k+|\beta|)}(x) \partial_\xi^\beta L_q(\xi) + R_{q,\varphi,k}(x, \xi),$$

where  $R_{q,\varphi,k}(x, \xi)$  denotes a symbol in  $S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$  satisfying estimates of the form

$$(A.8) \quad |\partial_x^\gamma \partial_\xi^\beta R_{q,\varphi,k}(x, \xi)| \leq C_{R,\varphi}^{k+1} p^{k+mq+\nu} \langle \xi \rangle^{-|\beta|},$$

where  $[q]$  is the integer part of  $q$ ,  $|\gamma|+|\beta| \leq C_n$ , where  $C_n$  denotes a constant depending on the ambient space dimension  $n$ ,  $C_{R,\varphi} > 0$  depending only on  $n$ ,  $C_a$ ,  $C_\Lambda$ ,  $C_\varphi$  (see Definition 3.3) and  $\nu$  is a positive constant independent of  $q$ ,  $p$ ,  $k$ .

In particular the operator  $R_{q,\varphi,k}(x, D)$  is bounded in  $L^2(\mathbb{R}^n)$  with the estimate

$$(A.9) \quad \|R_{q,\varphi,k}(x, D)u\|_0 \leq C_0^{k+1} p^{k+mq+\nu} \|u\|_0,$$

for any  $u \in \mathcal{S}(\mathbb{R}^n)$ .

*Proof.* The proof is carried out along the same lines as the proof of Lemma A.1. In particular we show that  $R_{q,\varphi,k}$  is a symbol in  $S_{1,0}^0$  in exactly the same way, since no explicit control of the derivatives is required.

We give only some indication about how to obtain (A.8).



Let us denote by  $R_{q,\varphi,k}$  the symbol in the last line of (A.4). Then we have

$$\begin{aligned} & \partial_x^\gamma \partial_\xi^\beta R_{q,\varphi,k}(x, \xi) \\ &= ([q]+1) \sum_{|\alpha|=[q]+1} \frac{1}{\alpha!} i^{-|\alpha|} \int \int_0^1 e^{iz\zeta} (1-s)^{[q]} \partial_\xi^{\alpha+\beta} L_q(\xi+s\zeta) \varphi_p^{(k+|\alpha|+|\gamma|)}(x-z) d\zeta dz ds. \end{aligned}$$

Since  $(1-\Delta_z)^M e^{iz\zeta} = e^{iz\zeta} (1+|\zeta|^2)^M$  we may write the above integral as

$$\frac{1}{\alpha!} \int \int_0^1 e^{iz\zeta} (1+|\zeta|^2)^{-M} (1-s)^{[q]} \partial_\xi^{\alpha+\beta} L_q(\xi+s\zeta) (1-\Delta_x)^M \varphi_p^{(k+|\alpha|+|\gamma|)}(x-z) d\zeta dz ds,$$

which makes the integral a convergent one provided  $M \geq \frac{n+1}{2}$ , since the function  $a$  has compact support.

We split the domain of integration into two regions:  $|\zeta| \leq \varepsilon|\xi|$  and  $|\zeta| \geq \varepsilon|\xi|$ , where  $\varepsilon$  denotes a positive number less than 1.

Consider the integral over the first region:

$$\begin{aligned} \text{(A.10)} \quad R_1(x, \xi) &= \frac{1}{\alpha!} \\ & \int_{|\zeta| \leq \varepsilon|\xi|} \int_0^1 e^{iz\zeta} (1+|\zeta|^2)^{-M} (1-s)^{[q]} \partial_\xi^{\alpha+\beta} L_q(\xi+s\zeta) (1-\Delta_x)^M \varphi_p^{(k+|\alpha|+|\gamma|)}(x-z) d\zeta dz ds. \end{aligned}$$

We observe that  $|\xi + s\zeta| \sim |\xi|$  as before and we choose  $M \sim \frac{n+1}{2}$ .

Let us now prove estimate (A.8) for  $R_1$ . We point out that  $|\alpha| + |\beta| = [q] + 1 + |\beta| \leq q + 1 + C_n \leq R_0(q + 1)$  if  $R_0$  is chosen large enough. Hence the derivatives on  $L_q$  are admissible according to Definition 3.3.

Moreover  $k + |\alpha| + |\gamma| + 2[\frac{n}{2}] + 2 \leq k + [q] + 1 + C_n + 2[\frac{n}{2}] + 2 \leq p + C_n + 2[\frac{n}{2}] + 2 \leq R(p + 1)$  if  $R$  is large enough. Then, arguing as before,

$$\begin{aligned} |R_1(x, \xi)| &\leq \frac{1}{\alpha!} C_\Lambda^{1+|\alpha|+|\beta|} C_\varphi^{k+|\alpha|+|\gamma|+2[\frac{n}{2}]+2+1} \langle \xi \rangle^{q-|\alpha|-|\beta|} p^{(k+2|\alpha|+|\beta|+|\gamma|+2[\frac{n}{2}]+2)} \\ & \quad \cdot \int (1+|\zeta|^2)^{-M} d\zeta \int_{x-\text{supp } a} dz \\ &\leq \frac{1}{\alpha!} C_\Lambda^{1+|\alpha|+|\beta|} C_\varphi^{k+|\alpha|+|\gamma|+2[\frac{n}{2}]+2+1} p^{m|\alpha|+k} \langle \xi \rangle^{-|\beta|} p^{C_n+2[\frac{n}{2}]+2} \\ & \quad \cdot \int (1+|\zeta|^2)^{-M} d\zeta \int_{x-\text{supp } a} dz \leq C_{R_1,\varphi}^{k+1} p^{k+mq+c} \langle \xi \rangle^{-|\beta|}, \end{aligned}$$

where  $c$  is a positive universal constant independent of  $q, p$ ,  $C_{R_1,\varphi}$  verifies the same conditions of  $C_{R,\varphi}$  in the statement of the lemma and we used the estimate

$$\frac{b^{|\alpha|}}{\alpha!} \leq e^{nb} \quad \text{for } b \in \mathbb{R}^+.$$

We point out that in deriving the above estimate we used the fact that  $m \geq 2$ . Furthermore a slightly better estimate could have been obtained by using Lemma 2.2 of [13] with a compactly supported cutoff function in a Gevrey class. However this is not particularly useful in our context.

Consider the integral over the second region:

$$(A.11) \quad R_2(x, \xi) = \frac{1}{\alpha!} \int_{|\zeta| \geq \varepsilon |\xi|} \int_0^1 e^{iz\zeta} (1+|\zeta|^2)^{-M} (1-s)^{[q]} \partial_\xi^{\alpha+\beta} L_q(\xi+s\zeta) (1-\Delta_x)^M \partial_x^{\gamma+\alpha} \varphi_p^{(k)}(x-z) d\zeta dz ds.$$

We choose  $M \sim \left[\frac{n}{2}\right] + 1 + \left[\frac{|\beta|}{2}\right] + 1$ . It is easy to see that  $R_2$  verifies the estimates for the class  $S_{1,0}^0$ . Let us show that  $R_2$  satisfies (A.8). We have that  $k+|\alpha|+|\gamma|+2M = k+|\alpha|+|\gamma|+2\left[\frac{n}{2}\right]+2+2\left[\frac{|\beta|}{2}\right]+2 \leq k+|\alpha|+|\gamma|+n+|\beta|+4 \leq k+q+C_n+n+4 \leq R_0(p+1)$  provided  $R_0$  is large enough. Then

$$\begin{aligned} |R_2(x, \xi)| &\leq \frac{1}{\alpha!} C_\Lambda^{1+|\alpha|+|\beta|} C_\varphi^{k+|\alpha|+|\gamma|+2\left[\frac{n}{2}\right]+4+2\left[\frac{|\beta|}{2}\right]} p^{k+2|\alpha|+|\gamma|+2|\beta|+4+2\left[\frac{n}{2}\right]} \\ &\cdot \int_0^1 \int_{|\zeta| \geq \varepsilon |\xi|} (1+|\zeta|^2)^{-\left[\frac{n}{2}\right]-\left[\frac{|\beta|}{2}\right]-2} \langle \xi + s\zeta \rangle^{q-|\alpha|-|\beta|} d\zeta ds \int_{x-\text{supp } a} dz \\ &\leq C_{R_2, \varphi}^{k+1} p^{k+mq+c_1} \langle \xi \rangle^{-|\beta|}, \end{aligned}$$

where  $c_1$  is a positive universal constant independent of  $q, p$ ,  $C_{R_2, \varphi}$  verifies the same conditions of  $C_{R, \varphi}$  in the statement of the lemma.

Finally (A.9) is proved as (A.3) in the preceding lemma.

This completes the proof of the lemma.  $\square$

Since in the proof of Theorem 1.4 we use iteratively the a priori estimate (2.1) which is applied to smooth functions with compact support, we need a slight modification of the previous lemma allowing for estimates of norms of non compactly supported functions.

**Lemma A.3.** *Let  $\varphi_p$  be an Ehrenpreis cutoff as that in (3.5), and let  $L_q \in \Lambda_q^p$ . Let  $k$  denote an integer such that  $mq+k \leq mp$ . Then*

$$(A.12) \quad \sigma(L_q \varphi_p^{(k)})(x, \xi) = \sum_{\ell=0}^{[q]+1} \sum_{|\beta|=\ell} \frac{1}{\beta!} \varphi_p^{(k+|\beta|)}(x) \partial_\xi^\beta L_q(\xi) + R_{q, \varphi, k}(x, \xi),$$

where  $R_{q, \varphi, k}(x, \xi)$  denotes a symbol in  $S_{1,0}^{-1}(\mathbb{R}^n \times \mathbb{R}^n)$  satisfying estimates of the form

$$(A.13) \quad |\partial_x^\gamma \partial_\xi^\beta R_{q, \varphi, k}(x, \xi)| \leq C_{R, \varphi}^{k+1} p^{k+mq+v} \langle \xi \rangle^{-1-|\beta|},$$

where  $[q]$  is the integer part of  $q$ ,  $|\gamma|+|\beta| \leq C_n$ , where  $C_n$  denotes a constant depending on the ambient space dimension  $n$ ,  $C_{R,\varphi} > 0$  depending only on  $n$ ,  $C_a$ ,  $C_\Lambda$ ,  $C_\varphi$  (see Definition 3.3) and  $\nu$  is a positive constant independent of  $q$ ,  $p$ ,  $k$ .

In particular the operator  $R_{q,\varphi,k}(x, D)$  is bounded from  $L^2(\mathbb{R}^n)$  to  $H^1(\mathbb{R}^n)$  with the estimate

$$(A.14) \quad \|R_{q,\varphi,k}(x, D)u\|_1 \leq C_0^{k+1} p^{k+qm+\nu} \|u\|_0,$$

for any  $u \in \mathcal{S}(\mathbb{R}^n)$ .

*Proof.* We point out that (A.12), (A.13) are proved in the same way as (A.7), (A.8) of the preceding lemma. Thus we have to prove (A.14).

To this end we note that

$$\|R_{q,\varphi,k}(x, D)u\|_1 \leq C \left( \|R_{q,\varphi,k}(x, D)u\|_0 + \sum_{j=1}^n \|D_j R_{q,\varphi,k}(x, D)u\|_0 \right).$$

As in the preceding lemma, using Theorem A.8, we see that

$$(A.15) \quad \|R_{q,\varphi,k}(x, D)u\|_0 \leq C_{00}^{k+1} p^{k+qm+\nu} \|u\|_0.$$

Let us consider then  $\|D_j R_{q,\varphi,k}(x, D)u\|_0$ ,  $1 \leq j \leq n$ . Now the symbol of  $D_j R_{q,\varphi,k}(x, D)$  is

$$\xi_j R_{q,\varphi,k}(x, \xi) + \frac{1}{i} \partial_{x_j} R_{q,\varphi,k}(x, \xi).$$

In view of (A.13) we have that

$$|\partial_x^\gamma \partial_\xi^\beta (\xi_j R_{q,\varphi,k}(x, \xi) + \frac{1}{i} \partial_{x_j} R_{q,\varphi,k}(x, \xi))| \leq C_j^{k+1} p^{k+mq+\nu} \langle \xi \rangle^{-|\beta|},$$

where  $|\gamma| + |\beta| \leq C_n - 1$ . By Theorem A.8 we conclude that

$$(A.16) \quad \|D_j R_{q,\varphi,k}(x, D)u\|_0 \leq C_{0j}^{k+1} p^{k+qm+\nu} \|u\|_0.$$

The estimates (A.15), (A.16) imply the lemma.  $\square$

**Lemma A.4.** Let  $L_q \in \Lambda_q^p$  and  $\beta$  a multiindex such that  $|\beta| \leq q$ . Then

$$(A.17) \quad \frac{1}{C_\Lambda^{|\beta|} p^{|\beta|}} \partial_\xi^\beta L_q \in \Lambda_{q-|\beta|}^p,$$

where  $C_\Lambda$  is the constant in (3.3).

*Proof.* Consider  $\partial_\xi^{\alpha+\beta} L_q$ , with  $|\alpha| \leq R_0(q - |\beta| + 1)$ . By (3.3) we have

$$|\partial_\xi^{\alpha+\beta} L_q| \leq C_\Lambda^{|\alpha|+|\beta|+1} p^{|\alpha|+|\beta|} \langle \xi \rangle^{q-|\alpha|-|\beta|},$$

since  $|\alpha + \beta| \leq R_0(q - |\beta| + 1) + q \leq R_0(q + 1)$ .  $\square$

In particular the above lemma gives that  $\partial_\xi^\beta L_q(\xi) = C_\Lambda^{|\beta|} p^{|\beta|} L_{q-|\beta|}(\xi)$ . We point out that this is a small abuse of notation since the left hand side of the latter equation depends on  $\beta$  and the right hand side depends on  $|\beta|$ . However both sides verify the same symbol estimates depending only on  $|\beta|$  and this is all that we need in the proofs.

We need also a slight extension of the above lemma.

**Lemma A.5.** *Let  $L_q \in \Lambda_q^p$  and  $\beta, \gamma$  multiindices,  $0 \leq \delta \leq 1$ , with  $|\gamma| \leq 2$ ,  $m_0 = \max\{|\beta| - |\gamma|, \delta\} > 0$  and  $|\gamma| \leq |\beta| \leq q$ . Then there exists a positive constant,  $K$ , dependent on  $C_\Lambda, R_0$  and independent of  $p, q, \beta$ , such that, for every  $L_q \in \Lambda_q^p$ ,*

$$(A.18) \quad \frac{1}{KC_\Lambda^{|\beta|} p^{|\beta|}} \langle \xi \rangle^{-\delta} \xi^\gamma \partial_\xi^\beta L_q \in \Lambda_{q-|\beta|+|\gamma|-\delta}^p,$$

where  $C_\Lambda$  is the constant in (3.3).

*Proof.* Consider  $\partial_\xi^\alpha \left( \langle \xi \rangle^{-\delta} \xi^\gamma \partial_\xi^\beta L_q \right)$ , with  $|\alpha| \leq R_0(q - |\beta| + |\gamma| - \delta + 1)$ . Let us show first that  $|\alpha| + |\beta| \leq R_0(q + 1)$ , so that we are able to estimate the  $|\alpha| + |\beta|$  derivatives of  $L_q$ . In fact

$$\begin{aligned} |\alpha| + |\beta| &\leq R_0(q - |\beta| + |\gamma| - \delta + 1) + |\beta| \\ &= R_0(q + 1) - R_0(|\beta| - |\gamma| + \delta) + |\beta| \\ &= R_0(q + 1) - (R_0 - 1)(|\beta| - |\gamma| + \delta) + (|\gamma| - \delta) \\ &\leq R_0(q + 1) - (R_0 - 1)m_0 + 2 - \delta \leq R_0(q + 1), \end{aligned}$$

provided  $R_0$  is large enough.

We have

$$\begin{aligned} \left| \partial_\xi^\alpha \left( \langle \xi \rangle^{-\delta} \xi^\gamma \partial_\xi^\beta L_q \right) \right| &\leq \sum_{\sum_j \mu_j = \alpha} \frac{\alpha!}{\mu_1! \mu_2! \mu_3!} |\partial_\xi^{\mu_1} \langle \xi \rangle^{-\delta}| |\partial_\xi^{\mu_2} \xi^\gamma| |\partial_\xi^{\mu_3 + \beta} L_q| \\ &\leq \sum_{\substack{\sum_j \mu_j = \alpha \\ \mu_2 \leq \gamma}} \frac{\alpha!}{\mu_1! \mu_2! \mu_3!} \frac{\gamma!}{(\gamma - \mu_2)!} C_\Lambda^{1+|\mu_3|+|\beta|} C_0^{|\mu_1|+1} \mu_1! \langle \xi \rangle^{-\delta-|\mu_1|} p^{|\beta|+|\mu_3|} \langle \xi \rangle^{q-|\beta|-|\mu_3|} |\xi^{\gamma-\mu_2}| \\ &\leq \sum_{\substack{\sum_j \mu_j = \alpha \\ \mu_2 \leq \gamma}} \binom{\gamma}{\mu_2} \frac{\alpha!}{(\alpha - \mu_1 - \mu_2)!} C_\Lambda^{1+|\beta|+|\mu_3|} C_0^{|\mu_1|+1} p^{|\beta|+|\mu_3|} \langle \xi \rangle^{q+|\gamma|-|\beta|-|\alpha|-\delta}. \end{aligned}$$

Now

$$\frac{\alpha!}{(\alpha - \mu_1 - \mu_2)!} \leq \alpha_1^{\mu_{11} + \mu_{21}} \dots \alpha_n^{\mu_{1n} + \mu_{2n}} \leq |\alpha|^{|\mu_1| + |\mu_2|} \leq (2R_0)^{|\mu_1| + |\mu_2|} p^{|\mu_1| + |\mu_2|},$$

since  $|\alpha| \leq R_0(q+1) \leq R_0(p+1) \leq 2R_0p$ . We also have

$$\binom{\gamma}{\mu_2} \leq n^{|\gamma|} \binom{|\gamma|}{|\mu_2|} \leq 2n^2.$$

Hence

$$\begin{aligned} & \left| \partial_{\xi}^{\alpha} \left( \langle \xi \rangle^{-\delta} \xi^{\gamma} \partial_{\xi}^{\beta} L_q \right) \right| \\ & \leq \sum_{\substack{\sum_j \mu_j = \alpha \\ \mu_2 \leq \gamma}} \binom{\gamma}{\mu_2} (2R_0)^{|\mu_1|+|\mu_2|} C_{\Lambda}^{-|\mu_1|-|\mu_2|} C_0^{|\mu_1|+1} C_{\Lambda}^{1+|\alpha|+|\beta|} p^{|\beta|+|\alpha|} \langle \xi \rangle^{q+|\gamma|-|\beta|-|\alpha|-\delta} \\ & \leq 2n^2 \left( \sum_{\mu_1 \leq \alpha} \sum_{\substack{\mu_2 \leq \alpha - \mu_1 \\ \mu_2 \leq \gamma}} (2R_0)^{|\mu_1|+|\mu_2|} C_{\Lambda}^{-|\mu_1|-|\mu_2|} C_0^{|\mu_1|+1} \right) \\ & \quad \cdot C_{\Lambda}^{1+|\alpha|+|\beta|} p^{|\beta|+|\alpha|} \langle \xi \rangle^{q+|\gamma|-|\beta|-|\alpha|-\delta} \\ & \leq 2n^2 C_0 \left( \sum_{\ell_1=0}^{|\alpha|} \binom{n-1+\ell_1}{n-1} \left( \frac{2R_0 C_0}{C_{\Lambda}} \right)^{\ell_1} \sum_{\ell_2 \leq 2} \binom{n-1+\ell_2}{n-1} \left( \frac{2R_0}{C_{\Lambda}} \right)^{\ell_2} \right) \\ & \quad \cdot C_{\Lambda}^{1+|\alpha|+|\beta|} p^{|\beta|+|\alpha|} \langle \xi \rangle^{q+|\gamma|-|\beta|-|\alpha|-\delta} \\ & \leq 2^{2n-1} n^2 C_0 \left( \sum_{\ell_1=0}^{|\alpha|} \left( \frac{4R_0 C_0}{C_{\Lambda}} \right)^{\ell_1} \sum_{\ell_2 \leq 2} \left( \frac{4R_0}{C_{\Lambda}} \right)^{\ell_2} \right) \\ & \quad \cdot C_{\Lambda}^{1+|\alpha|+|\beta|} p^{|\beta|+|\alpha|} \langle \xi \rangle^{q+|\gamma|-|\beta|-|\alpha|-\delta} \\ & \leq K C_{\Lambda}^{1+|\alpha|+|\beta|} p^{|\beta|+|\alpha|} \langle \xi \rangle^{q+|\gamma|-|\beta|-|\alpha|-\delta}, \end{aligned}$$

provided  $R_0$  is chosen large enough. This completes the proof of the lemma.  $\square$

**Lemma A.6.** *Let  $\theta_0$  denote a fixed positive number and  $\theta \in \mathbb{R}^+$ ,  $0 \leq \theta \leq \theta_0 < q$ . Then there exists a positive constant,  $K$ , independent of  $p, q$ , such that, for every  $L_q \in \Lambda_q^p$ ,*

$$(A.19) \quad \frac{1}{K} \langle \xi \rangle^{-\theta} L_q \in \Lambda_{q-\theta}^p.$$

*Proof.* Let  $\alpha$  be a multiindex such that  $|\alpha| \leq R_0(q - \theta + 1)$ . Then

$$\begin{aligned} \left| \partial_{\xi}^{\alpha} \left( \langle \xi \rangle^{-\theta} L_q \right) \right| & \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial_{\xi}^{\beta} \langle \xi \rangle^{-\theta}| |\partial_{\xi}^{\alpha-\beta} L_q| \\ & \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} C_{\Lambda}^{1+|\alpha|-|\beta|} C_0^{|\beta|+1} p^{|\alpha|-|\beta|} \beta! \langle \xi \rangle^{q-|\alpha|-\theta} \end{aligned}$$

$$\leq \sum_{\beta \leq \alpha} |\alpha|^{|\beta|} C_{\Lambda}^{1+|\alpha|-|\beta|} C_0^{|\beta|+1} p^{|\alpha|-|\beta|} \langle \xi \rangle^{q-|\alpha|-\theta},$$

since  $|\alpha| - |\beta| \leq R_0(q + 1)$ . Here we used the estimate  $|\partial_{\xi}^{\beta} \langle \xi \rangle^{-\theta}| \leq C_0^{|\beta|+1} \beta! \langle \xi \rangle^{-\theta-|\beta|}$ . From the above inequality we get

$$\begin{aligned} \frac{1}{K} \left| \partial_{\xi}^{\alpha} \left( \langle \xi \rangle^{-\theta} L_q \right) \right| &\leq C_{\Lambda}^{1+|\alpha|} p^{|\alpha|} \langle \xi \rangle^{q-|\alpha|-\theta} \frac{C_0}{K} \sum_{\beta \leq \alpha} \left( \frac{C_0}{C_{\Lambda}} \right)^{|\beta|} \frac{(R_0(p+1))^{|\beta|}}{p^{|\beta|}} \\ &\leq C_{\Lambda}^{1+|\alpha|} p^{|\alpha|} \langle \xi \rangle^{q-|\alpha|-\theta} \frac{C_0 C_1}{K} \sum_{\ell \geq 0} \left( \frac{2R_0 C_0 C_1}{C_{\Lambda}} \right)^{\ell}, \end{aligned}$$

where we applied (3.15) as  $\#\{\beta \mid |\beta| = \ell\} \leq C_1^{\ell+1}$  for a suitable constant  $C_1$ . Choosing  $K$  large enough we obtain the assertion.  $\square$

We also recall the  $L^2$  continuity theorem for pseudodifferential operators.

**Definition A.7.** For any  $m \in \mathbb{R}$ ,  $\rho, \delta \in \mathbb{R}$  with  $0 \leq \delta \leq \rho \leq 1$ ,  $\delta < 1$ , we denote by  $S_{\rho, \delta}^m(\mathbb{R}^n \times \mathbb{R}^n)$  the set of all the functions  $p(x, \xi) \in C^{\infty}(\mathbb{R}^{2n})$  such that for every multi-index  $\alpha, \beta$  there exists a positive constant  $C_{\alpha, \beta}$  for which

$$|\partial_{\xi}^{\alpha} \partial_x^{\beta} p(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|},$$

where  $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ .

We denote by  $OPS_{\rho, \delta}^m$  the class of the corresponding pseudodifferential operators  $P = p(x, D)$ .

We denote by  $S_{\rho, \delta}^m(\mathbb{R}^n)$  the class of symbols depending on  $\xi$  only.

It is trivial to see that the symbol class  $S_{\rho, \delta}^m$  equipped with the semi-norms

$$|p|_{\ell}^{(m)} = \max_{|\alpha+\beta| \leq \ell} \sup_{(x, \xi)} \{ |\partial_{\xi}^{\alpha} \partial_x^{\beta} p(x, \xi)| \langle \xi \rangle^{-(m-\rho|\alpha|+\delta|\beta|)} \}, \quad \ell \in \mathbb{N}$$

is a Fréchet space.

The Calderón-Vaillancourt theorem shows the  $L^2$ -continuity properties of the pseudodifferential operators in the above classes (see [5] or, for a more general setting, [17] Chap. 7, Th.1.6). We state below a formulation of such a theorem for pseudodifferential operators of order zero.

**Theorem A.8** (Calderón-Vaillancourt). *Let  $P = p(x, D) \in OPS_{\rho, \delta}^0$  with  $0 \leq \delta \leq \rho \leq 1$ ,  $\delta < 1$ . Then there exist a positive integer  $\ell$  and a positive constant  $M$  (depending only on  $n$ ) such that*

$$(A.20) \quad \|Pu\|_0 \leq M |p|_{\ell}^{(0)} \|u\|_0, \quad \text{for every } u \in L^2(\mathbb{R}^n).$$

*If the symbol of  $P$  is a function of  $\xi$  only, then  $\ell = n + 2$ .*

## REFERENCES

- [1] M. S. Baouendi and Ch. Goulaouic, *Nonanalytic-hypoellipticity for some degenerate operators*, Bull. A. M. S., **78** (1972), 483–486.
- [2] A. Bove and M. Mughetti, *Analytic hypoellipticity for sums of squares in the presence of symplectic non Treves strata*, J. Inst. Math. Jussieu **19**(6) (2020), 1877–1888.
- [3] A. Bove and M. Mughetti, *Analytic and Gevrey hypoellipticity for a class of pseudodifferential operators in one variable*, J. Differential Equations **255** (2013), no. 4, 728–758.
- [4] A. Bove and M. Mughetti, *Gevrey Regularity for a Class of Sums of Squares of Monomial Vector Fields*, Advances in Math. **373**(2020) 107323, 35pp.
- [5] A. P. Calderón and R. Vaillancourt, *A Class of Bounded Pseudo-Differential Operators*, Proc. Nat. Acad. Sci. USA **69** (1972), no. 5, 1185–1187.
- [6] G. Chinni, *On the sharp Gevrey regularity for a generalization of the Métivier operator*, Math. Ann. (2023), <https://doi.org/10.1007/s00208-022-02558-7>.
- [7] P. D. Cordaro and N. Hanges, *Hypoellipticity in spaces of ultradistributions—study of a model case*, Israel J. Math. **191** (2012), 771–789.
- [8] D. Cordaro and A. A. Himonas, *Global analytic hypoellipticity for a class of degenerate elliptic operators on the torus*, Math. Res. Letters **1** (1994), 501–510.
- [9] L. Ehrenpreis, *Solutions of some problems of division IV*, Amer. J. Math **82** (1960), 522–588.
- [10] M. Derridj and C. Zuily, *Sur la régularité Gevrey des opérateurs de Hörmander*, J. Math. Pures Appl. **52** (1973), 309–336.
- [11] M. Derridj, *On Gevrey vectors of L. Hörmander’s operators*, Transactions Amer. Math. Soc. **372** (2019), no. 6, 3845–3865.
- [12] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171.
- [13] L. Hörmander, *Uniqueness Theorems and Wave Front Sets for Solutions of Linear Differential Equations with Analytic Coefficients*, Communications Pure Appl. Math. **24** (1971), 671–704.
- [14] L. Hörmander, *The Analysis of Partial Differential Operators, I*, Springer Verlag, 1985.
- [15] L. Hörmander, *The Analysis of Partial Differential Operators, III*, Springer Verlag, 1985.
- [16] J.J. Kohn, *Hypoellipticity and loss of derivatives*, Annals of Mathematics **162**(2) (2005), 943–982.
- [17] H. Kumano-Go, *Pseudodifferential Operators*, MIT Press, Cambridge, Mass.–London, 1981.
- [18] G. Métivier, *Une Classe d’Opérateurs Non Hypoelliptiques Analytiques*, Indiana Univ. Math. J. **29** (1980), 823–860.
- [19] G. Métivier, *Non-hypoellipticité Analytique pour  $D_x^2 + (x^2 + y^2)D_y^2$* , C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 7, 401–404.
- [20] M. Mughetti, *Regularity properties of a double characteristics differential operator with complex lower order terms*, J. Pseudo-Differ. Oper. Appl. **5** (2014), no. 3, 343–358.
- [21] T. Ōkaji, *On the Gevrey index of some hypoelliptic operators*, J. Math. Kyoto Univ. **26**(3) (1986), 387–402.
- [22] O. A. Oleñnik, *On the analyticity of solutions of partial differential equations and systems*, Colloque International CNRS sur les Équations aux Dérivées Partielles

- Linéaires (Univ. Paris- Sud, Orsay, 1972), 272–285. Astérisque, 2 et 3. Societé Mathématique de France, Paris, 1973.
- [23] O. A. Oleïnik and E. V. Radkevič, *The analyticity of the solutions of linear partial differential equations*, (Russian) Mat. Sb. (N.S.), **90(132)** (1973), 592–606.
- [24] L. Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (3 - 4) (1976), 247–320.
- [25] F. Treves, *Symplectic geometry and analytic hypo-ellipticity*, in *Differential equations*, La Pietra 1996 (Florence), Proc. Sympos. Pure Math. **65**, Amer. Math. Soc., Providence, RI, 1999, 201-219.
- [26] F. Treves, *On the analyticity of solutions of sums of squares of vector fields*, Phase space analysis of partial differential equations, Bove, Colombini, Del Santo ed.'s, 315-329, Progr. Nonlinear Differential Equations Appl., **69**, Birkhäuser Boston, Boston, MA, 2006.
- [27] F. Treves, *Analytic Partial Differential Equations*, Grundlehren der Mathematischen Wissenschaften, vol. 359, Springer Verlag, 2022.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI BOLOGNA, PIAZZA DI PORTA SAN DONATO 5, BOLOGNA ITALY

*Email address:* antonio.bove@unibo.it

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI BOLOGNA, PIAZZA DI PORTA SAN DONATO 5, BOLOGNA ITALY

*Email address:* marco.mughetti@unibo.it