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# Quantum Dust Cores of Black Holes and Their Quasi-Normal Modes

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## ABSTRACT

The quantum description of a gravitationally collapsed ball of dust proposed by R. Casadio is characterised by a linear effective Misner–Sharp–Hernandez mass function describing a matter core hidden by the event horizon. After reviewing the original model and some of its refinements, we investigate the quasi-normal mode spectrum of the resulting spacetime and compare it with the Schwarzschild case. Computations are performed within the WKB approximation, based on the Padé approximants up to 13th order. Our analysis shows that deviations from the Schwarzschild spectrum are sensitive to the quantum nature of the core surface.

## 1 | Introduction

Black holes are some of the most challenging physical prediction of General Relativity (GR) [1]. Traditionally, they are described by stationary vacuum solutions of the Einstein equations, which are characterised by the presence of an event horizon and by their geodesic incompleteness. Spacetime singularities mark the breakdown of the classical theory of gravity, and there is a growing consensus that they should not show up in a complete, but still unknown, theory of quantum gravity [2]. Moreover, their trivial vacuum structure is likely altered if one takes into account the quantum nature of matter that collapses and forms astrophysical black holes.

Fundamental constituents of matter in the standard model of particle physics are represented by quantum excitations of fields that behave like localised objects (particles) in suitable regimes (when particle number is conserved), and any realistic modelling of black holes should somehow take into account these features. In this context, a quantum model was studied in Ref. [1] in which a black hole forms from the gravitational collapse of an isotropic distribution of dust, partitioned into nested layers of dust particles moving in the Schwarzschild spacetime

$$ds^2 = -\left(1 - \frac{2G_N m}{r}\right) dt^2 + \left(1 - \frac{2G_N m}{r}\right)^{-1} dr^2 + r^2 d\Omega, \quad (1.1)$$

where  $m = m(r)$  is the Misner–Sharp–Hernandez (MSH) fraction of the Arnowitt–Deser–Misner (ADM) mass inside a sphere of radius  $r = r(\tau)$  and  $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ .<sup>1</sup> Each dust particle carries the same proper mass  $\mu$ , while the  $i$ th layer has a MSH mass  $\mu_i$  for  $i = 0, \dots, N$ , where  $i = 0$  refers to the innermost core, and  $i = N$  labels the outermost shell. The cumulative mass  $M_i$  inside the  $i$ th layer is then given by

$$M_i = \sum_{j=0}^{i-1} \mu_j, \quad (1.2)$$

with  $M_1 = \mu_0$  and the total ADM mass  $M = M_{N+1}$ . During the collapse, each dust particle falls freely along radial time-like geodesics, whose equation for the shell  $r = R_i(t)$  reads

$$H_i \equiv \frac{P_i^2}{2\mu} - \frac{G_N \mu M_i}{R_i} = \frac{\mu}{2} \left( \frac{E_i^2}{\mu^2} - 1 \right) = \mathcal{E}_i, \quad (1.3)$$

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where  $H_i$  denotes the Hamiltonian of the system,  $P_i = \mu dR_i/d\tau$  is the radial momentum conjugated to  $R_i(\tau)$  and  $E_i$  the conserved momentum conjugated to  $t = t_i(\tau)$ .

The differences between our description of a ball of dust particles and other approaches to the quantisation of a dust ball are discussed in details in Ref. [1]. We just remark that our procedure amounts to quantising the trajectories of individual dust particles like is done for the electron in the quantum mechanics of the hydrogen atom, whereas canonical approaches like that in Ref. [3] quantise collective degrees of freedom, such as the radius of the ball or of shells. The relativistic action for geodesics of dust particles is their proper time (times the mass  $\mu$ ), which yields the mass-shell condition in the form of the Hamiltonian constraint (1.3) [4]. The canonical quantisation prescription therefore leads to a time-independent Schrödinger equation,

$$\hat{H}_i \psi_{n_i} = \mathcal{E}_i \psi_{n_i}, \quad (1.4)$$

whose solutions are given by the Hamiltonian eigenfunctions  $\psi_{n_i}$  corresponding to the eigenvalues

$$\mathcal{E}_i = \frac{\mu^3 M_i^2}{2 m_p^4 n_i^2}. \quad (1.5)$$

We also recall that the measure in this Hilbert space is chosen to ensure that  $\hat{H}_i$  are Hermitian (precisely like for the hydrogen atom) [1]. Since  $E_i^2 \geq 0$ , Equation (1.3) sets the bound  $n_i \geq N_i$ , where <sup>2</sup>

$$N_i = \frac{\mu M_i}{m_p^2} \quad (1.6)$$

represents the allowed ground state of each particle in the  $i$ th layer. The corresponding wave-function is given by

$$\psi_{N_i}(R_i) = \sqrt{\frac{\mu m_p}{\pi \ell_p^3 M_i^2}} \exp\left(-\frac{\mu R_i}{m_p \ell_p}\right) L_{N_i-1}^1\left(\frac{2\mu R_i}{m_p \ell_p}\right), \quad (1.7)$$

with  $L_{n-1}^1$  the generalised Laguerre polynomials and  $n = 1, 2, \dots$ . Finally, the wave-functions (1.7) are normalised in the scalar product, which makes  $\hat{H}_i$  Hermitian

$$\langle n_i | n_i' \rangle = 4\pi \int_0^\infty R_i^2 \psi_{n_i}^*(R_i) \psi_{n_i'}(R_i) dR_i = \delta_{n_i n_i'}. \quad (1.8)$$

For this ground state, one can compute both the expectation value

$$\bar{R}_i = \langle N_i | \hat{R}_i | N_i \rangle = \frac{3}{2} G_N M_i \quad (1.9)$$

and its uncertainty

$$\frac{\Delta R_i}{\bar{R}_i} = \frac{\sqrt{\langle N_i | \hat{R}_i^2 | N_i \rangle - \bar{R}_i^2}}{\bar{R}_i} = \frac{\sqrt{N_i^2 + 2}}{3N_i} \simeq \frac{1}{3}, \quad (1.10)$$

where we assumed  $N_i \gg 1$ . The finest layering compatible with the quantum uncertainty (1.10) is given by  $\bar{R}_{i+1} \simeq \bar{R}_i + \Delta R_i$ . The discrete MSH mass is therefore distributed as  $M_{i+1} \simeq 4M_i/3$ , which means that  $M_i$  grows linearly with respect to the radius  $R_i$

(regardless of the total number of layers  $N$ ). This result motivates the introduction of an effective energy density

$$\rho \simeq \frac{m_p}{6\pi \ell_p r^2}, \quad (1.11)$$

which yields a linear effective MSH mass

$$m(r) \simeq \frac{2m_p r}{3\ell_p}. \quad (1.12)$$

This mass function weakens the central singularity of the Schwarzschild solution and matches the total ADM mass at the surface of the outermost layer, as shown in the left panel of Figure 1. Equation (1.12) thus provides an effective description of the geometry of a collapsed core of radius  $R_s \simeq \bar{R}_N + \Delta R_N \simeq 3G_N M/2$ , shorter than, but still comparable to, the gravitational radius  $R_H = 2G_N M$  (more details about the effective energy-momentum tensor of the ground state and interior geometry can be found in Ref. [1]).

This brief summary of the model from Ref. [1] illustrates the derivation of its linear mass distribution (1.12), which regularises the singularity and represents an interesting playground for the calculation of quasi-normal modes (QNMs). We recall that black holes are characterised by quasi-normal frequencies of oscillation [6–8] and footprints of the collapsed dust distribution could then be observed in deviations from the characteristic spectrum of a classical Schwarzschild black hole. In the analysis of linear perturbations of a static and spherically symmetric gravitational background, QNMs are solutions of the form  $e^{-i\omega t} \xi_\omega(r)$  which satisfy the boundary conditions

$$\xi_\omega(r_*) \sim \begin{cases} \exp(-i\omega r_*), & \text{for } r_* \rightarrow -\infty \\ \exp(+i\omega r_*), & \text{for } r_* \rightarrow +\infty, \end{cases} \quad (1.13)$$

where  $r_*$  is the tortoise radial coordinate. The outer horizon forbids the system to be time-symmetric, hence the boundary problem to be Hermitian, and this results in complex frequency at infinity,

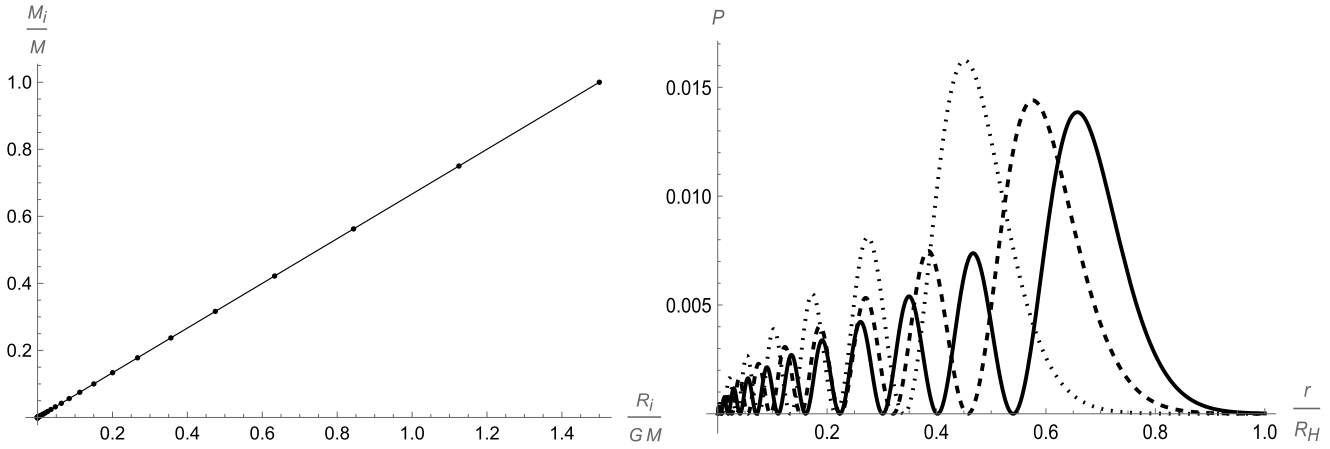
$$\omega = \omega_R + i\omega_I. \quad (1.14)$$

Since the boundary problem (1.13) admits only discrete values with imaginary part  $\omega_I < 0$ , quasi-normal modes describe damped oscillations with a decay timescale set by  $\omega_I^{-1}$ .

In Section 2, we will show that the linear mass profile can be improved by taking into account the effective quantum description of matter in the outermost layer or among all layers of the collapsed core [9]; QNMs are then computed for each different description of the core in Section 3, and conclusions are drawn in Section 4.

## 2 | Mass Distributions

The linear effective mass function (1.12) matches continuously with the outer Schwarzschild geometry of constant ADM mass  $M$  (for  $r \gtrsim R_s$ ). However, its first derivative is discontinuous at



**FIGURE 1** | Left panel: discrete mass function  $M_i$  (dots) for  $N = 100$  layers and its continuous approximation (1.12) (solid line). Right panel: probability densities (2.2) for  $N = 3$ ,  $\mu = m_p/10$ ,  $M = (440/3)\mu$ .

the core surface, with  $m'(r) > 0$  for  $r \rightarrow R_s^-$  [and  $m'(r) = 0$  for  $r > R_s$ ], which arguably conflicts with the quantum core being in (final) equilibrium in the ground state. We also remark that the outermost layer with (approximate) thickness  $\Delta R_N \simeq R_s/4$  contains one fourth of the total mass  $M$ ,

$$\mu_N = M_{N+1} - M_N \simeq \frac{M}{4}. \quad (2.1)$$

A better approximation for the effective geometry near the core surface can be obtained by considering more carefully the fuzziness of quantum layers, which is effectively described by the thickness  $\Delta R_i$  in Equation (1.10).

A more accurate description of the mass distribution inside each layer can be obtained from the probability density to localise a particle of the  $i$ th layer:

$$\mathcal{P}_i = 4\pi r^2 |\psi_{\eta_i}(r)|^2. \quad (2.2)$$

The corresponding effective mass distribution is then given by  $\eta_i = \mu_i \mathcal{P}_i$ , which also shows a non-zero probability to find a particle in a different layer  $j \neq i$ . In particular, the mass distribution in the outermost layer

$$\eta_N = \mu_N \mathcal{P}_N \quad (2.3)$$

does not depend on the number of layers  $N$ , and is again expected to match smoothly with the outer Schwarzschild geometry of total ADM mass  $M$ . This is only possible if the behaviour of  $\eta_N$  deviates from the linear form in (1.12), which also follows from the wave-functions (1.7).

Finally, we note that the same argument based on the wave-functions (1.7) implies that dust particles can be located at positions  $r > R_H$  with finite (albeit typically very small) probability.

## 2.1 | Quantum Mass Refinement

The wave-function (1.7) of each layer shows an oscillatory damped pattern that suggests an overlap among (at least) the nearest

neighbours. This effect turns the linear mass profile into a (slightly) parabolic mass distribution inside the ball (all the details can be found in Ref. [9]). In fact, particles from the  $i$ th layer have non-vanishing probability to be located inside the  $j$ th layer given by Equation (2.2) (see right panel of Figure 1).

The mass elements which take into account this quantum effect are defined as

$$\Delta\mu_{i,j} = \mu_i \int_{R_j}^{R_{j+1}} \mathcal{P}_i(r) dr, \quad i, j = 0, \dots, N. \quad (2.4)$$

The sum of all these contributions inside the  $j$ th layer gives the refined mass of that layer indicated as  $\Delta\mu_j$ . The total mass distribution  $\mathcal{M}_i$  is then obtained by cumulative summing the mass of each layer

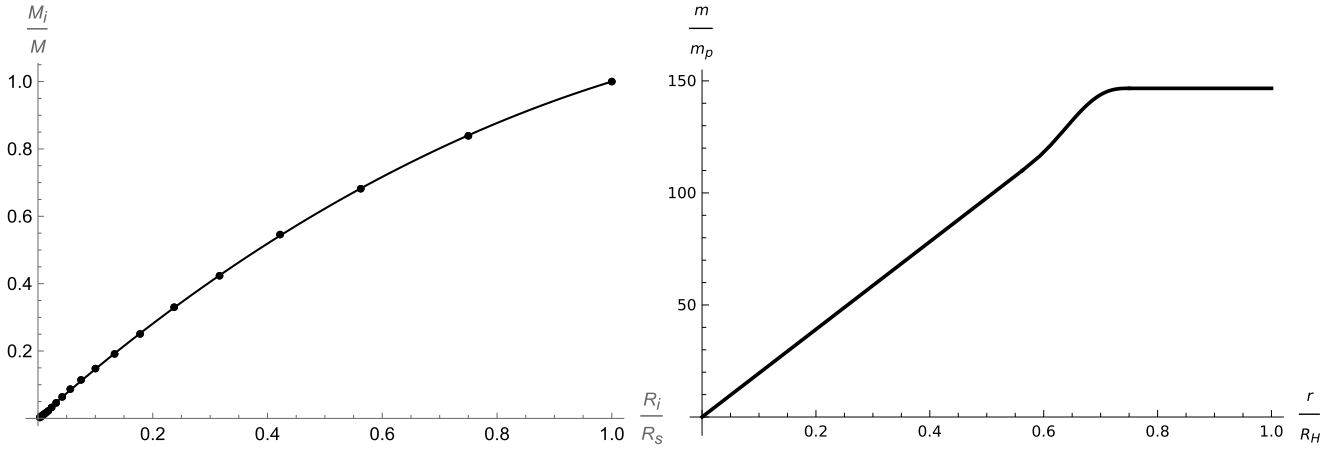
$$\mathcal{M}_i = \sum_{j=0}^i \Delta\mu_j, \quad i = 0, \dots, N. \quad (2.5)$$

The best fit of the curve obtained from the numerical calculation of  $\mathcal{M}_i$  shows the parabolic profile

$$m = M(ax + bx^c), \quad (2.6)$$

with  $x = r/R_H$  and  $a, b, c$  fitting coefficients [9].<sup>3</sup>This refined distribution preserves the integrability of the metric in the origin and does not give rise to any inner horizons. Values of  $\mathcal{M}_i$  and the parabolic profile are shown in the left panel of Figure 2.

It is important to recall that the ground state for the dust core that we consider requires a large number of dust particles in each layer, which means that  $M_i \gg m_p \gtrsim \mu$ . For a realistic astrophysical black hole, with  $M$  at least of the order of the solar mass and  $\mu$  of the order of the proton mass, the Laguerre polynomials in the wave-functions (1.7) are of order  $n \gtrsim 10^{18}$ , which is numerically intractable. For this reason, in the following, we shall perform all numerical calculations that involve Equation (1.7) only for values of  $M_i$  and  $\mu$  that are tractable, albeit far from realistic.



**FIGURE 2** | Left panel: quantum corrected MSH mass function for  $N = 20$  with  $M = 3000 m_p$  and  $\mu = m_p/10$ . Right panel: interpolating MSH mass function for a black hole of total ADM mass  $M \simeq 150 m_p$ ,  $\mu = m_p/10$  and  $R_H = 300 \ell_p$ .

## 2.2 | Boundary Layer

The above improved description of the effective mass function allows us to introduce equally improved descriptions of the outermost layer of dust based on the wave-function (1.7) for  $i = N$ .

First of all, for the linear mass distribution (1.12), we can consider a smoothed MSH mass for the outermost layer given by

$$\begin{aligned} m_{\text{lin}}(r \geq R_N) &= M_N + \int_{R_N}^r \eta_N(x) dx \\ &= M_N + 4\pi \mu_N \int_{R_N}^r x^2 |\psi_N(x)|^2 dx, \end{aligned} \quad (2.7)$$

with  $M_N = 3M/4$  and  $\mu_N = M/4$ . This function accounts for the linear mass profile up to the inner border  $r = \bar{R}_N$  of the outermost layer, but is also sensitive to the quantum nature of the dust particles which can be localised outside  $R_s$ , an effect particularly relevant in this (most massive) layer.

Similarly, for the parabolic refinement (2.6) reviewed in Section 2.1, we consider an MSH function for the outermost layer given by

$$m_{\text{par}}(r \geq R_N) = \mathcal{M}_N + 4\pi \mu_N \int_{R_N}^r x^2 |\psi_N(x)|^2 dx, \quad (2.8)$$

where  $\mathcal{M}_N$  and the mass of the outermost layer  $\mu_N$  are now obtained using the numerical mass profile (2.5).

In both cases above, the effective MSH mass approaches  $M$  more rapidly (and smoothly) than the simple linear function (1.12) in the transitional region of width  $\Delta R_N$ . We can also consider a mass function  $m = m(r)$ , which interpolates smoothly between the linear form (1.12) for  $r \lesssim r_0 \equiv \bar{R}_N$  and the outer Schwarzschild vacuum for  $r > r_1 \equiv \bar{R}_N + \Delta R_N = R_s$ . Clearly,  $r_0$  and  $r_1$  parametrise the boundaries of the outermost layer and we

can write the interpolating mass function as

$$m_{\text{int}}(r) = \begin{cases} \alpha r, & \text{for } r \leq r_0 \\ B(r), & \text{for } r_0 \leq r \leq r_1 \\ M, & \text{for } r \geq r_1, \end{cases} \quad (2.9)$$

where  $\alpha$  is a constant and  $B(r)$  a function to be determined. The right panel of Figure 2 shows the interpolating mass function (2.9), whose analytical expression can be found in Appendix A (see also Ref. [10]). As required, the interpolating mass function (2.9) matches the outer Schwarzschild geometry, with continuous first and second derivatives at  $r = R_s$ . This implies that quantum effects at the outermost layer are neglected in this approximation and the corresponding outer geometry is exactly given by the Schwarzschild vacuum. In this case, no effect can be expected on linear perturbations for  $r > R_H$ , including the QNMs that we are going to study in the next section.

## 3 | Quasi-Normal Modes

Let us briefly recall the theoretical framework of black hole QNMs. Their governing equation can be solved with various numerical methods, extensively reviewed in Ref. [11], but we shall here only employ the WKB approach.

For a generic static and spherical symmetric spacetime, the metric reads

$$ds^2 = -f(r) dt^2 + h(r) dr^2 + \sigma^2(r) d\Omega^2. \quad (3.1)$$

Scalar (massless) perturbations are described by solutions of the Klein–Gordon equation

$$\square \Phi = \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu] \Phi = 0, \quad (3.2)$$

and spherical symmetry and time independence of the metric allow for the factorisation

$$\Phi = e^{-i\omega t} \phi_{\omega\ell m}(r) Y_l^m, \quad (3.3)$$

where  $Y_\ell^m = Y_\ell^m(\theta, \phi)$  are spherical harmonics and  $\phi_{\omega\ell m}$  is the radial function with  $0 \leq l \leq n$  and  $-l \leq m \leq l$ . Equation (3.2) then yields

$$\omega^2 \phi_{\omega\ell m} + \frac{f}{h} \left[ \phi_{\omega\ell m}'' + \left( \frac{2\sigma'}{\sigma} + \frac{f'}{2f} - \frac{h'}{2h} \right) \phi_{\omega\ell m}' \right] - \frac{\ell(\ell+1)}{\sigma^2} f \phi_{\omega\ell m} = 0. \quad (3.4)$$

It is convenient to perform the substitution

$$\phi_{\omega\ell m}(r) = \left( \frac{h}{f} \right)^{1/4} \frac{\xi}{\sigma}, \quad (3.5)$$

with  $\xi = \xi_{\omega\ell m}(r)$ , that leads to

$$\omega^2 \xi = -\frac{f}{h} \left\{ \xi'' + \frac{1}{4} \left[ \frac{h''}{h} - \frac{f''}{f} - \frac{4\sigma''}{\sigma} + \frac{3}{4} \left( \frac{f'}{f} \right)^2 - \frac{5}{4} \left( \frac{h'}{h} \right)^2 - \frac{2\sigma'f'}{\sigma f} + \frac{2\sigma'h'}{\sigma h} + \frac{f'h'}{2fh} \right] \xi \right\} + \frac{\ell(\ell+1)}{\sigma^2} f \xi. \quad (3.6)$$

This can be greatly simplified by setting  $h = f^{-1}$ , using areal coordinates  $\sigma = r$ , and introducing the tortoise coordinate  $dr_* = dr/f$ . Equation (3.6), finally becomes

$$\frac{d^2 \xi(r)}{dr_*^2} + [\omega^2 - V(r)] \xi(r) = 0 \quad (3.7)$$

with

$$V(r) = \frac{\partial_{r_*}^2 r}{r} + \frac{\ell(\ell+1)}{r^2} f(r), \quad (3.8)$$

where  $r = r(r_*)$ . Similar equations hold for vector and tensor perturbations and a master (radial) equation can be cast in the form

$$\frac{d^2 \xi_s(r)}{dr_*^2} + Q_s(r) \xi_s(r) = 0, \quad (3.9)$$

with  $Q_s(r) = \omega^2 - V_s(r)$  and

$$V_s(r) = f(r) \left[ \frac{\ell(\ell+1)}{r^2} + \frac{f'(r)}{r} (1 - s^2) \right], \quad (3.10)$$

where the label  $s$  refers to scalar ( $s = 0$ ), vector ( $s = 1$ ) and tensor ( $s = 2$ ) perturbations.

This definition holds for  $s = 0, 1, 2$ , in the Schwarzschild spacetime. However, for  $s = 2$ , the tensor perturbations admit a multipole decomposition yielding an odd-parity (axial) and even-parity (polar) potential. The former, also called Regge–Wheeler (RW) potential, coincides with the one obtained from (3.10) for  $s = 2$ , namely

$$V_2^{(o)}(r) = \left( 1 - \frac{R_H}{r} \right) \left[ \frac{\ell(\ell+1)}{r^2} - \frac{3R_H}{r^2} \right], \quad (3.11)$$

while the latter, known as Zerilli potential, takes the form

$$V_2^{(e)}(r) = \frac{2}{r^3} \left( 1 - \frac{R_H}{r} \right) \times \frac{9G_N^3 M^3 + 3\lambda^2 G_N M r^2 + \lambda^2(1+\lambda)r^3 + 9G_N^2 M^2 \lambda r}{(3G_N M + \lambda r)^2}, \quad (3.12)$$

with  $\lambda \equiv (\ell - 1)(\ell + 2)/2$ .

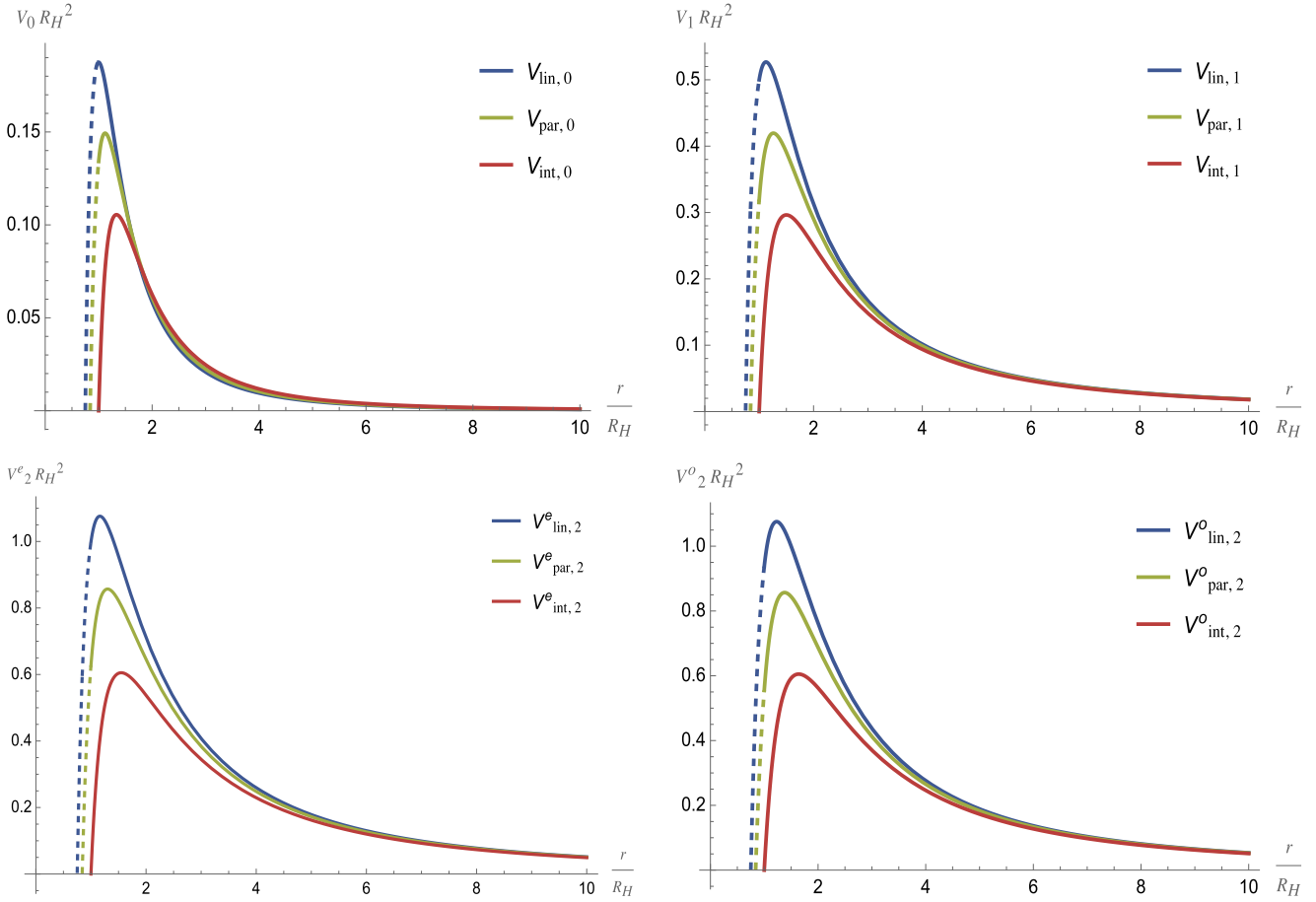
These potentials are shown in Figures 3 and 4, for the effective mass functions  $m_{\text{lin}}$ ,  $m_{\text{par}}$ , and  $m_{\text{int}}$  of the outermost layer of the dust core introduced in Section 2.2. In particular, we recall that the interpolating mass function (2.9) exactly reproduces the vacuum Schwarzschild geometry where QNMs have support. By comparing the plots in Figure 3 with those in Figure 4, it is clear that the differences with respect to the Schwarzschild case become smaller for larger values of the ADM mass  $M \gg \mu$ . This is expected from the asymptotic behaviour of the wavefunctions (1.7) for  $r \gg \bar{R}_i \sim M_i$ .

Having specified the potentials, one can solve the master equation (3.9) with the proper boundary conditions and obtain the QNMs frequencies. Several numerical methods can be employed, based on convergent algorithms, that require nontrivial analysis of the singular points of the master equation. Moreover, some methods involve different procedures for different master equations, which make them less convenient in general. Conversely, the WKB method proposed in Ref. [12] is suitable for various master equations and relative potentials and has been extended to third order in Refs. [13, 14] and lately improved to sixth and thirteenth order [11, 15]. We recall that in our model the black hole is spherically symmetric and the potential is asymptotically constant. Under these conditions, the main idea is to treat the master equation as a Schrödinger-like equation, with a barrier  $Q(r)$  with one peak and two roots, that divide the real line in three regions: I and III outside the turning points and II between them. The solutions in these regions have to be carefully connected with a Taylor expansion near the peak of the potential. In tortoise coordinates, region I corresponds to spatial infinity ( $r_* \rightarrow +\infty$ ), whereas region III corresponds to the horizon  $r = R_H$  ( $r_* \rightarrow -\infty$ ). The crucial part is to connect the amplitudes at infinity with those at the horizon, which is done starting from the usual WKB asymptotic expansion valid in regions I and III

$$\xi(r_*) \sim \exp \sum_{n=0}^{\infty} \frac{\epsilon^n S_n(r_*)}{\epsilon}, \quad (3.13)$$

where the parameter  $\epsilon$  keeps track of the expansion order. Once (3.13) is replaced into the master Equation (3.9) one obtains the expressions for the  $S_n$ . The first two terms  $S_0$  and  $S_1$  provide a solution of the form

$$\xi(r_*) \sim \begin{cases} A_\infty^{(\text{in})} e^{-ik_\infty r_*} + A_\infty^{(\text{out})} e^{+ik_\infty r_*}, & k_\infty^2 = \omega^2 - \lim_{r_* \rightarrow +\infty} V(r_*) \\ A_H^{(\text{in})} e^{+ik_H r_*} + A_H^{(\text{out})} e^{-ik_H r_*}, & k_H^2 = \omega^2 - \lim_{r_* \rightarrow -\infty} V(r_*), \end{cases} \quad (3.14)$$



**FIGURE 3** | Potentials for scalar perturbations  $V_0 R_H^2$  (top left), vector perturbations  $V_1 R_H^2$  (top right), even tensor perturbations  $V_2^{(e)} R_H^2$  (bottom left), and odd tensor perturbations  $V_2^{(o)} R_H^2$  (bottom right):  $V_{\text{lin}}$  for the linear mass function (2.7),  $V_{\text{par}}$  for the parabolic mass function (2.8), and  $V_{\text{int}}$  for the interpolating mass function (2.9). All plots with  $\mu = M/10 = m_p/10$ .

with  $A_H$  and  $A_\infty$  complex amplitudes at the horizon and at infinity, respectively. The amplitudes are linearly related according to

$$\begin{pmatrix} A_H^{(\text{out})} \\ A_H^{(\text{in})} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A_\infty^{(\text{out})} \\ A_\infty^{(\text{in})} \end{pmatrix} \quad (3.15)$$

and the next goal is to determine the matrix elements  $M_{ij}$ . The final step consists into matching WKB solutions in region I and III with the one in region II. This is done by Taylor expanding  $Q(r_*)$  around its maximum and solve Equation (3.9) with a combination of parabolic cylinder functions. The coefficients of these asymptotically expanded solutions, can be matched with those in the outer region and used to populate the matrix (3.15). The boundary conditions for  $\xi$  at  $r_* \rightarrow \pm\infty$  follow from requiring that no waves can escape from the horizon, that is,  $A_H^{(\text{in})} = 0$ , and that there are no incoming waves, that is,  $A_\infty^{(\text{in})} = 0$ .<sup>5</sup> These conditions on the amplitudes forces the respective coefficients  $M_{ij}$  to vanish, which finally allows us to derive the expression

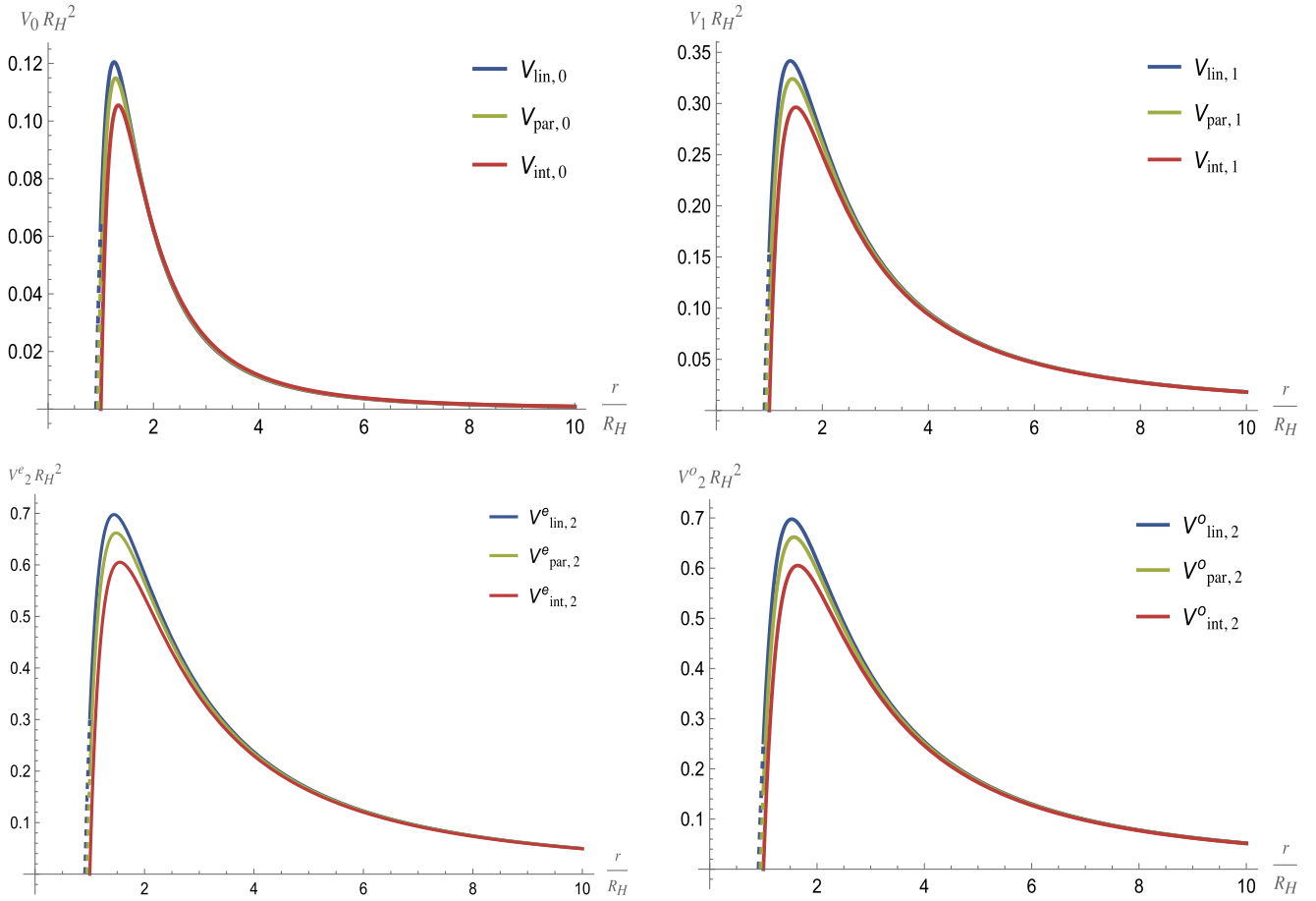
$$i \frac{\omega^2 - V_s(\bar{r}_*)}{\sqrt{-2V_s''(\bar{r}_*)}} = n + \frac{1}{2}, \quad n = 0, 1, 2, \dots, \quad (3.16)$$

with the integer  $n$  called the overtone number and  $\bar{r}_*$  denotes the location of the maximum of  $V_s$ . Equation (3.16) provides a way to compute the QNMs frequencies  $\omega$ .

In Refs. [11, 15], this approach has been extended to higher orders yielding:

$$i \frac{\omega^2 - V_s(\bar{r}_*)}{\sqrt{-2V_s''(\bar{r}_*)}} - \sum_{i=2}^k \Lambda_i = n + \frac{1}{2}, \quad n = 0, 1, 2, \dots, \quad (3.17)$$

with  $\Lambda_i$  that depends on the effective potential and its derivative and accounts for higher-order corrections. Such corrections are needed for achieving greater accuracy. For this reason, the authors of Ref. [11] extended the WKB method to the 13th order and introduced the Padé approximants that sensibly improved the precision for the frequencies, especially for  $n \leq \ell$ . On top of that, for a generic order  $k$  and a given Padé approximant  $P_{\bar{n}/\bar{m}}$  such that  $k = \bar{n} + \bar{m}$ , the best results are typically obtained for  $\bar{n} \sim \bar{m}$ . However, it is important to mention that higher-order corrections do not always correspond to a convergence of the method or to more accurate results. The degree of accuracy depends on the errors associated with the computation, that are usually estimated as the maximum deviation between the average frequency  $\bar{\omega}_k$  at order  $k$ , obtained from all Padé approximants



**FIGURE 4** | Potentials for scalar perturbations  $V_0 R_H^2$  (top left), vector perturbations  $V_1 R_H^2$  (top right), even tensor perturbations  $V_2^{(e)} R_H^2$  (bottom left), and odd tensor perturbations  $V_2^{(o)} R_H^2$  (bottom right):  $V_{lin}$  for the linear mass function (2.7),  $V_{par}$  for the parabolic mass function (2.8), and  $V_{int}$  for the interpolating mass function (2.9). All plots with  $M = 500 m_p$  and  $\mu = 1/100 m_p$ .

$P_{\tilde{n}/\tilde{m}}$  with  $\tilde{n} + \tilde{m} = k$ , and the individual values  $\omega_{\tilde{n}/\tilde{m}}$ . This choice tracks the stability of the solutions and how the values of the approximated frequencies change across different orders, which is reasonable from a convergence perspective, even if it is not mathematically rigorous. Nevertheless, this setup generally offers the best compromise between the overall validity of the method and the quality of the results, especially for higher overtone numbers [16].

Given the above remarks, we computed the QNMs using the Padé approximants to thirteenth order for all the models, except for the interpolating case (equivalent to the standard Schwarzschild vacuum), where we limited the analysis to fifth order. The numerical values of the QNM frequencies for  $M = m_p = 10 \mu$  (corresponding to the potentials in Figure 3) are displayed in Tables 1–4. Both linear and parabolic mass functions lead to small deviations for the frequencies from the Schwarzschild case, with the parabolic model yielding frequencies closer to the Schwarzschild ones, compared to the linear case, across all perturbation types. We recall that this case represents a more accurate internal mass distribution, with a smaller fraction of ADM mass near the surface of the core.

We also remark that the calculations for the interpolating case are performed to fifth order for practical considerations, since higher

orders of Padé approximants become more computationally demanding than advantageous. For some values of  $n$  and  $\ell$ ,<sup>6</sup> the polynomials involve very small numbers that compromise the numerical stability of their asymptotic convergence, rather than improving the precision. This is not a real drawback, as fifth-order frequencies are very accurate and there is no a priori reason to force higher-order corrections [16].

#### 4 | Conclusion and Outlooks

We started by reviewing the quantum dust core model developed in Ref. [1] and its effective energy density  $\rho$  in Equation (1.11). This effective density yields a linear MSH mass in the interior ( $r < R_s \simeq 3 R_H/4$ ) shown in Equation (1.12) and the left panel of Figure 1. Linearity makes the internal metric integrable, and provides a framework for investigating physically more acceptable black hole models [17], whose QNMs can be investigated.

However, the linear mass function does not account for the full quantum nature of the dust particles in the collapsed core. In particular, it discards the overlapping of the eigenfunctions (1.7) shown in the right panel of Figure 1, whose squared amplitude weighs the location of the particles inside each layer and yields the parabolic mass distribution [9] shown in Equation (2.5) and

TABLE 1 | Scalar perturbations.

	Schwarzschild/Interpolating	Linear	Parabolic
$n = 0, \ell = 0$	$0.111 - 0.105 i$	$0.148 - 0.138 i$	$0.132 - 0.125 i$
$n = 0, \ell = 1$	$0.293 - 0.0977 i$	$0.391 - 0.130 i$	$0.349 - 0.116 i$
$n = 1, \ell = 1$	$0.264 - 0.306 i$	$0.352 - 0.409 i$	$0.315 - 0.364 i$
$n = 0, \ell = 2$	$0.484 - 0.0968 i$	$0.645 - 0.129 i$	$0.575 - 0.115 i$
$n = 1, \ell = 2$	$0.464 - 0.296 i$	$0.618 - 0.394 i$	$0.552 - 0.352 i$
$n = 2, \ell = 2$	$0.431 - 0.509 i$	$0.574 - 0.678 i$	$0.512 - 0.605 i$
$n = 0, \ell = 3$	$0.675 - 0.0965 i$	$0.900 - 0.129 i$	$0.804 - 0.115 i$
$n = 1, \ell = 3$	$0.661 - 0.292 i$	$0.881 - 0.390 i$	$0.786 - 0.348 i$
$n = 2, \ell = 3$	$0.634 - 0.496 i$	$0.845 - 0.661 i$	$0.754 - 0.590 i$
$n = 3, \ell = 3$	$0.599 - 0.711 i$	$0.798 - 0.948 i$	$0.712 - 0.846 i$

TABLE 2 | Vector perturbations.

	Schwarzschild/Interpolating	Linear	Parabolic
$n = 0, \ell = 1$	$0.248 - 0.0925 i$	$0.331 - 0.123 i$	$0.295 - 0.110 i$
$n = 1, \ell = 1$	$0.215 - 0.294 i$	$0.285 - 0.392 i$	$0.255 - 0.349 i$
$n = 0, \ell = 2$	$0.458 - 0.0950 i$	$0.610 - 0.127 i$	$0.544 - 0.113 i$
$n = 1, \ell = 2$	$0.437 - 0.291 i$	$0.582 - 0.388 i$	$0.519 - 0.346 i$
$n = 2, \ell = 2$	$0.401 - 0.502 i$	$0.535 - 0.668 i$	$0.477 - 0.597 i$
$n = 0, \ell = 3$	$0.657 - 0.0956 i$	$0.876 - 0.127 i$	$0.782 - 0.113 i$
$n = 1, \ell = 3$	$0.642 - 0.290 i$	$0.856 - 0.386 i$	$0.764 - 0.345 i$
$n = 2, \ell = 3$	$0.614 - 0.492 i$	$0.818 - 0.656 i$	$0.730 - 0.586 i$
$n = 3, \ell = 3$	$0.578 - 0.706 i$	$0.770 - 0.941 i$	$0.688 - 0.840 i$

TABLE 3 | Odd tensor perturbations.

	Schwarzschild/Interpolating	Linear	Parabolic
$n = 0, \ell = 2$	$0.374 - 0.0890 i$	$0.498 - 0.118 i$	$0.445 - 0.106 i$
$n = 1, \ell = 2$	$0.347 - 0.274 i$	$0.461 - 0.365 i$	$0.412 - 0.326 i$
$n = 2, \ell = 2$	$0.298 - 0.477 i$	$0.397 - 0.638 i$	$0.355 - 0.568 i$
$n = 0, \ell = 3$	$0.599 - 0.0927 i$	$0.799 - 0.124 i$	$0.713 - 0.110 i$
$n = 1, \ell = 3$	$0.583 - 0.281 i$	$0.777 - 0.375 i$	$0.693 - 0.335 i$
$n = 2, \ell = 3$	$0.552 - 0.479 i$	$0.735 - 0.639 i$	$0.656 - 0.570 i$
$n = 3, \ell = 3$	$0.512 - 0.690 i$	$0.682 - 0.920 i$	$0.609 - 0.821 i$

TABLE 4 | Even tensor perturbations.

	Schwarzschild/ Interpolating	Linear	Parabolic
$n = 0, \ell = 2$	$0.374 - 0.0890 i$	$0.498 - 0.119 i$	$0.498 - 0.119 i$
$n = 1, \ell = 2$	$0.347 - 0.274 i$	$0.462 - 0.365 i$	$0.462 - 0.365 i$
$n = 2, \ell = 2$	$0.301 - 0.478 i$	$0.401 - 0.635 i$	$0.401 - 0.635 i$
$n = 0, \ell = 3$	$0.599 - 0.0927 i$	$0.799 - 0.124 i$	$0.799 - 0.124 i$
$n = 1, \ell = 3$	$0.583 - 0.281 i$	$0.777 - 0.375 i$	$0.777 - 0.375 i$
$n = 2, \ell = 3$	$0.552 - 0.479 i$	$0.736 - 0.639 i$	$0.736 - 0.639 i$
$n = 3, \ell = 3$	$0.512 - 0.690 i$	$0.682 - 0.920 i$	$0.682 - 0.920 i$

the left panel of Figure 2. In this work, we further refined the description of the outermost layer, to account for the quantum leaking of the dust particles into the region outside the horizon. This is the effect that can lead to modifications of (linear) perturbations in the exterior space. To show that neglecting this quantum leakage reproduces the Schwarzschild phenomenology, we also considered the mass function (2.9) that interpolates smoothly between the interior linear MSH mass and the total ADM mass at a value of the areal coordinate  $r = R_s < R_H$ .

Finally, we derived the QNMs potential (3.10) that accommodates scalar ( $s = 0$ ), vector ( $s = 1$ ) and odd tensor ( $s = 2$ ) perturbations, while the potential for even tensor perturbations is given in Equation (3.12). The corresponding QNM frequencies were determined from Equation (3.17) in the framework of the WKB approximation, following the discussion in Ref. [18]. The results are listed and compared to the Schwarzschild spectra in Tables 1–4. As expected, the interpolating mass function yields identical results to the Schwarzschild black hole, whereas the MSH mass functions obtained from the explicit wave-functions for dust particles in the outermost layer lead to sizeable, albeit typically very small, deviations. In particular, deviations obtained from the parabolic interior profile are usually smaller than those stemming from the linear case. This could already be predicted from the shape of the potentials shown in Figure 3, since the linear mass function leads to larger deviations from the potentials for the Schwarzschild geometry. It also agrees with the fact that the amount of dust in the outermost layer is larger for the linear profile than for the parabolic case, which reduces the total mass that can leak outside the horizon.

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### Conflicts of Interest

The authors declare no conflict of interest.

### Data Availability Statement

Data sharing was not applicable to this article as no data sets were generated or analyzed in this study.

### Endnotes

<sup>1</sup>We shall always use units with  $c = 1$  and often write the Planck constant  $\hbar = \ell_p m_p$  and the Newton constant  $G_N = \ell_p/m_p$ , where  $\ell_p$  and  $m_p$  are the Planck length and mass, respectively.

<sup>2</sup>See also Ref. [5].

<sup>3</sup>The profile does not significantly depend on the number  $N$  of layers since the largest fraction of the total mass is located in the outer layers.

<sup>4</sup>The second order term in Equation (3.9) is to be considered multiplied by  $\epsilon^2$ .

<sup>5</sup>These arguments are reversed if  $\omega_R < 0$ .

<sup>6</sup>Specifically when  $n \geq \ell$  where the method is less accurate.

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### Appendix A: Hermite Interpolation

We determine a smooth interpolating mass function of the form in Equation (2.9) requiring that  $B(r)$ ,  $B'(r)$  and  $B''(r)$  be continuous at the boundaries  $r = r_0$  and  $r = r_1$  to ensure the continuity of all components of the (effective) energy-momentum tensor at both ends. This results in the six conditions

$$\alpha r_0 = B(r_0), \quad M = B(r_1), \quad (\text{A.1})$$

$$\alpha = B'(r_0), \quad 0 = B'(r_1), \quad (\text{A.2})$$

$$0 = B''(r_0), \quad 0 = B''(r_1), \quad (\text{A.3})$$

which can hold for an ‘osculating polynomial’ of order  $K = 5$ .

Osculating polynomials are interpolating functions which pass through a set of points with specified derivatives [19, 20]. Given a set of  $n + 1$  points  $\{x_i\}_{i=0}^n$ , non-negative numbers  $\{m_i\}_{i=0}^n$ , with  $m_i \in \mathbb{N} \cup \{0\}$ , and values to interpolate  $\{f_j^{(k)}\}$ , with  $j = 0, \dots, n$  and  $k = 0, \dots, m_j$ , the osculating polynomial approximating the function  $f \in C^m([a, b])$  is the polynomial of least degree such that

$$f^{(k)}(x_j) = f_j^{(k)}, \quad \forall j = 0, \dots, n, \quad \forall k = 0, \dots, m_j. \quad (\text{A.4})$$

From these requirements, we have  $n + 1$  conditions for  $f^{(0)} \equiv f$ , and  $\sum_{j=0}^n m_j$  more conditions to be satisfied for the derivatives. A polynomial of degree

$$K = \sum_{j=0}^n m_j + n \quad (\text{A.5})$$

has  $K + 1$  coefficients that can be used to fulfil these requirements. A general theorem [19] states that a polynomial of degree  $K$  such that the conditions (A.4) are satisfied exist and is given by

$$f(x) = \sum_{j=0}^n \sum_{k=0}^{m_j} A_{jk}(x) f_j^{(k)}, \quad (\text{A.6})$$

where

$$A_{jk}(x) = p_j(x) \frac{(x - x_j)^k}{k!} \sum_{l=0}^{m_j-k} \frac{g_j^{(l)}(x_j)}{l!} (x - x_j)^l, \quad (\text{A.7})$$

with

$$p_j(x) = \prod_{l=0, l \neq j}^n (x - x_l)^{m_l+1} \quad (\text{A.8})$$

$$g_j(x) = \frac{1}{p_j(x)}. \quad (\text{A.9})$$

In the special case,  $m_j = 1$  for each  $j = 0, \dots, n$ , the polynomials  $f(x)$  are usually called Hermite polynomials and Equation (A.6) is called the Hermite formula.

In our case, we have conditions up to second derivatives at the two points  $x_0$  and  $x_1$ , so that  $m_0 = m_1 = 2$ . The general formula (A.6) hence simplifies to

$$B(x) = \sum_{j=0}^1 \sum_{k=0}^2 A_{jk}(x) B_j^{(k)}, \quad (\text{A.10})$$

where  $x = r/R_H$ ,  $x_0 = r_0/R_H$  and  $x_1 = r_1/R_H$  and

$$B_j = B^{(0)}(x_j) \quad B'_j = B^{(1)}(x_j) \quad B''_j = B^{(2)}(x_j). \quad (\text{A.11})$$

The conditions (A.1), (A.2) and (A.3) imply

$$B^{(1)}(x_1) = B^{(2)}(x_0) = B^{(2)}(x_1) = 0, \quad (\text{A.12})$$

and we can set

$$A_{02}(x) = A_{11}(x) = A_{12}(x) = 0. \quad (\text{A.13})$$

The full problem therefore simplifies to determine only the three functions  $A_{00}(x)$ ,  $A_{01}(x)$  and  $A_{10}(x)$ . Assuming  $r_0 = \bar{R}_N$  and  $r_1 = \bar{R}_N + \Delta R_N$ , we define

$$\Delta x = \frac{\Delta R_N}{R_H}, \quad (\text{A.14})$$

and find that these functions read

$$A_{00}(x) = -\frac{1}{\Delta x^3}(x - x_1)^3 - \frac{3}{\Delta x^4}(x - x_1)^3(x - x_0) - \frac{6}{\Delta x^5}(x - x_1)^3(x - x_0)^2 \quad (\text{A.15})$$

$$A_{01}(x) = -\frac{1}{\Delta x^3}(x - x_1)^3(x - x_0) - \frac{3}{\Delta x^4}(x - x_1)^3(x - x_0)^2 \quad (\text{A.16})$$

$$A_{10}(x) = +\frac{1}{\Delta x^3}(x - x_0)^3 - \frac{3}{\Delta x^4}(x - x_0)^3(x - x_1) + \frac{6}{\Delta x^5}(x - x_1)^2(x - x_0)^3. \quad (\text{A.17})$$

Finally, the complete interpolating function is given by

$$B(x) = -\frac{c_1}{\Delta x^3}(x - x_1)^3 + \frac{M}{\Delta x^3}(x - x_0)^3 - \left(\frac{3c_1}{\Delta x^4} + \frac{c_2}{\Delta x^3}\right)(x - x_1)^3(x - x_0) - \frac{3M}{\Delta x^4}(x - x_0)^3(x - x_1) - \left(\frac{6c_1}{\Delta x^5} + \frac{3c_2}{\Delta x^4}\right)(x - x_1)^3(x - x_0)^2 + \frac{6M}{\Delta x^5}(x - x_1)^2(x - x_0)^3, \quad (\text{A.18})$$

with  $c_1 = M_N$  and  $c_2 = c_1/x_0$ .