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THE DISTRIBUTIONAL DIVERGENCE OF HORIZONTAL VECTOR FIELDS VANISHING AT INFINITY ON CARNOT GROUPS

A. BALDI - F. MONTEFALCONE

We define a BV-type space in the setting of Carnot groups (i.e., simply connected Lie groups with stratified nilpotent Lie algebra) that allows one to characterize all distributions F for which there exists a continuous horizontal vector field Φ, vanishing at infinity, that solves the equation $\mathrm{div}_H \Phi = F$. This generalizes to the setting of Carnot groups some results by De Pauw and Pfeffer, [13], and by De Pauw and Torres, [14], for the Euclidean setting.

Contents

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1. Introduction

In their celebrated 2003 paper [7], Bourgain and Brezis studied a problem concerning the equation $\operatorname{div} Y = f$ for L^p-periodic functions f defined on $\mathbb{R}^n$. Among their results, they considered the limiting case $p = n$ and proved that there exists a vector field Y solving the equation and that belongs to L^∞. To attack the above problem, they started by using special vector fields of the form $Y = \nabla u$, thus considering the problem $\Delta u = f$. This method for $1 < p < \infty$ yields a solution $u \in W^{2,p}$ and, consequently, a solution $Y \in W^{1,p}$. Unfortunately, in the limiting case $p = n$, the fact that $Y \in W^{1,p}$ does not imply directly that ∇u belongs to L^∞, since $W^{1,p}$ is not contained in L^∞. Despite this, they proved that in a suitable class of periodic functions on $\mathbb{R}^n$ there exists indeed a *continuous* vector fields Y that solves the equation $\operatorname{div} Y = f$ and such that

$$\|Y\|_{L^\infty} \leq \mathscr{C}(n)\|f\|_{L^n}, \tag{1}$$

where $\mathscr{C}(n)$ denotes a dimensional constant. The continuity of Y is a key point in their proof, which relies on the Sobolev embedding of both spaces $W^{1,1}$ and BV into $L^{n/n-1}$, and on a duality argument. The proof itself was not constructive. As a matter of fact, the authors showed that there are no bounded linear operators K from the space of L^n-periodic functions to L^∞ such that $\operatorname{div}(Kf) = f$ in the distributional sense. Thus, inequality (1) cannot follow from a representation formula for solutions to the equation under study. After the paper [6] was written, a huge literature appeared concerning equations such as

$$\operatorname{div} Y = F. \tag{2}$$

Among them we quote [13], where the authors considered the problem in a more general framework, finding necessary and sufficient conditions on F in order to get a continuous weak solution of (2). Moreover, they introduced the notions of charge and strong charge, which originated from their researches on generalized Riemann integrals and Gauss-Green theorems; see [13] and references therein.

We remind the reader that a distribution $F \in \mathcal{D}'(\mathbb{R}^n)$ is said a *flux* if the equation (2) has a continuous solution, i.e., if there exists a vector field $Y \in$

$C(\mathbb{R}^n;\mathbb{R}^n)$ such that

$$F(\varphi) = -\int_{\mathbb{R}^n} \langle Y(x), \nabla\varphi(x)\rangle\, dx \qquad \forall\, \varphi \in \mathcal{D}(\mathbb{R}^n).$$

A linear functional $F : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is called a *charge* in $\mathbb{R}^n$ if $\lim_{i\to+\infty} F(\varphi_i) = 0$ for every sequence $\{\varphi_i\}_{i\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ such that

$$\lim_{i\to+\infty} \|\varphi_i\|_{L^1} = 0 \qquad \text{and} \qquad \sup_i(\|\nabla\varphi_i\|_{L^1} + \|\varphi_i\|_{L^\infty}) < \infty;$$

see Definition 2.3 in [13]. On the other hand, the linear functional $F : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is said a *strong charge* in $\mathbb{R}^n$ if $\lim_{i\to+\infty} F(\varphi_i) = 0$ for every sequence $\{\varphi_i\}_{i\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ such that $\lim_{i\to+\infty} \|\varphi_i\|_{L^1} = 0$ and $\sup_i \|\nabla\varphi_i\|_{L^1} < +\infty$. The linear spaces of all fluxes, charges and strong charges in $\mathbb{R}^n$ are denoted, respectively, by $\mathcal{F}(\mathbb{R}^n)$, $\mathbf{Ch}(\mathbb{R}^n)$, and $\mathbf{Ch}_s(\mathbb{R}^n)$. It is observed in [13] that, in principle, $\mathcal{F}(\mathbb{R}^n) \subset \mathbf{Ch}(\mathbb{R}^n) \subset \mathbf{Ch}_s(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ but in the paper the authors show that $\mathcal{F} = \mathbf{Ch}_s$.

We remark that an example of strong charge is given by any distribution associated with a function $f \in L^n_{loc}(\mathbb{R}^n)$: this shows the connection with the problem studied by Bourgain and Brezis. Later on, De Pauw and Torres, [14], characterized all functionals F acting linearly on the subspace of $L^{n/(n-1)}(\mathbb{R}^n)$ of all functions whose distributional gradient is a vector valued measure, under a suitable continuity assumption. The requirement on F is connected with the definition of charge vanishing at infinity (see Definition 3.1 in [14]). As a corollary of their characterization result, De Pauw and Torres proved that given $f \in L^n(\mathbb{R}^n)$ there exists $Y \in C_0(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} Y = f$ in the sense of distribution, where $C_0(\mathbb{R}^n, \mathbb{R}^n)$ denotes the space of all continuous vector fields vanishing at infinity.

Starting from the existence result of De Pauw and Torres and adapting Bourgain and Brezis' proof, Moonens and Picon proved in [25] that if $f \in L^n(\mathbb{R}^n)$, then there exists $\widetilde{Y} \in C_0(\mathbb{R}^n, \mathbb{R}^n)$ solving the equation $\operatorname{div} \widetilde{Y} = f$, *and such that*

$$\|\widetilde{Y}\|_{L^\infty} \le \mathscr{C}(n)\|f\|_{L^n},$$

where the constant $\mathscr{C}(n)$ is a dimensional constant independent of f.

In this paper we study, in the setting of Carnot groups (i.e., simply connected Lie groups $\mathbb{G}$, with stratified nilpotent Lie algebra $\mathfrak{g}$; see, e.g., [5], [15], [31]), an analogous of the equation (2), obtaining also a continuity estimate similar to the one above. Carnot groups are the simpler examples of sub-Riemannian manifolds and play a deep role in studying, in a sub-Riemannian setting, problems arising from differential geometry, geometric measure theory, subelliptic differential equations, optimal control theory, mathematical models in neurosciences and robotics. Roughly speaking, a sub-Riemannian structure on a smooth n-dimensional manifold M is given by a subbundle HM of the tangent bundle

TM, which defines a family of admissible directions at any point of M. The subbundle HM is called the *horizontal* bundle. If we endow each fiber H_xM of HM with a scalar product $\langle,\rangle_x$, there exists a naturally associated distance d on M, called Carnot-Carathéodory distance, defined as the infimum of the Riemannian length of all *horizontal curves* (i.e., any curve $\gamma : I \to M$ such that $\gamma'(t) \in H_{\gamma(t)}M$ for a.e. $t \in I$) joining two given points.

In any Carnot group $\mathbb{G}$, the horizontal subbundle $H\mathbb{G}$ is generated by left translation of the first layer of the stratification of the Lie algebra $\mathfrak{g}$, which can be identified with a linear subspace of the tangent space of the group at the identity. Moreover, through the Lie group exponential map, $\mathbb{G}$ can be identified with the Euclidean space $\mathbb{R}^n$, endowed with a polynomial group law, where $n = \dim \mathfrak{g}$. Notice that the Hausdorff dimension Q of a Carnot group $\mathbb{G}$ turns out to be strictly greater than its topological dimension.

Horizontal vector fields in Carnot groups (i.e., smooth sections of the horizontal subbundle $H\mathbb{G}$) are the natural counterpart of vector fields in Euclidean spaces, and there is a well understood notion of horizontal divergence, later denoted as div_H. This fact makes possible to study an equation of the type

$$\mathrm{div}_H\Phi = F. \tag{3}$$

More precisely, in this paper we study the notion of charge vanishing at infinity in the setting of Carnot groups, following the lines of [14], in connection with the solvability of the equation (3).

Our main result is stated in Theorem 5.6, where we prove that if $F \in \mathcal{D}'(\mathbb{G})$, then there exist continuous horizontal vector fields vanishing at infinity (see Section 2 for precise definitions) that solve (3) in the distributional sense if and only if F is a charge vanishing at infinity. As a corollary, if $F \in L^Q(\mathbb{G})$ (hence, it turns out that F can be regarded as a charge vanishing at infinity), there is a continuous solution of (3) vanishing at infinity that in addition satisfies the inequality

$$\|\Phi\|_{L^\infty} \le \mathscr{C}(Q)\|F\|_{L^Q}, \tag{4}$$

where $\mathscr{C}(Q)$ denotes a geometric constant, which is independent of F (see (42)).

The problem of the existence of an L^∞-solution Φ, and of an inequality like (4), could be formulated in the more general setting of the Rumin complex of intrinsic differential forms on Carnot groups. In fact, horizontal vector fields can be identified with intrinsic differential forms of degree $(n-1)$, so that an estimate like (4) can be seen as the first link of a chain of analogous inequalities for intrinsic differential forms of any degree. A similar result, for Rumin's differential forms of any degree, has been recently obtained in the setting of Heisenberg groups in [3]. Nevertheless, the formulation of the problem itself, in terms of differential forms of arbitrary degree in general Carnot groups, is not straightforward at all due to the lack of homogeneity of the Rumin's exterior differential (for an explanation of this phenomenon, see, e.g., [4] p.6). Thus, one of the motivations of our paper is to attack this kind of problem in general Carnot groups for horizontal vector fields (thought of as identified with intrinsic differential forms of degree $(n-1)$), where the Rumin's exterior differential turns out to be always homogeneous. Finally, it is worth also mentioning the very recent paper [24], where the notion of charge is studied in connection with the compactness of normal currents in Carnot groups.

The paper is organized as follows. Precise definitions and basic properties of Carnot groups are discussed in Section 2, together with the notions of horizontal vector measures and horizontal vector fields vanishing at infinity adapted for this setting; see Section 2.1. Then, in Section 2.2, we collect several results about BV functions in Carnot groups. In Section 3 we introduce and study another BV-like space, denoted by $BV^{Q/Q-1}(\mathbb{G})$ and defined as the set of all functions in $L^{Q/Q-1}(\mathbb{G})$ whose distributional gradient (regarded as a measure) has finite total variation. In Section 4 we study a closed subspace of the dual space of $BV^{Q/Q-1}(\mathbb{G})$, denoted by $\mathbf{Ch}_0(\mathbb{G})$. In particular, following the lines of [14], we prove that its dual is isomorphic to $BV^{Q/Q-1}(\mathbb{G})$. Section 5 contains our main result (see Theorem 5.6) concerning the equation $\mathrm{div}_H\Phi = F$ (meant in the distributional sense). In particular, we show that this equation admits as a solution a continuous horizontal vector field Φ vanishing at infinity if and only if $F \in \mathbf{Ch}_0(\mathbb{G})$. In addition, as a corollary, we prove an estimate of the type (4); see Corollary 5.7.

2. Notation and preliminary results

A *Carnot group* $\mathbb{G}$ *of step* κ is a simply connected Lie group whose Lie algebra $\mathfrak{g}$ is finite dimensional, say of dimension n, and admits a *step* κ *stratification*, i.e., there exist linear subspaces $V_1,...,V_\kappa$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1,V_j] = V_{j+1}, \quad V_\kappa \neq \{0\}, \quad V_j = \{0\} \text{ if } i > \kappa, \qquad (5)$$

where $[V_1,V_j]$ denotes the subspace of $\mathfrak{g}$ generated by all commutators of the form $[X,Y]$, with $X \in V_1$ and $Y \in V_j$ $(j \geq 1)$.

For any $j = 1,\dots,\kappa$, let $m_j := \dim V_j$ and $h_j := m_1 + \cdots + m_j$, where $h_0 = 0$ and, clearly, $h_\kappa = n$. Now choose a basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ of $\mathfrak{g}$ adapted to the stratification, i.e.,

$$\{\mathbf{e}_{h_{j-1}+1},\dots,\mathbf{e}_{h_j}\} \text{ is a basis of } V_j \text{ for any } j = 1,\dots,\kappa.$$

Let $X = \{X_1,\dots,X_n\}$ be the set of left-invariant vector fields of $\mathbb{G}$ such that $X_i(e) = \mathbf{e}_i$ $(i = 1,...,n)$, where e denotes the identity of $\mathbb{G}$. By the stratification hypothesis (5), all left-invariant vector fields of $\mathbb{G}$ are generated by iterated Lie brackets of the subset $\{X_1,\dots,X_{m_1}\}$: we will refer to $X_1,\dots,X_{m_1}$ as the *generating vector fields* of the group.

The exponential map is a one to one map from $\mathfrak{g}$ onto $\mathbb{G}$. Thus, any $x\in\mathbb{G}$ can be written in a unique way as $x=\exp(x_1X_1+\cdots+x_nX_n)$. Using these *exponential coordinates*, we shall identify x with the n-tuple $(x_1,\dots,x_n)\in\mathbb{R}^n$ and, accordingly, $\mathbb{G}$ with $(\mathbb{R}^n,\cdot)$. The explicit expression of the group operation "$\cdot$" follows from the Campbell-Baker-Hausdorff formula; see [5]. If $j=1,\dots,\kappa$, then set $x^j:=(x_{h_{j-1}+1},\dots,x_{h_j})\in\mathbb{R}^{m_j}$. Thus, we can also identify x with the κ-tuple $(x^1,\dots,x^\kappa)\in\mathbb{R}^{m_1}\times\ldots\times\mathbb{R}^{m_\kappa}=\mathbb{R}^n$.

Recall that there are two important families of group automorphisms: left translations and group dilations. For any $x\in\mathbb{G}$, the *left translation by* x, say $\tau_x:\mathbb{G}\longrightarrow\mathbb{G}$, is the map given by

$$\mathbb{G}\ni z\longmapsto\tau_x z:=x\cdot z.$$

For any $\lambda>0$, the *dilation* $\delta_\lambda:\mathbb{G}\longrightarrow\mathbb{G}$, is defined as

$$\delta_\lambda(x_1,...,x_n)=(\lambda^{d_1}x_1,...,\lambda^{d_n}x_n), \tag{6}$$

where $d_i\in\mathbb{N}$ $(i=1,...,n)$ denotes the *homogeneity* of the monomial x_i in $\mathbb{G}$ (see [15], Ch.1, par. C), which is given by

$$d_i=j \quad \text{whenever} \quad h_{j-1}+1\le i\le h_j \quad (j=1,...,\kappa). \tag{7}$$

In particular, note that $1=d_1=...=d_{m_1}<d_{m_1+1}=2\le...\le d_n=\kappa$.

The Lie algebra $\mathfrak{g}$ can always be equipped with a scalar product $\langle\cdot,\cdot\rangle$ for which $\{X_1,\dots,X_n\}$ is an orthonormal basis.

As customary, we also fix a smooth homogeneous norm $\|\cdot\|$ in $\mathbb{G}$ (see [31], p. 638) such that the gauge distance $d(x,y):=\|y^{-1}\cdot x\|$ is a left-invariant distance on $\mathbb{G}$, in fact equivalent to the "Carnot-Carathéodory distance" (see [1]). We set

$$B(x,r):=\{y\in\mathbb{G};\ d(x,y)<r\}$$

to denote the open r-ball centered at $x\in\mathbb{G}$. It is well-known that any Haar measure of a Carnot group $\mathbb{G}$ coincides, up to a constant factor, with the standard Lebesgue measure $\mathscr{L}^n$ on $\mathfrak{g}\cong\mathbb{R}^n$ (notice that we just write dx instead of $d\mathscr{L}^n(x)$ in the integrals). If $A\subset\mathbb{G}$ is a $\mathscr{L}^n$-measurable set, we will also set $|A|:=\mathscr{L}^n(A)$.

The *homogeneous dimension* Q of the group $\mathbb{G}$ is the number defined as

$$Q:=\sum_{j=1}^{\kappa} j\dim V_j. \tag{8}$$

Since for any $x \in \mathbb{G}$ and $r > 0$ we have

$$|B(x,r)| = |B(e,r)| = r^Q |B(e,1)|, \tag{9}$$

the integer Q turns out to be the Hausdorff dimension of the metric space $(\mathbb{G}, d)$.

Proposition 2.1. The group product "$\cdot$" has the form

$$x \cdot y = x + y + \mathcal{Q}(x,y) \qquad \text{for all } x, y \in \mathbb{R}^n, \tag{10}$$

where $\mathcal{Q} = (\mathcal{Q}_1, \dots, \mathcal{Q}_n) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and any $\mathcal{Q}_i$ is a homogeneous polynomial of degree d_i $(i = 1, ..., n)$ with respect to the intrinsic dilations (6), i.e.,

$$\mathcal{Q}_i(\delta_\lambda x, \delta_\lambda y) = \lambda^{d_i} \mathcal{Q}_i(x,y) \qquad \text{for all } x, y \in \mathbb{G}.$$

In addition, for every $x, y \in \mathbb{G}$ the following hold:

$$\mathcal{Q}_1(x,y) = ... = \mathcal{Q}_{m_1}(x,y) = 0; \tag{11}$$

$$\mathcal{Q}_j(x,0) = \mathcal{Q}_j(0,y) = 0 \quad \text{and} \quad \mathcal{Q}_j(x,x) = \mathcal{Q}_j(x,-x) = 0 \quad \text{for } m_1 < j \le n; \tag{12}$$

$$\mathcal{Q}_j(x,y) = \mathcal{Q}_j(x_1, \dots, x_{h_{i-1}}, y_1, \dots, y_{h_{i-1}}) \quad \text{for} \quad h_{i-1} \le j \le h_i \quad (i > 1). \tag{13}$$

It follows from Proposition 2.1 that $\delta_\lambda x \cdot \delta_\lambda y = \delta_\lambda (x \cdot y)$ for every $x, y \in \mathbb{G}$, and that the inverse x^{-1} of any $x = (x_1, \dots, x_n) \in \mathbb{G}$ has the form $x^{-1} = (-x_1, \dots, -x_n)$.

Proposition 2.2 (see, e.g., [19], Proposition 2.2)**.** The left-invariant vector fields $\{X_1, ..., X_n\}$ have polynomial coefficients and are of the form

$$X_j(x) = \partial_j + \sum_{i > h_l}^{n} q_{i,j}(x) \partial_i \qquad \text{for any } j = 1, \dots, n \text{ and } j \le h_l \quad (l = 1, ..., \kappa), \tag{14}$$

where $q_{i,j}(x) = \frac{\partial \mathcal{Q}_i}{\partial y_j}(x,y)\big|_{y=0}$.

In particular, if $h_{l-1} < j \le h_l$, then $q_{i,j}(x) = q_{i,j}(x_1, ..., x_{h_{l-1}})$ and $q_{i,j}(0) = 0$.

The subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$ spanned by the vector fields $\{X_1, \dots, X_{m_1}\}$ is called the *horizontal bundle* and plays a particularly important role in the theory. The fibers of $H\mathbb{G}$ are explicitly given by

$$H_x\mathbb{G} = \text{span}\,\{X_1(x), \dots, X_{m_1}(x)\} \qquad \forall x \in \mathbb{G}.$$

For simplicity of notation, we will henceforth set $m := m_1$.

A subriemannian structure is defined on $\mathbb{G}$ once one endows each fiber $H_x\mathbb{G}$ of the horizontal bundle $H\mathbb{G}$ with a scalar product $\langle\cdot,\cdot\rangle_x$; its associated norm is denoted as $|\cdot|_x$. When clear from the context, we will drop the subscript x, simply writing $\langle\cdot,\cdot\rangle$ and $|\cdot|$.

From now on, we shall assume that, at any $x\in\mathbb{G}$, the basis $\{X_1(x),\dots,X_m(x)\}$ is orthonormal (under the chosen scalar product).

Measurable sections of the horizontal bundle $H\mathbb{G}$ are called *horizontal sections* (or *horizontal vector fields*), and vectors in $H_x\mathbb{G}$ are called *horizontal vectors*.

Given a horizontal vector field[1] $\Phi:\mathbb{G}\to H\mathbb{G}$, and since a horizontal frame has already been fixed, we can write Φ in terms of its m components $\Phi_i:\mathbb{G}\to\mathbb{R}$ $(i=1,\dots,m)$ along the horizontal frame $\{X_1,\dots X_m\}$, so that

$$\Phi=\sum_{j=1}^m \phi_j X_j.$$

In other words, we can always assume that $\Phi=(\phi_1,...,\phi_m)$.

Now, let $f:\mathbb{G}\longrightarrow\mathbb{R}$ be a smooth function, say $f\in C^\infty(\mathbb{G})$. The horizontal gradient of f is the horizontal vector field D_Hf defined by

$$\langle D_Hf(x),X\rangle_x=df_x(X),\qquad \forall x\in\mathbb{G},\ \forall X\in H_x\mathbb{G}.$$

Clearly, with respect to the the horizontal frame, we can write $D_Hf=(X_1f,...,X_mf)$.

Moreover, if $\Phi=(\phi_1,\dots,\phi_m)$ is a smooth horizontal vector field, say $\Phi\in C^\infty(\mathbb{G},H\mathbb{G})$, its horizontal divergence $\mathrm{div}_H\,\Phi$ is, by definition, the real valued function

$$\mathrm{div}_H\,\Phi:=\sum_{j=1}^m X_j\phi_j. \tag{15}$$

The same symbols D_H and div_H will be adopted later, when working with the *weak* horizontal gradient and divergence operators (intended in the sense of distributions).

Recall that if $\Omega\subseteq\mathbb{G}$ is an open set, the space of continuous linear functionals on $C^\infty(\Omega)$ $(=:\mathcal{E}(\Omega))$ is denoted by $\mathcal{E}'(\Omega)$ and the space of continuous linear functionals on $C_c^\infty(\Omega)$ $(=:\mathcal{D}(\Omega))$ is denoted by $\mathcal{D}'(\Omega)$. Throughout the paper, we will use the notation $\langle\cdot|\cdot\rangle$ for the duality between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$ and also for the duality between $\mathcal{E}'(\Omega)$ and $\mathcal{E}(\Omega)$ (more generally, the same notation will be used for the duality between other function spaces defined below).

If $f:\mathbb{G}\longrightarrow\mathbb{R}$, we denote by ${}^{\mathrm{v}}f$ the function given by ${}^{\mathrm{v}}f(x):=f(x^{-1})$. Furthermore, if $T\in\mathcal{D}'(\mathbb{G})$, then ${}^{\mathrm{v}}T$ will denote the distribution defined by $\langle{}^{\mathrm{v}}T|\varphi\rangle:=\langle T|{}^{\mathrm{v}}\varphi\rangle$ for any test function $\varphi\in\mathcal{D}(\mathbb{G})$.

[1] In other words, if $\pi:T\mathbb{G}\to\mathbb{G}$ is the bundle projection map, then $\pi\circ\Phi$ is the identity map.

As in [15], we adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_n)$ is a multi–index, we set $X^I = X_1^{i_1} \dots X_n^{i_n}$. By the Poincaré-Birkhoff-Witt theorem (see, e.g., [9], I.2.7), the differential operators X^I form a basis for the algebra of left-invariant differential operators in $\mathbb{G}$. Furthermore, let $|I| := i_1 + \dots + i_n$ be the order of the differential operator X^I, and let $d(I) := d_1 i_1 + \dots + d_n i_n$ be its degree of homogeneity with respect to group dilations. From the Poincaré–Birkhoff-Witt theorem it follows, in particular, that any homogeneous linear differential operator in the horizontal derivatives can be expressed as a linear combination of the operators X^I of the special form above.

We now recall the notion of convolution in the setting of Carnot groups (see, e.g., [15]). If $f \in \mathcal{D}(\mathbb{G})$ and $g \in L^1_{\mathrm{loc}}(\mathbb{G})$, we set

$$f * g(x) := \int f(y) g(y^{-1} \cdot x)\, dy \qquad \forall x \in \mathbb{G}. \tag{16}$$

Furthermore, recall that if also g is a smooth function and P is a left-invariant differential operator, then

$$P(f * g) = f * Pg.$$

More generally, we remark that the convolution is well-defined whenever $f, g \in \mathcal{D}'(\mathbb{G})$, provided at least one of them has compact support. In this case, for any test function $\phi \in \mathcal{D}(\mathbb{G})$, the following identities hold:

$$\langle f * g | \phi \rangle = \langle g | {}^{\mathrm{v}} f * \phi \rangle \quad \text{and} \quad \langle f * g | \phi \rangle = \langle f | \phi * {}^{\mathrm{v}} g \rangle. \tag{17}$$

Suppose now that $f \in \mathcal{E}'(\mathbb{G})$ and $g \in \mathcal{D}'(\mathbb{G})$. If $\psi \in \mathcal{D}(\mathbb{G})$, then it can be shown that

$$\begin{aligned} \langle (X^I f) * g | \psi \rangle &= \langle X^I f | \psi * {}^{\mathrm{v}} g \rangle = (-1)^{|I|} \langle f | \psi * (X^I\, {}^{\mathrm{v}} g) \rangle \\ &= (-1)^{|I|} \langle f * {}^{\mathrm{v}} X^I\, {}^{\mathrm{v}} g | \psi \rangle. \end{aligned} \tag{18}$$

The following theorem can be found in [15] (see Proposition 1.18).

Theorem 2.3 (Hausdorff-Young inequality)**.** If $f \in L^p(\mathbb{G})$, $g \in L^q(\mathbb{G})$, $1 \le p, q, r \le \infty$, and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then $f * g \in L^r(\mathbb{G})$ and $\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}$.

Remark 2.4. If $T \in \mathcal{E}'(\mathbb{G})$, and P is a differential operator in $\mathbb{G}$, then $PT \in \mathcal{E}'(\mathbb{G})$, and it turns out that $\operatorname{supp} PT \subset \operatorname{supp} T$ (see [32], Exercise 24.3).

We collect in the next proposition a few basic properties of the convolution of two distributions.

Proposition 2.5. The following assertions hold.

1. If $T \in \mathcal{D}'(\mathbb{G})$ (or, $T \in \mathcal{E}'(\mathbb{G})$, respectively), then the convolution $\phi \mapsto \phi * T$ is a continuous linear map of $\mathcal{E}(\mathbb{G})$ (or, $\mathcal{D}(\mathbb{G})$, respectively) into $\mathcal{D}(\mathbb{G})$ (see [32], Theorem 27.3).

2. The convolution maps $\mathcal{E}(\mathbb{G}) \times \mathcal{D}'(\mathbb{G})$ (or, $\mathcal{D}(\mathbb{G}) \times \mathcal{E}'(\mathbb{G})$, respectively) into $\mathcal{D}(\mathbb{G})$ (see [32], p. 288).

3. The convolution $(S,T) \mapsto S * T$, defined as
$$\langle S * T | \phi \rangle_{\mathcal{D}',\mathcal{D}} = \langle S | \phi * {}^{\mathrm{v}}T \rangle_{\mathcal{E}',\mathcal{E}},$$
is a separately continuous bilinear map from $\mathcal{E}'(\mathbb{G}) \times \mathcal{D}'(\mathbb{G})$ into $\mathcal{D}'(\mathbb{G})$ (see [32], Theorem 27.6).

Let $J : \mathbb{G} \longrightarrow \mathbb{R}$ be a mollifier (for the group structure), i.e., $J \in C_c^\infty(\mathbb{G})$, $J \geq 0$, $\mathrm{supp}(J) \Subset B(e,1)$, and $\int_{\mathbb{G}} J(x)\, dx = 1$. Note that, if one starts from a standard mollifier J defined in $(\mathbb{R},+)$, then the function $J(\|x\|)$ turns out to be a mollifier in $\mathbb{G}$. Now, given a mollifier J, we define a family of approximations to the identity $\{J_\varepsilon\}_{\varepsilon>0}$ by setting
$$J_\varepsilon(x) := \frac{1}{\varepsilon^Q} J(\delta_{1/\varepsilon} x).$$

We remark explicitly that $J_\varepsilon(x) = {}^{\mathrm{v}}J_\varepsilon(x)$ for every $x \in \mathbb{G}$.

Let $1 \leq p < +\infty$. If $f \in L^p(\mathbb{G})$, then $J_\varepsilon * f \longrightarrow f$ in $L^p(\mathbb{G})$ as $\varepsilon \to 0$. Furthermore, since $f * J_\varepsilon = {}^{\mathrm{v}}({}^{\mathrm{v}}J_\varepsilon * {}^{\mathrm{v}}f) = {}^{\mathrm{v}}(J_\varepsilon * {}^{\mathrm{v}}f)$, the same assertions hold true for $f * J_\varepsilon$.

2.1. Vector Measures in $H\mathbb{G}$ and Riesz Theorem

Throughout we shall denote by $C_c(\mathbb{G}, H\mathbb{G})$ the class of continuous horizontal vector fields with compact support in $\mathbb{G}$, and by $C_0(\mathbb{G}, H\mathbb{G})$ its completion with respect to the uniform norm
$$\|\Phi\|_\infty = \sup\{|\Phi(x)|_x : x \in \mathbb{G}\},$$
where $\Phi : \mathbb{G} \longrightarrow H\mathbb{G}$. It turns out that $C_0(\mathbb{G}, H\mathbb{G})$, endowed with the uniform norm $\|\cdot\|_\infty$, is a Banach space. Furthermore, since the uniform limit of continuous functions is a continuous function, it follows that $\Phi \in C_0(\mathbb{G}, H\mathbb{G})$ if, and only if, Φ is continuous and for every $\varepsilon > 0$ there exists a compact set $\mathcal{K} \subset \mathbb{G}$ such that $|\Phi(x)|_x \leq \varepsilon$ whenever $x \in \mathbb{G} \setminus \mathcal{K}$.

We shall refer to the space $C_0(\mathbb{G}, H\mathbb{G})$ as the space of continuous horizontal vector fields *vanishing at infinity*. Exactly as in the Euclidean case, the linear subspace $\mathcal{D}(\mathbb{G}, H\mathbb{G})$ is dense in $C_0(\mathbb{G}, H\mathbb{G})$.

Now we need a substitute for the notion of vector-valued measure in Carnot groups (compare with [12], Definition 3.5).

Let $\gamma \in \mathcal{M}(\mathbb{G})$ be a Radon measure on $\mathbb{G}$ and let $\alpha : \mathbb{G} \to H\mathbb{G}$ be a (locally) bounded γ-measurable horizontal vector field. Hence, there is a naturally defined linear functional on $C_c(\mathbb{G}, H\mathbb{G})$ given by $T_{\alpha\gamma}(\Phi) := \int_{\mathbb{G}} \langle \Phi, \alpha \rangle d\gamma$ (clearly, $T_{\alpha\gamma}$ is bounded in $C_c(\mathbb{G}, H\mathbb{G})$ with respect to the L^∞-topology). As a consequence, we can define a notion of *vector measure* $\alpha\gamma$ *in* $H\mathbb{G}$ by setting

$$C_c(\mathbb{G}, H\mathbb{G}) \ni \Phi \longmapsto \int_{\mathbb{G}} \langle \Phi, d(\alpha\gamma) \rangle := T_{\alpha\gamma}(\Phi).$$

By density, this functional extends to a continuous linear functional in $C_0(\mathbb{G}, H\mathbb{G})$. In the sequel, we shall denote by $\mathcal{M}(\mathbb{G}, H\mathbb{G})$ the space of all vector measures on $\mathbb{G}$ (in the previous sense). As previously pointed out, we can write $\alpha = \sum_{i=1}^m \alpha_i X_i$, where the components $\alpha_i : \mathbb{G} \to \mathbb{R}$ $(i = 1, \dots, m)$ with respect to the horizontal frame are now (locally) bounded γ-measurable functions. Hence, the vector measure $\mu = \alpha\gamma$ can be written (in components) as $\mu = (\mu_1, \dots, \mu_m) = (\alpha_1, \dots, \alpha_m)\gamma$, and we get

$$T_\mu(\Phi) = \int_{\mathbb{G}} \langle \Phi, d\mu \rangle = \sum_{i=1}^m \int_{\mathbb{G}} \Phi_i(x) d\mu_i(x).$$

Since in Carnot groups the horizontal bundle has a global trivialization, we can always argue componentwise. Then it is not difficult to show that any $T \in C_0(\mathbb{G}, H\mathbb{G})^*$ can be represented by a vector measure μ in $H\mathbb{G}$ as

$$T(\Phi) = \int_{\mathbb{G}} \langle \Phi, d\mu \rangle \qquad \forall \Phi \in C_0(\mathbb{G}, H\mathbb{G}).$$

Moreover, due to the density of $\mathcal{D}(\mathbb{G}, H\mathbb{G})$ in $C_0(\mathbb{G}, H\mathbb{G})$, if we take $T \in \mathcal{D}(\mathbb{G}, H\mathbb{G})^*$ such that $\sup\{T(\Phi) : \Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G}),\ \|\Phi\|_\infty \le 1\} < +\infty$, we can extend uniquely T to an element of $C_0(\mathbb{G}, H\mathbb{G})^*$. Hence, any T turns out to be associated with a vector measure $\mu \in \mathcal{M}(\mathbb{G}, H\mathbb{G})$. We henceforth set

$$\|\mu\|_{\mathcal{M}} := \sup\{T(\Phi) : \Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G}),\ \|\Phi\|_\infty \le 1\} = \|T\|_{C_0^*}$$

(the symbol $\mathcal{M}$ will be omitted when clear by the context). The identification between the space $\mathcal{M}(\mathbb{G}, H\mathbb{G})$ of vector measures with finite mass and $C_0(\mathbb{G}, H\mathbb{G})^*$ can be proved using the map $\rho : \mathcal{M}(\mathbb{G}, H\mathbb{G}) \longrightarrow C_0(\mathbb{G}, H\mathbb{G})^*$ defined by

$$\rho(\mu)(\Phi) := \int_{\mathbb{G}} \langle \Phi, d\mu \rangle = T_\mu(\Phi) \qquad \forall \Phi \in C_0(\mathbb{G}, H\mathbb{G}).$$

2.2. Functions of bounded H-variation in Carnot groups

In this subsection we recall some known definitions and results concerning functions of "intrinsic bounded variation".

Let $\Omega \subseteq \mathbb{G}$ be an open set. Recall that a function $f : \Omega \longrightarrow \mathbb{R}$ is said to have *intrinsic bounded variation in* Ω, and in this case we write $f \in BV_H(\Omega)$, if $f \in L^1(\Omega)$ and

$$\|D_H f\|(\Omega) := \sup\left\{ \int_\Omega f\, \mathrm{div}_H \Phi\, dx \,:\, \Phi \in \mathcal{D}(\Omega, H\Omega),\ \|\Phi\|_\infty \leq 1 \right\} < +\infty,$$

where $\|\Phi\|_\infty = \sup\{|\Phi(x)|_x : x \in \Omega\}$.

The quantity $\|D_H f\|(\Omega)$ represents the *total horizontal variation* (or, H-*variation*) of the distributional horizontal gradient $D_H f$ in Ω.

Unless otherwise stated, throughout the paper we shall assume that $\Omega = \mathbb{G}$. In this case, the total H-variation of $D_H f$ in $\mathbb{G}$ will be simply denoted as $\|D_H f\|$.

Note that the preceding definition can easily be localized. To this aim, let $f \in L^1_{loc}(\Omega)$ and assume that $\|D_H f\|(V) < +\infty$ for every open subset $V \Subset \Omega$. In this case, we set $f \in BV_{H,loc}(\Omega)$ to denote the space of functions of locally bounded H-variation in Ω.

Of course, if $\mathbb{G}$ is commutative and equipped with the Euclidean metric, the previous definitions coincide with the classical ones. There is a wide literature on BV_H-functions in Carnot groups for which we refer, for instance, to [18], [20], [33], and references therein.

By adapting the classical Riesz representation theorem to our setting, one can prove the following "structure theorem".

Theorem 2.6. If $f \in BV_{H,loc}(\Omega)$, then $\|D_H f\|$ is a Radon measure on Ω. In addition, there exists a bounded $\|D_H f\|$-measurable horizontal section $\sigma_f : \Omega \to H\Omega$ such that $|\sigma_f(x)|_x = 1$ for $\|D_H f\|$-a.e. $x \in \Omega$, and the following holds

$$\int_\Omega f\, \mathrm{div}_H \Phi\, dx = -\int_\Omega \langle \Phi, \sigma_f \rangle\, d\|D_H f\| \qquad \forall \Phi \in \mathcal{D}(\Omega, H\Omega). \tag{19}$$

Let $C^1_H(\Omega)$ denote the linear space of functions $f : \Omega \longrightarrow \mathbb{R}$ such that the pointwise horizontal partial derivatives $X_1 f, \ldots, X_m f$ are continuous in Ω.

Remark 2.7. As in the Euclidean case, every function $f \in C^1_H(\Omega)$ belongs to $BV_{H,loc}(\Omega)$. This follows by integrating by parts. Indeed, we have

$$\int_\Omega f\, \mathrm{div}_H \Phi\, dx = -\int_\Omega \langle \Phi, D_H f \rangle\, dx,$$

which implies that $\|D_H f\|(\Omega) = \mathscr{L}^n \llcorner |D_H f|$, and

$$\sigma_f = \begin{cases} \frac{D_H f}{|D_H f|} & \text{if} \quad D_H f \neq 0 \\ 0 & \text{if} \quad D_H f = 0 \end{cases} \qquad \mathscr{L}^n\text{-}a.e.$$

Let $\Omega = \mathbb{G}$. According to the previous section's definition, $\mu = \sigma_f \|D_H f\|$ is a vector measure in $H\mathbb{G}$. Writing σ_f with respect to the horizontal frame as $\sigma_f = \sum_{i=1}^m \sigma_{f,i} X_i$, where the components $\sigma_{f,i} : \mathbb{G} \longrightarrow \mathbb{R}$ $(i = 1, \ldots, m)$ are bounded measurable functions, we have $\mu = (\sigma_{f,1}, \ldots, \sigma_{f,m})\|D_H f\|$. We shall set $[D_H f] := \mu$, so that (19) becomes

$$\int_{\mathbb{G}} f \,\mathrm{div}_H \Phi \, dx = -\int_{\mathbb{G}} \langle \Phi, d[D_H f]\rangle \qquad \forall \Phi \in \mathcal{D}(\mathbb{G}, H\Omega). \tag{20}$$

The following results are relevant in the theory of bounded H-variation functions in Carnot groups (for a proof we refer the reader to the literature quoted above).

The first one asserts that the (total) H-variation is lower semicontinuous with respect to the L^1_{loc}-convergence and follows because the map $f \mapsto \|D_H f\|(\cdot)$ is the supremum of a family of L^1-continuous functionals.

Theorem 2.8. Let $\Omega \subseteq \mathbb{G}$ be an open set. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $BV_H(\Omega)$ such that $f_k \longrightarrow f$ in $L^1_{\mathrm{loc}}(\Omega)$ as $k \to +\infty$. Then

$$\|D_H f\|(\Omega) \le \liminf_{k\to+\infty} \|D_H f_k\|(\Omega).$$

The next theorem, in the Euclidean setting, is better known as the "Anzellotti-Giaquinta approximation theorem".

Theorem 2.9. Let $\Omega \subseteq \mathbb{G}$ be an open set and let $f \in BV_H(\Omega)$. Then, there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset BV_H(\Omega) \cap C^\infty(\Omega)$ such that $f_k \longrightarrow f$ in $L^1(\Omega)$ as $k \to +\infty$, and

$$\lim_{k\to+\infty} \|D_H f_k\|(\Omega) = \|D_H f\|(\Omega).$$

If $E \subseteq \mathbb{G}$ is a Borel set, we set $P_H(E) := \|D_H \chi_E\|$, where χ_E is the characteristic function of E. More generally, if $\Omega \subseteq \mathbb{G}$ is an open set, we set $P_H(E, \Omega) := \|D_H \chi_E\|(\Omega)$. The quantities just defined are the H-perimeter of E in $\mathbb{G}$ and in Ω, respectively.

The next result is the coarea formula for functions of bounded H-variation (see, e.g., [18], [20]).

Theorem 2.10 (Coarea formula)**.** Let $f \in BV_H(\Omega)$ and set $E_t := \{x \in \Omega : f(x) > t\}$. Then, E_t has finite H-perimeter in Ω for a.e. $t \in \mathbb{R}$ and the following formula holds

$$\|D_H f\|(\Omega) = \int_{\mathbb{R}} P_H(E_t, \Omega)\, dt. \tag{21}$$

Conversely, if $f \in L^1(\Omega)$ and $\int_{\mathbb{R}} P_H(E_t, \Omega)\, dt < +\infty$, then $f \in BV_H(\Omega)$.

Finally, we have to recall a fundamental inequality, whose validity will be of central importance for our next results.

Remark 2.11 (Gagliardo-Nirenberg inequality). As is well-known, the classical Gagliardo-Nirenberg inequality has been generalized to Carnot groups by many authors (and with different aims); see, e.g., [11], [16], [17], [20], [23], [27]. More precisely, if $f \in \mathcal{D}(\mathbb{G})$, the inequality states that there exists a "geometric" constant $\mathscr{C}_{GN} = \mathscr{C}_{GN}(Q, \mathbb{G})$ such that

$$\|f\|_{L^{Q/Q-1}} \leq \mathscr{C}_{GN} \|D_H f\|_{L^1}. \tag{22}$$

The inequality (22) extends to functions in $BV_H(\mathbb{G})$ having compact support. In fact, arguing as in [22] (see Theorem 1.28), it is sufficient to approximate $f \in BV_H(\mathbb{G})$ with a sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{G})$ such that $f_j \longrightarrow f$ in $L^1(\mathbb{G})$ and $\|D_H f_j\| \longrightarrow \|D_H f\|$ as $j \to +\infty$. Then, by (22) the sequence is uniformly bounded in the $L^{Q/Q-1}$-norm and hence there exists a subsequence weakly convergent to some $f_0 \in L^{Q/Q-1}(\mathbb{G})$. But since $f_j \longrightarrow f$ in $L^1(\mathbb{G})$ as $j \to +\infty$, it follows that $f_j \rightharpoonup f = f_0$ in $L^{Q/Q-1}(\mathbb{G})$ as $j \to +\infty$ and the proof is achieved by using the weak lower semicontinuity of the $L^{Q/Q-1}$-norm (see, e.g., [10], Proposition 3.5).

3. The space $BV_H^{Q/Q-1}(\mathbb{G})$

We introduce another intrinsic BV_H-type space, which is in fact a subspace of $L^{Q/Q-1}(\mathbb{G})$, where Q denotes the homogeneous dimension (equal to the Hausdorff dimension) of $\mathbb{G}$; see (8). In the Euclidean setting this space was introduced and studied by De Pauw and Torres in [14].

Definition 3.1. The space $BV_H^{Q/Q-1}(\mathbb{G})$ is the set of functions $f \in L^{Q/Q-1}(\mathbb{G})$ whose distributional gradient $D_H f$ is a finite vector measure, i.e.,

$$\|D_H f\| := \|D_H f\|(\mathbb{G}) = \sup\left\{ \int_{\mathbb{G}} f \,\mathrm{div}_H \Phi \, dx \ :\ \Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G}),\ \|\Phi\|_\infty \leq 1 \right\} < +\infty.$$

The space $BV_H^{Q/Q-1}(\mathbb{G})$ is a Banach space when endowed with the norm

$$\|f\|_{L^{Q/Q-1}} + \|D_H f\|.$$

Note also that $BV_H^{Q/Q-1}(\mathbb{G}) \subset BV_{H,loc}(\mathbb{G})$.

The next result shows the lower semicontinuity of the H-variation with respect to the weak convergence in $L^{Q/Q-1}(\mathbb{G})$.

Theorem 3.2. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $BV_H^{Q/Q-1}(\mathbb{G})$ such that $f_k \rightharpoonup f$ in $L^{Q/Q-1}(\mathbb{G})$ as $k \to +\infty$. Then

$$\|D_H f\| \leq \liminf_{k\to+\infty} \|D_H f_k\|.$$

Proof. We consider the functional $\int_{\mathbb{G}} f \,\mathrm{div}_H \Phi \, dx$ with $\Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G})$ and $\|\Phi\|_\infty \leq 1$. Since $\mathrm{div}_H \Phi \in L^Q(\mathbb{G})$ and $f_k \rightharpoonup f$ in $L^{Q/Q-1}(\mathbb{G})$ as $k \to +\infty$, we have

$$\int_{\mathbb{G}} f \,\mathrm{div}_H \Phi \, dx = \lim_{k\to+\infty} \int_{\mathbb{G}} f_k \,\mathrm{div}_H \Phi \, dx.$$

By assumption, $\{f_k\}_{k\in\mathbb{N}} \subset BV_H^{Q/Q-1}(\mathbb{G})$, and hence $\int_{\mathbb{G}} f_k \,\mathrm{div}_H \Phi \, dx \leq \|D_H f_k\|$. Thus

$$\int_{\mathbb{G}} f \,\mathrm{div}_H \Phi \, dx \leq \liminf_{k\to+\infty} \|D_H f_k\|,$$

and the conclusion follows by taking the supremum on the left-hand side over all Φ in $\mathcal{D}(\mathbb{G}, H\mathbb{G})$ such that $\|\Phi\|_\infty \leq 1$. □

3.1. An approximation result for $BV_H^{Q/Q-1}(\mathbb{G})$

We start with an approximation result that yields as corollaries a Gagliardo-Nirenberg inequality for functions in $BV_H^{Q/Q-1}(\mathbb{G})$ and a compactness result in $BV_H^{Q/Q-1}(\mathbb{G})$. The results in this subsection generalize the corresponding Euclidean ones in [14].

Theorem 3.3. Let $f \in BV_H^{Q/Q-1}(\mathbb{G})$. Then, there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{G})$ such that:

(i) $f_j \rightharpoonup f$ in $L^{Q/Q-1}(\mathbb{G})$ as $j \to +\infty$ and $\sup_j \|D_H f_j\| < +\infty$.

In addition, the sequence $\{f_j\}_{j\in\mathbb{N}}$ satisfies:

(ii) $\lim_{j\to+\infty} \|D_H f_j\|_{L^1} = \|D_H f\|$.

Proof. The proof is divided in several steps.

Step 1. Consider a family of approximations to the identity $\{J_\varepsilon\}_{\varepsilon>0}$ (see Section 2) and remember that $J_\varepsilon = {}^{\mathrm{v}}J_\varepsilon$. Since $J_\varepsilon * f \longrightarrow f$ in $L^{Q/Q-1}(\mathbb{G})$ as $\varepsilon \to 0^+$, one has obviously $J_\varepsilon * f \rightharpoonup f$ in $L^{Q/Q-1}(\mathbb{G})$ as $\varepsilon \to 0^+$. In addition, it follows from (17) that if $\Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G})$ and $\|\Phi\|_\infty \leq 1$, then

$$\langle J_\varepsilon * f | X_i \Phi \rangle = \langle f | {}^{\mathrm{v}}J_\varepsilon * X_i \Phi \rangle = \langle f | J_\varepsilon * X_i \Phi \rangle = \langle f | X_i (J_\varepsilon * \Phi) \rangle.$$

Hence

$$\int_{\mathbb{G}} (J_\varepsilon * f) \operatorname{div}_H \Phi \, dx = \int_{\mathbb{G}} f \operatorname{div}_H (J_\varepsilon * \Phi) \, dx.$$

Now since $\|J_\varepsilon * \Phi\|_\infty \leq \|\Phi\|_\infty \leq 1$, taking the supremum on the right-hand side, we get

$$\int_{\mathbb{G}} (J_\varepsilon * f) \operatorname{div}_H \Phi \, dx \leq \|D_H f\|.$$

In turn, since $J_\varepsilon * f \in C^\infty(\mathbb{G})$, taking the supremum on the left-hand side over all $\Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G})$ such that $\|\Phi\|_\infty \leq 1$, we obtain

$$\|D_H(J_\varepsilon * f)\|_{L^1} = \|D_H(J_\varepsilon * f)\| \leq \|D_H f\| \tag{23}$$

for every $\varepsilon > 0$; see, e.g., Remark 2.7. So let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a strictly decreasing sequence such that $\varepsilon_k \to 0$ as $k \to +\infty$. Using the lower semicontinuity property in Theorem 3.2 together with (23), it follows eventually that

$$\lim_{k\to+\infty} \|D_H(J_{\varepsilon_k} * f)\|_{L^1} = \|D_H f\|. \tag{24}$$

Step 2. Starting from (24), it is clear that there must exist a subsequence $\left\{J_{\varepsilon_{k_j}} * f\right\}_{j\in\mathbb{N}}$ of $\{J_{\varepsilon_k} * f\}_{k\in\mathbb{N}}$ such that

$$\|D_H(J_{\varepsilon_{k_j}} * f)\|_{L^1} \leq \|D_H f\| + \frac{1}{j} \qquad \forall\, j \in \mathbb{N}. \tag{25}$$

Step 3. Let us fix a sequence of cut-off functions $\{g_i\}_{i\in\mathbb{N}} \subset \mathcal{D}(\mathbb{G})$ such that for any $i \in \mathbb{N}$ $\operatorname{supp}(g_i) \subset B(e, 2i)$, $g_i \equiv 1$ in $B(e,i)$, and $\sup_i \|D_H g_i\| < +\infty$. We have

$$D_H((J_{\varepsilon_{k_j}} * f) g_i) = g_i D_H(J_{\varepsilon_{k_j}} * f) + (J_{\varepsilon_{k_j}} * f) D_H g_i. \tag{26}$$

Let us start by estimating the second term of the right hand side above. Let $j \in \mathbb{N}$ be fixed. Since $J_{\varepsilon_{k_j}} * f \in L^{Q/Q-1}(\mathbb{G})$, it follows that

$$\begin{aligned}
\limsup_{i\to+\infty} \int_{\mathbb{G}} \left|(J_{\varepsilon_{k_j}} * f) D_H g_i\right| dx &= \limsup_{i\to+\infty} \int_{\mathbb{G}\setminus B(e,i)} \left|(J_{\varepsilon_{k_j}} * f) D_H g_i\right| dx \\
&\leq \limsup_{i\to+\infty} \left(\int_{\mathbb{G}\setminus B(e,i)} \left|(J_{\varepsilon_{k_j}} * f)\right|^{Q/Q-1} dx \right)^{Q-1/Q} \|D_H g_i\|_{L^Q} \\
&= 0.
\end{aligned}$$

With this estimate in mind, and by means of (25), it can be shown that there exists a strictly increasing sequence $\{i_j\}_{j\in\mathbb{N}}$ such that

$$\begin{aligned}\int_{\mathbb{G}}\left|D_H\left((J_{\varepsilon_{k_j}}*f)g_i\right)\right|dx &\leq \int_{\mathbb{G}}\left|D_H(J_{\varepsilon_{k_j}}*f)\right|dx+\frac{1}{j}\\ &\leq \|D_Hf\|+\frac{2}{j} \qquad \forall j\in\mathbb{N}.\end{aligned} \tag{27}$$

Step 4. Let us set

$$f_j := (J_{\varepsilon_{k_j}}*f)g_{i_j} \qquad \forall j\in\mathbb{N}.$$

From Step 3 it follows in particular that $\sup_j\|D_Hf_j\|<+\infty$. Let us to show that $f_j \rightharpoonup f$ in $L^{Q/Q-1}(\mathbb{G})$ as $j\to+\infty$. If we take $g\in L^Q(\mathbb{G})$, we have

$$\begin{aligned}\left|\int_{\mathbb{G}}g\left(f-(J_{\varepsilon_{k_j}}*f)g_{i_j}\right)dx\right| &\leq \int_{\mathbb{G}}|g|\left|f-(J_{\varepsilon_{k_j}}*f)\right|dx+\int_{\mathbb{G}}|g|\left|J_{\varepsilon_{k_j}}*f\right||1-g_{i_j}|\,dx\\ &\leq \|g\|_{L^Q}\|f-(J_{\varepsilon_{k_j}}*f)\|_{L^{Q/Q-1}}+\left(\int_{\mathbb{G}\setminus B(e,2i_j)}|g|^Q\,dx\right)^{1/Q}\|f\|_{L^{Q/Q-1}}.\end{aligned}$$

Since both addends of the right-hand side vanish as $j\to+\infty$, assertion (i) is proved. Finally, using the inequalities (27) together with the lower semicontinuity property in Theorem 3.2, it follows that $\lim_{j\to+\infty}\|D_Hf_j\|_{L^1} = \|D_Hf\|$, which proves (ii).

□

Corollary 3.4 (Gagliardo-Nirenberg inequality in $BV_H^{Q/Q-1}(\mathbb{G})$)**.** Let $f\in BV_H^{Q/Q-1}(\mathbb{G})$. Then

$$\|f\|_{L^{Q/Q-1}} \leq \mathscr{C}_{GN}\|D_Hf\|. \tag{28}$$

Proof. The proof follows by approximating f as in Theorem 3.3, using inequality (22) for functions in $\mathcal{D}(\mathbb{G})$, and then applying the weak lower semicontinuity of the $L^{Q/Q-1}$-norm.

□

Remark 3.5. Let $f\in BV_H^{Q/Q-1}(\mathbb{G})$. By (28) it follows that the H-variation $\|D_Hf\|$ is an equivalent norm to $\|f\|_{L^{Q/Q-1}}+\|D_Hf\|$. For this reason, in the sequel the H-variation will be taken as a norm and we shall set

$$\|f\|_{BV_H^{Q/Q-1}} := \|D_Hf\|.$$

Note also that (28) immediately implies the continuous embedding

$$BV_H(\mathbb{G}) \hookrightarrow BV_H^{Q/Q-1}(\mathbb{G}). \tag{29}$$

As a corollary of Theorem 3.3 and of the Gagliardo-Nirenberg inequality, we obtain the following compactness result.

Corollary 3.6 (compactness)**.** Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $BV_H^{Q/Q-1}(\mathbb{G})$ satisfying

$$\sup_k \|D_H f_k\| < +\infty.$$

Then, there exists a subsequence $\{f_{k_j}\}_{j\in\mathbb{N}}$ and a function $f \in BV_H^{Q/Q-1}(\mathbb{G})$ such that

$$f_{k_j} \rightharpoonup f \quad \textit{in} \quad L^{Q/Q-1}(\mathbb{G}) \quad \textit{as} \quad j \to +\infty.$$

.

Proof. Since $\sup_k \|D_H f_k\| < +\infty$, by Corollary 3.4 $\{f_k\}_{k\in\mathbb{N}}$ is equibounded in $L^{Q/Q-1}(\mathbb{G})$. Hence there exists a subsequence $\{f_{k_j}\}_{j\in\mathbb{N}}$ that weakly converges in $L^{Q/Q-1}(\mathbb{G})$ to some function f (see, e.g. [10], Theorem 3.18). By Theorem 3.2, $\|D_H f\| \le \liminf_{j\to+\infty} \|D_H f_{k_j}\|$. Thus, using the equiboundeness of $\|D_H f_{k_j}\|$, it follows that $f \in BV_H^{Q/Q-1}(\mathbb{G})$. □

4. Charges vanishing at infinity

In this section we shall define a subspace of $\left(BV_H^{Q/Q-1}(\mathbb{G})\right)^*$, denoted by $\mathbf{Ch}_0(\mathbb{G})$, and we shall investigate the relationship between its dual and the space $\left(BV_H^{Q/Q-1}(\mathbb{G})\right)$.

The results of this section will be used later, in order to define a divergence-type operator from $C_0(\mathbb{G}, H\mathbb{G})$ to $\mathbf{Ch}_0(\mathbb{G})$, which will turn out to be a bounded linear operator.

In rough terms, this operator will be the right substitute for the horizontal divergence operator div_H, when acting on $C_0(\mathbb{G}, H\mathbb{G})$, and we shall prove that is a surjective operator, which means that we can find a solution in $C_0(\mathbb{G}, H\mathbb{G})$ to the equation $\mathrm{div}_H \Phi = F$, whenever $F \in \mathbf{Ch}_0(\mathbb{G})$.

The presentation and results in this section are largely inspired by those in [14].

Definition 4.1. Given a sequence $\{f_j\}_{j\in\mathbb{N}}$ in $BV_H^{Q/Q-1}(\mathbb{G})$ we write

$$f_j \twoheadrightarrow 0 \qquad (j \to +\infty)$$

if and only if $f_j \rightharpoonup 0$ in $L^{Q/Q-1}(\mathbb{G})$ as $j \to +\infty$ and $\sup_j \|D_H f_j\| < +\infty$.

More generally, if $f \in BV_H^{Q/Q-1}(\mathbb{G})$, we write $f_j - f \twoheadrightarrow 0$ as $j \to +\infty$ whenever $f_j \rightharpoonup f$ in $L^{Q/Q-1}(\mathbb{G})$ as $j \to +\infty$ and $\sup_j \|D_H f_j\| < +\infty$.

Definition 4.2 (Charges vanishing at ∞)**.** Let $F : BV_H^{Q/Q-1}(\mathbb{G}) \longrightarrow \mathbb{R}$ be a linear functional. We say that F is a charge vanishing at ∞ if and only if

$$\langle F|f_j\rangle \xrightarrow[j \longrightarrow +\infty]{} 0$$

for any sequence $\{f_j\}_{j\in\mathbb{N}} \subset BV_H^{Q/Q-1}(\mathbb{G})$ such that $f_j \twoheadrightarrow 0$ as $j \to +\infty$.

From now on we shall denote by $\mathbf{Ch}_0(\mathbb{G})$ the class of all charges vanishing at ∞.

Remark 4.3. It is clear that $\mathbf{Ch}_0(\mathbb{G})$ is a (real) vector space. We set

$$\|F\|_{\mathbf{Ch}_0} := \sup\left\{\langle F|f\rangle \;:\; f \in BV_H^{Q/Q-1}(\mathbb{G}),\;\; \|D_H f\| \le 1\right\}.$$

Notice that $\|F\|_{\mathbf{Ch}_0} < +\infty$ whenever $F \in \mathbf{Ch}_0(\mathbb{G})$. In fact, there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset BV_H^{Q/Q-1}(\mathbb{G})$ with $\|D_H f_j\| \le 1$ such that $\langle F|f_j\rangle \longrightarrow \|F\|_{\mathbf{Ch}_0}$ as $j \to +\infty$. By Proposition 3.6, there exist $f \in BV_H^{Q/Q-1}(\mathbb{G})$ and a subsequence $\{f_{j_k}\}_{k\in\mathbb{N}}$ such that $f_{j_k} - f \twoheadrightarrow 0$ as $k \to +\infty$. As a consequence, $\langle F|f_{j_k} - f\rangle \longrightarrow 0$ as $k \to +\infty$. Thus

$$\langle F|f\rangle = \lim_{k\to+\infty} \langle F|f_{j_k}\rangle = \|F\|_{\mathbf{Ch}_0} < +\infty.$$

From this remark it follows that $\|\cdot\|_{\mathbf{Ch}_0}$ is a norm on $\mathbf{Ch}_0(\mathbb{G})$. We also observe that $\mathbf{Ch}_0(\mathbb{G}) \subset \left(BV_H^{Q/Q-1}(\mathbb{G})\right)^*$ and that for any $F \in \mathbf{Ch}_0(\mathbb{G})$ we have

$$\|F\|_{\mathbf{Ch}_0} = \|F\|_{\left(BV_H^{Q/Q-1}(\mathbb{G})\right)^*}.$$

Proposition 4.4. The space $\mathbf{Ch}_0(\mathbb{G})$ is a Banach space under the norm $\|\cdot\|_{\mathbf{Ch}_0}$.

Proof. We show that each Cauchy sequence $\{F_k\}_{k\in\mathbb{N}} \subset \mathbf{Ch}_0(\mathbb{G})$ converges to an element of $\mathbf{Ch}_0(\mathbb{G})$. To this end, note that $\{F_k\}_{k\in\mathbb{N}}$ has to converge to some $F \in \left(BV_H^{Q/Q-1}(\mathbb{G})\right)^*$, hence for any $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that $\|F - F_k\|_{\left(BV_H^{Q/Q-1}(\mathbb{G})\right)^*} < \varepsilon$ for any $k > k_\varepsilon$.

Let now $k > k_\varepsilon$ and let $\{f_j\}_{j\in\mathbb{N}} \subset BV_H^{Q/Q-1}(\mathbb{G})$ be any sequence such that $f_j \twoheadrightarrow 0$ as $j \to +\infty$. Furthermore, set

$$\mathscr{K} := \sup_j \|D_H f_j\|.$$

For every $j \in \mathbb{N}$

$$\begin{aligned} |\langle F|f_j\rangle| &\leq |\langle F-F_k|f_j\rangle| + |\langle F_k|f_j\rangle| \\ &\leq \mathscr{K}\,\|F-F_k\|_{\left(BV_H^{Q/Q-1}(\mathbb{G})\right)^*} + |\langle F_k|f_j\rangle| \\ &\leq \mathscr{K}\,\varepsilon + |\langle F_k|f_j\rangle|. \end{aligned}$$

In turn, this implies that

$$\limsup_{j\to+\infty} |\langle F|f_j\rangle| \leq \mathscr{K}\,\varepsilon.$$

From the arbitrariness of $\varepsilon > 0$ we get that $F \in \mathbf{Ch}_0(\mathbb{G})$. □

4.1. An example of charge vanishing at ∞

Since $BV_H^{Q/Q-1}(\mathbb{G}) \subset L^{Q/Q-1}(\mathbb{G})$, we can state the following definition.

Definition 4.5. For any $f \in L^Q(\mathbb{G})$, let $\Lambda(f) : BV_H^{Q/Q-1}(\mathbb{G}) \longrightarrow \mathbb{R}$ be the linear functional defined by

$$\langle \Lambda(f)|g\rangle := \int_{\mathbb{G}} fg\,dx.$$

Proposition 4.6. If $f \in L^Q(\mathbb{G})$, then $\Lambda(f) \in \mathbf{Ch}_0(\mathbb{G})$ and $\|\Lambda(f)\|_{\mathbf{Ch}_0} \leq \mathscr{C}_{GN}\|f\|_{L^Q}$. Thus, the linear operator $\Lambda : L^Q(\mathbb{G}) \longrightarrow \mathbf{Ch}_0(\mathbb{G})$ is a bounded linear operator whose norm is bounded by the Gagliardo-Nirenberg constant $\mathscr{C}_{GN}$.

Proof. Let $\{g_j\}_{j\in\mathbb{N}} \subset BV_H^{Q/Q-1}(\mathbb{G})$ be a sequence such that $g_j \twoheadrightarrow 0$ as $j \to +\infty$. In particular, this sequence weakly converges to 0 in $L^{Q/Q-1}(\mathbb{G})$. So we get that

$$\langle \Lambda(f)|g_j\rangle = \int_{\mathbb{G}} fg_j\,dx \xrightarrow[j\to+\infty]{} 0,$$

which shows that $\Lambda(f) \in \mathbf{Ch}_0(\mathbb{G})$. Moreover, for any $g \in BV_H^{Q/Q-1}(\mathbb{G})$ we have

$$|\langle \Lambda(f)|g\rangle| \leq \|f\|_{L^Q}\,\|g\|_{L^{Q/Q-1}} \leq \mathscr{C}_{GN}\|D_H g\|\,\|f\|_{L^Q},$$

where we have used Hölder inequality and the Gagliardo-Nirenberg inequality (28). Hence

$$\|\Lambda(f)\|_{\mathbf{Ch}_0} \leq \mathscr{C}_{GN}\|f\|_{L^Q}.$$

□

We would like to show that the image $\mathcal{R}(\Lambda)$ of Λ is dense in $\mathbf{Ch}_0(\mathbb{G})$ or, equivalently, that any charge vanishing at infinity can be approximated by a charge in $\mathcal{R}(\Lambda)$.

As already recalled in Section 2 (see, e.g., Proposition 2.5), we notice that in distribution theory the common way to define the convolution between a distribution F and a test function ϕ is as follows:

$$\langle F * \phi | \psi \rangle := \langle F | \psi * {}^{\mathrm{v}}\phi \rangle_{\mathcal{D}', \mathcal{D}} \qquad \forall\, \psi \in \mathcal{D}(\mathbb{G}).$$

Now, let $F \in \mathbf{Ch}_0(\mathbb{G})$ and $\phi \in \mathcal{D}(\mathbb{G})$: our aim is to define a new charge $F * \phi$.

More precisely, let $g \in BV_H^{Q/Q-1}(\mathbb{G})$ and $\phi \in \mathcal{D}(\mathbb{G})$. Arguing as in Step 1 of the proof of Proposition 3.3, we get that $g * {}^{\mathrm{v}}\phi \in BV_H^{Q/Q-1}(\mathbb{G})$ and that

$$\|D_H(f * \phi)\|_{L^1} \le \|D_H f\| \|\phi\|_{L^1}. \tag{30}$$

This motivates the following definition.

Definition 4.7. Let $F \in \mathbf{Ch}_0(\mathbb{G})$ and $\phi \in \mathcal{D}(\mathbb{G})$. We define the linear functional

$$F * \phi : BV_H^{Q/Q-1}(\mathbb{G}) \longrightarrow \mathbb{R}$$

by setting

$$BV_H^{Q/Q-1}(\mathbb{G}) \ni g \longmapsto \langle F * \phi | g \rangle := \langle F | g * {}^{\mathrm{v}}\phi \rangle.$$

Proposition 4.8. Let $F \in \mathbf{Ch}_0(\mathbb{G})$ and $\phi \in \mathcal{D}(\mathbb{G})$. Then $F * \phi \in \mathbf{Ch}_0(\mathbb{G}) \cap \mathcal{R}(\Lambda)$.

Proof. The proof follows almost verbatim the corresponding one in [14], Proposition 4.1, and we sketch it for the reader's convenience. When one restricts F to $\mathcal{D}(\mathbb{G})$, the restricted functional is a distribution. Thus, the convolution $F * \phi$ is a well-defined distribution, which is actually a smooth function. Thus, there must exist $f \in C^\infty(\mathbb{G})$ such that

$$\langle F * \phi | \psi \rangle = \int_{\mathbb{G}} f \psi \, dx \qquad \forall \psi \in \mathcal{D}(\mathbb{G}). \tag{31}$$

Moreover, it is not difficult to see that the function f belongs to $L^Q(\mathbb{G})$. In fact, let $\{\psi_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{G})$ be a sequence such that $\psi_j \rightharpoonup 0$ in $L^{Q/Q-1}$ and $\|\psi_j\|_{L^{Q/Q-1}} \xrightarrow[j\to+\infty]{} 0$.

We clearly have the following:

$$\begin{aligned}
\sup_j \|D_H(\psi_j * {}^{\mathrm{v}}\phi)\| &= \sup_j \|D_H(\psi_j * {}^{\mathrm{v}}\phi)\|_{L^1} = \sup_j \|\psi_j * D_H({}^{\mathrm{v}}\phi)\|_{L^1} \\
&\le \sup_j \|\psi_j\|_{L^{Q/Q-1}} \|D_H({}^{\mathrm{v}}\phi)\|_{L^{Q/Q+1}} \\
&= \sup_j \|\psi_j\|_{L^{Q/Q-1}} \|D_H \phi\|_{L^{Q/Q+1}} < +\infty,
\end{aligned}$$

where we have used the Hausdorff-Young inequality (see Theorem 2.3). Moreover, for any $g \in L^Q(\mathbb{G})$ we have

$$\int_{\mathbb{G}} g(\psi_j * \phi)\, dx = \int_{\mathbb{G}} \psi_j (g * \phi)\, dx.$$

Since $g * \phi \in L^Q(\mathbb{G})$ and $\psi_j \rightharpoonup 0$ weakly in $L^{Q/Q-1}(\mathbb{G})$ as $j \to +\infty$, the right-hand side of the last equality tends to 0 as $j \to +\infty$. In particular, this implies that $\psi_j * \phi \twoheadrightarrow 0$ as $j \to +\infty$ and that

$$\langle F | \psi_j * \phi \rangle = \langle F * \phi | \psi_j \rangle \xrightarrow[j \to +\infty]{} 0.$$

Thus, the linear functional $F * \phi$ turns out to be continuous in $\mathcal{D}(\mathbb{G})$ (with respect to the topology of $L^{Q/Q-1}(\mathbb{G})$). The density of $\mathcal{D}(\mathbb{G})$ in $L^{Q/Q-1}(\mathbb{G})$ implies that $F * \phi$ can be uniquely extended to a bounded linear functional on $L^{Q/Q-1}(\mathbb{G})$. Thus, it follows from the Riesz representation theorem that $f \in L^Q(\mathbb{G})$.

Note that since $f \in L^Q(\mathbb{G})$, Proposition 4.6 implies that $\Lambda(f) \in \mathbf{Ch}_0(\mathbb{G})$.

We are left to show that $\Lambda(f) = F * \phi$, which means that $F * \phi \in \mathcal{R}(\Lambda)$.

In fact, this is equivalent to show that equation (31) holds true whenever $\psi \in BV_H^{Q/Q-1}(\mathbb{G})$.
By Theorem 3.3 we can take a sequence $\{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{D}(\mathbb{G})$ such that $\psi_j \twoheadrightarrow \psi$ as $j \to +\infty$. Hence, from (31) we get that $\langle F * \phi | \psi_j \rangle = \int_{\mathbb{G}} f \psi_j\, dx$ for every $j \in \mathbb{N}$ and

$$\int_{\mathbb{G}} f \psi_j\, dx \xrightarrow[j \to +\infty]{} \int_{\mathbb{G}} f \psi\, dx.$$

We also observe that

$$\langle F * \phi | \psi_j \rangle = \langle F | \psi_j * {}^{\mathrm{v}}\phi \rangle \xrightarrow[j \to +\infty]{} \langle F | \psi * {}^{\mathrm{v}}\phi \rangle = \langle F * \phi | \psi \rangle,$$

which is true because $F \in \mathbf{Ch}_0(\mathbb{G})$ and $\psi_j * {}^{\mathrm{v}}\phi \twoheadrightarrow \psi * {}^{\mathrm{v}}\phi$ as $j \to +\infty$. As a consequence

$$\langle F * \phi | \psi \rangle = \int_{\mathbb{G}} f \psi\, dx \qquad \forall\, \psi \in BV_H^{Q/Q-1}(\mathbb{G}),$$

as wished.

$\square$

Let $\{J_\varepsilon\}_{\varepsilon > 0}$ be a family of approximations to the identity associated with a symmetric kernel (i.e., $J_\varepsilon(x) = {}^{\mathrm{v}}J_\varepsilon(x)$ for every $x \in \mathbb{G}$). Let $\{\varepsilon_k\}_{k \in \mathbb{N}}$ be a strictly decreasing sequence such that $\varepsilon_k \to 0^+$ as $k \to +\infty$.

Proposition 4.9. Let $F \in \mathbf{Ch}_0(\mathbb{G})$ and let $\{J_{\varepsilon_k}\}_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{G})$ be as above. Then

$$\|F - F * J_{\varepsilon_k}\|_{\mathbf{Ch}_0} \xrightarrow[k\to+\infty]{} 0.$$

We omit this proof since it looks very similar to the corresponding one in [14] (see Proposition 4.2).

Remark 4.10. An immediate consequence of this approximation result is the density of $\mathcal{R}(\Lambda)$ in the space $\mathbf{Ch}_0(\mathbb{G})$ of all charges vanishing at ∞.

Remark 4.11. For any $\Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G})$ with $\|\Phi\|_\infty \leq 1$, let us consider the charge $\Lambda(\mathrm{div}_H\Phi)$. Since

$$\langle \Lambda(\mathrm{div}_H\Phi)|g\rangle = \int_{\mathbb{G}} g\, \mathrm{div}_H\Phi\, dx \qquad \forall g \in BV_H^{Q/Q-1}(\mathbb{G}),$$

we infer that $\langle \Lambda(\mathrm{div}_H\Phi)|g\rangle \leq \|D_H g\|$. Thus, if $g \in BV_H^{Q/Q-1}(\mathbb{G})$ and $\|D_H g\| \leq 1$, we immediately get that

$$\|\Lambda(\mathrm{div}_H\Phi)\|_{\mathbf{Ch}_0} \leq 1\,. \tag{32}$$

Proposition 4.12. There exists a linear bijective operator $ev : BV_H^{Q/Q-1}(\mathbb{G}) \longrightarrow \mathbf{Ch}_0^*(\mathbb{G})$, given by

$$\langle ev(f)|F\rangle := \langle F|f\rangle \qquad \forall f \in BV_H^{Q/Q-1}(\mathbb{G}) \quad \forall F \in \mathbf{Ch}_0(\mathbb{G}).$$

Proof. It is obvious that ev is a linear operator. Furthermore, since

$$|\langle ev(f)|F\rangle| = |\langle F|f\rangle| \leq \|F\|_{\mathbf{Ch}_0}\|D_H f\|,$$

it follows that the operator ev maps $BV_H^{Q/Q-1}(\mathbb{G})$ onto $\mathbf{Ch}_0^*(\mathbb{G})$. In order to show that ev is also injective, let $f \in BV_H^{Q/Q-1}(\mathbb{G})$ be such that $ev(f) = 0$. Thus, if we take $g \in \mathcal{D}(\mathbb{G})$ together with its corresponding charge $\Lambda(g)$, we get that

$$0 = \langle ev(f)|\Lambda(g)\rangle = \langle \Lambda(g)|f\rangle = \int_{\mathbb{G}} fg\, dx \quad \forall g \in \mathcal{D}(\mathbb{G}).$$

Since $f \in L^1_{loc}(\mathbb{G})$, it follows that $f(x) = 0$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{G}$. Therefore, f turns out to be identically zero (as a function in $BV_H^{Q/Q-1}(\mathbb{G})$).

To prove that ev is surjective, we select $\alpha \in \mathbf{Ch}_0^*(\mathbb{G})$. By using Proposition 4.6 it follows that the composition $\alpha \circ \Lambda$ belongs to the space $(L^Q(\mathbb{G}))^*$. Hence, by the Riesz representation theorem there exists a unique $h \in L^{Q/Q-1}(\mathbb{G})$ for which

$$\langle \alpha|\Lambda(f)\rangle = \langle \alpha \circ \Lambda|f\rangle = \int_{\mathbb{G}} hf\, dx \qquad \forall f \in L^Q(\mathbb{G}).$$

We need to show that $h \in BV_H^{Q/Q-1}(\mathbb{G})$. To this aim, we apply the preceding equality to $\mathrm{div}_H\Phi$, whenever $\Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G})$ and $\|\Phi\|_\infty \le 1$. Thus

$$\langle \alpha | \Lambda(\mathrm{div}_H\Phi)\rangle = \int_{\mathbb{G}} h\, \mathrm{div}_H\Phi\, dx.$$

Hence, we get that

$$\int_{\mathbb{G}} h\, \mathrm{div}_H\Phi\, dx = \langle \alpha | \Lambda(\mathrm{div}_H\Phi)\rangle \le \|\alpha\|_{\mathbf{Ch}_0^*}\|\Lambda(\mathrm{div}_H\Phi)\|_{\mathbf{Ch}_0} \le \|\alpha\|_{\mathbf{Ch}_0^*},$$

where the last inequality follows from (32). Taking the supremum on the left hand side over all $\Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G})$ such that $\|\Phi\|_\infty \le 1$, we get that $\|D_H h\| < +\infty$. If follows that $h \in BV_H^{Q/Q-1}(\mathbb{G})$ and that

$$\langle ev(h) | \Lambda(f)\rangle = \langle \Lambda(f) | h\rangle = \int_{\mathbb{G}} hf\, dx = \langle \alpha | \Lambda(f)\rangle,$$

for every $f \in L^Q(\mathbb{G})$. Using that $\mathcal{R}(\Lambda)$ is dense in $\mathbf{Ch}_0(\mathbb{G})$, we finally get that $ev(h) = \alpha$, as wished. $\square$

Notice that the map *ev* is in fact an isomorphism of Banach spaces.

5. Bourgain-Brezis's duality argument for the getting the estimate (4)

In order to prove inequality (1), Bourgain and Brezis pass from an operator to its adjoint and conversely. A similar method is used in [13], [14], [25], and [26].

To begin with, if $f \in L^Q(\mathbb{G})$ we have to explain in which sense we want to solve the equation

$$\mathrm{div}_H\Phi = f$$

in our setting, finding a solution $\Phi \in C_0(\mathbb{G}, H\mathbb{G})$ such that

$$\|\Phi\|_{L^\infty} \le \mathscr{C}(Q)\|f\|_{L^Q},$$

where $\mathscr{C}(Q)$ is a geometric constant.

The results in this section generalize both Theorem 6.1 in [14] and Theorem 3.1 in [25] to sub-Riemannian Carnot groups.

5.1. A charge associated with a divergence operator

Also in Carnot groups, we can define the notion of *flux*. More precisely, we say that a distribution $F \in \mathcal{D}'(\mathbb{G})$ is a flux if the equation $\mathrm{div}_H Y = F$ has a continuous solution, i.e., if there exists a horizontal vector field $Y \in C(\mathbb{G};H\mathbb{G})$ such that

$$F(\varphi) = -\int_{\mathbb{G}} \langle Y(x), D_H\varphi(x)\rangle\, dx \qquad \forall\, \varphi \in \mathcal{D}(\mathbb{G}).$$

We now have to define a linear operator $\Gamma : C_0(\mathbb{G},H\mathbb{G}) \longrightarrow \mathbf{Ch}_0(\mathbb{G})$ such that the charge $\Gamma(\Phi)$, for any given $\Phi \in C_0(\mathbb{G},H\mathbb{G})$, can be thought of as the (distributional) horizontal divergence of Φ.

We start by observing that for any $f \in BV_H^{Q/Q-1}(\mathbb{G}) \subset BV_{H,loc}(\mathbb{G})$ the structure theorem implies that

$$\int_{\mathbb{G}} f\,\mathrm{div}_H\Phi\, dx = -\int_{\mathbb{G}} \langle \Phi, d[D_H f]\rangle \qquad \forall\, \Phi \in \mathcal{D}(\mathbb{G},H\mathbb{G}). \tag{33}$$

We give the following definition.

Definition 5.1. For any $\Phi \in C_0(\mathbb{G},H\mathbb{G})$, let $\Gamma(\Phi) : BV_H^{Q/Q-1}(\mathbb{G}) \longrightarrow \mathbb{R}$ be the linear functional defined as

$$\langle \Gamma(\Phi), g\rangle := -\int_{\mathbb{G}} \langle \Phi, d[D_H g]\rangle \qquad \forall\, g \in BV_H^{Q/Q-1}(\mathbb{G}).$$

Proposition 5.2. If $\Phi \in C_0(\mathbb{G},H\mathbb{G})$, then $\Gamma(\Phi) \in \mathbf{Ch}_0(\mathbb{G})$ and $\|\Gamma(\Phi)\|_{\mathbf{Ch}_0} \leq \|\Phi\|_\infty$.

As a consequence, the linear operator

$$\Gamma : C_0(\mathbb{G},H\mathbb{G}) \longrightarrow \mathbf{Ch}_0(\mathbb{G})$$

is a bounded linear operator.

Proof. Let $\Phi \in C_0(\mathbb{G},H\mathbb{G})$ and let $\{g_j\}_{j\in\mathbb{N}} \subset BV_H^{Q/Q-1}(\mathbb{G})$ be a sequence such that $g_j \twoheadrightarrow 0$ as $j \to +\infty$. For any $\varepsilon > 0$, let $\Psi \in \mathcal{D}(\mathbb{G},H\mathbb{G})$ be such that $\|\Phi - \Psi\|_\infty < \varepsilon$. Moreover, let us set $\mathscr{K} := \sup_j \|D_H g_j\|$. We have

$$\begin{aligned} \left|\langle \Gamma(\Phi)|g_j\rangle\right| &\leq \left|\int_{\mathbb{G}} \langle (\Phi - \Psi), d[D_H g_j]\rangle\, dx\right| + \left|\int_{\mathbb{G}} \mathrm{div}_H\Psi\, g_j\, dx\right| \\ &\leq \mathscr{K}\varepsilon + \left|\int_{\mathbb{G}} \mathrm{div}_H\Psi\, g_j\, dx\right|. \end{aligned}$$

Since $\mathrm{div}_H\Psi$ is a smooth compactly supported function, we get that $\mathrm{div}_H\Psi \in L^Q(\mathbb{G})$ and hence the second integral goes to 0 as $j \to +\infty$. As a consequence

$$\limsup_{j\to+\infty} \left|\langle \Gamma(\Phi)|g_j\rangle\right| \leq \mathscr{K}\varepsilon.$$

Thus, the first claim follows from the arbitrariness of $\varepsilon > 0$.

It is also clear that for any $g \in BV_H^{Q/Q-1}(\mathbb{G})$ the following inequality holds

$$|\langle \Gamma(\Phi)|g\rangle| \leq \|\Phi\|_\infty \|D_H g\|.$$

This implies the second claim and achieves the proof.

□

Remark 5.3. For any $\Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G})$, let us consider the charges $\Lambda(\mathrm{div}_H \Phi)$ and $\Gamma(\Phi)$. It is immediate to see that

$$\langle \Lambda(\mathrm{div}_H \Phi)|g\rangle = \int_{\mathbb{G}} g\,\mathrm{div}_H \Phi\, dx = \langle \Gamma(\Phi)|g\rangle \qquad \forall g \in \mathcal{D}(\mathbb{G}).$$

Thus, using Theorem 3.3 we get that they coincide as functionals on $BV_H^{Q/Q-1}(\mathbb{G})$.

Keeping this in mind, what we shall prove in Corollary 5.7 is that for any $f \in L^Q(\mathbb{G})$ there exists a continuous vector field vanishing at infinity $\Phi \in C_0(\mathbb{G}, H\mathbb{G})$ such that

$$\Gamma(\Phi) = \Lambda(f)$$

in the sense that

$$-\int_{\mathbb{G}} \langle \Phi, d[D_H g]\rangle = \int_{\mathbb{G}} f\, g\, dx \qquad \forall g \in BV_H^{Q/Q-1}(\mathbb{G}). \tag{34}$$

This will be a consequence of Theorem 5.6 below.

Following the original idea of Bourgain and Brezis, as in [14] we need to characterize the adjoint Γ^* of Γ.

We first consider the map

$$-D_H : BV_H^{Q/Q-1}(\mathbb{G}) \longrightarrow \mathcal{M}(\mathbb{G}, H\mathbb{G}),$$

where $D_H g := [D_H g]$. Moreover, let $\rho : \mathcal{M}(\mathbb{G}, H\mathbb{G}) \longrightarrow C_0(\mathbb{G}, H\mathbb{G})^*$ be such that

$$\rho(\mu)(v) = \int_{\mathbb{G}} \langle v, d\mu\rangle = T_\mu(v) \qquad \forall v \in C_0(\mathbb{G}, H\mathbb{G}).$$

The map $\Gamma^* : \mathbf{Ch}_0^*(\mathbb{G}) \to C_0(\mathbb{G}, H\mathbb{G})^*$ makes the following diagram commutative

$$\begin{tikzcd} BV_H^{Q/Q-1}(\mathbb{G}) \arrow[r, "-D_H"] \arrow[d, "ev"'] & \mathcal{M}(\mathbb{G}, H\mathbb{G}) \arrow[d, "\rho"] \\ \mathbf{Ch}_0^*(\mathbb{G}) \arrow[r, "\Gamma^*"'] & C_0(\mathbb{G}, H\mathbb{G})^*. \end{tikzcd}$$

Indeed, let $\alpha \in \mathbf{Ch}_0^*(\mathbb{G})$ and $\Phi \in C_0(\mathbb{G}, H\mathbb{G})$. Furthermore, let $g = ev^{-1}(\alpha)$. Hence

$$\begin{aligned}\langle \Gamma^*(\alpha)|\Phi\rangle_{C_0^*,C_0} &= \langle \alpha|\Gamma(\Phi)\rangle_{\mathbf{Ch}_0^*,\mathbf{Ch}_0} = \langle ev(g)|\Gamma(\Phi)\rangle \\ &= \langle \Gamma(\Phi)|g\rangle = -\int_{\mathbb{G}} \langle \Phi, d[D_H g]\rangle.\end{aligned} \tag{35}$$

Thus, up to the identifications $C_0(\mathbb{G}, H\mathbb{G}) \cong \mathcal{M}(\mathbb{G}, H\mathbb{G})$ and $\mathbf{Ch}_0^*(\mathbb{G}) \cong BV_H^{Q/Q-1}(\mathbb{G})$, since Γ is the distributional horizontal divergence of Φ, then Γ^* is (minus) the distributional horizontal gradient $-D_H g$ of g.

Proposition 5.4. The range $\mathcal{R}(\Gamma^*)$ of the adjoint operator

$$\Gamma^* : \mathbf{Ch}_0^*(\mathbb{G}) \to C_0(\mathbb{G}, H\mathbb{G})^*$$

is closed in $\mathbf{Ch}_0^*(\mathbb{G})$.

Proof. Let $\{\alpha_j\}_{j\in\mathbb{N}} \subset \mathbf{Ch}_0^*(\mathbb{G})$ be a sequence such that

$$\Gamma^*(\alpha_j) \xrightarrow[j\to+\infty]{} T,$$

for some $T \in C_0(\mathbb{G}, H\mathbb{G})^*$. Let $\{g_j\}_{j\in\mathbb{N}}$ be the corresponding sequence in $BV_H^{Q/Q-1}(\mathbb{G})$, where we have set $g_j := ev^{-1}(\alpha_j)$ for any $j \in \mathbb{N}$. The sequence $\{\Gamma^*(\alpha_j)\}_{j\in\mathbb{N}}$ is bounded, being convergent. Hence, by (35) also $\|D_H g_j\|$ is bounded and we get that $\sup_j \|D_H f_j\| < +\infty$. From Proposition 3.6 we get that there exist $g \in BV_H^{Q/Q-1}(\mathbb{G})$ and a subsequence $\{g_{j_k}\}_{k\in\mathbb{N}} \subset BV_H^{Q/Q-1}(\mathbb{G})$ such that $g_{j_k} \twoheadrightarrow g$ as $k \to +\infty$. Setting $\alpha = ev(g)$, we have

$$\begin{aligned}\langle T|\Phi\rangle_{C_0^*,C_0} &= \lim_{k\to+\infty} \langle \Gamma^*(\alpha_{j_k})|\Phi\rangle = -\lim_{k\to+\infty} \int_{\mathbb{G}} \langle \Phi, d[D_H g_{j_k}]\rangle \\ &= \lim_{k\to+\infty} \int_{\mathbb{G}} g_{j_k} \operatorname{div}_H \Phi\, dx = \int_{\mathbb{G}} g \operatorname{div}_H \Phi\, dx \\ &= -\int_{\mathbb{G}} \langle \Phi, d[D_H g]\rangle = \langle \Gamma^*(\alpha)|\Phi\rangle_{C_0^*,C_0}\end{aligned}$$

for any $\Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G})$.

By the density of $\mathcal{D}(\mathbb{G}, H\mathbb{G})$ in $C_0(\mathbb{G}, H\mathbb{G})$, we get that $T = \Gamma^*(\alpha)$, which achieves the proof.

□

As a corollary, keeping in mind Proposition II.18 in [10] we have the following:

Corollary 5.5. The range $\mathcal{R}(\Gamma)$ of Γ is closed in $\mathbf{Ch}_0(\mathbb{G})$.

5.2. Main results

We are in a position to solve the problem

$$\Gamma(\Phi) = F,$$

whenever $\Phi \in C_0(\mathbb{G}, H\mathbb{G})$ and $F \in \mathcal{D}'(\mathbb{G})$. More precisely, the following holds:

Theorem 5.6. Let $F \in \mathcal{D}'(\mathbb{G})$. Then, there exists $\Phi \in C_0(\mathbb{G}, H\mathbb{G})$ such that

$$\Gamma(\Phi) = F \tag{36}$$

if, and only if, $F \in \mathbf{Ch}_0(\mathbb{G})$.

In addition, if $F \in \mathbf{Ch}_0(\mathbb{G})$ there exists a solution $\Phi \in C_0(\mathbb{G}, H\mathbb{G})$ of (36) such that

$$\|\Phi\|_\infty \le 2\|F\|_{\mathbf{Ch}_0}. \tag{37}$$

Proof. **Step 1 (proof of (36)).** The necessity part follows from Proposition 5.2.

Furthermore, since in Corollary 5.5 we have proved that $\mathcal{R}(\Gamma)$ is closed, the sufficiency part will be proved once we have shown that $\mathcal{R}(\Gamma)$ is dense in $\mathbf{Ch}_0(\mathbb{G})$.

To show that $\mathcal{R}(\Gamma)$ is dense in $\mathbf{Ch}_0(\mathbb{G})$ we use a standard consequence of the Hahn-Banach theorem; see [10], Corollary I.8. We assume that $\alpha \in \mathbf{Ch}_0^*(\mathbb{G})$ vanishes on all of $\mathcal{R}(\Gamma)$. Thus, we have to show that α must vanish everywhere on $\mathbf{Ch}_0(\mathbb{G})$.

To this aim, let $\alpha \in \mathbf{Ch}_0^*(\mathbb{G})$ be such that $\langle \alpha | \Gamma(\Phi) \rangle = 0$ for every $\Phi \in C_0(\mathbb{G}, H\mathbb{G})$. By Proposition 4.12, there exists a unique $g \in BV_H^{Q/Q-1}(\mathbb{G})$ such that $\alpha = ev(g)$. Then

$$0 = \langle ev(g) | \Gamma(\Phi) \rangle = \langle \Gamma(\Phi) | g \rangle = -\int_{\mathbb{G}} \langle \Phi, d[D_H g] \rangle \qquad \forall \Phi \in C_0(\mathbb{G}, H\mathbb{G}).$$

This implies that $D_H g = 0$ and in turn that $g = 0$, since $g \in BV_H^{Q/Q-1}(\mathbb{G})$.

Step 2 (proof of the second part). Let $F \in \mathbf{Ch}_0(\mathbb{G})$. We show that it is possible to find a solution of (36) that satisfies *also* the estimate (37).

Again, we use the original idea by Bourgain and Brezis for periodic functions, already used in the Euclidean setting by [13] and [25], under more general assumptions.

For the sake of simplicity, we will set here $X = C_0(\mathbb{G}, H\mathbb{G})$.

Let $F \in \mathbf{Ch}_0(\mathbb{G})$ be such that $\|F\|_{\mathbf{Ch}_0} > 0$, and define two convex subsets by setting

$$\mathcal{U} := \{\Phi \in X \,:\, \Gamma(\Phi) = F\}, \qquad \mathcal{V} := \{\Phi \in X \,:\, \|\Phi\|_\infty < 2\|F\|_{\mathbf{Ch}_0}\}.$$

From Step 1 we get that $\mathcal{U} \neq \emptyset$. Moreover, $\mathcal{V} \neq \emptyset$ because $\Phi = 0$ clearly belongs to $\mathcal{V}$.

Claim: *We claim that $\mathcal{U} \cap \mathcal{V} \neq \emptyset$.*

If we could show that the claim is true, then the proof would be complete since we would have found a solution Φ of (36) that satisfies also the estimate $\|\Phi\|_\infty < 2\|F\|_{\mathbf{Ch}_0}$.

Thus, we are left to prove the claim. By contradiction, we assume that

$$\mathcal{U} \cap \mathcal{V} = \emptyset. \tag{38}$$

By the first geometric form of the Hahn-Banach theorem (see, e.g., Theorem 1.6 in [10]) we get that there exist $T \in X^*$ and $t \in \mathbb{R}$ such that:

$$\langle T|\Phi\rangle \geq t \quad \forall \Phi \in \mathcal{U} \quad \text{and} \quad \langle T|\Phi\rangle \leq t \quad \forall \Phi \in \mathcal{V}. \tag{39}$$

Note that $t > 0$, since $\Phi = 0 \in \mathcal{V}$. Moreover, we observe that $\operatorname{Ker}(\Gamma) \subset \operatorname{Ker}(T)$.

In fact, let $\Phi_0 \in \operatorname{Ker}(\Gamma)$ and $\Phi \in \mathcal{U}$. Then, for every $s \in \mathbb{R}$ we have $\Phi + s\Phi_0 \in \mathcal{U}$.

As a consequence, from the inequality $\langle T|\Phi + s\Phi_0\rangle \geq t$ we should have

$$s\langle T|\Phi_0\rangle \geq t - \langle T|\Phi\rangle \qquad \forall s \in \mathbb{R}.$$

But this does not hold unless $\langle T|\Phi_0\rangle = 0$. Hence $\Phi_0 \in \operatorname{Ker}(T)$. Being surjective, Γ is also open by the open mapping theorem. Therefore, it turns out that Γ is a quotient map. Hence there exists $\alpha \in \mathbf{Ch}_0(\mathbb{G})^*$ such that $T = \alpha \circ \Gamma$. Now, take $\tilde{g} = ev^{-1}(\alpha) \in BV_H^{Q/Q-1}(\mathbb{G})$. Then, for any $\Phi \in X$ we have

$$\begin{aligned} -\int_{\mathbb{G}} \langle \Phi, d[D_H\tilde{g}]\rangle &= \langle \Gamma(\Phi)|\tilde{g}\rangle = \langle ev(\tilde{g})|\Gamma(\Phi)\rangle = \langle \alpha|\Gamma(\Phi)\rangle_{\mathbf{Ch}_0^*,\mathbf{Ch}_0} \\ &= (\alpha \circ \Gamma)(\Phi) = \langle T|\Phi\rangle_{C_0^*,C_0}. \end{aligned} \tag{40}$$

On the other hand, let $\Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G})$ be such that $\|\Phi\|_\infty \leq 1$ and choose $\varepsilon > 0$ such that $1+\varepsilon < 2$. Hence $\Psi := (1+\varepsilon)\|F\|_{\mathbf{Ch}_0}\Phi \in \mathcal{V}$. In addition, we have

$$\begin{aligned} \int_{\mathbb{G}} \tilde{g}\operatorname{div}_H\Phi\, dx &= -\int_{\mathbb{G}} \langle \Phi, d[D_H\tilde{g}]\rangle = -\frac{1}{(1+\varepsilon)\|F\|_{\mathbf{Ch}_0}} \int_{\mathbb{G}} \langle \Psi, d[D_H\tilde{g}]\rangle \\ &\underset{\text{by (40)}}{=} \frac{1}{(1+\varepsilon)\|F\|_{\mathbf{Ch}_0}} \langle T|\Psi\rangle \underset{\Psi\in\mathcal{V}}{\leq} \frac{t}{(1+\varepsilon)\|F\|_{\mathbf{Ch}_0}}. \end{aligned}$$

In particular, by taking the supremum on all $\Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G}) \subset X$ such that $\|\Phi\|_\infty \leq 1$, we get that

$$\|D_H\tilde{g}\| \leq \frac{t}{(1+\varepsilon)\|F\|_{\mathbf{Ch}_0}}.$$

Let $\Phi \in \mathcal{U}$. Using the last estimate together with (39) and (40), we get that

$$t \leq \langle T|\Phi\rangle = \langle \Gamma(\Phi)|\tilde{g}\rangle \underset{\Phi\in\mathcal{U}}{=} \langle F|\tilde{g}\rangle \leq \|F\|_{\mathbf{Ch}_0}\|D_H\tilde{g}\| \leq \frac{t}{(1+\varepsilon)}.$$

But this cannot be true, since we have seen that t is positive. This contradiction shows our claim and concludes the proof.

$\square$

By Step 1 of the above proof, we have that Γ is surjective. Since Γ is also continuous (see Proposition 5.2), by the open mapping theorem there exists a positive constant $\mathscr{C} > 0$ such that

$$\|\Phi\|_\infty \leq \mathscr{C}\|F\|_{\mathbf{Ch}_0},$$

for any solution $\Phi \in C_0(\mathbb{G}, H\mathbb{G})$ of (36). Therefore, the second part of Theorem 5.6 would follow straightforwardly *for any solution* $\Phi \in C_0(\mathbb{G}, H\mathbb{G})$ of (36), if one were satisfied with a generic constant. On the contrary, we have been able to get an estimate with an explicit constant, but paying the price that the estimate holds for *some* Φ. We also note that the constant 2 does not play any role here. The proof would work as well with a constant as close to 1 as one wants.

As an immediate corollary of Theorem 5.6, for any $f \in L^Q(\mathbb{G})$ we have the following estimate with a geometric constant, which depends only on the homogeneous dimension.

Clearly, the equation $\Gamma(\Phi) = \Lambda(f)$ is meant here as specified in (34).

Corollary 5.7. For any $f \in L^Q(\mathbb{G})$ there exists a solution $\Phi \in C_0(\mathbb{G}, H\mathbb{G})$ of

$$\Gamma(\Phi) = \Lambda(f) \tag{41}$$

satisfying the inequality

$$\|\Phi\|_\infty \leq 2\,\mathscr{C}_{GN}\|f\|_{L^Q}, \tag{42}$$

where $\mathscr{C}_{GN}$ is the constant appearing in (28).

Proof. Recall that $\Lambda(f) \in \mathbf{Ch}_0(\mathbb{G})$ for any $f \in L^Q(\mathbb{G})$. Thus, from Theorem 5.6 we get that there exists a solution $\Phi \in C_0(\mathbb{G}, H\mathbb{G})$ satisfying

$$\|\Phi\|_\infty \leq 2\|\Lambda(f)\|_{\mathbf{Ch}_0}.$$

Finally, (42) follows from Proposition 4.6. $\square$

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A. BALDI
Dipartimento di Matematica
Università di Bologna
e-mail: `annalisa.baldi2@unibo.it`

F. MONTEFALCONE
Dipartimento di Matematica "Tullio Levi-Civita"
Università di Padova
e-mail: `montefal@math.unipd.it`