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CORRIGENDUM AND ADDENDUM TO "POLARIZED PARALLEL TRANSPORT AND UNIRULED DIVISORS ON GENERALIZED KUMMER VARIETIES"

GIOVANNI MONGARDI AND GIANLUCA PACIENZA

Abstract. We correct the statement of the main result of [MP] and provide some further precisions.

The goal of this short note is to state correctly the main result of [MP]. For the definitions, the notations and the motivations we refer the reader to [MP]. The correct statement is the following:

Theorem 0.1. *Let $n \geq 1$ be an integer. Let $\mathfrak{M} = \cup_{d>0}\mathfrak{M}_{2d}$ be the union of the moduli spaces $\mathfrak{M}_{2d}$ of projective irreducible holomorphic symplectic varieties of $K_n(A)$-type polarized by a line bundle of degree $2d$. For all $(X, H) \in \mathfrak{M}$, outside at most a finite number of connected components determined by the monodromy orbit of H, the linear system $|mH|$, for some m, contains a uniruled divisor covered by rational curves of primitive class.*

Let q be the Beauville-Bogomolov quadratic form on $H^2(X,\mathbb{Z})$. This induces an embedding $H^2(X,\mathbb{Z}) \hookrightarrow H_2(X,\mathbb{Z}), H \mapsto H^\vee$. By abuse of notation we denote again by q the quadratic form on $H_2(X,\mathbb{Z})$.

Remark 0.2. The statement above insures precisely existence of uniruled divisors covered by primitive rational curves if there exist integers p, g and ϵ such that $p \geq g$ and $\epsilon = 0$ or 1 with

(i) the class $\alpha := \frac{H^\vee}{\mathrm{div}(H)} \in H_2(X,\mathbb{Z})$ can be written as $\gamma + (2g-\epsilon)\eta$ with η in the monodromy orbit of the class of the exceptional curve on a $K_n(A)$;
(ii) $\gamma \in \eta^\perp$, $q(\gamma) = 2p-2$ (hence $q(\alpha) = 2p - 2 - \frac{(2g-\epsilon)^2}{2n+2}$).

Remark 0.3.
(i) It follows from Proposition 2.1 that if $q(\alpha) > n+1$, then a multiple of H is uniruled by primitive rational curves of class α.
(ii) If $\rho(X) \geq 2$ then X always contains an ample uniruled divisor covered by primitive rational curves (cf. Corollary 2.3).
(iii) If $n \leq 5$ then the conclusion of the theorem holds for *all* the connected components of $\mathfrak{M}$ (cf. Remark 2.4).
(iv) If $n+1$ is a power of a prime number, then by [Mark, Mon2], the monodromy group is maximal. Therefore it suffices to check that the square $q(\alpha)$ is of the form $2p-2-\frac{(2g-\epsilon)^2}{2n+2}$, with $p \geq g$.

The original proof was based on 3 ingredients: the first was a deformation theoretic statement, saying that rational curves whose deformations cover a divisor in irreducible holomorphic symplectic manifolds are non-obstructed (see [CP, Corollary 3.5]). The second is the characterization of polarized parallel transport operators on polarized irreducible holomorphic symplectic varieties (X, H) of $K_n(A)$-type (see [MP, Theorem 1.1]) which allows to obtain an explicit description of

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the polarized deformation equivalence (see [MP, Theorem 4.2]). These two ingredients are true. The third argument consists in the construction of explicit examples of uniruled divisors on the generalized Kummer variety associated to a polarised abelian surface (A, H_A) with $NS(A) = \mathbb{Z}H_A$ such that $p_a(H_A) \geq g \geq 2$. The construction is also correct, but the examples that we provided cannot yield all the possible primitive polarizations, as we tacitely and erroneously assumed in [MP]. Even without taking the monodromy orbit into account, this is simply because it may happen that the number $2p - 2 - \frac{(2g-\epsilon)^2}{2n+2}, \epsilon = 0, 1$, is positive even with $p < g$, which obviously renders our geometric argument empty. Indeed the rational curves are constructed as $\mathfrak{g}^1_n$ on the normalization of a nodal curve of geometric genus g lying in the hyperplane linear system $|H_A|$, which is supposed to have $p_a(H_A) = p$. We also take the occasion of this note to provide the full proof (see Proposition 1.1) of a technical point which we claimed in [MP, Section 4.2] to follow from a dimension count as in [Voi15, Example 4.1, 3)]. The statement is correct, but the argument cannot be the same as in [Voi15, Example 4.1, 3)] because we deal here with a locally closed subset (the Severi variety) of a complete linear system, and not with the full complete linear system.

The $K3^{[n]}$-type case, initially treated in [CP], is subjet to the same considerations and will be treated in [CMP].

We realized our mistake after the appearance of [OSY], which provides counterexamples in the $K3^{[n]}$-case which apply exactly in all the cases not covered by the similar geometric constructions for the Hilbert scheme of points on a general projective $K3$.Contrary to the $K3^{[n]}$-type case as far as we know there are no known counterexamples to the existence of uniruled divisors ruled by a primitive curve class in the $K_n(A)$-type case. Nevertheless we have no reasons to believe that the $K_n(A)$-type case could be exempt from this type of sporadic pathologies.

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1. Existence of uniruled divisors on $K_n(A)$

In [MP, Section 4.2 "Examples"] we claimed that "the natural map from $\widetilde{\mathscr{C}}^1_{g+1} \to A^{[g+1]}$ is finite onto its image" invoking a dimension count made in [Voi15, Example 4.1, 3)]. However the same argument cannot work because we do not work with the full continuous system, but with a locally closed subset (the Severi variety). Hence we take the occasion to provide a full proof of that statement in the following.

Proposition 1.1. *Let g be an integer ≥ 2 and (A, H_A) be a general polarized abelian surface with $p_a(H_A) =: p \geq g$. Then $A^{[g+1]}$ contains a uniruled divisor covered by the $\mathfrak{g}^1_{g+1}$ on nodal genus g curves in the continuous system $\{H_A\}$.*

Proof. To prove the statement we can actually work over a very general polarized abelian surface, so let us suppose that $NS(A) = \mathbb{Z}H_A$. We will prove this statement by induction on g. It is sufficient to show it on the symmetric product of A.

Observe that, by [KLM, Thm. 1.1], for all $2 \leq g \leq p_a(H_A)$, that the Severi variety parametrizing nodal genus g curves inside $\{H_A\}$ is non-empty of the expected dimension g.

It is sufficient to show the claim on the symmetric product $A^{(g+1)}$ of A. More precisely, we will prove the following statement: there exists an irreducible component V of the (Zariski closure of the) Severi variety parametrizing nodal genus g curves inside $\{H_A\}$ such that, if $\mathscr{C}_V \to V$ denotes the universal curve and $\mathscr{C}_V^{(g+1)} \to V$ the relative symmetric product, the natural morphism

$$\mathscr{C}_V^{(g+1)} \to A^{(g+1)}$$

is generically finite onto its image. Note that this is equivalent to saying that $(g+1)$ generic points on a generic curve of the family lie only on a finite number of curves of the family.

Indeed as

$$\dim \mathscr{C}_V^{(g+1)} = \mathrm{reldim}(\mathscr{C}_V^{(g+1)}) + \dim V = (g+1) + g = 2g+1$$

it follows that the image is a divisor inside $A^{(g+1)}$. Since the k-th symmetric product of a curve is uniruled for k greater than the genus of the curve, as a by-product we have that such divisor is uniruled.

Note also that positive dimensional fibers of the morphism $\mathscr{C}_V^{(g+1)} \to A^{(g+1)}$ cannot lie in a fiber of $\mathscr{C}_V^{(g+1)} \to V$, as $C_t^{(g+1)}$ injects into $A^{(g+1)}$ for every $t \in V$.

We start with the case $g = 2$. Let C be one of the (finitely many) nodal curves of geometric genus 2 inside the linear system $|H_A|$. In this case the points of the component V of the Severi variety containing C are given by all the translates of C. The third symmetric product $C^{(3)}$ injects as a 3-dimensional subvariety inside $A^{(3)}$. The action of A on $C^{(3)}$ by translation has no positive-dimensional stabilizer (as A is general, hence simple). Therefore the orbit of $C^{(3)}$ under this action is a divisor. Using the notation above such divisor is the image of $\mathscr{C}_V^{(2+1)}$ in $A^{(3)}$.

By inductive hypothesis, there exists an irreducible component W of the (Zariski closure of the) Severi variety parametrizing nodal genus $g-1$ curves inside $\{H_A\}$ such that, if $\mathscr{C}_W \to W$ denotes the universal curve and $\mathscr{C}_W^{(g)} \to W$ the relative symmetric product, the natural morphism

$$\mathscr{C}_W^{(g)} \to A^{(g)}$$

is generically finite onto its image.

Now let V be (the Zariski closure of) an irreducible component of the Severi variety of nodal genus g curves in $\{H_A\}$ obtained by smoothing one node of the curves in W (which can be done by the regularity of the Severi variety, [CS, Example 1.3]). By construction $W \subset V$. Let $\mathscr{C}_V \to V$ be the universal curve. Its restriction over W yields a map $\mathscr{C}_W \to W$. Let D be the image of the morphism

$$\mathscr{C}_V^{(g+1)} \to A^{(g+1)}.$$

Observe that D contains the image D_W of

$$\mathscr{C}_W^{(g+1)} \to A^{(g+1)}.$$

We claim that by the inductive hypothesis D_W has codimension 2, or, equivalenty, that the morphism $\mathscr{C}_W^{(g+1)} \to A^{(g+1)}$ is generically finite onto its image. Indeed if $\xi = x_1 + \ldots + x_{g+1}$ is a generic point of the image, then, say, $x_1 + \ldots + x_g$ is a generic point of the image of the morphism $\mathscr{C}_W^{(g)} \to A^{(g)}$. By the inductive hypothesis the points $x_1, \ldots, x_g$ lie on finitely many curves of the family W, *a fortiori* that will be true for $x_1, \ldots, x_g, x_{g+1}$ and the claim follows.

We want to prove that D contains D_W strictly. If this were not the case, by irreducibility, we would have $D = D_W$. Let $U \subset D$ be an open subset over which the morphisms $\mathscr{C}_W^{(g+1)} \to A^{(g+1)}$ and $\mathscr{C}_V^{(g+1)} \to A^{(g+1)}$ are smooth and let $p_1 + p_2 + \cdots + p_{g+1}$ be a point in U. Let C be a nodal genus g curve in V containing these points. Let us fix the first g points $p_1, \ldots, p_g$. By induction these points are contained inside a finite number of curves of genus $g-1$ belonging to W. Let

$B_1, \dots, B_m$ be all such curves. Let $U_C \subset C$ be an open subset such that for all $q \in U_C$ we have $p_1 + \dots + p_g + q \in U$. As we have seen above $p_1, \dots, p_g, q$ lie on finitely many curves of genus $g-1$ belonging to W, and these curves must be $B_1, \dots, B_m$. Therefore, as q varies in U_C, we deduce that U_C is a subset of a finite union of genus $(g-1)$ curves. As C is irreducible, there is an i such that $C = B_i$, which is clearly a contradiction. Therefore D must strictly contain D_W and be a divisor, which is necessarily uniruled.

□

The rest of the proof remains the same and we refer the reader to [MP] for the details.

2. Where it does not work

In this section we prove that, for every dimension, there is at most a finite number of components of the moduli space of polarized manifolds (X, H) of $K_n(A)$-type where the strategy of the previous section does not work.

The uniruled divisors we constructed have a cohomology class which is a multiple of $H_A - (2g)\tau$ (or $H_A - (2g-1)\tau$) where $2p - 2 = H_A^2$ and H_A is the primitive polarization on the abelian surface. We have the following:

Proposition 2.1. *Let X be a projective irreducible holomorphic symplectic variety of $K_n(A)$-type. Let $C \in H_2(X, \mathbb{Z}) \cap N_1(X)$ be a primitive class such that its square $q(C)$ with respect to the Beauville-Bogomolov form is $> n+1$. Then, the class C is deformation equivalent to the class of one of the curves constructed in the previous section.*

Proof. We know by [MP, Theorem 4.2] that C is deformation equivalent to either $H_A - 2g\tau$ or $H_A - (2g-1)\tau$, with $g \leq n+1$. If $q(C) > n+1$, the square of $H_A - (2n+2)\tau$ is positive, that is $H_A^2 = 2p - 2$ with $p > n+1$. Thus, $p > n+1 \geq g$ which means that $H_A - 2g\tau$ can be represented by the class of a $\mathfrak{g}^1_{g+1}$ on a nodal curve in $\{H_A\}$. □

Corollary 2.2. *Let $\mathscr{M}_n$ be the moduli space of all polarized manifolds of $K_n(A)$-type with n fixed. Then, the number of components of $\mathscr{M}_n$ whose general points (X, H) do not have a uniruled divisor ruled by a rational curve of primitive class is at most finite.*

Proof. The components of $\mathscr{M}_n$ are in bijective correspondence with the monodromy orbits of a given class of positive square in $L_n := U^3 \oplus (-2n-2) \cong H^2(X, \mathbb{Z})$, see [Ono, Thm. 2.8].

For a fixed square of H, there is a finite number of orbits (computed again in [Ono, Thm. 2.8]), so it follows that if X has a uniruled divisor when $q(H)$ is big enough, our claim will hold. The dual curve to H is given by $H/div(H)$, where $div(H)$ is the divisibility of H which is the positive generator of the ideal $q(H, H^2(X, \mathbb{Z}))$. The divisibility is at most $2n+2$, therefore if $q(H) \geq (2n+2)^2(n+1)$ the dual curve has square at least $n+1$, so that Proposition 2.1 applies and our claim follows. □

Corollary 2.3. *Let X be a projective manifold of $K_n(A)$-type with Picard rank at least two. Then X has an ample divisor ruled by primitive rational curves.*

Proof. Since X is projective and has Picard rank at least two, its Picard lattice is indefinite and contains primitive elements of positive arbitrary Beauville-Bogomolov square, and so does the ample cone. Let H be an ample divisor such that $q(H) \geq (2n+2)^2(n+1)$. Let C be its dual curve in $H_2(X, \mathbb{Z})$. As the divisibility of H is at most $2n+2$ it follows that $q(C) \geq n+1$ and Proposition 2.1 yields our claim. □

Remark 2.4. The estimate of Proposition 2.1 is definitely not sharp, indeed all primitive curves of positive square on manifolds of $K_n(A)$-type with $n \leq 5$ are deformation equivalent to the curves we construct in Proposition 1.1. Indeed, by [MP, Theorem 4.2] we can suppose that our pair (X, C) with $q(C) > 0$ is $(K_n(A), H_A - \mu\tau)$ with $0 \leq \mu \leq n+1$ and A is an abelian surface of genus p. The class $H_A - \mu\tau$ is given by the class of the rational curves constructed in Proposition 1.1, which have class $H_A - 2g\tau$, with the eventual addition of a tail of class τ, so that $2g \leq n+2$. By contradiction let us suppose that $g > p$ and $n \leq 5$. We have $q(H_A - 2g\tau) = 2p - 2 - 2\frac{g^2}{n+1} \leq 2p - 2 - 2\frac{(p+1)^2}{n+1} \leq 2p - 2 - 2\frac{(p+1)^2}{6}$. However, the last value is never positive, hence $q(H_A - 2g\tau)$ cannot be positive and we reach a contradiction. Analogously, for $C = H_A - (2g-1)\tau$, we have $q(C) \leq \frac{20p - 25 - 4p^2}{12}$ with $g \geq p+1$ and $2g \leq n+2$, which is again not positive.

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Alma Mater studiorum Universitá di Bologna Dipartimento di Matematica, Piazza di Porta San Donato 5, Bologna, 40126 Italia

Email address: `giovanni.mongardi2@unibo.it`

Institut Elie Cartan de Lorraine, Université de Lorraine, B.P. 70239, F-54506 Vandoeuvre-lés-Nancy Cedex France

Email address: `gianluca.pacienza@univ-lorraine.fr`