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# CORRIGENDUM AND ADDENDUM TO “POLARIZED PARALLEL TRANSPORT AND UNIRULED DIVISORS ON GENERALIZED KUMMER VARIETIES”

GIOVANNI MONGARDI AND GIANLUCA PACIENZA

ABSTRACT. We correct the statement of the main result of [MP] and provide some further pre-  
cisions.

The goal of this short note is to state correctly the main result of [MP]. For the definitions, the notations and the motivations we refer the reader to [MP]. The correct statement is the following:

**Theorem 0.1.** *Let  $n \geq 1$  be an integer. Let  $\mathfrak{M} = \cup_{d>0} \mathfrak{M}_{2d}$  be the union of the moduli spaces  $\mathfrak{M}_{2d}$  of projective irreducible holomorphic symplectic varieties of  $K_n(A)$ -type polarized by a line bundle of degree  $2d$ . For all  $(X, H) \in \mathfrak{M}$ , outside at most a finite number of connected components determined by the monodromy orbit of  $H$ , the linear system  $|mH|$ , for some  $m$ , contains a uniruled divisor covered by rational curves of primitive class.*

Let  $q$  be the Beauville-Bogomolov quadratic form on  $H^2(X, \mathbb{Z})$ . This induces an embedding  $H^2(X, \mathbb{Z}) \hookrightarrow H_2(X, \mathbb{Z}), H \mapsto H^\vee$ . By abuse of notation we denote again by  $q$  the quadratic form on  $H_2(X, \mathbb{Z})$ .

**Remark 0.2.** The statement above insures precisely existence of uniruled divisors covered by primitive rational curves if there exist integers  $p, g$  and  $\epsilon$  such that  $p \geq g$  and  $\epsilon = 0$  or  $1$  with

- (i) the class  $\alpha := \frac{H^\vee}{\text{div}(H)} \in H_2(X, \mathbb{Z})$  can be written as  $\gamma + (2g - \epsilon)\eta$  with  $\eta$  in the monodromy orbit of the class of the exceptional curve on a  $K_n(A)$ ;
- (ii)  $\gamma \in \eta^\perp$ ,  $q(\gamma) = 2p - 2$  (hence  $q(\alpha) = 2p - 2 - \frac{(2g-\epsilon)^2}{2n+2}$ ).

**Remark 0.3.** (i) It follows from Proposition 2.1 that if  $q(\alpha) > n + 1$ , then a multiple of  $H$  is uniruled by primitive rational curves of class  $\alpha$ .  
(ii) If  $\rho(X) \geq 2$  then  $X$  always contains an ample uniruled divisor covered by primitive rational curves (cf. Corollary 2.3).  
(iii) If  $n \leq 5$  then the conclusion of the theorem holds for *all* the connected components of  $\mathfrak{M}$  (cf. Remark 2.4).  
(iv) If  $n + 1$  is a power of a prime number, then by [Mark, Mon2], the monodromy group is maximal. Therefore it suffices to check that the square  $q(\alpha)$  is of the form  $2p - 2 - \frac{(2g-\epsilon)^2}{2n+2}$ , with  $p \geq g$ .

The original proof was based on 3 ingredients: the first was a deformation theoretic statement, saying that rational curves whose deformations cover a divisor in irreducible holomorphic symplectic manifolds are non-obstructed (see [CP, Corollary 3.5]). The second is the characterization of polarized parallel transport operators on polarized irreducible holomorphic symplectic varieties  $(X, H)$  of  $K_n(A)$ -type (see [MP, Theorem 1.1]) which allows to obtain an explicit description of

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the polarized deformation equivalence (see [MP, Theorem 4.2]). These two ingredients are true. The third argument consists in the construction of explicit examples of uniruled divisors on the generalized Kummer variety associated to a polarised abelian surface  $(A, H_A)$  with  $NS(A) = \mathbb{Z}H_A$  such that  $p_a(H_A) \geq g \geq 2$ . The construction is also correct, but the examples that we provided cannot yield all the possible primitive polarizations, as we tacitly and erroneously assumed in [MP]. Even without taking the monodromy orbit into account, this is simply because it may happen that the number  $2p - 2 - \frac{(2g-\epsilon)^2}{2n+2}, \epsilon = 0, 1$ , is positive even with  $p < g$ , which obviously renders our geometric argument empty. Indeed the rational curves are constructed as  $\mathfrak{g}_n^1$  on the normalization of a nodal curve of geometric genus  $g$  lying in the hyperplane linear system  $|H_A|$ , which is supposed to have  $p_a(H_A) = p$ . We also take the occasion of this note to provide the full proof (see Proposition 1.1) of a technical point which we claimed in [MP, Section 4.2] to follow from a dimension count as in [Voi15, Example 4.1, 3]). The statement is correct, but the argument cannot be the same as in [Voi15, Example 4.1, 3]) because we deal here with a locally closed subset (the Severi variety) of a complete linear system, and not with the full complete linear system.

The  $K3^{[n]}$ -type case, initially treated in [CP], is subject to the same considerations and will be treated in [CMP].

We realized our mistake after the appearance of [OSY], which provides counterexamples in the  $K3^{[n]}$ -case which apply exactly in all the cases not covered by the similar geometric constructions for the Hilbert scheme of points on a general projective  $K3$ . Contrary to the  $K3^{[n]}$ -type case as far as we know there are no known counterexamples to the existence of uniruled divisors ruled by a primitive curve class in the  $K_n(A)$ -type case. Nevertheless we have no reasons to believe that the  $K_n(A)$ -type case could be exempt from this type of sporadic pathologies.

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#### 1. EXISTENCE OF UNIRULED DIVISORS ON $K_n(A)$

In [MP, Section 4.2 “Examples”] we claimed that “the natural map from  $\widetilde{\mathcal{C}}_{g+1}^1 \rightarrow A^{[g+1]}$  is finite onto its image” invoking a dimension count made in [Voi15, Example 4.1, 3]). However the same argument cannot work because we do not work with the full continuous system, but with a locally closed subset (the Severi variety). Hence we take the occasion to provide a full proof of that statement in the following.

**Proposition 1.1.** *Let  $g$  be an integer  $\geq 2$  and  $(A, H_A)$  be a general polarized abelian surface with  $p_a(H_A) =: p \geq g$ . Then  $A^{[g+1]}$  contains a uniruled divisor covered by the  $\mathfrak{g}_{g+1}^1$  on nodal genus  $g$  curves in the continuous system  $\{H_A\}$ .*

*Proof.* To prove the statement we can actually work over a very general polarized abelian surface, so let us suppose that  $NS(A) = \mathbb{Z}H_A$ . We will prove this statement by induction on  $g$ . It is sufficient to show it on the symmetric product of  $A$ .

Observe that, by [KLM, Thm. 1.1], for all  $2 \leq g \leq p_a(H_A)$ , that the Severi variety parametrizing nodal genus  $g$  curves inside  $\{H_A\}$  is non-empty of the expected dimension  $g$ .

It is sufficient to show the claim on the symmetric product  $A^{(g+1)}$  of  $A$ . More precisely, we will prove the following statement: there exists an irreducible component  $V$  of the (Zariski closure of the) Severi variety parametrizing nodal genus  $g$  curves inside  $\{H_A\}$  such that, if  $\mathcal{C}_V \rightarrow V$  denotes the universal curve and  $\mathcal{C}_V^{(g+1)} \rightarrow V$  the relative symmetric product, the natural morphism

$$\mathcal{C}_V^{(g+1)} \rightarrow A^{(g+1)}$$

is generically finite onto its image. Note that this is equivalent to saying that  $(g+1)$  generic points on a generic curve of the family lie only on a finite number of curves of the family.

Indeed as

$$\dim \mathcal{C}_V^{(g+1)} = \text{reldim}(\mathcal{C}_V^{(g+1)}) + \dim V = (g+1) + g = 2g+1$$

it follows that the image is a divisor inside  $A^{(g+1)}$ . Since the  $k$ -th symmetric product of a curve is uniruled for  $k$  greater than the genus of the curve, as a by-product we have that such divisor is uniruled.

Note also that positive dimensional fibers of the morphism  $\mathcal{C}_V^{(g+1)} \rightarrow A^{(g+1)}$  cannot lie in a fiber of  $\mathcal{C}_V^{(g+1)} \rightarrow V$ , as  $C_t^{(g+1)}$  injects into  $A^{(g+1)}$  for every  $t \in V$ .

We start with the case  $g = 2$ . Let  $C$  be one of the (finitely many) nodal curves of geometric genus 2 inside the linear system  $|H_A|$ . In this case the points of the component  $V$  of the Severi variety containing  $C$  are given by all the translates of  $C$ . The third symmetric product  $C^{(3)}$  injects as a 3-dimensional subvariety inside  $A^{(3)}$ . The action of  $A$  on  $C^{(3)}$  by translation has no positive-dimensional stabilizer (as  $A$  is general, hence simple). Therefore the orbit of  $C^{(3)}$  under this action is a divisor. Using the notation above such divisor is the image of  $\mathcal{C}_V^{(2+1)}$  in  $A^{(3)}$ .

By inductive hypothesis, there exists an irreducible component  $W$  of the (Zariski closure of the) Severi variety parametrizing nodal genus  $g-1$  curves inside  $\{H_A\}$  such that, if  $\mathcal{C}_W \rightarrow W$  denotes the universal curve and  $\mathcal{C}_W^{(g)} \rightarrow W$  the relative symmetric product, the natural morphism

$$\mathcal{C}_W^{(g)} \rightarrow A^{(g)}$$

is generically finite onto its image.

Now let  $V$  be (the Zariski closure of) an irreducible component of the Severi variety of nodal genus  $g$  curves in  $\{H_A\}$  obtained by smoothing one node of the curves in  $W$  (which can be done by the regularity of the Severi variety, [CS, Example 1.3]). By construction  $W \subset V$ . Let  $\mathcal{C}_V \rightarrow V$  be the universal curve. Its restriction over  $W$  yields a map  $\mathcal{C}_W \rightarrow W$ . Let  $D$  be the image of the morphism

$$\mathcal{C}_V^{(g+1)} \rightarrow A^{(g+1)}.$$

Observe that  $D$  contains the image  $D_W$  of

$$\mathcal{C}_W^{(g+1)} \rightarrow A^{(g+1)}.$$

We claim that by the inductive hypothesis  $D_W$  has codimension 2, or, equivalently, that the morphism  $\mathcal{C}_W^{(g+1)} \rightarrow A^{(g+1)}$  is generically finite onto its image. Indeed if  $\xi = x_1 + \dots + x_{g+1}$  is a generic point of the image, then, say,  $x_1 + \dots + x_g$  is a generic point of the image of the morphism  $\mathcal{C}_W^{(g)} \rightarrow A^{(g)}$ . By the inductive hypothesis the points  $x_1, \dots, x_g$  lie on finitely many curves of the family  $W$ , *a fortiori* that will be true for  $x_1, \dots, x_g, x_{g+1}$  and the claim follows.

We want to prove that  $D$  contains  $D_W$  strictly. If this were not the case, by irreducibility, we would have  $D = D_W$ . Let  $U \subset D$  be an open subset over which the morphisms  $\mathcal{C}_W^{(g+1)} \rightarrow A^{(g+1)}$  and  $\mathcal{C}_V^{(g+1)} \rightarrow A^{(g+1)}$  are smooth and let  $p_1 + p_2 + \dots + p_{g+1}$  be a point in  $U$ . Let  $C$  be a nodal genus  $g$  curve in  $V$  containing these points. Let us fix the first  $g$  points  $p_1, \dots, p_g$ . By induction these points are contained inside a finite number of curves of genus  $g-1$  belonging to  $W$ . Let

$B_1, \dots, B_m$  be all such curves. Let  $U_C \subset C$  be an open subset such that for all  $q \in U_C$  we have  $p_1 + \dots + p_g + q \in U$ . As we have seen above  $p_1, \dots, p_g, q$  lie on finitely many curves of genus  $g - 1$  belonging to  $W$ , and these curves must be  $B_1, \dots, B_m$ . Therefore, as  $q$  varies in  $U_C$ , we deduce that  $U_C$  is a subset of a finite union of genus  $(g - 1)$  curves. As  $C$  is irreducible, there is an  $i$  such that  $C = B_i$ , which is clearly a contradiction. Therefore  $D$  must strictly contain  $D_W$  and be a divisor, which is necessarily uniruled.

□

The rest of the proof remains the same and we refer the reader to [MP] for the details.

## 2. WHERE IT DOES NOT WORK

In this section we prove that, for every dimension, there is at most a finite number of components of the moduli space of polarized manifolds  $(X, H)$  of  $K_n(A)$ -type where the strategy of the previous section does not work.

The uniruled divisors we constructed have a cohomology class which is a multiple of  $H_A - (2g)\tau$  (or  $H_A - (2g - 1)\tau$ ) where  $2p - 2 = H_A^2$  and  $H_A$  is the primitive polarization on the abelian surface. We have the following:

**Proposition 2.1.** *Let  $X$  be a projective irreducible holomorphic symplectic variety of  $K_n(A)$ -type. Let  $C \in H_2(X, \mathbb{Z}) \cap N_1(X)$  be a primitive class such that its square  $q(C)$  with respect to the Beauville-Bogomolov form is  $> n + 1$ . Then, the class  $C$  is deformation equivalent to the class of one of the curves constructed in the previous section.*

*Proof.* We know by [MP, Theorem 4.2] that  $C$  is deformation equivalent to either  $H_A - 2g\tau$  or  $H_A - (2g - 1)\tau$ , with  $g \leq n + 1$ . If  $q(C) > n + 1$ , the square of  $H_A - (2n + 2)\tau$  is positive, that is  $H_A^2 = 2p - 2$  with  $p > n + 1$ . Thus,  $p > n + 1 \geq g$  which means that  $H_A - 2g\tau$  can be represented by the class of a  $\mathfrak{g}_{g+1}^1$  on a nodal curve in  $\{H_A\}$ . □

**Corollary 2.2.** *Let  $\mathcal{M}_n$  be the moduli space of all polarized manifolds of  $K_n(A)$ -type with  $n$  fixed. Then, the number of components of  $\mathcal{M}_n$  whose general points  $(X, H)$  do not have a uniruled divisor ruled by a rational curve of primitive class is at most finite.*

*Proof.* The components of  $\mathcal{M}_n$  are in bijective correspondence with the monodromy orbits of a given class of positive square in  $L_n := U^3 \oplus (-2n - 2) \cong H^2(X, \mathbb{Z})$ , see [Ono, Thm. 2.8].

For a fixed square of  $H$ , there is a finite number of orbits (computed again in [Ono, Thm. 2.8]), so it follows that if  $X$  has a uniruled divisor when  $q(H)$  is big enough, our claim will hold. The dual curve to  $H$  is given by  $H/\text{div}(H)$ , where  $\text{div}(H)$  is the divisibility of  $H$  which is the positive generator of the ideal  $q(H, H^2(X, \mathbb{Z}))$ . The divisibility is at most  $2n + 2$ , therefore if  $q(H) \geq (2n + 2)^2(n + 1)$  the dual curve has square at least  $n + 1$ , so that Proposition 2.1 applies and our claim follows. □

**Corollary 2.3.** *Let  $X$  be a projective manifold of  $K_n(A)$ -type with Picard rank at least two. Then  $X$  has an ample divisor ruled by primitive rational curves.*

*Proof.* Since  $X$  is projective and has Picard rank at least two, its Picard lattice is indefinite and contains primitive elements of positive arbitrary Beauville-Bogomolov square, and so does the ample cone. Let  $H$  be an ample divisor such that  $q(H) \geq (2n + 2)^2(n + 1)$ . Let  $C$  be its dual curve in  $H_2(X, \mathbb{Z})$ . As the divisibility of  $H$  is at most  $2n + 2$  it follows that  $q(C) \geq n + 1$  and Proposition 2.1 yields our claim. □

**Remark 2.4.** The estimate of Proposition 2.1 is definitely not sharp, indeed all primitive curves of positive square on manifolds of  $K_n(A)$ -type with  $n \leq 5$  are deformation equivalent to the curves we construct in Proposition 1.1. Indeed, by [MP, Theorem 4.2] we can suppose that our pair  $(X, C)$  with  $q(C) > 0$  is  $(K_n(A), H_A - \mu\tau)$  with  $0 \leq \mu \leq n + 1$  and  $A$  is an abelian surface of genus  $p$ . The class  $H_A - \mu\tau$  is given by the class of the rational curves constructed in Proposition 1.1, which have class  $H_A - 2g\tau$ , with the eventual addition of a tail of class  $\tau$ , so that  $2g \leq n + 2$ . By contradiction let us suppose that  $g > p$  and  $n \leq 5$ . We have  $q(H_A - 2g\tau) = 2p - 2 - 2\frac{g^2}{n+1} \leq 2p - 2 - 2\frac{(p+1)^2}{n+1} \leq 2p - 2 - 2\frac{(p+1)^2}{6}$ . However, the last value is never positive, hence  $q(H_A - 2g\tau)$  cannot be positive and we reach a contradiction. Analogously, for  $C = H_A - (2g - 1)\tau$ , we have  $q(C) \leq \frac{20p - 25 - 4p^2}{12}$  with  $g \geq p + 1$  and  $2g \leq n + 2$ , which is again not positive.

## REFERENCES

- [CP] F. Charles, G. Pacienza, Families of rational curves on holomorphic symplectic varieties, [arXiv:1401.4071v2](#) [[math.AG](#)].
- [CMP] F. Charles, G. Mongardi, G. Pacienza, Families of rational curves on holomorphic symplectic varieties of  $K3^{[n]}$ -type, forthcoming.
- [CS] L. Chiantini, E. Sernesi, Nodal curves on surfaces of general type, *Math. Annalen* 307 (1997), 41-56.
- [KLM] A. L. Knutsen, M. Lelli-Chiesa, and G. Mongardi, Severi varieties and Brill-Noether theory of curves on abelian surfaces, *Journal für die reine und angewandte Mathematik (Crelles Journal)*, Vol. 749 (2019), Pages 161–200.
- [KLM2] A.-L. Knutsen, M. Lelli-Chiesa, G. Mongardi, Wall divisors and algebraically coisotropic subvarieties of irreducible holomorphic symplectic manifolds, *Trans. Amer. Math. Soc.* vol. 371 (2019) no. 2, 1403–1438.
- [LS] H. Lange, E. Sernesi, Severi varieties and branch curves of abelian surfaces of type (1,3), *International J. of Math.* 13 (2002), 227-244.
- [Mon2] G. Mongardi, On the monodromy of irreducible symplectic manifolds, *Alg. Geom.* (2016) no. 3, 385–391.
- [Mark] E. Markman, The monodromy of generalized Kummer varieties and algebraic cycles on their intermediate Jacobians, [arXiv:1805.11574](#).
- [MP] G. Mongardi, G. Pacienza, Polarized parallel transport and uniruled divisors on deformations of generalized Kummer varieties, *International Mathematics Research Notices* Volume 2018 (2018), no. 11, Pages 3606–3620.
- [OSY] G. Oberdieck, J. Shen and Q. Yin, Rational curves in the Fano varieties of cubic 4-folds and Gromov-Witten invariants, [arXiv:1805.07001v1](#).
- [Ono] C. Onorati, Connected components of moduli spaces of generalized Kummer varieties, [arXiv:1608.06465](#)
- [Voi15] C. Voisin, Remarks and questions on coisotropic subvarieties and 0-cycles of hyper-Kähler varieties, “K3 Surfaces and Their Moduli”, *Progress in Math*, Proceedings of the Schiermonnikoog conference 2014, Birkhäuser.

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