N = 2 quantum chiral superfields and quantum super bundles

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$N = 2$ quantum chiral superfields and quantum super bundles

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Abstract

We give the superalgebra of $N = 2$ chiral (and antichiral) quantum superfields realized as a subalgebra of the quantum supergroup $\mathrm{SL}_q(4|2)$. The multiplication law in the quantum supergroup induces a coaction on the set of chiral superfields. We also realize the quantum deformation of the chiral Minkowski superspace as a quantum principal bundle.

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1 Introduction

It is well known that the $N = 1$ superconformal superspace, in its complexified version [1, 2], is the superflag $Fl(2|0, 2|1, 4|1)$, on which the conformal supergroup $SL(4|1)$ acts naturally. The space $\mathbb{C}^{4|1}$, underlying the defining representation of $SL(4|1)$, is the space of supertwistors [3, 4].

Dealing with the complexified version has the advantage of seeing this structure, while the conditions for the real form can be imposed later on [2]. It is also well known, and differently from the non super case, that not all the superflags are projective superspaces (take for example the super Grassmannian $Gr(1|1, 2|2)$ [1]) and indeed the projective cases are rare among these superspaces, though a new approach to this question was taken in [5]. For the super Grassmannians only the extreme cases $Gr(p|0, m|n)$ or $Gr(p|n, m|n)$ are superprojective and are both embedded into the projective superspace $\mathbb{P}^{M|N}$ for suitable M and N see [8]. These super Grassmannians are dual to each other and are the antichiral and chiral superspaces respectively.

The superflag $Fl(2|0, 2|1, 4|1)$ can be embedded in the product

$$Fl(2|0, 2|1, 4|1) \subset Gr(2|0, 4|1) \times Gr(2|1, 4|1),$$

and using the super Segre embedding [6] the superflag is embedded into the projective superspace $\mathbb{P}^{80|64}$ [7, 8].

For $N = 2$ we can reproduce the same situation with

$$Fl(2|0, 2|2, 4|2) \subset Gr(2|0, 4|2) \times Gr(2|2, 4|2),$$

but this superflag is too big. The scalar superfields associated to it have too many field components to be useful in the formulation of supersymmetric field theories. Still, the antichiral $Gr(2|0, 4|2)$ and chiral $Gr(2|2, 4|2)$ superspaces do have physical applications so it is useful to study them. They are both embedded in $\mathbb{P}^{8|8}$.

There is a third super Grassmannian, $Gr(2|1, 4|2)$, which is not projective but that has physical applications. It is the harmonic superspace of [9, 10]. This example, together with more general ones, were studied from this point of view in the series of papers [11, 12, 13, 14, 15, 16, 17].

Here we will consider only the (anti)chiral superspace (also considered in [18]). Our aim is to quantize it by substituting the supergroup $SL(4|2)$ by the quantum group $SL_q(4|2)$ (in the sense of Manin [19]) and trying to define appropriately the quantum super Grassmannian as an homogeneous

superspace. This was done for $N = 1$ in [7, 8]. As we will see, the $N = 2$ case has its own peculiarities.

This program could, in principle be proposed for general homogeneous superspaces, not necessarily superprojective. But the projectivity gives us an advantage: the algebra associated to the projective embedding (super Plücker embedding) can be seen both, as a quotient algebra of the projective superspace $\mathbb{P}^{8|8}$ modulo some homogeneous polynomial relations (super Plücker relations) as well as a graded subring of the superring $\mathbb{C}[\mathrm{SL}(4|2)]$, encoding its projective embedding (see also [20, 21, 22]). We will see this in detail in Section 3. One can then define a quantum super Grassmannian as a certain subalgebra of the super Hopf algebra $\mathrm{SL}_q(4|2)$. If done correctly, the subalgebra must represent a quantum homogeneous superspace for $\mathrm{SL}_q(4|2)$, that is, the coproduct in $\mathrm{SL}_q(4|2)$ induces a coaction on the quantum super Grassmannian.

The chiral Minkowski $N = 2$ superspace $\mathbb{M}$ emerges naturally in this context as the big cell in the Grassmannian $Gr(2|0, 4|2)$. The $N = 1$ case was extensively studied in [8], Chapter 4. However, as remarked above, the $N = 2$ SUSY has its own peculiarities, which make the theory richer. We view the big cell in $Gr(2|0, 4|2)$ as the subsupermanifold containing certain $2|0$ subspaces and we realize it as the set S of pairs of vectors in $\mathbb{C}^{4|2}$ modulo the natural right $\mathrm{GL}(2)$ action, which accounts for basis change. Hence, we construct $\mathbb{M}$ as the quotient of S modulo the ordinary general linear group $\mathrm{GL}(2)$. The quantization of $\mathbb{M}$ is obtained, as expected, as the subsuperring of a localization of $\mathrm{SL}_q(4|2)$, generated by the quantum coinvariants with respect to the coaction of quantum $\mathrm{GL}_q(2)$ (see [8], Chapter 4 for the $N = 1$ case). The presentation of this quantum superring via generators and relations, makes an essential use of the commutation relations among the quantum determinants appearing in the definition of the quantum $Gr(2|0, 4|2)$ and the Plücker relations. Moreover, the quantum Minkowski space, $\mathbb{M}_q$, is isomorphic to the quantum Manin superalgebra, that is, the quantum super bialgebra of matrices, as described in [19]. This fact is highly non obvious, it depends on the quite involved commutation relations of quantum determinants and it shows how this framework is natural and suitable for more exploration, as we detail below.

The chiral Minkowski $N = 2$ superspace, being a quotient, appears then naturally also as a principal bundle for the action of $\mathrm{GL}(2)$. There is an extensive literature regarding the quantization of principal bundles (see [22,

23, 24, 25, 26] and references therein). In particular the notion of Hopf-Galois extension [27] appears to be the right one to formulate, in the affine setting, the theory of principal bundles to obtain their quantum deformations.

We hence proceed to define Hopf-Galois extensions in the SUSY framework and prove that the chiral Minkowski $N = 2$ superspace $\mathbb{M}$ is the base for a principal bundle S for the supergroup GL(2), by realizing it as a trivial Hopf-Galois extension (see also [28] for a more geometric, yet equivalent, view on super principal bundles). Next, we construct a quantum deformation $\mathbb{M}_q$ of $\mathbb{M}$, by taking advantage of our previous realization and show that $\mathbb{M}_q$ is the quantum space, base for the quantum principal bundle S_q, for $\mathrm{GL}_q(2)$.

We plan to explore, in a forthcoming paper, the construction of covariant differential calculi on the quantum chiral Minkowski $N = 2$ superspace and then proceed towards the realization of a theory in a curved background.

The paper is organized as follows.

In Section 2 we describe the super Plücker embedding of the super Grassmannian and its presentation in terms of generators and relations.

In Section 3 we give the classical super Grassmannian as a subalgebra of the coordinate superalgebra of SL(4|2).

In Section 4 we briefly describe the big cell of the super Grassmannian, the $N = 2$, $D = 4$ Minkowski superspace.

In Section 5 we pass to define the super Grassmannian as a subsuperalgebra of $\mathrm{SL}_q(4|2)$, computing the commutation relations of the generators and the quantum super Plücker relations that they satisfy.

In Section 6 we give the coaction of $\mathrm{SL}_q(4|2)$ on the quantum super Grassmannian defined in the previous section, proving that it is a quantum homogeneous superspace.

In the last section, Section 7, we construct the $N = 2$ ordinary chiral Minkowski superspace $\mathbb{M}$ and its quantization $\mathbb{M}_q$, realized both first as homogeneous spaces for the action of the ordinary (quantum) general linear group in dimension 2, then as bases for (quantum) principal bundles for GL(2) and $\mathrm{GL}_q(2)$ respectively.

2 The super Plücker embedding

We are going to give the embedding of $Gr(2|0, 4|2)$ in the projective superspace $\mathbb{P}^{8|8}$. Let $E = \bigwedge^2 \mathbb{C}^{4|2}$ and $\{e_1, \ldots, e_4, \epsilon_5, \epsilon_6\}$ an homogeneous basis for

$\mathbb{C}^{4|2}$, we then have a basis for E as

$$e_i \wedge e_j \quad 1 \leq i < j \leq 4, \quad \epsilon_5 \wedge \epsilon_5, \quad \epsilon_6 \wedge \epsilon_6, \quad \epsilon_5 \wedge \epsilon_6, \qquad \text{(even)},$$
$$e_k \wedge \epsilon_5, \quad e_k \wedge \epsilon_6 \quad 1 \leq k \leq 4, \qquad\qquad\qquad\qquad \text{(odd)}.$$

So $E \simeq \mathbb{C}^{9|8}$ and $\mathbb{P}(E) \simeq \mathbb{P}^{8|8}$. An element of E is given as

$$Q = q + \lambda_5 \wedge \epsilon_5 + \lambda_6 \wedge \epsilon_6 + a_{55}\epsilon_5 \wedge \epsilon_5 + a_{66}\epsilon_6 \wedge \epsilon_6 + a_{56}\epsilon_5 \wedge \epsilon_6 \,,$$

with

$$q = q_{ij}e_i \wedge e_j, \quad \lambda_m = \lambda_{im}e_i, \qquad i,j = 1,\ldots,4, \quad m = 5,6\,.$$

The element q is decomposable if $q = a \wedge b$, where

$$a = r + \xi_5\epsilon_5 + \xi_6\epsilon_6, \qquad b = s + \eta_5\epsilon_5 + \eta_6\epsilon_6,$$

with $r = r_i e_i$, $s = s_i e_i$.

One obtains the following relations

$$q = r \wedge s$$
$$\lambda_5 = \xi_5 s - \eta_5 r, \qquad\qquad \lambda_6 = \xi_6 s - \eta_6 r,$$
$$a_{55} = \xi_5\eta_5, \qquad\qquad a_{66} = \xi_6\eta_6, \qquad\qquad a_{56} = \xi_5\eta_6 + \xi_6\eta_5\,, \quad (1)$$

which imply

$$q \wedge q = 0,$$
$$q \wedge \lambda_5 = 0, \qquad\qquad q \wedge \lambda_6 = 0,$$
$$\lambda_5 \wedge \lambda_5 = -2a_{55}q, \qquad \lambda_6 \wedge \lambda_6 = -2a_{66}q, \qquad \lambda_5 \wedge \lambda_6 = -a_{56}q,$$
$$\lambda_5 a_{55} = 0, \qquad\qquad \lambda_6 a_{66} = 0,$$
$$\lambda_5 a_{66} = -\lambda_6 a_{56}, \qquad \lambda_6 a_{55} = -\lambda_5 a_{56},$$
$$a_{55}^2 = 0, \qquad\qquad a_{66}^2 = 0, \qquad\qquad a_{56}a_{56} = -2a_{55}a_{66},$$
$$a_{55}a_{56} = 0, \qquad\qquad a_{66}a_{56} = 0\,. \qquad\qquad\qquad\qquad (2)$$

Relations (2) are the super Plücker relations. We can write them in coordinates in the following way (always $1 \leq i < j < k \leq 4$ and $5 \leq n \leq 6$):

$$q_{12}q_{34} - q_{13}q_{24} + q_{14}q_{23} = 0, \quad \text{(Plücker relation)}$$

$$q_{ij}\lambda_{kn} - q_{ik}\lambda_{jn} + q_{jk}\lambda_{in} = 0,$$

$$\lambda_{in}\lambda_{jn} = a_{nn}q_{ij}, \qquad \lambda_{i5}\lambda_{j6} + \lambda_{i6}\lambda_{j5} = a_{56}q_{ij},$$

$$\lambda_{in}a_{nn} = 0, \qquad \lambda_{i5}a_{66} = -\lambda_{i6}a_{56} \qquad \lambda_{i6}a_{55} = -\lambda_{i5}a_{56}$$

$$a_{nn}^2 = 0 \qquad a_{55}a_{56} = 0, \qquad a_{66}a_{56} = 0$$

$$a_{56}a_{56} = -2a_{55}a_{66}. \tag{3}$$

For $N = 1$ these relations were given in [7], were the relation $a_{55}^2 = 0$ was missing but implicitly assumed. As we can see, for $N = 2$ extra relations appear. The super Plücker relations for arbitrary N are given in [5], page 17. They coincide with ours by a change in the notation due to the appearance of a sign, because of a different convention on row/column vectors, hence the consequent change of sign of the supertranspose.

We will denote as $\mathcal{I}_P$ the ideal generated by them in the affine superspace $\mathbb{A}^{9|8}$ (with generators $q_{ij}, a_{nm}, \lambda_{kn}$). They are homogeneous quadratic equations, so they are defined in the projective space $\mathbb{P}^{8|8}$.

Let us denote $\mathrm{Gr} = Gr(2|0, 4|2)$ and consider the super Plücker map

$$\mathrm{Gr} \longrightarrow \mathbb{P}^{8|8}$$

$$\mathrm{span}\{a, b\} \longrightarrow [a \wedge b],$$

We have the following

Proposition 2.1. *The superring associated to the image of* Gr *under the super Plücker embedding is*

$$\mathbb{C}[\mathrm{Gr}] \cong \mathbb{C}[q_{ij}, a_{nm}, \lambda_{kn}]/\mathcal{I}_P,$$

that is, the relations in $\mathcal{I}_P$ *are all the relations satisfied by the generators* $q_{ij}, a_{nm}, \lambda_{kn}$. *Then* $Gr(2|0, 4|2)$, *is a projective supervariety.*

Proof. This is proven for arbitrary N (and further generalizations) in [5], Theorem 6, denoted as the "algebraic case".

∎

3 The classical picture

As stated in the introduction, one can see $\mathbb{C}[\mathrm{Gr}]$ as a subalgebra of $\mathbb{C}[\mathrm{SL}(4|2)]$. Let us display the generators of this algebra in matrix form

$$\left(\begin{array}{cccc|cc} g_{11} & g_{12} & g_{13} & g_{14} & \gamma_{15} & \gamma_{16} \\ g_{21} & g_{22} & g_{23} & g_{24} & \gamma_{25} & \gamma_{26} \\ g_{31} & g_{32} & g_{33} & g_{34} & \gamma_{35} & \gamma_{36} \\ g_{41} & g_{42} & g_{43} & g_{44} & \gamma_{45} & \gamma_{46} \\ \hline \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & g_{55} & g_{56} \\ \gamma_{61} & \gamma_{62} & \gamma_{63} & \gamma_{64} & g_{65} & g_{66} \end{array}\right) \tag{4}$$

then

$$\mathbb{C}[\mathrm{SL}(4|2)] = \mathbb{C}[g_{ij}, g_{mn}, \gamma_{im}, \gamma_{nj}]/(\mathrm{Ber} - 1)\,,$$

where Ber is the Berezinian of the matrix and $1 \leq i, j \leq 4$ and $5 \leq m, n \leq 6$.

Proposition 3.1. *The superring*

$$\mathbb{C}[\mathrm{Gr}] \cong \mathbb{C}[q_{ij}, a_{nm}, \lambda_{kn}]/\mathcal{I}_P\,,$$

is generated, as a subring of $\mathbb{C}[\mathrm{SL}(4|2)]$ by the elements

$$y_{ij} = g_{i1}g_{j2} - g_{i2}g_{j1}, \qquad \eta_{kn} = g_{k1}\gamma_{n2} - g_{k2}\gamma_{n1}$$
$$x_{55} = \gamma_{51}\gamma_{52}, \qquad x_{66} = \gamma_{61}\gamma_{62} \qquad x_{56} = \gamma_{51}\gamma_{62} + \gamma_{61}\gamma_{52}$$

with the homomorphism

$$\begin{array}{ccc} \mathbb{C}[\mathrm{Gr}] & \longrightarrow & \mathbb{C}[\mathrm{SL}(4|2)] \\ q_{ij}, \lambda_{kn} & \longrightarrow & y_{ij}, \eta_{kn} \\ a_{55}, a_{66}, a_{56} & \longrightarrow & x_{55}, x_{66}, x_{56} \end{array}$$

Proof. The proof uses an argument similar to the one used to obtain (1). Instead of taking the vectors a and b we have to take the first two columns of the matrix (4). ∎

4 The big cell

A $(2|0)$ subspace of $\mathbb{C}^{4|2}$ is given as the linear span of two even vectors[2]

$$V(A) = \operatorname{span} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \\ u_4 & v_4 \\ \hline \mu_1 & \nu_1 \\ \mu_2 & \nu_2 \end{pmatrix} \qquad u_i, v_i \in A_0, \ \mu_i, \nu_i \in A_1 \,, \tag{5}$$

where A is any superalgebra. Clearly there is a right action of $\mathrm{GL}_2(A)$ over $V(A)$ (change of basis). We assume now that

$$\det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \quad \text{invertible in} \quad A_0 \tag{6}$$

This is a supervariety, which is an affine open set into the affine superspace $\mathbb{A}^{8|4}$. It is represented by the superring:

$$\mathbb{C}[S] = \mathbb{C}[a_{ij}, \alpha_{kl}][T]/((a_{11}a_{22} - a_{12}a_{21})T - 1), \ 1 \le i < j \le 4, \ 5 \le k < l \le 6$$

The condition of invertibility of the determinant function $a_{11}a_{22} - a_{12}a_{21}$ accounts for the condition in (6).

Then, using the right action of $\mathrm{GL}_2(A)$ we can bring (5) to the standard form

$$V(A) = \operatorname{span} \begin{pmatrix} \mathbb{1}_{2\times 2} \\ P_{2\times 2} \\ \psi^t_{2\times 2} \end{pmatrix} .$$

P and ψ are even and odd coordinates in the open subset of Gr characterized by (6) called the big cell of Gr. As for the $N = 1$ case, one can show that the subgroup of $\mathrm{SL}(4|2)$ that leaves invariant the big cell contains the (complexified) $N = 2$ super Poincaré group times the R-symmetry (dilations for $N = 1$). In fact, we call Gr the antichiral[3] conformal superspace, while the big cell is the antichiral Minkowski superspace. In this respect we do not

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[2]Here we are using implicitly the formalism of the *functor of points* to describe a super variety [29].

[3]See [8] to see why this space is the antichiral one.

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coincide with the notation of [11]-[17], where they call directly Minkowski superspace to the Grassmannian.

We do not extend here on this construction, but the condition (6) will be also used in the quantum setting.

5 The quantum Grassmannian

We follow [19] to define the quantum group $\mathrm{SL}_q(r|s)$.

Definition 5.1. The quantum matrix superalgebra $\mathrm{M}_q(r|s)$ is defined as

$$\mathrm{M}_q(r|s) =_{\text{def}} \mathbb{C}_q\langle z_{ij}, \xi_{kl}\rangle/\mathcal{I}_M$$

where $\mathbb{C}_q\langle z_{ij}, \xi_{kl}\rangle$ denotes the free superalgebra over $\mathbb{C}_q = \mathbb{C}[q, q^{-1}]$ generated by the even variables

$$z_{ij}, \qquad \text{for} \quad 1 \leq i, j \leq r \quad \text{or} \quad r+1 \leq i, j \leq r+s.$$

and by the odd variables

$$\xi_{kl} \qquad \begin{aligned} &\text{for} \quad 1 \leq k \leq r, \quad r+1 \leq l \leq r+s \\ &\text{or} \quad r+1 \leq k \leq r+s, \quad 1 \leq l \leq r, \end{aligned}$$

satisfying the relations $\xi_{kl}^2 = 0$ and $\mathcal{I}_M$ is an ideal that we describe below. We can visualize the generators as a matrix

$$\begin{pmatrix} z_{r\times r} & \xi_{r\times s} \\ \xi_{s\times r} & z_{s\times s} \end{pmatrix}. \tag{7}$$

It is convenient sometimes to have a common notation for even and odd variables.

$$a_{ij} = \begin{cases} z_{ij} & 1 \leq i, j \leq r, \text{ or} \quad r+1 \leq i, j \leq r+s, \\ \\ \xi_{ij} & 1 \leq i \leq r, \quad r+1 \leq j \leq r+s, \text{ or} \\ & r+1 \leq i \leq r+s, \quad 1 \leq j \leq r. \end{cases}$$

We assign a parity to the indices: $p(i) = 0$ if $1 \leq i \leq r$ and $p(i) = 1$ if $r+1 \leq i \leq r+s$. The parity of a_{ij} is $\pi(a_{ij}) = p(i) + p(j) \bmod 2$. Then, the ideal $\mathcal{I}_M$ is generated by the relations [19]:

$$a_{ij}a_{il} = (-1)^{\pi(a_{ij})\pi(a_{il})}q^{(-1)^{p(i)+1}}a_{il}a_{ij}, \qquad\qquad\qquad \text{for } j < l$$

$$a_{ij}a_{kj} = (-1)^{\pi(a_{ij})\pi(a_{kj})}q^{(-1)^{p(j)+1}}a_{kj}a_{ij}, \qquad\qquad\qquad \text{for } i < k$$

$$a_{ij}a_{kl} = (-1)^{\pi(a_{ij})\pi(a_{kl})}a_{kl}a_{ij}, \qquad\qquad\qquad \begin{aligned}&\text{for } i < k,\ j > l\\ &\text{or } i > k,\ j < l\end{aligned}$$

$$a_{ij}a_{kl} - (-1)^{\pi(a_{ij})\pi(a_{kl})}a_{kl}a_{ij} = (-1)^{\pi(a_{ij})\pi(a_{kl})}(q^{-1} - q)a_{kj}a_{il},$$
$$\text{for } i < k, j < l$$
$$(8)$$

■

There is also a comultiplication

$$\mathrm{M}_q(m|n) \xrightarrow{\ \Delta\ } \mathrm{M}_q(m|n) \otimes \mathrm{M}_q(m|n)$$

$\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$ and a counit $\varepsilon(a_{ij}) = \delta_{ij}$.

One can restrict to $\mathrm{SL}_q(r|s)$ by setting the quantum Berezinian to 1. The antipode is the usual one (see [19] or [8], Appendix E). Then $\mathrm{SL}_q(m|n)$ is a super Hopf algebra.

We can now define the quantum Grassmannian Gr_q mimicking Proposition 3.1:

Definition 5.2. The quantum super Grassmannian $\mathrm{Gr}_q := Gr_q(2|0, 4|2)$ is the subalgebra of $\mathrm{SL}_q(4|2)$ generated by the elements

$$D_{ij} := a_{i1}a_{j2} - q^{-1}a_{i2}a_{j1} \qquad D_{in} := a_{i1}a_{n2} - q^{-1}a_{i2}a_{n1}$$
$$D_{55} := a_{51}a_{52} \qquad\qquad D_{66} := a_{61}a_{62}$$
$$D_{56} = a_{51}a_{62} - q^{-1}a_{52}a_{61}$$

with $1 \leq i < j \leq 4$ and $n = 5, 6$.

■

We want to give a presentation in terms of generators and relations, as in Proposition 2.1 for the classical case. Note that, first of all, we have to compute the commutation rules among the D's. After some (tedious) calculations we arrive at:

- Let $1 \leq i, j, k, l \leq 6$ be not all distinct, and D_{ij}, D_{kl} not both odd. Then

$$D_{ij} D_{kl} = q^{-1} D_{kl} D_{ij}, \qquad (i,j) < (k,l), \ i < j, k < l, \qquad (9)$$

where the ordering '$<$' of pairs is the lexicographical ordering.

- Let $1 \leq i, j, k, l \leq 6$ be all distinct, and D_{ij}, D_{kl} not both odd and $D_{ij}, D_{kl} \neq D_{56}$. Then

$$
\begin{aligned}
D_{ij} D_{kl} &= q^{-2} D_{kl} D_{ij}, & 1 \leq i < j < k < l \leq 6, \\
D_{ij} D_{kl} &= q^{-2} D_{kl} D_{ij} - (q^{-1} - q) D_{ik} D_{jl} & 1 \leq i < k < j < l \leq 6, \\
D_{ij} D_{kl} &= D_{kl} D_{ij} & 1 \leq i < k < l < j \leq 6, \quad (10)
\end{aligned}
$$

- Let $1 \leq i < j \leq 4$, $5 \leq n \leq m \leq 6$. Then

$$
\begin{aligned}
D_{in} D_{jn} &= -q^{-1} D_{jn} D_{in} - (q^{-1} - q) D_{ij} D_{nn} = -q D_{jn} D_{in}, \\
D_{ij} D_{nm} &= q^{-2} D_{nm} D_{ij}, \\
D_{i5} D_{j6} &= -q^{-2} D_{j6} D_{i5} - (q^{-1} - q) D_{ij} D_{56}, \\
D_{i6} D_{j5} &= -D_{j5} D_{i6}, \\
D_{i5} D_{i6} &= -q^{-1} D_{i6} D_{i5}, \\
D_{i5} D_{i6} &= -q^{-1} D_{i6} D_{i5}, \\
D_{55} D_{66} &= q^{-2} D_{66} D_{55}, \\
D_{55} D_{56} &= 0.
\end{aligned}
\qquad (11)
$$

The Plücker relations are modified. One has for $1 \leq i < j < k \leq 4$ and $n = 5, 6$:

$$D_{12}D_{34} - q^{-1}D_{13}D_{24} + q^{-2}D_{14}D_{23} = 0,$$
$$D_{ij}D_{kn} - q^{-1}D_{ik}D_{jn} + q^{-2}D_{jk}D_{in} = 0,$$
$$D_{i5}D_{j6} + q^{-1}D_{i6}D_{j5} = qD_{ij}D_{56},$$
$$D_{in}D_{jn} = qD_{ij}D_{nn},$$
$$D_{in}D_{nn} = 0,$$
$$D_{i5}D_{66} = -q^{-1}D_{i6}D_{56},$$
$$D_{i6}D_{55} = -q^2 D_{i5}D_{56}$$
$$D_{nn}^2 = 0,$$
$$D_{55}D_{56} = 0,$$
$$D_{66}D_{56} = 0$$
$$D_{56}D_{56} = (q^{-1} - 3q)D_{55}D_{66}. \tag{12}$$

The first relation in (11) has been simplified with the use of the fourth relation in (12).

We have the following

Proposition 5.3. *The quantum Grassmannian superring* $\mathrm{Gr}_q = Gr_q(2|0, 4|2)$ *is given in terms of generators and relations as*

$$\mathrm{Gr}_q = \mathbb{C}_q\langle X_{ij}, X_{mn}, X_{im}\rangle, \mathcal{I}_q, \qquad 1 \leq i < j \leq 4; \quad 5 \leq m \leq n \leq 6,$$

where $\mathcal{I}_q$ is the ideal generated by the commutation relations (9),(10),(11) and the quantum super Plücker relations (12).

Proof. We give a sketch of the argument, whose idea is expressed in [20] Theorem 5.4 and also in [8] Chapter 4.

The super Plücker relations are all the relations satisfied by the quantum determinants: suppose that there is an extra relation R. Then $R = (q - 1)R^{(1)}$. Then $R^{(1)}$ may be of the form $R^{(1)} = (q - 1)R^{(2)}$ or $R^{(1)}$ mod $(q - 1)$ not identically 0. In the second case, since $\mathrm{SL}_q(m|n)$ is an algebra without torsion, we would have an additional classical Plücker relation, which cannot be. In the first case we have the same possibilities for $R^{(2)}$. At the end of the procedure we will obtain $R^{(n)} = 0$, that would be a new classical Plücker condition. But we know that this is not possible. ∎

6 The quantum super Grassmannian as a quantum homogeneous superspace

To finish the interpretation of the quantum super Grassmannian as an homogeneous superspace under the quantum supergroup $\mathrm{SL}_q(4|2)$ we have to see how it is the coaction on Gr_q. This is done in the following

Proposition 6.1. *The restriction of the comultiplication in* $\mathrm{SL}_q(4|2)$

$$\mathrm{SL}_q(4|2) \xrightarrow{\ \Delta\ } \mathrm{SL}_q(4|2) \otimes \mathrm{SL}_q(4|2)$$
$$a_{ij} \longrightarrow \Delta(a_{ij}) = \textstyle\sum_{k=1}^{6} a_{ik} \otimes a_{kj}$$

to the subalgebra Gr_q *is of the form*[4]

$$\mathrm{Gr}_q \xrightarrow{\ \Delta\ } \mathrm{SL}_q(4|2) \otimes \mathrm{Gr}_q .$$

Proof. The coaction property is guaranteed by the associativity of the coproduct, so we only have to check that

$$D_{ij}, D_{im}, D_{mn} \in \mathrm{SL}_q(4|2) \otimes \mathrm{Gr}_q .$$

Let us denote as $D_{ij}^{kl} = a_{ik}a_{jl} - q^{-1}a_{il}a_{jk}$, so in the previous notation $D_{ij} = D_{ij}^{12}$. After some calculations one can prove

1. Let us call P the condition $1 \leq k, l \leq 6$ and at least one of the two indices is less that 5. For $1 \leq i < j \leq 4$:

$$
\Delta(D_{ij}) = \sum_{P \cap (k<l)} D_{ij}^{kl} \otimes D_{kl}^{12} - \left(a_{i5}a_{j6} + q^{-1}a_{i6}a_{j5}\right) \otimes D_{56}
$$
$$
- (1 + q^{-2}) \sum_{5 \leq k \leq 6} a_{ik}a_{jk} \otimes D_{kk} .
$$

2. For $1 \leq i \leq 4$ and $5 \leq m \leq 6$:

---

[4]We denote with Δ both, the comultiplication and its restriction to Gr_q in order not to burden the notation. The meaning should be clear from the context.

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$$\Delta(D_{im}) = \sum_{\substack{k<5 \\ k<l}} a_{ik}a_{ml} \otimes D_{kl} - q^{-1} \sum_{\substack{k<5 \\ l<k}} a_{ik}a_{ml} \otimes D_{lk}$$

$$+ (a_{i5}a_{m6} + q^{-1}a_{i6}a_{m5}) \otimes D_{56}$$

$$+ (1+q^{-2}) \sum_{5 \leq k \leq 6} a_{ik}a_{mk} \otimes D_{kk} + q^{-1} \sum_{\substack{k \geq 5 \\ l<5}} a_{ik}a_{ml} \otimes D_{lk}\,.$$

3. For $5 \leq m, n \leq 6$:

$$\Delta(D_{56}) = \sum_{\substack{k<5 \\ k<l}} a_{5k}a_{6l} \otimes D_{kl} - q^{-1} \sum_{\substack{k<5 \\ l<k}} a_{5k}a_{6l} \otimes D_{lk}$$

$$+ (a_{55}a_{66} + q^{-1}a_{56}a_{65}) \otimes D_{56}$$

$$+ (1+q^{-2}) \sum_{5 \leq k \leq 6} a_{5k}a_{6k} \otimes D_{kk} + q^{-1} \sum_{\substack{k \geq 5 \\ l<5}} a_{5k}a_{6l} \otimes D_{lk}\,,$$

and

$$\Delta(D_{nn}) = \sum_{1 \leq k < l \leq 6} a_{nk}a_{nl} \otimes D_{kl} + \sum_{5 \leq k \leq 6} a_{nk}^2 \otimes D_{kk}\,.$$

This proves our statement.

■

7 Quantum super bundles: quantum chiral Minkowski superspace

In this section we want to reinterpret our construction in the framework of quantum principal bundles, as in [22] and references therein. We shall concentrate our attention on the local picture, that is, we want to look at the quantization of a super bundle $S \longrightarrow \mathbb{C}^{4|4}$, with base space the chiral Minkowski superspace $\mathbb{C}^{4|4}$, which we interpret as the big cell into the Grassmannian supermanifold Gr (see also Sec. 4).

We shall not develop a full theory of quantum principal super bundles, but we will recall the key definitions in order to put in the correct framework our construction.

We start with the classical definition.

Definition 7.1. Let X and M be topological spaces, P a topological group and $\wp : X \longrightarrow M$ a continuous function. We say that $(X, M, \wp, P)$ is a P-*principal bundle* (or *principal bundle* for short) with total space X and base M, if the following conditions hold

1. $\wp$ is surjective.

2. P acts freely from the right on X.

3. P acts transitively on the fiber $\wp^{-1}(m)$ of each point $m \in M$.

4. X is locally trivial over M, i.e. there is an open covering $M = \cup U_i$ and homeomorphisms $\sigma_i : \wp^{-1}(U_i) \longrightarrow U_i \times P$ that are P-equivariant i.e., $\sigma_i(up) = \sigma_i(u)p$, $u \in U_i$, $p \in P$.

If $X \cong M \times P$ we say that the bundle is globally trivial.

∎

We can then define algebraic, analytic or smooth P-principal bundles, by the taking objects and morphisms in the appropriate categories. There is clearly no obstacle in writing the same definition in the super context, provided we exert some care in the definition of surjectivity (see [30], Section 8.1 for details). We would like, however, to take a different route.

We turn to the notion of Hopf-Galois extension, that is most fruitful for the quantization. Our definition in the super category is the same as for the ordinary one (see [27] for more details in the ordinary category).

Definition 7.2. Let (H, Δ, ϵ, S) be a Hopf superalgebra and A be an H-comodule superalgebra with coaction $\delta : A \longrightarrow A \otimes H$. Let

$$B := A^{\text{coinv } H} := \{a \in A \,|\, \delta(a) = a \otimes 1\} \,. \tag{13}$$

The extension A of the superalgebra B is called H-Hopf-Galois (or simply Hopf-Galois) if the map

$$\chi : A \otimes_B A \longrightarrow A \otimes H, \qquad \chi = (m_A \otimes \text{id})(\text{id} \otimes_B \delta)$$

called the canonical map, is bijective (m_A denotes the multiplication in A).

The extension $B = A^{\text{coinv } H} \subset A$ is called H-*principal comodule super-algebra* if it is Hopf-Galois and A is H-equivariantly projective as a left B-supermodule, i.e., there exists a left B-supermodule and right H-comodule

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morphism $s : A \rightarrow B \otimes A$ that is a section of the (restricted) product $m : B \otimes A \rightarrow A$.

In particular if H is a Hopf algebra with bijective antipode over a field, the condition of equivariant projectivity of A is equivalent to that of faithful flatness of A (see [31], [32]). ∎

We now follow [22] Sec. 2, in giving the definition of quantum principal bundle, though it differs slightly from the one given in the literature, which also requires the existence of a differential structure (see e.g. [33] Ch. 5). We plan to explore such structures in a forthcoming paper.

Definition 7.3. We define *quantum principal bundle* a pair (A, B), where A is an H-Hopf Galois extension and A is H-equivariantly projective as a left B-supermodule, that is, A is an H-principal comodule superalgebra.

∎

In the ordinary case, the notion of affine H-principal bundle coincides with Definition 7.1 when we take $H = \mathcal{O}(P)$, $A = \mathcal{O}(X)$ and $B = \mathcal{O}(M)$, where $\mathcal{O}(X)$ denotes the algebra of functions on X (algebraic, differential, holomorphic, etc). The Hopf-Galois condition is equivalent to saying that the action of P on X is free and the equivariance property means that the bundle is locally trivial.

We assume, in the algebraic setting, that all our varieties are affine.

There is a special case of Hopf-Galois extensions, corresponding to a globally trivial principal bundle. In this case the technical conditions of Definition 7.3 are automatically satisfied. We shall focus on this case leaving aside the general one.

Definition 7.4. Let H be a Hopf superalgebra and A an H-comodule superalgebra. The algebra extension $A^{\mathrm{coinv}\, H} \subset A$ is called a *cleft extension* if there is a right H-comodule map $j : H \rightarrow A$, called *cleaving map*, that is convolution invertible, i.e. there exists a map $h : H \rightarrow A$ such that the convolution product $j \star h$ satisfies,

$$j \star h := m_A \circ (j \otimes h) \circ \Delta(f) = \epsilon(f) \cdot 1$$

or, in Sweedler notation $\Delta(f) = \sum f_1 \otimes f_2$,

$$\sum j(f_1) h(f_2) = \epsilon(f) \cdot 1$$

for all $f \in H$. The map h is the convolution inverse of j.

An extension $A^{\mathrm{coinv}\,H} \subset A$ is called a trivial extension if there exists such map.

Notice that when j is an algebra map, its convolution inverse is just $h = j \circ S^{-1}$.

$\blacksquare$

A trivial extension is actually a Hopf-Galois extension and a principal bundle. When j is an algebra map, we have an algebra isomorphism $A \cong B \# H$ (see [27], Sections 4.1 and 7.2 for the smash product '$\#$'), which in the classical case means that we have a trivial bundle (see [27] Chapter 8 and [33] Sec. 5.1.2).

We now examine an example with physical significance coming from our previous treatment. Consider the set of $4 \times 2 \,|\, 2 \times 2$ supermatrices with complex entries

$$
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
a_{31} & a_{32} \\
a_{41} & a_{41} \\
\alpha_{51} & \alpha_{52} \\
\alpha_{61} & \alpha_{62}
\end{pmatrix}. \tag{14}
$$

This can be seen as the affine superspace $\mathbf{A}^{8|4}$ described by the coordinate superalgebra $\mathbb{C}[a_{ij}, \alpha_{kl}]$ with $i = 1, \ldots, 4$, $j, l = 1, 2$, $k = 5, 6$. As in the ordinary setting, we can view elements in $\mathbf{A}^{8|4}$ as $2|0$ subspaces of $\mathbb{C}^{4|2}$:

$$
W = \mathrm{span}\{a_1, a_2\} \subset \mathbb{C}^{4|2}
$$

In this way, W may also be viewed as an element in Gr.

In the superspace $\mathbf{A}^{8|4}$ consider the open subset S consisting of matrices such that the minor formed with a_{ij}, $i, j = 1, 2$ is invertible. This open subset S is described by its coordinate superalgebra:

$$
\mathbb{C}[S] = \mathbb{C}[a_{ij}, \alpha_{kl}][T]/((a_{11}a_{22} - a_{12}a_{21})T - 1)
$$

We have a right action of $\mathrm{GL}_2(\mathbb{C})$ on S corresponding to the change of basis of such subspaces:

$$
\mathrm{span}\{a_1, a_2\}, g \mapsto \mathrm{span}\{a_1 \cdot g, a_2 \cdot g\}, \qquad g \in \mathrm{GL}_2(\mathbb{C}).
$$

Proposition 7.5. *Let the notation be as above. The quotient of S by the right $\mathrm{GL}_2(\mathbb{C})$ action is an affine superspace $\mathbb{M}$ of dimension $4|4$, the $N = 2$ chiral Minkowski superspace $\mathbb{M}$.*

Proof. We can write:

$$\mathbb{M} = \left\{ (a_1, a_2),\, a_1, a_2 \in \mathbb{C}^{4|2} \mid \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0 \right\} \Big/ \mathrm{GL}_2(\mathbb{C}) \tag{15}$$

In the quotient $\mathbb{M}$ we can choose a (unique) representative (u, v) for (a_1, a_2) of the form:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ u_1 \\ u_2 \\ \nu_3 \\ \nu_4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ v_1 \\ v_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} \right\}, \tag{16}$$

so $\mathbb{M}$ is $\mathbb{C}^{4|4}$.

∎

We notice that $\mathbb{M}$ is naturally identified with the dense open set of the Grassmannian Gr in the Plücker embedding, determined by the invertibility of the coordinate q_{12} in $\mathbb{P}^{8|8}$.

We now would like to retrieve a set of global coordinates for $\mathbb{M}$ starting from the global coordinates a_{ij} for S. Let $\mathbb{C}[\mathrm{GL}_2] = \mathbb{C}[g_{ij}][T]/((g_{11}g_{22} - g_{12}g_{21})T - 1)$ be the coordinate algebra for the algebraic group $\mathrm{GL}_2(\mathbb{C})$. Let us write heuristically the equation relating the generators of $\mathbb{C}[S]$, $\mathbb{C}[\mathrm{GL}_2]$ and the polynomial superalgebra $\mathbb{C}[\mathbb{M}] := \mathbb{C}[u_{ij}, \nu_{kl}]$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ \alpha_{51} & \alpha_{52} \\ \alpha_{61} & \alpha_{62} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ u_{31} & u_{32} \\ u_{41} & u_{42} \\ \nu_{51} & \nu_{52} \\ \nu_{61} & \nu_{62} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{17}$$

We obtain immediately:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and then with a short calculation,

$$u_{i1} = -d_{2i}d_{12}^{-1} \qquad u_{i2} = d_{1i}d_{12}^{-1}$$
$$\nu_{k1} = -d_{2k}d_{12}^{-1} \qquad \nu_{k2} = d_{1k}d_{12}^{-1}$$

for $i = 3, 4$ and $k = 5, 6$, where

$$d_{rs} := a_{r1}a_{s2} - a_{r2}a_{s1} \qquad r < s\,.$$

Proposition 7.6. *Let the notation be as above.*

1. *The complex supermanifold S is diffeomorphic to the supermanifold $\mathbb{C}^{4|4} \times \mathrm{GL}_2(\mathbb{C})$:*

$$S \xrightarrow{\ \psi\ } \mathbb{C}^{4|4} \times \mathrm{GL}_2(\mathbb{C})\,,$$

   with

$$\psi^*(g_{ij}) = a_{ij}$$
$$\psi^*(u_{i1}) = -d_{2i}d_{12}^{-1}, \qquad\qquad \psi^*(u_{i2}) = d_{1i}d_{12}^{-1}.$$
$$\psi^*(\nu_{k1}) = -d_{2k}d_{12}^{-1} \qquad\qquad \psi^*(\nu_{k2}) = d_{1k}d_{12}^{-1}$$

2. *The diffeomorphism ψ is $\mathrm{GL}_2(\mathbb{C})$-equivariant with respect to the right $\mathrm{GL}_2(\mathbb{C})$ action, hence $S/\mathrm{GL}_2(\mathbb{C}) \cong \mathbb{C}^{4|4}$.*

Proof. We notice that ψ is invertible, ψ^{-1} is given by:

$$(\psi^{-1})^*(a_{ij}) = g_{ij}$$

and the rest follows from equation (17).

The right equivariance of ψ is a simple calculation, taking into account that the determinants d_{ij} transform as $d_{ij} \det g'$, were $g' \in \mathrm{GL}_2(\mathbb{C})$. ∎

Lemma 7.7. *The coordinate superalgebra $\mathbb{C}[\mathbb{M}] := \mathbb{C}[u_{ij}, \nu_{kl}]$ is isomorphic to $\mathbb{C}[S]^{\mathrm{coinv}\,\mathbb{C}[\mathrm{GL}_2]}$ the coinvariants in $\mathbb{C}[S]$ with respect to the $\mathbb{C}[\mathrm{GL}_2]$ right coaction δ:*

$$
\mathbb{C}[S] \xrightarrow{\ \delta\ } \mathbb{C}[S] \otimes \mathbb{C}[\mathrm{GL}_2]
$$

$$
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
a_{31} & a_{32} \\
a_{41} & a_{42} \\
\alpha_{51} & \alpha_{52} \\
\alpha_{61} & \alpha_{62}
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
a_{31} & a_{32} \\
a_{41} & a_{42} \\
\alpha_{51} & \alpha_{52} \\
\alpha_{61} & \alpha_{62}
\end{pmatrix}
\otimes
\begin{pmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{pmatrix}
$$

Proof. In our heuristic calculation we computed expressions for the co-ordinates on $\mathbb{M}$. We claim that these are coinvariants, so we need to show $\delta(c) = c \otimes 1$ for any $c \in \{u_{ij}, \nu_{kl}\}$. A little calculation gives us

$$\delta(d_{rs}) = d_{rs} \otimes (g_{11}g_{22} - g_{12}g_{21}) \;\Rightarrow\; \delta(d_{rs}d_{12}^{-1}) = d_{rs}d_{12}^{-1} \otimes 1$$

which proves our claim.

On the other hand, the space of functions on S that are invariant under the right translation of the group can be identified with the space of functions on the quotient $S/\mathrm{GL}_2(\mathbb{C})$. Since we have global coordinates in $\mathbb{M}$, any other invariant will be a function of these coordinates. In the Hopf algebra language, this means that $\{u_{ij}, \nu_{kl}\}$ are the only independent coinvariants.

∎

Proposition 7.8. *Let the notation be as above. The natural projection $p : S \longrightarrow S/\mathrm{GL}_2(\mathbb{C})$ is a trivial principal bundle on the chiral superspace.*

Proof. We will show that $\mathbb{C}[S]$ is a trivial $\mathbb{C}[\mathrm{GL}_2]$-Hopf Galois extension of $\mathbb{C}[\mathbb{M}]$. In our previous lemma we proved that $\mathbb{C}[\mathbb{M}] \cong \mathbb{C}[S]^{\mathrm{coinv}\,\mathbb{C}[\mathrm{GL}_2]}$, so if we give an algebra cleaving map we will have proven our proposition. We define

$$\mathbb{C}[\mathrm{GL}_2] \xrightarrow{\;\;j\;\;} \mathbb{C}[S]$$

$$g_{ij} \longrightarrow a_{ij}\,.$$

We leave to the reader the easy check that j is convolution invertible with convolution inverse

$$h = j \circ S\,.$$

Moreover, the calculation below shows that j is a $\mathbb{C}[\mathrm{GL}_2(\mathbb{C})]$-comodule map,

$$(\delta \circ j)(g_{ij}) = \delta(a_{ij}) = \sum a_{ik} \otimes g_{kj}.$$

$$((j \otimes \mathrm{id}) \circ \Delta)(g_{ij}) = (j \otimes \mathrm{id})\left(\sum g_{ik} \otimes g_{kj}\right) = \sum a_{ik} \otimes g_{kj}.$$

$$\Rightarrow \quad \delta \circ j = (j \otimes \mathrm{id}) \circ \Delta.$$

This proves the result.

∎

We now go to the quantum setting, where we lose the geometric interpretation and we retain only the algebraic point of view. Hence a quantum principal super bundle over an affine base is just understood as a Hopf-Galois extension with the properties mentioned in Definition 7.3.

We want to study the quantization of the example studied above. Let $\mathbb{C}_q[S]$ be the quantization of the superalgebra $\mathbb{C}[S]$ obtained by taking the Manin relations (8) among the entries still denoted as a_{ij} and α_{kl}, with $i, j = 1, \ldots, 4$ and $k, l = 5, 6$.

Definition 7.9. The $N = 2$ quantum chiral Mikowski superspace, $\mathbb{C}_q[\mathbb{M}]$, is the superalgebra generated by the elements

$$\tilde{u}_{i1} = -q^{-1}D_{2i}D_{12}^{-1}, \qquad \tilde{u}_{i2} = D_{1i}D_{12}^{-1},$$
$$\tilde{\nu}_{k1} = -q^{-1}D_{2k}D_{12}^{-1}, \qquad \tilde{\nu}_{k2} = D_{1k}D_{12}^{-1},$$

for $i = 3, 4$ and $k = 5, 6$ in $\mathbb{C}_q[\text{Gr}]$, where

$$D_{rs} := a_{r1}a_{s2} - q^{-1}a_{r2}a_{s1}, \qquad r < s.$$

■

Using our previous computations for commutation relations among D_{rs}'s we get the following commutation relations among $\tilde{u}_{ij}$'s and $\tilde{\nu}_{kl}$'s:

$$
\begin{aligned}
&\tilde{u}_{i2}\tilde{u}_{i1} = q^{-1}\tilde{u}_{i1}\tilde{u}_{i2}, && i = 3, 4, \\
&\tilde{\nu}_{k1}\tilde{\nu}_{k2} = -q^{-1}\tilde{\nu}_{k2}\tilde{\nu}_{k1}, && k = 5, 6, \\
&\tilde{\nu}_{5l}\tilde{\nu}_{6l} = -q^{-1}\tilde{\nu}_{6l}\tilde{\nu}_{5l}, && l = 1, 2, \\
&\tilde{u}_{3j}\tilde{u}_{4j} = q^{-1}\tilde{u}_{4j}\tilde{u}_{3j}, && j = 1, 2, \\
&\tilde{u}_{ij}\tilde{\nu}_{kj} = q^{-1}\tilde{\nu}_{kj}\tilde{u}_{ij}, && j = 1, 2 \quad i = 3, 4 \quad k = 5, 6, \\
&\tilde{u}_{i1}\tilde{\nu}_{k2} = \tilde{\nu}_{k2}\tilde{u}_{i1}, && i = 3, 4 \quad k = 5, 6, \\
&\tilde{u}_{31}\tilde{u}_{42} = \tilde{u}_{42}\tilde{u}_{31}, && \\
&\tilde{\nu}_{51}\tilde{\nu}_{62} = -\tilde{\nu}_{62}\tilde{\nu}_{51}, && \\
&\tilde{u}_{32}\tilde{u}_{41} - \tilde{u}_{41}\tilde{u}_{32} = (q^{-1} - q)\tilde{u}_{42}\tilde{u}_{31}, && \\
&\tilde{u}_{i2}\tilde{\nu}_{k1} - \tilde{\nu}_{k1}\tilde{u}_{i2} = (q^{-1} - q)\tilde{\nu}_{k2}\tilde{u}_{i1}, && i = 3, 4 \quad k = 5, 6, \\
&\tilde{\nu}_{52}\tilde{\nu}_{61} + \tilde{\nu}_{61}\tilde{\nu}_{52} = -(q^{-1} - q)\tilde{\nu}_{62}\tilde{\nu}_{51} \,. && (18)
\end{aligned}
$$

We have the following

Proposition 7.10. *The quantum chiral Minkowski superspace $\mathbb{C}_q[M]$ is iso-morphic to the quantum superalgebra of matrices $\mathrm{M}_q(2|2)$ (Definition 5.1).*

Proof. We define the map $\beta : \mathbb{C}_q[M] \longrightarrow \mathrm{M}_q(2|2)$ by giving it on the generators as follows:

$$\beta(\tilde{u}_{ij}) := z_{rs} \qquad \text{where} \ \ r = i - 2 \ \ \text{and} \ \ s = \begin{cases} 1 \ \ \text{if} \ \ j = 2, \\ 2 \ \ \text{if} \ \ j = 1, \end{cases}$$

$$\beta(\tilde{\nu}_{kl}) := \xi_{mn} \qquad \text{where} \ \ m = k - 2 \ \ \text{and} \ \ n = \begin{cases} 1 \ \ \text{if} \ \ l = 2, \\ 2 \ \ \text{if} \ \ l = 1. \end{cases}$$

It is clearly bijective. Comparing the commutation relations (8) with (18) it follows that β is an isomorphism.

∎

We now want to show the main result for this section.

Theorem 7.11. *The quantum superalgebra $\mathbb{C}_q[S]$ is a trivial quantum principal super bundle on the quantum chiral Minkowski superspace.*

Proof. We need to show that $\mathbb{C}_q[S]$ is a trivial Hopf-Galois extension of $\mathbb{C}_q[M]$. We will proceed similarly to the classical case. It is easy to see that the quantum version of Lemma 7.7 also holds. It is enough to check that

$$\delta_q(D_{rs}) = D_{rs} \otimes \left(g_{11}g_{22} - q^{-1}g_{12}g_{21}\right).$$

Therefore, we need to give an algebra cleaving map $j_q : \mathbb{C}_q[\mathrm{GL}_2(\mathbb{C})] \longrightarrow \mathbb{C}_q[S]$. Define:

$$j_q(g_{ij}) := a_{ij}, \qquad h_q = j_q \circ S_q,$$

so $h_q : \mathbb{C}_q[\mathrm{GL}_2(\mathbb{C})] \longrightarrow \mathbb{C}_q[S]$,

$$h_q(g_{11}) := D^{-1}a_{22} \qquad h_q(g_{12}) := -qD^{-1}a_{12}$$

$$h_q(g_{21}) := -q^{-1}D^{-1}a_{21} \qquad h_q(g_{22}) := D^{-1}a_{11},$$

where $D := a_{11}a_{22} - q^{-1}a_{12}a_{21}$. One can observe that:

$$j_q \star h_q = \varepsilon.1 = h_q \star j_q,$$

where $\star$ denotes the convolution product, i.e j_q is convolution invertible. Moreover, a similar calculation to the one given in Proposition 7.8 shows that j_q is a $\mathbb{C}_q[\mathrm{GL}_2]$-comodule map, i.e. $\delta_q \circ j_q = (j_q \otimes \mathrm{id}) \circ \Delta$. Therefore, j_q is an algebra cleaving map and $\mathbb{C}_q[M] \subset \mathbb{C}_q[S]$ is a trivial extension. Hence the theorem is proven.

∎

Acknowledgements

References

[1] Y. Manin. *Gauge field theory and complex geometry.* Springer Verlag, (1988). (Original Russian edition in 1984).

[2] V. S. Varadarajan, *Supersymmetry for mathematicians: an introduction.* Courant Lecture Notes, 1. AMS (2004).

[3] R. Penrose. *Twistor algebra.* Journal of Mathematical Physics. **8** 345 - 366 (1967).

[4] A. Ferber. *Supertwistors and conformal supersymmetry.* Nuclear Physics B. **132** 55 - 64 (1978).

[5] E. Shemyakova, T. Voronov. *On super Plücker embedding and cluster algebras.* aXiv:1906.12011 [math.DG]. To appear in Selecta Mathematica.

[6] R. Fioresi, E. Latini, M. A. Lledó, F. A. Nadal, *The Segre embedding of the quantum conformal superspace.* Advances in Theoretical and Mathematical Physics **22**, n.8, 1939 – 2000 (2018).

[7] D. Cervantes, R. Fioresi, M. A. Lledo, *The quantum chiral Minkowski and conformal superspaces*, Advances in Theoretical and Mathematical Physics. **15** no.2 565 - 620 (2011).

[8] R. Fioresi, M. A. Lledó, *The Minkowski and conformal superspaces: the classical and the quantum pictures*, World Scientific Publishing, Singapore, (2015).

[9] A. S. Galperin, E. A. Ivanov, S. Kalitzin, V. I. Ogievetsky, E. S. Sokatchev, *Unconstrained $N = 2$ matter, Yang Mills and supergravity theories in harmonic superspace.* Classical and Quantum Gravity **1** 469 - 498(1984)

[10] A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky, E. S. Sokatchev, *Harmonic superspace.* Cambridge University Press. Cambridge (2001).

[11] P. S. Howe, M. I. Leeming, *Harmonic superspaces in low dimensions.* Classical and Quantum Gravity **11** 2843 - 2853 (1994).

[12] G. G. Hartwell, P. S. Howe, *A superspace survey.* Classical and Quantum Gravity **12** 1823 - 1881 (1995).

[13] G. G. Hartwell, P. S. Howe, *(N, p, q) harmonic superspace.* International Journal of Modern Physics A **10** 3901-3920 (1995).

[14] P. Heslop, P. S. Howe, *On harmonic superspaces and superconformal fields in four dimensions.* Classical and Quantum Gravity **17** 3743 - 3768 (2000).

[15] P. Heslop, P. S. Howe, *Harmonic Superspaces and Superconformal Fields.* JHEP Proceedings 'Nonperturbative Quantum Effects 2000' (2000).

[16] P. Heslop, *Superfield representations of superconformal groups.* Classical and Quantum Gravity **19** 303 - 345 (2002).

[17] G. G. Hartwell, P. S. Howe, *(N, p, q) harmonic superspace.* International Journal of Modern Physics A **10** 3901-3920 (1995).

[18] A. Menkara. *Superprojective embeddings of extended supersymmetry.* Master Thesis. Máster en Física Avanzada, Universitat de València (2018).

[19] Y. Manin, *Multiparametric quantum deformation of the general linear supergroup.* Communications in Mathematical Physics. **123** (1989) 163-175.

[20] R. Fioresi, *Quantum deformation of the grassmannian manifold*, Journal of Algebra, **214** no. 2 (1999) 418-447.

[21] N. Ciccoli, R. Fioresi, F. Gavarini, *Quantization of Projective Homogeneous Spaces and Duality Principle*, J. Noncommut. Geom., 2 (2008) 449–496.

[22] P. Aschieri, R. Fioresi, E. Latini, *Quantum Principal Bundles on Projective Bases*, Communications in Mathematical Physics, **382**(3) 1-34 (2021)

[23] P. M. Hajac, *Strong Connections on Quantum Principal Bundles*, Communications in Mathematical Physic. **182**(3) 579-617 (1996).

[24] P. M. Hajac, W. Pusz, Z. Skoda, *Localizations for Construction of Quantum Coset Spaces Noncommutative Geometry and Quantum Groups*, Warszawa, Polish Academy of Sciences, Banach Center Publications **61** 2003.

[25] P. M. Hajac, U. Krähmer, R. Matthes, B. Zielinski, *Piecewise principal comodule algebras* Journal of Noncommutative Geometry **5** 591-614 (2011).

[26] R. Ó Buachalla, *Quantum bundle description of quantum projective spaces*, Communications in Mathematical Physics. **316** 345-373 (2012).

[27] S. Montgomery, *Hopf Algebras and their actions on rings*, CBMS **82**, American Mathematical Society, Providence, RI, (1993).

[28] E. Keßler, *Supergeometry, super Riemann surfaces and the superconformal action functional.* Lecture Notes in Mathematics. Springer (2020).

[29] P. Deligne, J.W. Morgan *Notes on Supersymmetry (following J. Bernstein)* in *Quantum fields and strings: a course for mathematicians.* (P. Deligne, P. Etingof, D. S. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D.R. Morrison and E. Witten, eds.) Vol 1. American Mathematical Society, Providence, (1999).

[30] C. Carmeli, L. Caston and R. Fioresi *Mathematical foundations of supersymmetry*, with an appendix with I. Dimitrov, EMS Serie soif Lectures in Mathematics. European Mathematical Society, Zurich (2011).

[31] P. Schauenburg and H.J. Schneider, *On generalized Hopf Galois extensions*, J. Pure App. Algebra, 202, (2005), 168-194.

[32] P. Aschieri, P. Bieliavsky, C. Pagani, A. Schenkel, *Noncommutative Principal Bundles Through Twist Deformation*, Comm. Math. Phys., Vol. 352 (2017) 287-344.

[33] E. J. Beggs, S. Majid, *Quantum Riemannian Geometry*, Springer, 2020.